COMBINATORICS I.

Based on:
F. Roberts: Applied Combinatorics,
L. Lovász: Combinatorial Problems and Exercises and on
the handouts of E. Győri and D. Miklós

With Appendices of previous BSM students:
Eli Melaas: An analysis of generating functions
Chris Domenicali: Linear homogeneous recurrence relations
Halcyon Derks: Principle of Inclusion/exclusion and the sieve

1. BASIC COUNTING RULES

Product rule: If something can happen in n_1 ways, and no matter how the first thing happens, a second thing can happen in n_2 ways, and so on, no matter how the first $k-1$ things happen, a k-th thing can happen in n_k ways, then all the k things together can happen in $n_1 \times n_2 \times \ldots \times n_k$ ways.

Example 1. A local telephone number is given by a sequence of six digits. How many different telephone numbers are there if the first digit cannot be 0?

Answer: $9 \times 10 \times 10 \times 10 \times 10 \times 10 = 900,000$.

Example 2. The population of a town is 30,000. If each resident has three initials; is it true that there must be at least two individuals with the same initials?

Answer: Yes, since $30,000 > 26 \times 26 \times 26$.

Example 3. The number of subsets of an n-set is 2^n. (First we decide if the first element of the n set belongs to the subset or not, then we decide if the second element of the n-set belongs to the subset or not, etc.)

Sum rule: If one event can occur in n, ways, a second event can occur in n_2 (different) ways, and so on, a k-th event can occur in n_k (still different) ways then (exactly) one if the events can occur in $n_1 + n_2 + \ldots + n_k$ ways.

Example 4. A committee is to be chosen from among 8 mathematicians, 10 physicists, 12 physicians. If the committee is to have two members of different backgrounds, how many such committees can be chosen?

Answer: $8 \times 10 + 8 \times 12 + 10 \times 12 = 296$.

Example 5. See Ex. 2 if each resident has one, two or three initials.

Answer: Yes, since $30,000 > 26 + 26 \times 26 + 26 \times 26 \times 26$.

A *permutation* of a set of n elements is an arrangement of the elements of the set in order. The number of permutations of an n-set is given by

$$n \times (n-1) \times \ldots \times 1 = n! \qquad \text{(product rule)}.$$

Example 6. How many permutations of $\{1, 2, 3, 4, 5\}$

(a) are there? $(5!)$;
(b) begin with 5? $(4!)$;
(c) begin with an odd number? $(3 \times 4!)$.

Given an n-set, suppose that we want to pick out r elements and arrange them in order. Such an arrangement is called an *r-permutation* of the n-set. The number $P(n, r)$ or r-permutations of an n-set is given by

$$n \times (n-1) \times \ldots \times (n-r+1) \qquad \text{(product rule)}.$$

Example 7. Let $A = \{0, 1, 2, 3, 4, 5, 6\}$.

(a) Find the number of sequences of length 3 using elements of A.
(b) Repeat (a) if no element of A is to be used twice.
(c) Repeat (a) if the first element of the sequence is 4.
(d) Repeat (a) if the first element of the sequence is 4 and no element of A is used twice.

Answers: (a) $7 \times 7 \times 7$; (b) $7 \times 6 \times 5$; (c) $(1\times)7 \times 7$; (d) $(1\times)6 \times 5$.

An r-combination of an n-set is a selection of r elements from the set. Order does not count. (I.e. an r-combination is an r-element subset.) $\binom{n}{r}$ will denote the number of r-combinations of an n-set. Notice that $P(n, r) = \binom{n}{r} \times r!$ (product rule) and so $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Theorem 1.

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Proof. Notice that the number of r-subsets of an n-set that contains the "first" element of the n-set is $\binom{n-1}{r-1}$ and the number of r- subsets not containing the "first" element is $\binom{n-1}{r}$. Hence the sum rule yields the desired equality.

(You may prove the equality by means of algebraic manipulations, as well.)

Example 8. A committee is to be chosen from a set of 7 women and 4 men. How many ways are there to form the committee if

(a) the committee has 5 people, 3 women and 2 men?
(b) the committee can be any size (except empty) but it must have equal numbers of women and men?
(c) the committee has 4 people and one of them must be Mr. Smith?
(d) the committee has 4 people, 2 of each sex and Mr. and Mrs. Smith cannot both be on the committee?

Answers: (a) $\binom{7}{3} \times \binom{4}{2}$;
(b) $\binom{7}{1} \times \binom{4}{1} + \binom{7}{2} \times \binom{4}{2} + \binom{7}{3} \times \binom{4}{3} + \binom{7}{4} \times \binom{4}{4} = \binom{11}{4} - 1$;
(c) $\binom{7}{0} \times \binom{3}{3} + \binom{7}{1} \times \binom{3}{2} + \binom{7}{2} \times \binom{3}{1} + \binom{7}{3} \times \binom{3}{0} = \binom{10}{3}$;
(d) $\binom{6}{2} \times \binom{3}{1} + \binom{6}{1} \times \binom{3}{2} + \binom{6}{2} \times \binom{3}{2} = \binom{7}{2} \times \binom{4}{2} - \binom{6}{1} \times \binom{3}{1}$.

If we are choosing an *r-permutation* out of an n-set *with replacement* then we say that we are *sampling with replacement.* The product rule gives us that the number of r-permutations of an n-set with replacement is n^r.

Similarly, we may speak of *r-combinations* of an n-set *with replacement or repetition.* For example, the 3-combinations of a 2-set $\{0, 1\}$ with replacement are $\{0, 0, 0\}$, $\{0, 0, 1\}$, $\{0, 1, 1\}$, $\{1, 1, 1\}$.

Theorem 2. *The number of k-combinations of an n-set with repetition is $\binom{n+k-1}{k}$.*

Proof. We are looking for the number of ways one can choose k objects out of n types of objects if repetition is allowed. *In other words: How many different ways can one choose a bunch of k flowers if n types of flowers are available, each in sufficiently large supply? This problem later frequently will be referred as the florist shop problem.*

Assume that the flowers are sold in a florist shop where the containers of the n types of flowers are lined up along a corridor. The customer buying the k flowers walks along the containers and at every moment either he picks a piece of flower or steps to the next container. He begins his walk at the first container and finishes at the nth one, thus making $n-1$ steps and k picks altogether.

E.g. if the containers contain carnations, roses, lilies and tulips (thus 4 type of flowers) in this given order, the customer buys altogether 10 pieces and we denote by p and s the "pick" and "step" moves, the *pppssppppsppp* sequence corresponds to the choice of 3 carnations, 4 lilies and three tulips, while the sequence *sppppspppppps* to the choice of 4 roses and 6 lilies.

Thus, the total number of choices of k pieces of flowers out of the given n types is the same as the number of choices of the k "picks" out of the $n-1+k=n+k-1$ total number of moves, or equivalently, as the number of choices of the $n-1$ "steps" out of the $n-1+k=n+k-1$ total number of moves. This number is

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

The same result may be obtained by a different way: the number of choices of the k "picks" and $n-1$ "steps" is the same as the number of permutations of these altogether $n+k-1$ items, k and $n-1$ of which are identical; assuming for a minute they are all different the number of permutations is $(n+k-1)!$, but any reordering of the k picks and $n-1$ moves give us in fact the same permutation. Thus the final number of choices is

$$\frac{(n+k-1)!}{k!\cdot(n-1)!} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

Sample problems.

Choosing a sample of r elements from a set of n elements is summarized in the following table.

Order counts?	Repetition allowed?	The sample is called:	Number of ways to choose the sample:
No	No	r-combination	$\binom{n}{r}$
Yes	No	r-permutation	$P(n,r)=\frac{n!}{(n-r)!}$
No	Yes	r-combination with replacement	$\binom{n+r-1}{r}$
Yes	Yes	r-permutation with replacement	n^r

Placement or occupancy problems.

Problems of placing balls into cells are called occupancy problems. In occupancy problems, it makes a big difference whether or not the balls are distinguishable, whether or not the cells are distinguishable and whether or not the cells can be empty. The possible cases of occupancy problems are summarized in the following table.

	Distinguished balls?	Distinguished cells?	Can cells be empty	Number of ways to place n balls into k cells	Reference
Case 1.	Yes	No	No	$S(n,k)$	definition
Case 2.	Yes	No	Yes	$\sum_{i=1}^{k} S(n,i)$	Case 1
Case 3.	Yes	Yes	No	$k!\ S(n,k)$	Case 1
Case 4.	Yes	Yes	Yes	k^n	product rule
Case 5.	No	Yes	Yes	$\binom{k+n-1}{n}$	Thm.2
Case 6.	No	Yes	No	$\binom{n-1}{k-1}$	Case 5
Case 7.	No	No	Yes	Number of partitions of n into k or fewer parts	C.Berge, Principles of Combinatorics, Acad.Press, 1971
Case 8.	No	No	No	Number of partitions of n into exactly k parts	

The number $S(n,k)$ defined in Case 1 (and to be determined later) is called a *Stirling number of the second kind.*

Consider the particular occupancy problem of distributing n distinguishable balls into k distinguishable cells so that we distribute n_1 balls into the first cell, n_2 into the second cell, and so on, n_k into the k-th cell. Let $\binom{n}{n_1,n_2,\ldots,n_k}$ denote the number of ways this can be done. Notice that $\binom{n}{n_1,n_2,\ldots,n_k} = \binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \ldots \times \binom{n-n_1-n_2-\ldots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!\ldots n_k!}$. The numbers $\binom{n}{n_1,n_2,\ldots,n_k}$ are called multinomial coefficients. (The numbers $\binom{n}{r}$ are called binomial coefficients.)

Theorem 3. *Suppose that we have n objects, n_1 of type 1, n_2 of type 2, ..., n_k of type k, with $n_1 + n_2 + \ldots + n_k = n$ of course. Suppose that objects of the same type are indistinguishable. Then the number of distinguishable permutations of these objects is* $\binom{n}{n_1,n_2,\ldots,n_k}$.

Proof. We have n places to fill in the permutation and we assign n_1 of these to the objects of type 1, n_2 to the objects of type 2, and so on.

Theorem 4. (Binomial expansion.) *For $n \geq 0$,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \ .$$

Proof. In multiplying out, we pick one term from each factor $(a+b)$. Note that to obtain $a^k b^{n-k}$, we need to choose k of the factors from which to choose a.

Theorem 5. *(a)* $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n \quad$ *for $n \geq 0$,*
(b) $\binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n} = 0 \quad$ *for $n \geq 1$.*

Proof. Binomial expansion of $(1+1)^n$ and $(1+(-1))^n$.

Exercises

1. How many $m \times n$ matrices are there each of whose entries is 0 or 1?

2. How many numbers are there which have five digits, each being a number in $\{1, 2, \ldots, 9\}$ and either having all digits odd or having all digits even?

3. In how many ways can we get a sum of 3 or a sum of 4 when two dice are rolled?

4. Ten job applicants have been invited for interviews, five having been told to come in the morning and five having been told to come in the afternoon. In how many different orders can the interviews be scheduled?

5. Let $A = \{a, b, c, d, e, f, g\}$.
(a) Find the number of sequences of length 4 using elements of A.
(b) Repeat part (a) if no letter is repeated.
(c) Repeat part (a) if the first letter in the sequence is b.
(d) Repeat part (b) if the first letter is b and the last is d.

6. A pizza shop advertises that it offers over 500 varieties of pizza. The local consumer protection bureau is suspicious. At the pizza shop, it is possible to have a choice of any combination of the following toppings: pepperoni, mushrooms, peppers, sardines, sausage, anchovies, salami, onions, bacon. Is the pizza shop telling the truth in its advertisements?

7. An ice cream parlor offers 29 different flavors.
(a) How many different triple cones are possible if each scoop on the cone has to be a different flavor?
(b) Repeat part (a) if the scoops do not have to be different flavors.
(We assume that the order of the scoops does not count.)

8. Repeat Ex. 7 if the order of the scoops does count.

9. Show (in two different ways if possible) that

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k} \qquad \left(= \binom{n}{n-m, m-k, k} \right) \ .$$

10. Show (in two different ways if possible) that

$$\binom{n}{0} + \binom{n+1}{1} + \ldots + \binom{n+r}{r} = \binom{n+r+1}{r} \ .$$

11. In how many different ways can we choose 12 cans of soup if there are 5 different varieties available?

12. Suppose that a codeword of length 8 consists of letters A, B, C, D, or E and cannot start with A. How many such codewords are there?

13. Compute (a) $S(n, 0)$;
(b) $S(n, 1)$;
(c) $S(n, 2)$;
(d) $S(n, n-1)$;
(e) $S(n, n)$.

14. Find the number of ways to pair off 10 police officers into partners for a patrol.

15. Find the number of ways to assign 6 jobs to 4 workers so that each job gets a worker and each worker gets at least one job.

16. Find the number of ways to partition a set of 25 elements into exactly 4 subsets.

17. Show by a combinatorial argument that

$$S(n+1, k) = \binom{n}{0} S(0, k-1) + \binom{n}{1} S(1, k-1) + \ldots + \binom{n}{n} S(n, k-1).$$

18. In how many ways can we partition n into exactly k parts if the order counts?

19. A code is being written using the four letters a, b, c and d. How many 12 digit codewords are there which use exactly 3 of each letter?

20. Of 15 computer programs to be run in a day, 5 of them are short, 4 are long, and 6 are of intermediate length. If the 15 programs are all distinguishable, in how many different orders can they be run so that
(a) all the short programs are run at the beginning?
(b) all the programs of the same length class are run consecutively?

21. How many ways are there to distribute 8 patients to 5 doctors?

22. How many numbers less then 1 million contain the digit 2?

23. How many 5-letter "words" either start with f or do not have the letter f?

24. (a) In a 6-cylinder engine, the even-numbered cylinders are on the left and the odd-numbered ones are on the right. A good firing order is a permutation of the numbers 1 to 6 in which right and left sides are alternated. How many possible good firing orders are there which starts with a left cylinder?
(b) Repeat part (a) for a $2n$-cylinder engine.

25. If a campus telephone extension has four digits, how many different extensions are there with no repeated digits
(a) if the first digit cannot be 0?
(b) if the first digit cannot be 0 and the second cannot be 1?

26. A value function on a set A assigns 0 or 1 to each subset of A. How many different value functions are there on a set A of n elements?

27. How many odd numbers between 1,000 and 9,999 have distinct digits?

28. Show that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \ldots + \binom{n}{r}\binom{m}{0}.$$

29. In how many ways can we choose 8 bottles of soda if there are 4 brands available?

30. In checking the work of a proofreader, we look for 5 kinds of misprints in a textbook. In how many ways can we find 12 misprints?

31. In Ex. 30, suppose that we do not distinguish the types of misprints but we do keep a record of the page on which a misprint occurred. In how many ways can we find 25 misprints in 75 pages?

32. Show (by a combinatorial argument) that

$$S(n,k) = kS(n-1,k) + S(n-1,k-1).$$

33. How many ways are there to form a sequence of 10 letters from 4 a's, 4 b's, 4 c's and 4 d's if each letter must appear at least twice?

2. GENERATING FUNCTIONS

Suppose that we are interested in computing the k-th term in a sequence $a_0, a_1, \ldots$ of numbers. The *(ordinary) generating function* for the sequence (a_k) is defined to be

$$G(x) = \sum_k a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots \ .$$

The sum is finite if the sequence is finite, infinite if the sequence is infinite. In the latter case, we will think of x having been chosen so that the power series $\sum_{k=0}^{\infty} a_k x^k$ converges.

Ex. 1. Suppose that $a_k = \binom{n}{k}$ for $k = 0, 1, \ldots, n$. Then the ordinary generating function for the sequence (a_k) is

$$G(x) = \binom{n}{0} + \binom{n}{1} x + \ldots + \binom{n}{n} x^n = (1+x)^n \ .$$

Ex. 2. Find the ordinary generating function for the sequence $\frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!} \ldots,$

$$G(x) = \frac{1}{2!} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \ldots = \frac{1}{x^2}\left(\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots\right) =$$
$$\frac{1}{x^2}\left(e^x - 1 - x\right) \ .$$

Ex. 3. Let a_k be the number of labeled graphs with n vertices and k edges. Find the number of labeled graphs with n vertices using the ordinary generating function for the sequence (a_k).

$$a_k = \binom{\binom{n}{2}}{k}, \quad G_n(x) = \binom{\binom{n}{2}}{0} + \binom{\binom{n}{2}}{1} x + \ldots + \binom{\binom{n}{2}}{\binom{n}{2}} x^{\binom{n}{2}} = (1+x)^{\binom{n}{2}} \ .$$

The number of labeled graphs with n vertices is $\sum_{k=0}^{\binom{n}{2}} a_k = G_n(1) = 2^{\binom{n}{2}}$.

Theorem 1. *Suppose that $A(x), B(x)$ and $C(x)$ are the ordinary generating functions for the sequences (a_k), (b_k) and (c_k), respectively. Then*

(i) $C(x) = A(x) + B(x)$ *iff* $c_k = a_k + b_k$ *for* $k = 0, 1, 2, \ldots$.

(ii) $C(x) = A(x)B(x)$ *iff* $c_k = a_0 b_k + a_1 b_{k-1} + \ldots + a_k b_0$ *for* $k = 0, 1, 2, \ldots$.

((c_k) is called the convolution of the sequences (a_k) and (b_k).)

Ex. 4. Suppose that $G(x) = \frac{1+x+x^2+x^3}{1-x}$ is the ordinary generating function for a sequence (a_k). Find a_k.

We can write $G(x) = (1+x+x^2+x^3)\frac{1}{1-x}$. Now $1+x+x^2+x^3$ is the ordinary generating function for the sequence $1,1,1,1,0,0,0\ldots$ and $\frac{1}{1-x}$ for the sequence $1,1,1,\ldots$. Thus, $G(x)$ is the ordinary generating function for the convolution of these two sequences and so

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 3, \quad a_3 = 4, \quad a_4 = 4, \quad a_5 = 4, \quad a_6 = 4, \quad \ldots .$$

Theorem 2. *Suppose that we have p types of objects with n_i indistinguishable objects of type i for $i = 1, 2, \ldots, p$. The number of ways of picking k objects is given by the coefficient of x^k in the generating function*

$$(1+x+x^2+\ldots+x^{n_1})(1+x+x^2+\ldots+x^{n_2})\ldots(1+x+x^2+\ldots+x^{n_p}) .$$

If we have each type in infinite supply then the number of ways picking k objects is the coefficient of x^k in the generating function

$$\underbrace{(1+x+x^2+\ldots)(1+x+x^2+\ldots)\ldots(1+x+x^2+\ldots)}_{p \text{ terms}} = \frac{1}{(1-x)^p} .$$

Ex. 5. In doing a sampling survey, suppose that we have divided the possible men to be interviewed into r categories and similarly for the women. Suppose that in our group, we have two men from each category and one woman from each category. In how many ways can we pick a sample of k people? (People of the same sex are distinguished iff they belong to different categories.)

The generating function is given by

$$\begin{aligned} G(x) &= \underbrace{(1+x+x^2)(1+x+x^2)\ldots(1+x+x^2)}_{r \text{ terms}}\ \underbrace{(1+x)(1+x)\ldots(1+x)}_{r \text{ terms}} = \\ &= (1+x+x^2)^r(1+x)^r . \end{aligned}$$

The number of ways to select k people is the coefficient of x^k.

In order to expand out the generating function $(1-x)^{-p}$ (Theorem 2, second part), it will be useful to find the Maclaurin series for the function $f(x) = (1+x)^s$, where s is an arbitrary real number. We have

$$\begin{aligned} f'(x) &= s(1+x)^{s-1} \\ f''(x) &= s(s-1)(1+x)^{s-2} \\ &\vdots \\ f^{(k)}(x) &= s(s-1)\ldots(s-k+1)(1+x)^{s-k} . \end{aligned}$$

Thus we have the following theorem.

Theorem 3. (Binomial Theorem)

$$(1+x)^s = \binom{s}{0} + \binom{s}{1}x + \binom{s}{2}x^2 + \ldots \ ,$$

where the generalized binomial coefficient $\binom{s}{k}$ is defined for any real number s and non-negative integer k by

$$\binom{s}{k} = \begin{cases} \frac{s(s-1)\ldots(s-k+1)}{k!} & \text{if } k > 0 \ , \\ 1 & \text{if } k = 0 \ . \end{cases}$$

Theorem 4. *If there are p types of objects, then the number of ways to choose k objects if we are allowed unlimited repetition of each type is given by*

$$\binom{p+k-1}{k} \ .$$

Proof. The generating function is $G(x) = (1-x)^{-p}$ by Theorem 2. Apply the binomial theorem with $-x$ in place of x and with $s = -p$. We have

$$G(x) = \sum_{k=0}^{\infty} \binom{-p}{k}(-x)^k \ .$$

The coefficient of $x^k \quad (k > 0)$ is

$$\binom{-p}{k}(-1)^k = \frac{(-p)(-p-1)\ldots(-p-k+1)}{k!}(-1)^k =$$
$$= \frac{p(p+1)\ldots(p+k-1)}{k!} = \binom{p+k-1}{k} \ ,$$

which gives the coefficient of x^0, as well.

Ex. 6. Three (distinguishable) experts rate a job candidate on a scale of 1 to 6. In how many ways can the total of the ratings add up to 12?

The generating function to consider is $G(x) = (x + x^2 + \ldots + x^6)^3$, we want the coefficient of x^{12}.

Note that

$$G(x) = x^3(1+x+\ldots+x^5)^3 = x^3\frac{(1-x^6)^3}{(1-x)^3} = (x^3 - 3x^9 + 3x^{15} - x^{21})(1-x)^{-3} \ .$$

Applying the binomial theorem (or directly Theorem 4), the coefficient of x^{12} is

$$\binom{3+9-1}{9} - 3\binom{3+3-1}{3} = \binom{11}{9} - 3\binom{5}{3} = 25 \ .$$

The *exponential generating function* for a sequence (a_k) is the function $H(x) = \sum\limits_{k=0}^{\infty} a_k \frac{x^k}{k!}$.

Theorem 5. *Suppose that we have p types of objects with n_i indistinguishable objects of type i ($i = 1, 2, \ldots, p$). The number of permutations of length k (with up to n_i objects of type i) is the coefficient of $\frac{x^k}{k!}$ in the exponential generating function*

$$\left(1 + x + \frac{x^2}{2} + \ldots + \frac{x^{n_1}}{n_1!}\right)\left(1 + x + \frac{x^2}{2} + \ldots + \frac{x^{n_2}}{n_2!}\right)\ldots\left(1 + x + \frac{x^2}{2} + \ldots + \frac{x^{n_p}}{n_p!}\right) .$$

Theorem 6.

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

Proof. Let $T(n,k)$ $(= k!\,S(n,k))$ denote the number of distributions of n distinguishable balls into k *distinguishable* cells. The exponential generating function for $T(n,k)$ is

$$H(x) = (x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots)^k = (e^x - 1)^k$$

and $T(n,k)$ is the coefficient of $\frac{x^n}{n!}$. Now

$$H(x) = \sum_{i=0}^{k}\binom{k}{i}(-1)^i e^{(k-i)x} = \sum_{i=0}^{k}\binom{k}{i}(-1)^i\sum_{n=0}^{\infty}\frac{1}{n!}(k-i)^n x^n =$$
$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n \Longrightarrow T(n,k) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n .$$

Exercises

1. In each of the following, set up the appropriate generating function and indicate what coefficient you are looking for. But do not calculate the answer.
   (a) In how many ways can 5 letters be picked from the letters a, b, c, d if b, c and d can be picked at most once and a, if picked, must be picked 4 times?
   (b) In how many ways are there to choose 10 voters from a group of 5 Republicans, 5 Democrats, and 7 Independents if we want at least 3 Independents and any two voters of the same political persuasion are indistinguishable?
   (c) How many ways are there to distribute 15 indistinguishable balls into 10 distinguishable cells?
   (d) Repeat part (c) if no cell can be empty.
   (e) A survey team divides the possible people to interview into 5 groups depending on age and independently into 4 groups depending on geographic location. In how many ways can 8 people be chosen to interview if 2 people are distinguished only if they belong to different age groups, live in different geographic locations, or are opposite sex?
   (f) In how many ways can 8 binary digits be picked if each must be picked an even number of times?

(g) In checking the work of a proof reader, we look for 4 types of proof reading errors. In how many ways can we find 40 errors?

(h) In part (g), suppose that we do not distinguish the types of errors, but we do keep a record of the page on which an error occurred. In how many ways can we find 40 errors in 100 pages?

(i) Find the number of solutions to the equation

$$x_1 + x_2 + x_3 = 12$$

in which each x_i is a nonnegative integer and $x_i \leq 6$.

(j) How many codewords of three letters can be built from the letters a, b, c, d if b, c and d can only be picked once?

(k) How many 10-digit numbers contain at most three 0's, at most three 1's and at most four 2's.

(l) If n is a fixed even number, find the number of n-digit words from the alphabet $\{0, 1, 2, 3\}$ in each of which the number of 0's and the number of 1's is even.

(m) In how many ways can 200 identical terminals be divided among four computer rooms so that each room will have 20 or 40 or 60 or 80 or 100 terminals?

(n) A codeword consists of at least one of each of the digits 0,1,2, and 3 and has length 5. How many such codewords are there?

(o) In how many ways can $3n$ letters be selected from $2n$ A's, $2n$ B's, and $2n$ C's?

(p) In how many ways can a total of 100 be obtained if 50 dice are rolled?

2. Suppose that there are p kinds of objects with n_i indistinguishable objects of type i. Suppose we can pick all or none of each kind. Set up a generating function for computing the number of ways to choose k objects.

3. Suppose that there are p kinds of objects, each in infinite supply. Let a_k be the number of distinguishable ways of choosing k objects if only an even number (including 0) of each kind of object can be taken. Set up a generating function for the sequence (a_k) and solve for a_k.

4. Three people each roll a die once. In how many ways can the score add up to 11?

5. Suppose that there are p different kinds of objects, each in infinite supply. Let a_k be the number of permutations of k objects chosen from these objects. Find a_k explicitly by using exponential generating functions.

6. A small company wants to buy five vehicles including at most two pickup trucks, at most two station wagons, at most two passenger cars, and at least one van but at most two vans. How many ways are there to buy 5 vehicles if any two vehicles of the same type are indistinguishable?

7. Suppose that there are p different kinds of objects, each in infinite supply. Let a_k be the number of ways to pick k of the objects if we must pick at least one of each kind. Set up a generating function for (a_k) and find a_k for all k.

8. Let p_n^r be the number of partitions of the integer n into exactly r parts where order counts. Set up a generating function for the sequence (p_n^r) and find p_n^r.

3. RECURRENCE RELATIONS

4.1. Some examples

A *recurrence relation* is a formula reducing later values of a sequence of numbers to earlier ones. Note that a recurrence in general has many solutions, i.e., sequences satisfying it. However, there will be a unique solution when sufficiently many initial conditions are specified.

Ex. 1. A codeword from the alphabet $\{0, 1, 2, 3\}$ is legitimate iff it has an even number of 0's. How many legitimate codewords of length k are there?

Let a_k be the answer. We derive a recurrence for a_k. Observe that $4^k - a_k$ is the number of illegitimate k-digit codewords. Consider a legitimate $(k+1)$-digit codeword. If it starts with 1,2 or 3 then the last k digits form a legitimate codeword of length k and if it starts with 0 then they form an illegitimate codeword of length k. Thus

$$a_{k+1} = 3a_k + (4^k - a_k) = 2a_k + 4^k \ .$$

We have initial condition $a_1 = 3$.

Ex. 2. Suppose that we have n lines in "general position", i.e., no two are parallel and no three intersect in the same point. Into how many regions do these lines divide the plane?

For the number $f(n)$ of regions, we have

$$f(n+1) = f(n) + (n+1)$$

and the initial condition

$$f(1) = 2 \ .$$

Using the recurrence, we obtain that

$$f(n) = 2 + 2 + 3 + 4 + \ldots + n = 1 + \frac{n(n+1)}{2} \ .$$

Ex. 3. Suppose that we study the prolific breeding of rabbits. We start with one pair of adult rabbits (of opposite sex). Assume that each pair of adult rabbits produce one pair of young (of opposite sex) each month. A newborn pair become adults in two months, at which time they also produce their first pair of young. Assume the rabbits never die. Let F_k denote the number of rabbit pairs present at the beginning of the kth month. Then we have

$$F_0 = 1, \quad F_1 = 1, \quad F_2 = 2, \quad F_3 = 3, \quad F_4 = 5, \quad \ldots$$

and recurrence

$$F_k = F_{k-1} + F_{k-2} \ .$$

The numbers F_k are called the Fibonacci numbers.

Ex. 4. Let n objects be labeled $1, 2, \ldots, n$. An arrangement (permutation) in which object i is not placed in the ith place for any i is called a derangement. Let D_n be the number of derangements of n objects.

A derangement of n objects involves a choice of first element and then an ordering of the remaining $n-1$. The first element can be any of $n-1$ different elements: $2, 3, \ldots, n$.

Suppose that k is put first. Then either 1 appears in the k-th spot or it does not. If 1 appears in the k-th spot then there are D_{n-2} ways to order $2, 3, \ldots, k-1, k+1, \ldots, n$ so none appears in its proper place. Suppose next that 1 does not appear in the k-th spot. Now if we replace 1 with k then we obtain a derangement of the objects $2, 3, \ldots, n$. There are D_{n-1} such derangements. Thus, we have

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

and the initial conditions are

$$D_1 = 0 \ , \qquad D_2 = 1 \ .$$

Ex. 5. Generalizing Example 1, let a codeword from the alphabet $\{0, 1, 2, 3\}$ be legitimate iff it has an even number of 0's *and* an even number of 3's. Let a_k denote the number of legitimate codewords of length k. To find a_k, it turns out to be useful to consider other possibilities for a word of length k. Let b_k be the number of k-digit words with an even number of 0's and an odd number of 3's, c_k the number with an odd number of 0's and an even number of 3's, and d_k the number with an odd number of 0's and an odd number of 3's. Note that

$$a_1 = 2, \quad b_1 = 1, \quad c_1 = 1, \quad d_1 = 0$$

and

$$d_k = 4^k - a_k - b_k - c_k \ .$$

It is easy to get the recurrences

$$\begin{aligned} a_{k+1} &= 2a_k + b_k + c_k \ , \\ b_{k+1} &= a_k + 2b_k + d_k = b_k - c_k + 4^k \ , \\ c_{k+1} &= a_k + 2c_k + d_k = c_k - b_k + 4^k \ . \end{aligned}$$

We have found three relations which can be used simultaneously to compute the desired numbers.

4.2. The method of characteristic roots

Consider the recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_p a_{n-p} \qquad n \geq p \ ,$$

where $c_1, c_2, \ldots, c_p$ are constants and $c_p \neq 0$. Such recurrences are called *linear homogeneous recurrence relations with constant coefficients*. The recurrence in Ex. 3 is an example of such recurrences. The recurrence above has a unique solution once we specify the values of the first p terms, $a_0, a_1, \ldots, a_{p-1}$; these values for the initial conditions. A recurrence has many solutions in general if the initial conditions are disregarded. Some of these solutions will be sequences of form

$$\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^n, \ldots \; .$$

To find the values of α, let us substitute x^k for a_k and solve for x. We get

$$x^n - c_1 x^{n-1} - c_2 x^{n-2} - \ldots - c_p x^{n-p} = 0$$

or dividing by x^{n-p},

$$x^p - c, x^{p-1} - c_2 x^{p-2} - \ldots - c_p = 0 \; .$$

It is called the *characteristic equation* of the recurrence. The roots of it are called *characteristic roots.*

Theorem 1. *Suppose that a linear homogeneous recurrence with constant coefficients has characteristic roots $\alpha_1, \alpha_2, \ldots, \alpha_p$. Then if $\lambda_1, \lambda_2, \ldots, \lambda_p$ are constants, every expression of the form*

$$a_n = \lambda, \alpha_1^n + \lambda_2 \alpha_2^n + \ldots + \lambda_p \alpha_p^n$$

is a solution to the recurrence. Moreover, if the characteristic roots are distinct, then every solution to the recurrence has the form above for some constants $\lambda_1, \ldots, \lambda_p$. We call the expression above the general solution of the recurrence.

Theorem 1 implies that if the characteristic roots are distinct then to find the unique solution of the recurrence subject to initial conditions $a_0, a_1, \ldots, a_{p-1}$, we simply need to find values for $\lambda_1, \ldots, \lambda_p$ in the general solution.

Ex. 3. (Revisited.) We have the recurrence

$$F_k = F_{k-1} + F_{k-2}$$

with initial conditions $F_0 = 1,\; F_1 = 1$. The characteristic equation is given by

$$x^2 - x - 1 = 0 \; ,$$

the characteristic roots are given by

$$\alpha_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \alpha_2 = \frac{1-\sqrt{5}}{2} \; .$$

Because $\alpha_1 \neq \alpha_2$, the general solution is

$$\lambda_1 \left(\frac{1+\sqrt{5}}{2}\right)^k + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right)^k \; .$$

The initial conditions give us the two equations

$$\lambda_1 + \lambda_2 = 1 \; ,$$

$$\lambda_1 \left(\frac{1+\sqrt{5}}{2}\right) + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 \; .$$

Solving these equations for λ, and λ_2, we get

$$\lambda_1 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right) \; , \qquad \lambda_2 = -\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right) \; .$$

Hence

$$F_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}} \; .$$

Theorem 2. *Suppose that a linear homogeneous recurrence with constant coefficients has characteristic roots $\alpha_1, \alpha_2, \ldots, \alpha_q$ with α_i having multiplicity n_i. Then the general solution is given by*

$$\begin{aligned} a_n = \lambda_{10}\alpha_1^n + \lambda_{11}n\alpha_1^n + \ldots + \lambda_{1(n_1-1)}n^{n_1-1}\alpha_1^n + \ldots + \\ + \lambda_{q0}\alpha_q^n + \lambda_{q1}n\alpha_q^n + \ldots + \lambda_{q(n_q-1)}n^{n_q-1}\alpha_q^n \end{aligned}$$

for some constants $\lambda_{10}, \lambda_{11}, \ldots, \lambda_{1(n_1-1)}, \ldots, \lambda_{q0}, \lambda_{q1}, \ldots, \lambda_{q(n_q-1)}$.

Ex. 6. Solve the recurrence

$$\begin{aligned} a_n &= 6a_{n-1} - 9a_{n-2} \\ a_0 &= 1 \ , \quad a_1 = 2 \ . \end{aligned}$$

The characteristic equation is $x^2 - 6x + 9 = 0$, 3 is a double root. The general solution is $\lambda_{10}3^n + \lambda_{11}n3^n$. The initial conditions give us the equations

$$\begin{aligned} &\lambda_{10} = 1 \ , \\ &3\lambda_{10} + 3\lambda_{11} = 2 \quad \Longrightarrow \quad \lambda_{11} = -\frac{1}{3} \ . \end{aligned}$$

Hence the solution is given by $a_n = 3^n - n3^{n-1}$.

4.3. Solving recurrences using generating functions

To illustrate the method, let us revisit

Ex. 1. We have

$$\begin{aligned} a_{k+1} &= 2a_k + 4^k \\ a_1 &= 3 \ . \end{aligned}$$

¿From the recurrence, we can derive $a_0 = 1$ even though a_0 is not defined. Multiply both sides of the recurrence by x^k and sum, obtaining

$$\sum_{k=0}^{\infty} a_{k+1}x^k = 2\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} 4^k x^k \ .$$

Hence, for the ordinary generating function $G(x) = \sum_{k=0}^{\infty} a_k x^k$, we obtain

$$\begin{aligned} \frac{1}{x}\sum_{k=0}^{\infty} a_{k+1}x^{k+1} &= 2G(x) + \sum_{k=0}^{\infty}(4x)^k \ , \\ \frac{1}{x}G(x) - \frac{1}{x} &= 2G(x) + \frac{1}{1-4x} \ . \end{aligned}$$

Solving for $G(x)$, we obtain

$$G(x) = \frac{1}{1-2x} + \frac{x}{(1-2x)(1-4x)} \ .$$

Expanding the second term by the method of partial fractions, we get

$$G(x) = \frac{1/2}{1-2x} + \frac{1/2}{1-4x} \; .$$

Expanding it, we obtain

$$G(x) = \frac{1}{2}\sum_{k=0}^{\infty}(2x)^k + \frac{1}{2}\sum_{k=0}^{\infty}(4x)^k \; ,$$

and so

$$a_k = \frac{1}{2}\cdot 2^k + \frac{1}{2}\cdot 4^k \; .$$

Ex. 4. (Revisited) We have

$$\begin{aligned} D_{n+1} &= n(D_n + D_{n-1}) \\ D_1 &= 0 \; , \qquad D_2 = 1 \; . \end{aligned}$$

Use the exponential generating functions

$$H(x) = \sum_{n=0}^{\infty} D_n \frac{x^n}{n!} \qquad (D_0 = 1 \quad \text{from the recurrence}).$$

Multiply the recurrence $D_{n+1} = n(D_n + D_{n-1})$ by $\frac{x^{n+1}}{(n+1)!}$ and sum, obtaining

$$\begin{aligned} \sum_{n=0}^{\infty} D_{n+1}\frac{x^{n+1}}{(n+1)!} &= \sum_{n=0}^{\infty} nD_n\frac{x^{n+1}}{(n+1)!} + \sum_{n=0}^{\infty} nD_{n-1}\frac{x^{n+1}}{(n+1)!} \; , \\ H(x) - 1 &= x\sum_{n=0}^{\infty}(n+1)D_n\frac{x^n}{(n+1)!} - \sum_{n=0}^{\infty}(D_n - nD_{n-1})\frac{x^{n+1}}{(n+1)!} \; . \end{aligned}$$

Using that $D_n - nD_{n-1} = (-1)^n$ (prove by induction!), we get

$$\begin{aligned} H(x) - 1 &= xH(x) + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{n+1}{(n+1)!} = xH(x) + e^{-x} - 1 \\ H(x) &= e^{-x}(1-x)^{-1} = (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots)(1 + x + x^2 + \ldots) \\ H(x) &= \sum_{n=0}^{\infty} x^n \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n\frac{1}{n!}\right) \; . \end{aligned}$$

Thus

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n\frac{1}{n!}\right) \; .$$

Ex. 5. (Revisited). We have the recurrences

$$\begin{aligned} a_{k+1} &= 2a_k + b_k + c_k \; , \\ b_{k+1} &= b_k - c_k + 4^k \; , \qquad a_1 = 2,\; b_1 = 1,\; c_1 = 1 \; . \\ c_{k+1} &= c_k - b_k + 4^k \; , \end{aligned}$$

¿From the recurrences we get $a_0 = 1, b_0 = c_0 = 0$. Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$, $B(x) = \sum_{k=0}^{\infty} b_k x^k$, $C(x) = \sum_{k=0}^{\infty} c_k x^k$. Multiplying the recurrences by x^k and summing, we get some equations for these generating functions.

$$\begin{aligned}
\sum_{k=0}^{\infty} a_{k+1} x^k &= 2\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k + \sum_{k=0}^{\infty} c_k x^k \ , \\
\sum_{k=0}^{\infty} b_{k+1} x^k &= \sum_{k=0}^{\infty} b_k x^k - \sum_{k=0}^{\infty} c_k x^k + \sum_{k=0}^{\infty} 4^k x^k \ , \\
\sum_{k=0}^{\infty} c_{k+1} x^k &= \sum_{k=0}^{\infty} c_k x^k - \sum_{k=0}^{\infty} b_k x^k + \sum_{k=0}^{\infty} 4^k x^k \ .
\end{aligned}$$

We obtain

$$\begin{aligned}
\frac{1}{x}(A(x) - a_0) &= 2A(x) + B(x) + C(x) \ , \\
\frac{1}{x}(B(x) - b_0) &= B(x) - C(x) + \frac{1}{1-4x} \ , \\
\frac{1}{x}(C(x) - c_0) &= C(x) - B(x) + \frac{1}{1-4x} \ .
\end{aligned}$$

If follows that

$$\begin{aligned}
B(x) = C(x) &= \frac{x}{1-4x} \ , \\
A(x) &= \frac{2x^2 - 4x + 1}{(1-2x)(1-4x)} \ .
\end{aligned}$$

Using partial fractions, we get that

$$\begin{aligned}
B(x) = C(x) &= \sum_{k=0}^{\infty} 4^k x^{k+1} \ , \\
A(x) &= 1 + \sum_{k=0}^{\infty} 4^k x^{k+1} + \sum_{k=0}^{\infty} 2^k x^{k+1} \ .
\end{aligned}$$

Thus $a_k = 4^{k-1} + 2^{k-1}$ for $k > 0$.

Exercises

1. A codeword from the alphabet $\{0, 1, 2\}$ is legitimate iff no two 0's appear consecutively. Find a recurrence for the number b_n of legitimate codewords of length n.

2. Determine a recurrence for $f(n)$ if $f(n)$ is the number of ways that $2n$ tennis players can be paired off in n matches.

3*. If F_n is the nth Fibonacci number, find a simple expression for

$$F_1 + F_2 + \ldots + F_n \ ,$$

which involves F_p for only one p.

4. Find the number of codewords of length k from an alphabet $\{a, b, c, d, e\}$ if b occurs an even number of times.

5. Suppose that we have 4 forint stamps, 6 forint stamps and 10 forint stamps, each in infinite supply. Let $f(n)$ denote the number of ways of obtaining n forints of postage if the order in which we put on stamps counts. Derive a recurrence for $f(n)$ if $n > 10$.

6. A codeword from the alphabet $\{0, 1, 2\}$ is legitimate iff there is an even number of 0's and an odd number of 1's. Find simultaneous recurrences from which it is possible to compute the number of legitimate codewords of length n.

7. Determine a recurrence for $f(n)$, the number of regions into which the plane is divided by n circles each pair of which intersect in exactly two points and no three of which have a common point.

8. Solve the following recurrence relations under the given initial conditions

(a)	$a_k = 10a_{k-1} - 16a_{k-2}$	$a_0 = 0,\ a_1 = 1$
(b)	$b_{n+1} = -b_n + 2b_{n-1}$	$b_0 = b_1 = 1$
(c)	$c_k = 14c_{k-1} - 49c_{k-2}$	$c_0 = 0,\ c_1 = 10$
(d)	$d_n = 2d_{n-1} - d_{n-2}$	$d_0 = 1,\ d_1 = 2$
(e)	$e_n = 4e_{n-2}$	$e_0 = 10,\ e_1 = 2$
(f)	$f_n = 8f_{n-1} - 15f_{n-2}$	$f_0 = 0,\ f_1 = 2$
(g)	$g_n = 10g_{n-1} - 25g_{n-2}$	$g_0 = 1,\ g_1 = 2$
(h)	$h_{k+2} = 2h_{k+1} - h_k + 2^k$	$h_0 = 2,\ h_1 = 1$
(i)	$i_{n+1} = 2ni_n + 2i_n + 2$	$i_0 = 1$
(j)	$j_{n+1} = 3j_n + 1$	$j_1 = 1$
(k)	$k_n = k_{n-1} + n + 6$	$k_0 = 0$
(l)	$a_{n+1} = a_n + b_n + c_n$	
	$b_{n+1} = 4^n - c_n$	$a_1 = b_1 = c_1 = 1$
	$c_{n+1} = 4^n - b_n$	
(m)	$a_{n+1} = 2a_n + b_n + 2^n$	$a_0 = 2$
	$b_{n+1} = -a_n + b_n + 3^{n+1}$	$b_0 = 0$.

4. THE PRINCIPLE OF INCLUSION AND EXCLUSION

Inclusion-Exclusion Formula. *Let $A_1, A_2, \ldots, A_n \subseteq S$ where S is a finite set, and let*

$$A_I = \bigcap_{i \in I} A_i \qquad \text{for} \quad I \subseteq \{1, 2, \ldots, n\} \qquad (A_\emptyset = S)\ .$$

Then

$$|S - (A_1 \cup A_2 \cup \ldots \cup A_n)| = \sum_{I \subseteq \{1,2,\ldots n\}} (-1)^{|I|} |A_I|\ .$$

Proof. If $x \in S - (A_1 \cup A_2 \cup \ldots \cup A_n)$ then it is counted once. If x is contained in exactly k sets A_i then it is counted $\binom{k}{0} - \binom{k}{1} + \binom{k}{2} + \ldots + (-1)^k \binom{k}{k} = (1-1)^k = 0$ times.

Ch. 3. Ex. 6 (Revisited). We recall this problem in a new interpretation: Three distinguishable experts rate a job candidate on a scale of 0 to 5. In how many ways can the total of the ratings add up to 9?

Consider the case when the experts rate on a scale of 0 to infinite and let S be the set of all the ways how the total of ratings add up to 9. (Notice that $|S| = \binom{3+9-1}{9} = \binom{11}{9}$.) Let A_i be the subset of S such that the i-th expert's rating is at least 6. Now we are looking for $|S \cup (A_1 \cup A_2 \cup A_3)|$. By the principle of inclusion and exclusion,

$$|S - (A_1 \cup A_2 \cup A_3) = |S| - \sum_{i=1}^{3} |A_i| + \sum_{1 \le i < j \le 3} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3| =$$

$$= \binom{11}{9} - 3\binom{3+3-1}{3} + 0 - 0 = \binom{11}{9} - 3\binom{5}{3} = 25\ .$$

Exercises

1. How many integers between 1 and 10,000 inclusive are divisible by none of 5, 7 and 11?
2. How many integers between 1 and 500 inclusive are divisible by none of 2, 3 and 5?
3. A total of 5 misprints occur on 4 pages of a book. What is the number of distributions of these misprints such that each of these pages has at least one misprint?
4. Eight accidents occur during a week. Write an expression for the number of distributions of the accidents such that there is at least one accident each day.
5. Use the principle of inclusion and exclusion to count the number of ways to choose
   (a) 12 elements from a set of 5 a's, 5 b's, and 5 c's;
   (b) 8 elements from a set of 3 a's, 3 b's and 4 c's;
   (c) 9 elements from a set of 3 a's, 4 b's and 5 c's.

6. Use inclusion and exclusion to find the number of solutions to the equation

(a) $x_1 + x_2 + x_3 = 16$;
(b) $x_1 + x_2 + x_3 + x_4 = 18$;

in which each x_i is a nonnegative integer and $x_i \leq 7$.

7. Solve Ex. 6 if each x_i is strictly positive.

8. Find the number of permutations of the set $\{1, 2, \ldots, n\}$ in which the patterns

(a) $124, 35$
(b) $12, 23, 24, \ldots, (n-1)n$ do not appear.

9. Find the number of n-digit codewords from the alphabet $\{0, 1, \ldots, 9\}$ in which the digits 1,2, and 3 each appears at least once.

10. Find the number of ways the letters $a, a, b, b, c, c, d, d, d$ can be arranged so that two letters of the same kind never appear consecutively.

5. INTRODUCTION TO GRAPH THEORY

A *graph* is an ordered pair (V, E) of disjoint sets such that E is a subset of the set of unordered pairs of the elements of V. The elements of V and E are called *vertices* and *edges* of the graph, resp. If $G = (V, E)$ is a graph then $V = V(G)$ and $E = E(G)$ are the vertex set and the edge set of G, resp. An edge $\{x, y\}$ is said to join the vertices x and y and denoted by xy. If $xy \in E(G)$ then x and y are said to be *adjacent* (in G) and the vertices x and y are *incident* with the edge xy.

We say that the graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If G' contains all edges of G joining two vertices in V' then G' is called the subgraph *induced* or *spanned* by V' and denoted by $G[V']$. If $V' = V$ then G' is said to be a *spanning subgraph* of G.

Two graphs are *isomorphic* if there is a one-to-one correspondence between their vertex-sets that preserves adjacency.

The set of vertices adjacent to a vertex $V \in V(G)$ is denoted by $\Gamma_G(v)$ or $N_G(v)$. The *degree* of a vertex v is given by $D_G(v) = |\Gamma_G(v)|$. If every vertex of G has degree k then G is said to be *k-regular.* A graph is *regular* if it is k-regular for some k. A vertex of degree zero is said to be an *isolated vertex.*

Since each edge has two endvertices (the vertices incident to the edge), the sum of the degrees is exactly twice the number of edges:

$$\sum_{v \in V(G)} d_G(v) = 2|E(G)| \ .$$

If the edges are ordered pairs of vertices then we get the notion of *directed graph* or *digraph.* An ordered pair (x, y) is said to be an edge directed from x to y and denoted by xy.

A *walk* in a directed or undirected graph is a sequence $(x_0, e_1, x_1, e_2, x_2, \ldots, x_{k-1}, e_k, x_k)$ in which $x_0, x_1, \ldots, x_k$ are vertices and e_i is an edge from x_{i-1} to x_i $(i = 1, 2, \ldots, k)$. The *length* of the walk above is k. A walk is a *trail* if no edge is used more than once. A walk is a *path* if no vertex is used more than once. A walk $(x_0, x_1, x_2, \ldots, x_{k-1})$ is a *circuit* or *cycle* if $x_0, x_1, x_2, \ldots, x_{k-1}$ are distinct vertices and $x_k = x_0$.

A graph G is a *bipartite graph* with vertex classes V_1 and V_2 if $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and each edge joins a vertex of V_1 to a vertex of V_2.

A graph is *connected* if for any two vertices x and y, there is a path from x to y. A maximal connected subgraph is a *component* of the graph. A digraph is *strongly connected* if for any ordered pair (x, y) of vertices there is a path from x to y. A maximal strongly connected subgraph is a *strongly connected component* of the digraph.

A *coloring* of the vertices of a graph G is an assignment of colors to the vertices in such a way that adjacent vertices get distinct colors. The *(vertex) coloring number* or *chromatic number* $\chi(G)$ is the minimal number of colors in a (vertex) coloring of G. Notice that a graph G is bipartite if and only if $\chi(G) \leq 2$.

The *complete graph* of n vertices, denoted K_n, is the graph in which every pair of vertices is joined by an edge. Clearly, $\chi(K_n) = n$ and if a graph G contains K_m as a subgraph then $\chi(G) \geq m$.

A *tree* is a connected graph with no cycles. A (not necessarily connected) graph without any cycle is a *forest* or an *acyclic* graph. Notice a forest is a graph whose every component is a tree.

Lemma 1. *A tree of at least two vertices has at least two vertices of degree one.*

Proof. The terminal vertices of a maximal path have degree one.

Theorem 1. *A tree of n vertices has $n-1$ edges.*

Proof. We prove the theorem by induction on n. For $n = 1$, the statement is obvious. Suppose that the statement is true for any tree of $n-1$ vertices. Consider a tree T of n vertices. Delete a vertex v of degree one. The resulted graph $T[V(T) - \{v\}]$ is still acyclic and connected since for any two vertices x and y in $V(T) - \{v\}$, the path joining x and y in T could not use vertex v. Thus $T - v$ is a tree and has $n-2$ edges by the inductional hypothesis. Hence $|E(T)| = (n-2) + 1 = n-1$.

Theorem 2. *A connected graph of n vertices and $n-1$ edges is a tree.*

Proof. We prove the theorem by induction on n. For $n = 1$, the statement is trivial. Suppose that the statement is true for any graph of $n-1$ vertices and let G be a connected graph of n vertices and $n-1$ edges where $n \geq 2$. Then G does not contain any isolated vertex and so $d(x) \geq 1$ for $x \in V(G)$. On the other side, $\sum\limits_{x \in V(G)} d(x) = 2n-2$ and so G contains at least two vertices of degree one. Deleting one of them, we obtain a connected graph of $n-1$ vertices and $n-2$ edges which is acyclic by the inductional hypothesis. Since a vertex of degree one cannot be contained in a cycle, it implies that G is acyclic, as well, and so G is a tree.

Theorem 3. *An acyclic graph of n vertices and $n-1$ edge is a tree.*

Proof. Let G be an acyclic graph (i.e. a forest) of n vertices and $n-1$ edges. Its components $G_1, G_2, \ldots, G_p$ are trees and so $|E(G_i)| = |V(G_i)| - 1$ by Theorem 1. Thus

$$|E(G)| = \sum_{i=1}^{p} |E(G_i)| = \sum_{i=1}^{p} (|V(G_i)| - 1) = n - p \ ,$$

which implies that $p = 1$ and so G is connected.

A spanning subgraph of a graph G that is a tree is called a *spanning tree* of G.

Theorem 4. *If G is a connected graph then it has a spanning tree.*

Proof. Notice that a connected spanning subgraph of G with minimum number of edges is a spanning tree.

Finally, we will discuss some basic counting results of labeled graphs. We speak about *labeled graphs* if the vertices of the graphs are labeled with, say, the first n positive integers. In that case two — otherwise isomorphic — graphs will be distinguishable simple due to the fact that there edges run between vertices of different labels. It turns out to be much easier to give the number of labeled graphs with a certain property then the same question for simple (non-labeled) graphs.

Fact. *The number of labeled graphs on n vertices is $2^{\binom{n}{2}}$, while the number of labeled graphs on n vertices with k edges is $\binom{\binom{n}{2}}{k}$.*

A rather more difficult question is the number of labeled trees. The answer is given by the Cayley Formula, which we give below.

Problem. How many different trees do we have on n labeled vertices?

Lemma. *Assume that $d_1, d_2, \dots, d_n$ are all ≥ 1 and $\sum_{i=1}^n d_i = 2n-2$. Then the number of trees on the given (labeled) vertices $\{v_1, v_2, \dots, v_n\}$ such that vertex v_i has degree d_i is equal to $\frac{(n-2)!}{(d_1-1)!\cdots(d_n-1)!}$.*

Proof. Use induction on n. Base cases are trivial. Since $\sum_{i=1}^n d_i = 2n-2 < 2n$ we must have a vertex of degree one, say $d_i = 1$. We may assume $d_n = 1$ and remove v_n. Assume it was connected to v_j and so the removal of it will result another tree on $n-1$ vertices with degrees $d_1, \dots, d_{j-1}, d_j - 1, d_{j+1}, \dots, d_{n-1}$. Similarly, for any given trees on $\{v_1, \dots, v_{n-1}\}$ with degrees $d_1, \dots, d_{j-1}, d_j - 1, d_{j+1}, \dots, d_{n-1}$ we can join v_j to the newly added v_n resulting a tree on $\{v_1, \dots, v_n\}$ with degrees $d_1, \dots, d_n$. By induction, the number of trees on $\{v_1, \dots, v_{n-1}\}$ with degrees $d_1, \dots, d_{j-1}, d_j - 1, d_{j+1}, \dots, d_{n-1}$ is

$$\frac{(n-3)!}{(d_1-1)!\cdots(d_{j-1}-1)!(d_j-2)!(d_{j+1}-1)!\cdots(d_{n-1}-1)!} = \frac{(d_j-1)(n-3)!}{(d_1-1)!\cdots(d_n-1)!}.$$

The formula is valid for $d_j = 0$ as well. Thus the number of trees on vertices $\{v_1, \dots, v_n\}$ with degrees $d_1, \dots, d_n$ is equal to

$$\begin{aligned}\sum_{j=1}^{n-1} \frac{(d_j-1)(n-3)!}{(d_1-1)!\cdots(d_n-1)!} &= \left(\sum_{j=1}^{n-1}(d_j-1)\right)\frac{(n-3)!}{(d_1-1)!\cdots(d_n-1)!} \\ &= \frac{(n-2)(n-3)!}{(d_1-1)!\cdots(d_n-1)!}\end{aligned}$$

Theorem 5. *(***Cayley Formula) The number of labeled trees on n vertices is n^{n-2}.**

Proof. It is equal to

$$\begin{aligned}\sum_{\substack{d_1,\dots,d_n \geq 1 \\ d_1+\cdots+d_n = 2n-2}} \frac{(n-2)!}{(d_1-1)!\cdots(d_n-1)!} = \sum_{\substack{k_1,\dots,k_n \geq 0 \\ k_1+\cdots+k_n = n-2}} \frac{(n-2)!}{k_1!\cdots k_n!} &= \underbrace{(1+1+\cdots+1)}^{n-2} \\ &= n^{n-2}.\end{aligned}$$

Another way to deduce the Cayley Formula is via Prüfer Code. Let T be a tree on vertices $v_1, v_2, \dots, v_n$. Assign a code over the numbers $\{1, 2, \dots, n\}$ to this tree by the following procedure:

Delete the endpoint (vertex of degree 1) of the tree having he smallest index and write down the index of its (only) neighbor (therefore you will register a number between 1 and n). Repeat this procedure with the resulting tree, until a tree with only one endpoint remains. Note, that the vertex with label n may never be deleted (every tree has at least two endpoints), this will be the label of the last, remaining vertex, and therefore n will be the last, $n-1$st member of the sequence.

On can see, that this sequence, called the *Prüfer Code* of the tree T uniquely characterizes T. On the other hand, given any sequence $(a_1, a_2, \ldots, a_{n-1})$ such that $1 \leq a_i \leq n$, $a_{n-1} = n$ there is a unique tree with this Prüfer code. Therefore, the number of the labeled trees agrees with the number of such sequences, which is clearly n^{n-2}.

Exercises

1. Show that the complement of a disconnected graph is connected. (The *complement* of a graph G is the graph $\overline{G}$ defined by $V(\overline{G}) = V(G)$, $E(\overline{G}) = \{xy: x, y \in V(G), x \neq y, \ \ xy \notin E(G)\}$.)

2. Are there two non-isomorphic graphs of 5 vertices with degree sequence 1,2,2,2, and 3?

3. Let G be a graph of $2n$ vertices such that $d(x) \geq n$ for $x \in V(G)$. Show that G is connected. What is the situation if we have $d(x) \geq n-1$ only?

4*. If $d(x) \geq 3$ for $x \in V(G)$ then G contains a cycle of even length.

5. Let $d_{\text{out}}(x)$ and $d_{\text{in}}(x)$ denote the number of edges directed from x to another vertex and from another vertex to x in a digraph D, respectively. Show that for any vertex set $W \subset V(D)$, the number of edges leading from W to $V(D) - W$ equals the number of edges leading from $V(D) - W$ to W if $d_{\text{out}}(x) = d_{\text{in}}(x)$ for $x \in V(D)$.

6*. Show that every digraph D contains an independent vertex set W such that any vertex of D is reachable from W along a (directed) path of length at most two. (A vertex set W is *independent* if there is no edge joining two vertices in W.)

7. Prove that a connected regular bipartite graph is 2-connected, i.e. it does not contain any vertex whose deletion results a disconnected graph.

8. Show that $\chi(G) + \chi(\overline{G}) \leq n+1$ for any graph G of n vertices.

9. Show that $\chi(G) \cdot \chi(\overline{G}) \geq n$ for any graph G of n vertices.

10. Show that every graph contains two vertices of equal degree.

11*. Determine all graphs with one pair of vertices of equal degree.

12. Show that in a graph G, there exists a set of cycles such that each edge of G belongs to exactly one of these cycles iff every vertex has even degree.

13. Are there two non-isomorphic 2-regular graphs of n vertices?

14. Prove that G contains a circuit if $d(x) \geq 2$ for $x \in V(G)$.

15. Prove that G contains a circuit of length at least $k+1$ if $d(x) \geq k$ for $x \in V(G)$.

16. Prove that a graph G has at least $\binom{\chi(G)}{2}$ edges.

17. The *vertex independence number* $\alpha(G)$ is the size of the largest independent vertex set in G. If G has n vertices show that

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n - \alpha(G) + 1 \ .$$

18. Prove that an acyclic graph G is a tree if and only if joining any two non-adjacent vertices of G we get exactly one cycle. What if G is not necessarily acyclic?

19. Show that a graph G is a tree iff it is connected, however deleting any edge of G, it will be disconnected.

20. Prove that if a graph G of n vertices has k components then $|E(G)| \geq n - k$.

21. Prove that deleting an edge of a tree, we obtain a graph of exactly two components.

22. Find the number of labeled trees of six vertices, four having degree 2.

23. Find the number of labeled trees of five vertices, exactly three of them having degree one.

24. Let G be a graph of n vertices. Prove that a connected acyclic subgraph G_0 of G with $n - 1$ edges is a spanning tree of G. What if G_0 is not necessarily connected?

25. Construct a tree having exactly one isomorphism with itself, the identity.

26. Prove that a graph of n vertices and at least n edges contains a cycle.

27. Show that G is a tree iff for any two vertices x and y of G, there is exactly one path between x and y.

28. Let G be a graph of n vertices and m edges consisting of k components. Prove that G contains (at least) $m - n + k$ cycles.

29. Prove that any acyclic subgraph of a connected graph can be completed into a spanning tree of the graph.

30. Let G be a graph of n vertices and let H be an acyclic subgraph of G. Prove that if H has $n - 1$ edges then G is connected.

31. Find the chromatic number of a tree.

32. Do we have a graph G on n vertices such that G and $\overline{G}$ are isomorphic for $n = 3, 4, 5, 6, 7$ and 8?

6. THE PIGEONHOLE PRINCIPLE AND RAMSEY THEORY

Some versions of pigeonhole principle.

Proposition 1. *If $k+1$ pigeons are placed into k pigeonholes then at least one pigeonhole will contain two or more pigeons.*

Proposition 2. *If m pigeons are placed into k pigeonholes then at least one pigeonhole will contain at least $\lfloor \frac{m-1}{k} \rfloor + 1$ pigeons.*

Proposition 3. *Given a set of real numbers, there is always a number in the set whose value is at least as large (as small) as the average value of the numbers in the set.*

Ex. 1. There are 15 minicomputers and 10 printers in a computer lab. At most 10 computers are in use at one time. Every 5 minutes, some subset of computers requests printers. We want to connect each computer to some of the printers so that we should use as few connections as possible but we should be always sure that a computer will have a printer to use. (At most one computer can use a printer at a time.) How many connections are needed?

Note that if there are fewer than 60 connections then there will be some printers connected to at most 5 computers (Prop.3). If the remaining 10 computers were used at one time, there would be only 9 printers left for them. Thus, at least 60 connections are required. On the other hand, it can be shown that if the i-th printer is connected to the i-th, $(i+1)$-st, $\ldots$, $(i+5)$-th computers ($i = 1, \ldots, 10$) then these 60 connections have the desired properties.

Ex. 2. Show that if $n+1$ numbers are selected from the set $\{1, 2, 3, \ldots, 2n\}$ then one of these will divide another one of them.

Take n "pigeonholes". Put the selected numbers of form $(2k-1)2^{\alpha}$ into the k-th pigeonhole ($1 \leq k \leq n$). Then at least one pigeonhole will contain at least two numbers and one of these will divide another one of these.

Definition. The *graph Ramsey number* $R(G_1, G_2)$ is the smallest n such that for every graph G of n vertices, either G contains a subgraph (isomorphic to) G_1 or $\overline{G}$ contains a subgraph (isomorphic to) G_2. If the two forbidden subgraphs, $G_1 = K_k$ and $G_2 = K_l$ are complete graphs of size k and l, respectively, we denote the corresponding Ramsey number by $R(k, l)$ for $k, l \geq 2$

Theorem 4. *For every $k, l \geq 3$ the following inequality holds:*

$$R(k,l) \leq R(k-1,l) + R(k,l-1) \tag{1}$$

.

Proof. Take a graph G on $n = R(k-1,l) + R(k,l-1)$ vertices and fix any point x of it. By the pigeonhole principle there must be either at least R(k-1,l) vertices through x or at least $R(k,l-1)$ non-vertices from x (otherwise the total number of vertices and non-vertices via x — which is naturally $n-1$ — would be less then or equal to $[R(k-1,l)-1]+[R(k,l)-1 = n-2)$. In the first case consider the other endvertices of the edges through x and the graph spanned by them. In that graph either there will be a K_{k-1} which together with x and the edges from x to these vertices would form a K_k in the bigger graph, or there would be K_l in the complement of it, giving a K_l in the complement of the original, bigger graph G. The second case can be handled similarly.

With this theorem and the easy observation: $R(2,k) = R(k,2) = 2$ we get that $R(k,l) \leq \binom{k+l-2}{k-1} = \binom{k+l-2}{l-1}$, where the proof is by induction, the induction step being the theorem above and the base cases are $R(2,k) = R(k,2) = 2$.

Therefore a few upper bounds for the Ramsey numbers are given by the table below

k,l	2	3	4	5	6
2	2*	3*	4*	5*	6*
3	3*	6*	10	15	21
4	4*	10	20		
5	5*	15			
6	6*	21			

In this table the values denoted by a * are exact values. However, a better, exact estimate on $R(3,4) = R(4,3)$ can be given, namely 9. This value itself will give better estimates for the other members of the table by (1):

k,l	2	3	4	5	6
2	2*	3*	4*	5*	6*
3	3*	6*	9^a	14^b	20
4	4*	9^a	18^c		
5	5*	14^b			
6	6*	20			

In this table the values for $R(3,4) = 9$, $R(3,5) = 14$ and $R(4,4) = 18$ are exact, as the following figures on the next page shows.

The validity of the statements about the Figures 8.3 and 8.4 are easy to check by hand. It is also not very difficult to see that the graph on Figure 8.5 does not contain a K_4. Notice an interesting property of this graph: it's a so called self-complementary graph, that is the graph and its complement are the same (isomorphic). One can easily check it by redrawing the graph around a circle where the order of the vertices is 1,4,7,10,13,16,2,5,8,11,14,17,3,6,9,12,15 and only the edges which do not appear in G will be drawn. You will get back exactly the picture of Figure 8.5 Therefore, $\overline{G}$ will not contain a K_4 either.

A possible generalization of the original Ramsey definition is the following:

Definition. Let $R_t(G_1, G_2, G_3, \ldots, G_t) = R(G_1, G_2, G_3, \ldots, G_t)$ denote the smallest n such that for every coloring of the edges of the complete graph K_n on n vertices with t colors there will be a color, say i, such that the i colored edges will contain a subgraph (isomorphic to) G_i.

Again, the first question is the existence of $R_t(G_1, G_2, G_3, \ldots, G_t$, which can be proven easily by induction on t. $R(G_1, G_2)$ exists, since it is bounded above by $R(k, l)$, where k and l are the number of the vertices of the graphs G_1 and G_2, respectively. On the other hand, it is easy to see that

$$R_t(G_1, G_2, \ldots, G_{t-1}, G_t) \leq R_{t-1}(G_1, G_2, \ldots, G_{t-2}, R(G_{t-1}, G_t)),$$

where the later expression does exits by the induction hypothesis.

Similarly to the "classic" Ramsey definition, we may define $R(k_1, k_2, \ldots, k_t)$ by $R(K_{k_1}, K_{k_2}, \ldots, K_{k_t})$, where all $k_i \geq 2$.

We've seen an upper estimate on it in the previous paragraph. However, a much better estimation can be given on $R(k_1, k_2, k_3, \ldots, k_t)$:

Theorem 5. *For every $k_1, k_2, \ldots, k_t \geq 3$ we have*

$$\begin{aligned} R(k_1, k_2, k_3, \ldots, k_t) \leq R(k_1 - 1, k_2, k_3, \ldots, k_t) + R(k_1, k_2 - 1, k_3, \ldots, k_t) + \cdots \\ + R(k_1, k_2, k_3, \ldots, k_t - 1) \end{aligned}$$

Proof. It goes quite similar to the two color Ramsey case. Assume a graph has at least $R(k_1 - 1, k_2, k_3, \ldots, k_t) + R(k_1, k_2 - 1, k_3, \ldots, k_t) + \cdots + R(k_1, k_2, k_3, \ldots, k_t - 1) = m$ vertices and the edges are t colored. Picking any vertex x and considering the $m-1$ edges leaving that vertex, by the pigeonhole principle we will have a color class, say i, such that the number of the ith colored edges leaving x will be at least $R(k_1, k_2, \ldots, k_i - 1, \ldots, k_t)$. Now, either there will a complete K_{k_j} with all edges colored j for some $j \neq i$ among the other end vertices of these edges or a complete $K_{k_i - 1}$ with all edges ith colored, which together with x and all edges connecting this subgraph to x (colored i) will result a complete K_{k_i} all edges colored i, both cases completing the proof.

Corollary For every $p_1, p_2, \ldots, p_t \geq 1$ we have

$$R_t(p_1 + 1, p_2 + 1, \ldots, p_t + 1) \leq \binom{p_1 + p_2 + \cdots + p_t}{p_1, p_2, \ldots, p_t}.$$

Proof Goes by induction on t and for fixed value of t on the sum $p_1 + p_2 + \cdots + p_t$. The base cases are the cases when any of the p_i's are equal to 1. However, it is easy to see, that for $p_i = 1$

$$\begin{aligned} R_t(p_1+1,\ldots,p_i+1,\ldots,p_t+1) &= R_t(p_1+1,\ldots,2,\ldots,p_t+1) \\ &= R_{t-1}(p_1+1,\ldots,p_{i-1},p_{i+1},\ldots,p_t+1) \\ &\leq \binom{p_1+p_2+\cdots,+p_{i-1}+p_{i+1}+\cdots+p_t}{p_1,p_2,\ldots,p_{i-1},p_{i+1},\ldots p_t}, \end{aligned}$$

which is much less then

$$\binom{p_1+p_2+\cdots,+p_{i-1}+1+p_{i+1}+\cdots+p_t}{p_1,p_2,\ldots,p_{i-1},1,p_{i+1},\ldots p_t}.$$

If all p_i's are at least 2, then the previous Theorem may be applied and the result simply comes from the fact that

$$\sum_{i=1}^{t}\binom{p_1+p_2+\cdots+(p_i-1)+\cdots+p_t}{p_1,p_2,\ldots,(p_i-1),\ldots,p_t} = \binom{p_1+p_2+\cdots+p_t}{p_1,p_2,\ldots,p_t}$$

where this later equality is an easy combinatorial exercise. (Both sides are the answer to the question: how many different ways one can order p_1 identical plus p_2 identical plus $\cdots$ plus p_t identical objects, but on the left hand side we distinguish the cases according to what type the first object in the order is.)

Now apply the results to the simplest cases. By the above argument (or triviality) $R(3,2,2) = 3$ and $R(3,3,2) = 6$. What kind of upper bound can we get on $R(3,3,3)$? The immediate application of the corollary would give:

$$R(3,3,3) \leq \binom{(3-1)+(3-1)+(3-1)}{3-1,3-1,3-1} = \binom{6}{2,2,2} = 90$$

However, the immediate applications of the theorem and the first observation of the proof of the corollary, (2) immediately improves this result to

$$R(3,3,3) \leq 3 \times R(3,3,2) = 3 \times R(3,3) = 3 \times 6 = 18$$

And this is still not the best result.

Claim. $R(3,3,3) \leq 17$

Proof Take a graph G on 17 vertices, edges colored by three colors, say blue, red and green. Take any vertex x of it and focus on the 16 edges leaving this vertex. Due to the pigeonhole principle there will be at least 6 of them of the same color, say blue. Look at the 6 other end vertices of these blue edges and considered the colored edges among them. If any of those are blue, that edge plus the two others going from its vertices to x will form a blue triangle. Otherwise all the edges of this complete K_6 are colored red an green, therefore — by an earlier result — there will either be a red triangle or a blue triangle.

On the other hand, Claude Berge in *Berge: Graphs and Hypergraphs* shows a graph on 16 vertices, edges colored with three colors and no monocolored triangle. This example then shows that

$$R(3,3,3) = 17.$$

Finally let us see a few examples for the graph Ramsey numbers:

Ex. 3. Determine the graph Ramsey number $R(P_3, K_4)$.

First, notice that if G does not contain P_3 then G cannot contain a vertex of degree 2 or more. So if G is maximal with respect to this property then it has $\left[\frac{1}{2}|V(G)|\right]$ independent (pairwise disjoint) edges. (Maximal means that if we add any edge then the resulting graph contains a P_3.) Taking these maximal graphs G with $|V(G)| \le 6$, it is easy to see that $\overline{G}$ does not contain K_4. (E.g. if $|V(G)| = 6$ then $G =$ and the possible K_4 in $\overline{G}$ must contain both endvertices of the three edges of G by pigeonhole principle, a contradiction. If $|V(G)| \le 5$ then it can be obtained from by vertex deletion and so $P_3 \not\subseteq G$, $K_4 \not\subseteq \overline{G}$. However if $|V(G)| = 7$ then the unique maximal graph without P_3 is $G =$ and $\overline{G}$ contains a K_4 in this case. So $R(P_3, K_4) = 7$.

Ex. 4. Determine the graph Ramsey number $R(P_4, P_4)$.

For $n = |V(G)| = 4$, let $G =$. Then $\overline{G} =$ and so $P_4 \not\subseteq G$, $P_4 \not\subseteq \overline{G}$, $R(P_4, P_4) > 4$.

For $n = 5$, we can find two maximal graphs without P_4: $G_1 =$ and $G_2 =$. (You have to prove that there is no other maximal P_4-free graph of 5 vertices!) And both $\overline{G}_1$ and $\overline{G}_2$ contain P_4 as a subgraph (verify!), i.e. $R(P_4, P_4) = 5$.

Exercises

1. If a graph has 100 vertices and 7 connected components, what can you say about the largest component? The smallest?
2. A tennis player preparing for a tournament wants to practice by playing at least one match a day over a period of 50 days, but not more than 75 matches in all. Show that during those 50 days, there is a period of consecutive days during which the player plays exactly 24 matches. Is the same statement true with 30 matches?
3. An employee's time clock show that he worked 81 hours over a period of 10 days. Show that on some pair of consecutive days, the employee worked at least 17 hours.
4. Given a sequence of p integers $a_1, a_2, \ldots, a_p$, show that there exists consecutive terms in the sequence whose sum is divisible by p.
5. A computer is used for 300 hours over-period of 15 days. Show that on some period of 3 consecutive days, the computer was used at least 60 hours.
6. A social worker has 77 days to make his rounds. He wants to make at least one visit a day, and has 132 visits to make. Is there a period of consecutive days in which he makes 21 visits? Why?
7. There are 25 executives in a corporation sharing 12 secretaries. Every hour, some group of the executives needs secretarial help. We never expect more than 12 executives to require secretarial help at any given time. We give each secretary a list of the executives he or she is working for, and make sure that each executive is on at least one secretary's list. If the number of names on each of the lists is added up, the total is 95. Show that it is possible that at some hour some executive might not be able to obtain secretial help.
8. The *graph Ramsey number* $R(G_1, G_2)$ is the smallest n such that for every graph G of n vertices, either G contains a subgraph (isomorphic to) G_1 or $\overline{G}$ contains a subgraph (isomorphic to) G_2. Suppose that P_k and C_k is the path and cycle of k vertices, respectively.

   (a) Find $R(P_3, P_4)$.

(b) Find $R(P_3, C_4)$.
(c) Find $R(P_4, C_4)$.
(d) Find $R(C_4, C_4)$.

9*. Let T_m be an arbitrary, but fixed tree of m vertices. Show that

$$R(T_m, K_n) = 1 + (m-1)(n-1) \ .$$

10. Show that

$$R_t(p_1+1, p_2+1, \ldots, p_t+1) \le \binom{p_1+p_2+\ldots+p_t}{p_1, p_2, \ldots, p_t} \ .$$

11. Show that a sequence of n^2+1 real numbers always has a (not necessarily strictly) monotone subsequence of length at least $n+1$. Is the same statement true for a sequence of length n^2?

7. EXPERIMENTAL DESIGNS

Due to the lack of time I skip here the justification of the importance of these designs in the real life. Only I mention that in testing different agricultural or industrial products, during the statistical planning of the tests they play important roles. We will deal here with orthogonal Latin squares, Balanced Incomplete Block Desings ((BIBD's) and a special type of them, Steiner triple systems.

Orthogonal Latin Squares

Definition. A *Latin square* is a $k \times k$ matrix whose elements are chosen from a set of n elements (like $S = \{a_1, a_2, \dots, a_n\}$) such that every row and every column of the matrix contains each of these n elements exactly once.

In many cases we identify the n elements set with the set of the first n positive integers, i.e. $S = \{1, 2, \dots, n\}$.

With this identification an example of an orthogonal Latin square is shown below:

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & k-1 & k \\ 2 & 3 & 4 & \cdots & k & 1 \\ 3 & 4 & 5 & \cdots & 1 & 2 \\ & & & \vdots & & \\ k-1 & k & 1 & \cdots & k-3 & k-2 \\ k & 1 & 2 & \cdots & k-2 & k-1 \end{bmatrix}$$

Definition. Similarly, a *Latin rectangle* is an $\ell \times k$ matrix $(k \geq \ell)$ whose elements are from a k element set (again, normally identified with $\{1, 2, \dots, k\}$) such that every row of the matrix contains each of these elements (each of $1 \leq i \leq k$) exactly once and every column of the matrix contains each of these element (each of $1 \leq i \leq k$) at most once.

An example is shown below:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 6 & 4 & 5 & 3 & 1 & 2 \\ 5 & 1 & 2 & 6 & 3 & 4 \end{bmatrix}$$

Definition. Two distinct Latin squares $A = (a_{ij})$ and $B = (b_{ij})$ are called *orthogonal* iff the n^2 ordered pairs (a_{ij}, b_{ij}) are all different.

Thus the two 4×4 Latin squares below are orthogonal:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 1 & 2 & 3 \\ 3 & 2 & 1 & 4 \\ 1 & 4 & 3 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

while the two 4×4 Latin squares below are not orthogonal:

$$\begin{bmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 4 & 3 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

since the pair $(2,4)$ appears twice, at positions 2,2 and 3,3.

More generally, if $A^{(1)}, A^{(2)}, \dots, A^{(r)}$ are distinct $n \times n$ Latin squares, they are said to form an *orthogonal family* if every pair of them is orthogonal.

The main questions here are the following:

1. does there exist a pair of orthogonal Latin squares for every k,
2. or in general, how big orthogonal family of Latin squares do we have for a given size k.
3. is it always possible to augment a Latin rectangle into Latin square?

We will skip the last question here, since the graph theoretical background needed to handle this question is not discussed during this course.

It is easy to see, that the only orthogonal Latin squares of order 2 are

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and they are not orthogonal.

On the other hand, the two 3×3 orthogonal Latin squares

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

(and up to a certain symmetry they are the only pair of orthogonal Latin squares of order 3) and previously we have seen pairs of orthogonal Latin squares of order 4 as well.

Theorem. If there is an orthogonal family of r Latin squares of order n, then $r \leq n-1$.

Proof Assume $A^{(1)}, A^{(2)}, \dots, A^{(r)}$ is a family of orthogonal Latin squares of order n. Assume that the squares contains as elements the numbers $1, 2, \dots, n$. We claim that if carry out the following operation on one (and exactly one) of the r squares: for a fixed pair $1 \leq i < j \leq k$ change each "i" element of the square into "j" and each "j" element of the square into "i" then the family remains a family of r orthogonal Latin squares. (Check!!)

With the help of this operation one may change first $A^{(1)}$ such that its first row becomes $1, 2, \dots, n$, then change $A^{(2)}$ the same way, and so on. Finally we will get another family $B^{(1)}, B^{(2)}, \dots, B^{(r)}$ of Latin squares each having it's first row equal to $1, 2, \dots, n$. Then for every pair of them forming the pairs of the matrix elements of the same position all the pairs (i, i), $1 \leq i \leq n$ will already occur in the first row. Therefore no pair of these r matrices may have the same element in the same position (apart of the first row). E.g. the first elements of the second rows of these matrices are all different, they come from the set $\{2, 3, \ldots, n\}$, therefore the number of them is at most $n-1$. QED

Theorem. For every prime factor $n = p^k$ there is a family of $n-1$ orthogonal Latin squares.

Sketch of Proof It is a well known algebraic result that for every prime power $n = p^k$ (and only for prime powers) there is a unique so called *field*, an algebraic structure which behaves with respects to the operations addition and multiplication (and their inverses, subtractions and division) basically like th filed of the rational numbers. They are called Galois fields are denoted by $\mathrm{GF}(p^k)$.

With the help of this Galois field we will define the $n-1$ orthogonal Latin squares. Let the elements of the field be $b_1, b_2, \ldots, b_n$, where b_1 is the multiplicative identity of the field (think of 1 in the field of the rational numbers) and b_n is the additive identity (think of 0 of the field of the rational numbers). Now define for every $e = 1, 2, \ldots, n-1$ a Latin square $A^{(e)} = \left(a_{ij}^{(e)}\right)$ by

$$a_{ij}^{(e)} = (b_e \times b_i) + b_j.$$

One can check, that the family of matrices obtained this way form a family of $n-1$ orthogonal Latin squares of size $n = p^k$.

Theorem. Assume that there is an orthogonal family of r Latin squares of order n and another orthogonal family of r Latin squares of order m. Then there is another orthogonal family of r Latin squares of order nm.

Proof Let $A^{(1)}, A^{(2)}, \ldots, A^{(r)}$ be the orthogonal family of Latin squares of order n and $B^{(1)}, B^{(2)}, \ldots, B^{(r)}$ be the orthogonal family of Latin squares of order m. For a given element $a_{ij}^{(e)}$of $A^{(e)}$, let $\left(a_{ij}^{(e)}, B^{(e)}\right)$ be $\times m$ matrix, whose k, l, entry is the pair (note that this Latin square will not consist of numbers, rather from pairs as symbols) $\left(a_{ij}^{(e)}, b_{kl}^{(e)}\right)$. Form from this $m \times m$ matrices an $nm \times nm$ matrix putting them together according to the arrangement

$$C^{(e)} = \begin{bmatrix} \left(a_{11}^{(e)}, B^{(e)}\right) & \left(a_{12}^{(e)}, B^{(e)}\right) & \cdots & \left(a_{1n}^{(e)}, B^{(e)}\right) \\ \left(a_{21}^{(e)}, B^{(e)}\right) & \left(a_{22}^{(e)}, B^{(e)}\right) & \cdots & \left(a_{2n}^{(e)}, B^{(e)}\right) \\ & & \cdots & \\ \left(a_{n1}^{(e)}, B^{(e)}\right) & \left(a_{n2}^{(e)}, B^{(e)}\right) & \cdots & \left(a_{nn}^{(e)}, B^{(e)}\right) \end{bmatrix}$$

Corollary. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ is the prime power decomposition of the integer $n > 1$, then there is an orthogonal family of Latin squares of order n of size

$$\min\left\{\left(p_1^{\alpha_1} - 1\right), \left(p_2^{\alpha_2} - 1\right) \cdots \left(p_n^{\alpha_n} - 1\right)\right\}.$$

Proof For every prime power p^{α_i} we have a family of $p^{\alpha_i} - 1$ orthogonal Latin squares of order p^{α_i}. Then the repeated application of the previous theorem immediately finishes the proof of the recent theorem.

Corollary If for an $n > 1$ either 2 does not divide n or a higher then first power of 2 divides n, then there is at least a pair of orthogonal Latin squares of order n.

Proof is immediate by the previous results.

By the above theorems only about the numbers of the form $n = 2k$, k odd are we not able to decide if there is a pair of orthogonal Latin squares of the given size. It turned out, that for $n = 2$ and 6 there are no pairs of orthogonal Latin squares, while for all other numbers there is.

Balanced Incomplete Block Designs

Definition. A (b, v, r, k, λ)-design is a collection of b pieces of k-uniform (k sized) subset (called *blocks*) of a v element set, such that each element of the set is in r blocks and each pair of the elements of the set are in λ blocks. If $k < v$ the design is called a *balanced incomplete lock design* (in short BIBD) since each block consists of fewer than the total number of available elements.

In the forthcoming we will see that for a given set of parameters we do not necessarily have a BIBD.

Examples.

$$\{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\}$$

is a $(7, 7, 3, 3, 1)$-design, while

$$\{|[1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 1\}, \{4, 1, 2\}\}$$

is a $(4, 4, 3, 3, 2)$-design.

We have numerous necessary conditions for the parameters of the BIBD's.

Theorem. For a (b, v, k, r, λ) design we have $bk = vr$ and $r(k-1) = \lambda(v-1)$.

Proof The first equation comes from counting the pairs (blocks, elements of the sets in the blocks). One way is to count that each of the b blocks contains k elements. The other way is, that each of the v element are in exactly r blocks.

For the second equation fix an element x. Then first count the pairs (pairs of elements (x, y), blocks containing the pair), (counting a pair each time it occurs in a block). One way is: the element x is in r blocks and each of these blocks contain $k - 1$ elements apart of x. Another way is that there are $v - 1$ elements apart of x, and each of those pairs are contained in λ pairs. QED

Corollary. If we only assume about a design that it is over v elements, each of the b blocks contain k elements and each pair of the elements are contained in λ blocks, then the design will be balanced, that is every element will be contained in exactly $r = \lambda(v-1)/(k-1)$ blocks.

The proof of the above theorem shows that if r_x denote the number of blocks the element x is contained in, then

$$r_x = \frac{\lambda(v-1)}{(k-1)}$$

where the right hand side of the equations are independent of x.

The following basic theorem of the balanced incomplete block designs is given without proof.

Theorem. In a (b, v, k, r, λ) design we have $b \geq v$. (Fisher's inequality)

Steiner triple systems

Definition. A $(b, v, 3, r, 1)$ BIBD is called a Steiner triple system.

Steiner triple systems are important examples of BIBD because 1.) they are relatively small, but applicable systems, 2.) we can give necessary and sufficient conditions on the other parameters for the existence of the Steiner triple systems.

¿From the general equalities $bk = vr$ and $r(k-1) = \lambda(v-1)$ obtained for the BIBDS substituting the values $k = 3$ and $\lambda = 1$ we get $r = (v-1)/(k-1) = (v-1)/2$ and therefore $3b = v(v-1)/2$. The later implies that

$$b = \frac{v(v-1)}{6}$$

Therefore v must be odd and $v(v-1)$ must be divisible by 6. A careful investigation shows that Steiner triple system only may exist if $v = 3$ or $v = 6n+1$ or $6n+3$ for $k \geq 1$.

These conditions turned out to be sufficient as well.

Theorem of Kirkman A Steiner triple system over v elements exists iff $v = 3$ or $v = 6n+1$ or $6n+3$ for $k \geq 1$

Finally the last theorem of these chapter is:

Theorem. If there is a Steiner triple system S_1 over v_1 elements and a Steiner triple system S_2 over v_2 elements, then their is a Steiner triple system over v_1v_2 elements.

Proof is by construction. If S_1 over the set $\{a_1, a_2, \ldots, a_{v_1}\}$ and S_2 is over the set $\{b_1, b_2, \ldots, b_{v_2}\}$ then let the new Steiner triple system S be defined over the v_1v_2 elements c_{ij}, $1 \leq i \leq v_1$, $1 \leq j \leq v_2$ where each c_{ij} represents the pair (a_i, b_j). A triple $\{c_{ir}, c_{js}, c_{kt}\}$ will be in S iff either $r = s = t$ and $\{a_i, a_j, a_k\}$ is in S_1 or $i = j = k$ and $\{b_r, b_s, b_t\}$ is in S_2 or both $\{a_i, a_j, a_k\}$ is in S_1 and $\{b_r, b_s, b_t\}$ is in S_2. The proof of the fact that the obtained system S is a Steiner triple system is left to the reader.

ANSWERS OR HINTS TO EXERCISES

1. BASIC COUNTING RULES

1. 2^{mn};

2. $5^5 + 4^5$;

3. $1 + 2 = 3$ if the dice are indistinguishable, $2 + 3 = 5$ if the dice are distinguishable;

4. $(5!)^2$;

5. (a) 7^4; (b) $\frac{7!}{3!}$; (c) 7^3; (d) $5 \times 4 = 20$;

6. Yes, since $2^9 > 500$;

7. (a) $\binom{29}{3}$; (b) $\binom{29+3-1}{3}$;

8. (a) $29 \times 28 \times 27$; (b) 29^3;

11. $\binom{5+12-1}{12}$;

12. 4×5^7;

13. (a) 0 if $n > 0$, 1 if $n = 0$; (b) 1; (c) $2^{n-1} - 1$; (d) $\binom{n}{2}$; (e) 1;

14. $\binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2} \times \frac{1}{5!}$;

15. $4!\ S(6, 4)$;

16. $S(25, 4)$;

18. $\binom{n-1}{k-1}$;

19. $\frac{12!}{(3!)^4}$;

20. (a) $5! \times 10!$; (b) $(5! \times 4! \times 6!) \times 3!$;

21. $\binom{5+8-1}{8}$ if the patients are not distinguished, 5^8 if the patients are distinguished;

22. $10^6 - 9^6$;

23. $26^4 + 25^5$;

24. (a) $(3!)$; (b) $(n!)^2$;

25. (a) $9 \times 9 \times 8 \times 7$; (b) $1 \times 9 \times 8 \times 7 + 8 \times 8 \times 8 \times 7$;

26. 2^{2^n};

27. $5 \times 8 \times 8 \times 7$;

29. $\binom{4+8-1}{8}$;

30. $\binom{5+12-1}{12}$;

31. $\binom{75+25-1}{25}$;

33. $\binom{4}{1} \times \left(\frac{10!}{4!2!2!2!}\right) + \binom{4}{2} \times \left(\frac{10!}{3!3!2!2!}\right)$.

2. INTRODUCTION TO GRAPH THEORY

1. Prove that any two vertices are connected by a path of length at most 2.

2. Yes.

3. Show that any two vertices are connected by a path of length at most 2. For $d(x) \geq n-1$, give a counterexample.

4. Prove that G contains a cycle and a path joining two vertices of the cycle whose other vertices are not on the cycle.

5. Use the equality $\sum_{x \in W} d_{\text{out}}(x) = \sum_{x \in W} d_{\text{in}}(x)$.

6. First find a vertex set $V_0 = \{v_1, v_2, \ldots, v_r\}$ such that (i) any other vertex is reachable from V_0 along an edge, (ii) $v_i v_j \notin E(D)$ for $1 \leq i < j \leq r$.

7. If the graph is k-regular then use divisibility by k.

8. E.g. prove the statement by induction on n.

9. Notice that if G and $\overline{G}$ are colored then the ordered color pairs give a coloring of K_n.

10. If not then the degrees are $0, 1, 2, \ldots, n-1$ where n is the number of vertices of the graph. However 0 and $n-1$ exclude each other.

11. Notice that if a graph has exactly one pair of vertices of equal degree then deleting a vertex of degree 0 and a vertex of degree $n-2$ or a vertex of degree 1 and a vertex of degree $n-1$ we obtain a graph of $n-2$ vertices with the same property.

12. Find a cycle and delete the edges of it.

13. Yes, if $n \geq 6$.

14. Take a longest path.

15. Take a longest path.

16. Color G with $\chi(G)$ colors. Notice that any two color classes are joined by an edge.

17. Use the fact that in a coloring of G, a color class is an independent vertex set.

18. (a) If we get a cycle joining any two non-adjacent vertices, then the graph is connected. Adding an edge to a tree result in exactly one cycle since any two vertices are joined by exactly one path in a tree. (b) If G is not complete then we do not have to assume acyclicness.

19. If the deletion of an edge results in a disconnected graph then it is not contained in any cycle (provided that the original graph was connected!).

20. Prove and use that a connected graph of n vertices has (at least) $n-1$ edges.

21. If the deletion of an edge results in a connected graph then it was contained in a cycle.

22. $\binom{6}{2} \times 4!$

23. $5 \times 4 \times 3$

24. A tree of $n-1$ edges has n vertices, so G_0 is a *spanning* tree of G. An acyclic (but not necessarily connected) graph of $n-1$ edges has at least n vertices, so G_0 is a *spanning* subgraph of G and an acyclic graph of n vertices and $n-1$ edges is a tree.

25. E.g.

26. Take a spanning tree if the graph is connected. If it is disconnected then take a component G_0 with $|E(G_0)| \geq |V(G_0)|$.

27. The existence of the paths implies the connectedness, the uniqueness implies the acyclicness of G. On the other side, the connectedness implies the existence of the paths and the acyclicness implies the uniqueness.

28. Prove that a connected graph of n vertices and m edges contains at least $m-n+1$ cycles. (Take a spanning tree and add the remaining edges to it one by one.)

29. Take a maximal acyclic subgraph containing the given one.

30. Prove that H is a spanning tree of G.

31. 2 if it has at least 2 vertices.

32. For 3, 6 and 7 such a graph does not exist, while for 4, 5 and 8 there is such a graph.

3. GENERATING FUNCTIONS

1. (a) $(1+x)^3(1+x^4)$ coefficient of x^5
(b) $(1+x+x^2+x^3+x^4+x^5)^2(x^3+x^4+x^5+x^6+x^7)$ coefficient of x^{10}
(c) $(1+x+x^2+\dots)^{10}$ coefficient of x^{15}
(d) $(x+x^2+x^3+\dots)^{10}$ coefficient of x^{15}
(e) $(1+x+x^2+\dots)^{40}$ coefficient of x^8
(f) $(1+x^2+x^4+\dots)^2$ coefficient of x^8 (if order does not count)
$(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots)^2$ coefficient of $\frac{x^8}{8!}$ (if order does count)
(g) $(1+x+x^2+\dots)^4$ coefficient of x^{40}
(h) $(1+x+x^2+\dots)^{100}$ coefficient of x^{40}
(i) $(1+x+x^2+x^3+x^4+x^5+x^6)^3$ coefficient of x^{12}
(j) $(1+x)^3(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots)$ coefficient of $\frac{x^3}{3!}$
(k) $(1+x+\frac{x^2}{2!}+\frac{x^3}{3!})^2(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots)^7$ coefficient of x^{10}
(l) $(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots)^2(1+x+\frac{x^2}{2!}+\dots)^2$ coefficient of $\frac{x^n}{n!}$
(m) $(x^{20}+x^{40}+x^{60}+x^{80}+x^{100})^4$ coefficient of x^{200}
(n) $(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots)^4$ coefficient of x^5
(o) $(1+x+x^2+\dots+x^{2n})^3$ coefficient of x^{3n}
(p) $(x+x^2+x^3+x^4+x^5+x^6)^{50}$ coefficient of x^{100} (the dice are distinguishable)

2. $\prod_{i=1}^{p}(1+x^{n_i})$ coefficient of x^k

3. $(1+x^2+x^4+\dots)^p$ coefficient of $x^k = a_k$

$$a_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \binom{p+m-1}{m} & \text{if } k = 2m \end{cases}$$

4. RECURRENCE RELATIONS

1. $b_n = 2b_{n-1} + 2b_{n-2}$
2. $f(n) = (2n-1)f(n-1)$
3. $F_{n+2} - 2$
4. $\frac{5^k+3^k}{2}$ (the recurrence is $a_{k+1} = 3a_k + 5^k$)
5. $f(n) = f(n-4) + f(n-6) + f(n-10)$
6. $a_k = \#$ codewords with an even number of 0's, odd number of 1's
$b_k = \#$ codewords with an even number of 0's, even number of 1's
$c_k = \#$ codewords with an odd number of 0's, even number of 1's
$3^k - a_k - b_k - c_k = \#$ codewords with an odd number of 0's, odd number of 1's

$$\begin{array}{lll} a_{k+1} = (3^k - a_k - b_k - c_k) + b_k + a_k & a_1 = 1 & (a_0 = 0) \\ b_{k+1} = c_k - a_k - b_k & b_1 = 1 & (b_0 = 0) \\ c_{k+1} = b_k + (3^k - a_k - b_k - c_k) + c_k & c_1 = 1 & (c_0 = 0) \end{array}$$

7. $f(n+1) = f(n) + 2n \qquad f(1) = 2$
8. (a) $a_k = \frac{1}{6}(8^k - 2^k)$; (b) $b_n = 1$; (c) $c_k = 10k\ 7^{k-1}$; (d) $d_n = n+1$; (e) $e_{2k} = 10\cdot 4^k$; $e_{2k+1} = 2^{2k+1}$; (f) $f_n = 5^n - 3^n$; (g) $g_n = 5^n - 3n\cdot 5^{n-1}$; (h) $h_k = 2^k - 2k + 1$; (i) the exponential generating function is $H(x) = \frac{2e^x-1}{1-2x}$, $i_k = 2^k k! + \frac{2^k k!}{1!} + \frac{2^{k-1}k!}{2!} + \ldots + \frac{2k!}{k!}$; (j) $j_n = \frac{1}{2}(3^n - 1)$; (k) $k_n = \frac{n^2+13n}{2}$; (l) $a_n = \frac{1}{15}(2^{2n+1} + 10 + 3(-1)^n)$, $b_n = c_n = \frac{1}{5}(4^n + (-1)^{n+1})$; (m) $a_n = 3^n + 2^n$, $b_n = 3^n - 2^n$.

5. THE PRINCIPLE OF INCLUSION AND EXCLUSION

1. 6233;

2. 134;

3. $4^5 - \binom{4}{1}3^5 + \binom{4}{2}2^5 - \binom{4}{3}1^5 + 0$ if the misprints are distinguished, $\binom{4+5-1}{5} - \binom{4}{1}\binom{3+5-1}{5} + \binom{4}{2}\binom{2+5-1}{5} - \binom{4}{3}\binom{1+5-1}{5} + 0$ if the misprints are not distinguished;

4. $7^8 - \binom{7}{1}6^8 + \binom{7}{2}5^8 - \ldots + \binom{7}{6}1^8 - 0$ if the accidents are distinguished, $\binom{7+8-1}{8} - \binom{7}{1}\binom{6+8-1}{8} + \binom{7}{2}\binom{5+8-1}{8} - \ldots + \binom{7}{6}\binom{1+8-1}{8} - 0$ if the accidents are not distinguished;

5. (a) $\binom{3+12-1}{12} - \binom{3}{1}\binom{3+6-1}{6} + \binom{3}{2}\binom{3}{0}$; (b) $\binom{3+8-1}{8} - 2\binom{3+4-1}{4} - \binom{3+3-1}{3} + \binom{3}{0}$; (c) $\binom{3+9-1}{9} - \binom{3+5-1}{5} - \binom{3+4-1}{4} - \binom{3+3-1}{3} + \binom{3}{0}$;

6. (a) $\binom{3+16-1}{16} - 3\binom{3+8-1}{8} + 3\binom{3}{0}$; (b) $\binom{4+18-1}{18} - 4\binom{4+10-1}{10} + \binom{4}{2}\binom{4+2-1}{2}$;

7. (a) $\binom{3+13-1}{13} - 3\binom{3+6-1}{6}$; (b) $\binom{4+14-1}{14} - 4\binom{4+7-1}{7} + \binom{4}{2}\binom{4}{0}$;

8. (a) $n! - (n-2)! - (n-1)! + (n-3)!$; (b) $n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \ldots + (-1)^{n-1}\binom{n-1}{n-1}1!$;

9. $10^n - \binom{3}{1}9^n + \binom{3}{2}8^n - \binom{3}{3}7^n$;

10. $\binom{9}{2,2,2,3} - 3\binom{8}{1,2,2,3} - \left[\binom{8}{2,2,2,2} - \binom{7}{2,2,2,1}\right] + \binom{3}{2}\binom{7}{1,1,2,3} + 3\left[\binom{7}{1,2,2,2} - \binom{6}{1,2,2,1}\right] - \binom{6}{1,1,1,3} - \binom{3}{2}\left[\binom{6}{1,1,2,2} - \binom{5}{1,1,2,1}\right] + \left[\binom{5}{1,1,1,2} - \binom{4}{1,1,1,1}\right]$.

6. THE PIGEONHOLE PRINCIPLE AND RAMSEY THEORY

1. The largest component has at least 15 vertices, the smallest one has at most 14 vertices.

2. Let a_i denote the number of matches on the first i days ($i = 1, 2, \ldots, 50$). Study the set of a_i's. (Pigeonholes: $\{1, 25\}, \{2, 23\}, \ldots, \{24, 48\}, \{49, 73\}, \{50, 74\}, \{51, 75\}, \{52\}, \{53\}, \ldots, \{72\}$.)

3. Let a_i denote the total number of hours the employee worked on the $(2i-1)$-st and the $2i$-th days ($i = 1, 2, \ldots, 5$), Estimate the maximum a_i.

4. Consider the remainder of $b_i = a_1 + a_2 + \ldots + a_i$ for $i = 1, 2, \ldots, p$.

5. Let a_i denote the total number of hours the computer was used on the $(3i-2)$-nd, $(3i-1)$-st and $3i$-th days ($i = 1, 2, \ldots, s$). Estimate the maximum a_i.

6. See Example 2.

7. See Example 1.

8. (a) 4; (b) 4; (c) 5; (d) 6.

9. Induction on n. Try to "build" a T_m in color one.

10. Induction on $p_1 + p_2 + \ldots + p_t$. Prove and use that

$$R_t(p_1, \ldots, p_t) \le R_t(p_1 - 1, p_2, \ldots, p_t) + R_t(p_1, p_2 - 1, \ldots, p_t) + \ldots + R_t(p_1, p_2, \ldots, p_t - 1) \ .$$

11. Hint: to every member of the sequence of length $n^2 + 1$ assign a pair of positive integers. The first item of the pair will be the length of the longest monotone increasing subsequence beginning with the given number, while the second item the length of the longest monotone decreasing subsequence. Show that we have no two numbers with the same pair. Give a counterexample for length n^2.