HOMEWORK SET #3 / CO1A / Spring 2020

1. Find the exponential convolution of two sequences, i.e., for two given sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ define the sequence $(c_n)_{n=0}^{\infty} = (a_n) \star (b_n)$ such that if $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$, $g(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$ and $h(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n$ then $h(x) = f(x) \times g(x)$.
2. Suppose that there are p different kinds of objects, each in infinite supply. Let a_k be the number of ways to pick k of the objects if we must pick at least one of each kind. Set up a generating function for (a_k) and find a_k for all k.
3. Let p_n^r be the number of partitions of the integer n into exactly r parts where order counts. Set up a generating function for the sequence (p_n^r) and find p_n^r.
4. A codeword from the alphabet $\{a, b, c, d, e, f\}$ is legitimate iff a and c appear even and b and d appear odd number of times (no constrain on e and f). Find the number of legitimate codewords of length k.
5. In each of the following, denote by a_n (with an appropriate choice of n) the answer to the question, set up the appropriate generating function for the sequences a_n, indicate what coefficient (which value of n) you are looking for, and finally calculate the answer to the question.
   a. In how many different ways can 4n letters be selected from $2n$ A's, $2n$ B's and $2n$ C's without order?
   b. Find the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 = 18$ where for each x_i $1 \le x_i \le 7$ (the order of the variables counts).
   c. How many ways can we place 25 indistinguishable balls into 8 distinguishable cells with no empty cell?
   d. In how many ways can a total of 80 be obtained if 50 distinguishable dice are rolled?
6. What is the coefficient of x^{100} in the product $(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots) \times (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots) \times (x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots)$?
7. A codeword over the alphabet $\{0, 1, 2, 3\}$ consists of at least one of each of the digits 0,1,2 and 3, and has length 20. How many such codewords are there?
8. Suppose that $G(x) = \frac{x}{x^2 - 3x + 2}$ is the ordinary generating function of the sequence (a_k). Find a_k. (Hint: factor $x^2 - 3x + 2$ into linear terms and find $G(x)$ as $\frac{a}{\text{first linear term}} + \frac{b}{\text{second linear term}}$)