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A tree of n vertices has $n-1$ edges.

Proof We prove the theorem by induction on n. For $n=1$, the statement is obvious. Suppose that the statement is true for any tree of $n-1$ vertices. Consider a tree T of n vertices. Delete a vertex v of degree one. The resulted graph $T[V(T)-\{v\}]$ is still acyclic and connected since for any two vertices x and y in $V(T)-\{v\}$, the path joining x and y in T could not use vertex v. Thus $T-v$ is a tree and has $n-2$ edges by the inductional hypothesis. Hence $|E(T)|=(n-2)+1=n-1$.

Trees, alternate definitions

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Theorem

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Proof We prove the theorem by induction on n. For $n=1$, the statement is trivial. Suppose that the statement is true for any graph of $n-1$ vertices and let G be a connected graph of n vertices and $n-1$ edges where $n \geq 2$. Then G does not contain any isolated vertex and so $d(x) \geq 1$ for $x \in V(G)$. On the other side, $\sum_{x \in V(G)} d(x) = 2n-2$ and so G contains at least two vertices of degree one. Deleting one of them, we obtain a connected graph of $n-1$ vertices and $n-2$ edges which is acyclic by the inductional hypothesis. Since a vertex of degree one cannot be contained in a cycle, it implies that G is acyclic, as well, and so G is a tree.

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Proof Let G be an acyclic graph of n vertices and $n-1$ edges. Its components $G_1, G_2, \ldots, G_p$ are trees and so $|E(G_i)| = |V(G_i)| - 1$. Thus

$$|E(G)| = \sum_{i=1}^{p} |E(G_i)| = \sum_{i=1}^{p} (|V(G_i)| - 1) = n - p \ ,$$

which implies that $p = 1$ and so G is connected.

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Notice a forest is a graph whose every component is a tree.

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Proof Notice that a connected spanning subgraph of G with minimum number of edges is a spanning tree.

Labelled graphs

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We speak about *labeled graphs* if the vertices of the graphs are labeled with, say, the first n positive integers. In that case two — otherwise isomorphic — graphs will be distinguishable simple due to the fact that there edges run between vertices of different labels. It turns out to be much easier to give the number of labeled graphs with a certain property then the same question for simple (non-labeled) graphs.

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A rather more difficult question is the number of labeled trees. The answer is given by the Cayley Formula, which we give below.
Problem. How many different trees do we have on n labeled vertices?

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Lemma

Assume that $d_1, d_2, \ldots, d_n$ are all ≥ 1 and $\sum_{i=1}^{n} d_i = 2n-2$. Then the number of trees on the given (labeled) vertices $\{v_1, v_2, \ldots, v_n\}$ such that vertex v_i has degree d_i is equal to $\frac{(n-2)!}{(d_1-1)!\cdots(d_n-1)!}$.

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Proof Use induction on n. Base cases are trivial. Since $\sum_{i=1}^{n} d_i = 2n - 2 < 2n$ we must have a vertex of degree one, say $d_i = 1$.

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$$\frac{(n-3)!}{(d_1-1)!\cdots(d_{j-1}-1)!(d_j-2)!(d_{j+1}-1)!\cdots(d_{n-1}-1)!} = \tag{1}$$

$$= \frac{(d_j-1)(n-3)!}{(d_1-1)!\cdots(d_n-1)!}. \tag{2}$$

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The formula is valid for $d_j = 0$ as well.

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$$\begin{aligned}\sum_{j=1}^{n-1} \frac{(d_j - 1)(n-3)!}{(d_1 - 1)! \cdots (d_n - 1)!} &= \left(\sum_{j=1}^{n-1} (d_j - 1)\right) \frac{(n-3)!}{(d_1 - 1)! \cdots (d_n - 1)!} \\ &= \frac{(n-2)(n-3)!}{(d_1 - 1)! \cdots (d_n - 1)!}\end{aligned} \tag{3}$$

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Theorem

*(***Cayley Formula) The number of labeled trees on n vertices is n^{n-2}.**

Proof It is equal to

$$\sum_{\substack{d_1,\ldots,d_n\geq 1\\ d_1+\cdots+d_n=2n-2}} \frac{(n-2)!}{(d_1-1)!\cdots(d_n-1)!} = \sum_{\substack{k_1,\ldots,k_n\geq 0\\ k_1+\cdots+k_n=n-2}} \frac{(n-2)!}{k_1!\cdots k_n!}$$

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$$= \underbrace{(1+1+\cdots+1)}^{n-2} = n^{n-2}. \quad (4)$$