BOLYAI SOCIETY
MATHEMATICAL STUDIES, 14

A Panorama of Hungarian Mathematics in the Twentieth Century, pp. 373–382.

Theodore von Kármán

STUART S. ANTMAN

Theodore von Kármán (szöllőskislaki Kármán Tódor) was born in Budapest in 1881 and died in Aachen in 1963. In 1902 he received his undergraduate degree in Engineering from the Royal Joseph University of Polytechnics and Economics in Budapest. In 1908, under the direction of the eminent fluid-dynamicist Ludwig Prandtl, he received his doctorate from the University of Göttingen for his work on the buckling of columns. He served there as a Privatdozent under Prandtl until 1913, when he became Professor of Aeronautics and Mechanics at the Technical University of Aachen. In 1929 he left for the California Institute of Technology in Pasadena, where he spent the rest of his life.

Von Kármán's degrees were in engineering, his academic appointments were in engineering, and virtually all of his research was devoted to engineering science and to practical questions about the design of aircraft and missiles. He was an adept experimentalist. He always identified himself as an engineer. He became a celebrity as an engineer in the United States. And yet, von Kármán had marked mathematical ability, he was intimately associated with the great mathematicians of Göttingen and respected by them (they seemed to view him as mathematics' favorite engineer (see {38}*), many of his research papers were regarded as applied mathematics par excellence, he effectively exploited his reputation as a consummate engineer to promote the mathematical training of engineers, and he greatly influenced work in applied mathematics. (The noted fluid-dynamicist W. R. Sears, a student of von Kármán, wrote, "It was clear to those of us who worked close to him that mathematics—applied mathematics—was his true love." {39, p. 176}.)

*In this article, all reference numbers, enclosed in brackets, correspond to the list at the end of this article.

In this article, I discuss a small sampling of von Kármán's scientific work that could be regarded as applied mathematics when it was published. (Discussions of his contributions to technology and of his role as administrator, government consultant, and public figure can be found in {9, 12, 13, 32}.)

The von Kármán equations for plates. At the invitation of Felix Klein {32, pp. 52–53}, von Kármán {15} prepared the 75-page article *Festigkeitsprobleme in Maschinenbau* {15} for the *Encyklopädie der mathematischen Wissenschaften* edited by Klein. (That this invitation was made when von Kármán had just received his doctorate testifies to the esteem with which he was held by the mathematical community at Göttingen.) This survey of structural mechanics, i.e., the mechanics of deformable rods and shells, derived the governing differential equations (mostly linear) and analyzed some specific problems for them.

Von Kármán began his very brief treatment of the deformation of elastic plates with a discussion of the Kirchhoff theory, which characterizes the small transverse displacement w of a thin (homogeneous, isotropic) plate (of constant thickness) under the action of a transverse force of intensity f per unit area as the solution of

$$D\Delta^2 w = f$$

where

$$\Delta^2 u := \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}$$

is the two-dimensional biharmonic operator acting on a function u, and where D is a positive constant accounting for the stiffness of the plate; D is proportional to the cube of the thickness h. Von Kármán then observed that this model is valid only if w is small relative to the thickness of the plate. To construct a theory capable of describing larger displacements, von Kármán replaced the linear relations between the in-plane strains and the displacements with the correct nonlinear relations, but retained other geometric simplifications, and took the relation between stress and strain to be linear. By this process, in the span of one page, he came up with the celebrated von Kármán equations for plates:

$$D\Delta^2 w - h[\Phi, w] = f,$$

$$\Delta^2 \Phi = -\frac{1}{2}E[w, w]$$

where

$$[u, v] := \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}$$

is the Monge–Ampère operator acting on the functions u and v, and E is the elastic modulus. The function Φ is a stress function whose second derivatives deliver the resultant contact forces (stress resultants) in the plane of the plate.

Although the beauty of the von Kármán equations inherent in the presence of the biharmonic and Monge–Ampère operators could not fail to attract mathematicians, their semilinearity put these equations beyond the analytic resources available at the time. Indeed, in his influential expository paper {28} of 1940, von Kármán called upon mathematicians to bring their still primitive tools of nonlinear analysis to bear on these equations. (It seems to me that von Kármán presented these equations to the mathematical community in 1940 with an assurance as to their value that was lacking in 1910.) In the meantime, von Kármán {24, 26, 29} had demonstrated the crucial role of nonlinearity in the buckling of shells. (The problems discussed in these three papers continue to provide challenges for analysis.)

Friedrichs and Stoker {10} answered the call in 1941. Their lengthy work, which influenced the development of bifurcation theory in the United States, was the first rigorous mathematical analysis of the von Kármán equations. (Friedrichs, a student of Courant's at Göttingen, had been sent by Courant to work with von Kármán at Aachen {38}.) In the mid-1950's began an intensive analysis of existence, multiplicity, and bifurcation of solutions to boundary-value problems for the von Kármán and related equations (see {6, 7, 43}). The fascinating role of these equations as an inspiration for Rabinowitz's {37} global bifurcation and continuation theory is detailed in {1}.

That the von Kármán equations, obtained by an ad hoc combination of theory with insight, represent an improvement over the traditional Kirchhoff theory has inspired several directions of research in shell theory and in its mathematical analysis: (i) The derivation of the von Kármán and related equations systematically (albeit formally) as the leading term of an asymptotic expansion in a thickness parameter {5, 6, 8}. (ii) The derivation of "geometrically exact" equations for the large motion of shells {2, 11, 35} (which do not rely on any geometric approximation and which describe new phenomena. Underlying von Kármán's derivation of his equation are approximations analogous to replacing the sin function by its cubic approximation.) (iii) The still largely open problem of deriving sharp estimates for the errors between solutions of equations for shells and those for the 3-dimensional theory {3}.

Throughout his scientific career, von Kármán maintained a research interest in problems of solid mechanics. His work on the buckling of elastic structures has become a standard part of the engineering theory of elastic stability. His work on plasticity and plastic buckling have had an important influence on modern developments {14}. But von Kármán's main research and engineering efforts after 1914 were increasingly directed towards fluid dynamics.

The von Kármán vortex street. Von Kármán received his first recognition in fluid dynamics when he explained the failure of a student of Prandtl's, despite herculean efforts, to get rid of oscillations in the experimental measurements of pressure on the surface of a circular cylinder obstructing the flow of a steady stream of water {32, pp. 62 ff.}. Von Kármán first supposed that the oscillations are in fact present, and that they are caused by water rolling up into two trails of vortices (eddies) breaking off from the top and bottom of the cylinder. (Many years earlier, Helmholtz had observed the formation of vortices in the flow past a flat plate.) When the assumption that the vortices were shed simultaneously led to unacceptable instabilities, von Kármán assumed that they were shed alternately. He then determined the spacings of these alternating vortices that are stable. Specifically, he severely idealized the problem {16}: He considered the 2-dimensional irrotational flow of an invisicid incompressible fluid produced by two parallel rows of equally spaced vortices, with one row of vortices rotating in one direction and the other row in the opposite direction, and with each vortex of one row opposite a midpoint of a pair of vortices of the other row. Since all the rotation is concentrated at the singular points holding the vortices, the flow is irrotational away from them. Consequently, the conjugate of the complex velocity is the derivative of meromorphic function determined by the poles at the vortices. Von Kármán was able to ignore the source of the vortices, the cylinder, by regarding it as shifted to infinity. In other words, he was studying a steady state that could conceivably exist away from the source. He analyzed the linear stability of the flow by perturbing the locations of the vortices. Remarkably, the stable dispositions conform well to what was observed in experiment. For accessible discussions of the physical and mathematical setting of this work see {34, 36, 41}.

This work provided an explanation of a major and hitherto unknown source of drag. The collapse of the Tacoma Narrows bridge in 1940 (discussed in detail in {31}) is attributed to the resonant forcing produced by a similar vortex structure that was shed by solid fences when the bridge was subjected to a steady transverse wind.

The statistical theory of turbulence. Von Kármán, like Prandtl, had long been concerned with the puzzling phenomena of turbulence, making important contributions in {17, 19}. His most notable contribution to the subject was to endow the statistical theory of turbulence initiated by G. I. Taylor with a rich and useful mathematical structure. In the words of S. Goldstein {12, p. 349}, " ... [H]e dealt mainly with a general systematic development of [Taylor's theory in {21, 22, 23}], the last with L. Howarth. Von Kármán pointed out that the correlations between two velocity components at any two points at a distance r apart are the components of a tensor, which is a function of the vector distance between the points. In the case of isotropy, the correlation divided by the mean square velocity depends on just two scalar functions of the distance r and the time t. In an incompressible fluid, the equation of continuity yields a relation between these two scalar functions, so only one is involved. If the triple products of components of velocities at the two points are neglected, an equation can then be derived from the equations of motion for changes in this single scalar, which can be used to obtain information about the rate of decay of the turbulence. The triple correlations were first neglected in this way, but this is incorrect, as G. I. Taylor pointed out. Von Kármán in fact explictly stated that if this is incorrect the vortex filaments would have a permanent tendency to be stretched or compressed along the axis of vorticity, and thought this was not the case; Taylor pointed out that the facts showed that it was, there being a tendency for the vortex filaments to stretch on the average. Von Kármán and Howarth showed that the triple correlation tensor also involves only one scalar function for the case of isotropy for an incompressible fluid, and that the correlation between pressure and velocity is zero in this case. A partial differential equation connecting the double and triple correlation functions was then derived, and equations for the dissipation of energy and vorticity deduced." Throughout the next 15 years, von Kármán continued to contribute novel ideas to the subject of turbulence. For a technical account of some of this work see {4}.

Mathematical methods in engineering. Following in the footsteps of his father Mór (Moritz), whose role in modernizing the Hungarian educational system earned him a 'von', von Kármán did much to modernize the mathematical training of engineers in the United States and elsewhere. In the 1930's the mathematical sophistication of American engineers was far inferior to that which von Kármán picked up in Göttingen and which he found valuable in his own work. In pushing for a far richer (but not too rich an) exposure to real mathematics for engineers, von Kármán demonstrated the same political astuteness that served him so well in dealing with bureaucracies as a public figure: In two publications {25, 30} in the 1940's directed to engineers on the role of mathematics in engineering, he prominently identified himself as an engineer and put 'engineer' or 'engineering' in the titles. (These works have a flavor different from that of {18, 28} directed to mathematicians.) In these works he cited stereotypical criticisms of pure mathematicians: They are concerned with proving the existence of solutions to equations that every engineer knows to have solutions on physical grounds, and if mathematicians were ever to solve specific problems, they would employ the simplest possible geometries (just as von Kármán did for his vortex street). Having thus demonstrated that he was not a sycophant of mathematics, he was then positioned to advocate effectively for the enrichment of engineers' actual mathematical education and also for the incorporation of mathematical notions in their scientific courses. (He was thus trying to prevent American engineering students from experiencing his own unhappy exposure to engineering sciences at the Royal Joseph University, about which he said, "The conventional courses, such as hydraulics, electricity, steam engineering, or structures, were taught like baking or carpentry, with little regard for the understanding of nature's laws which underlie the sciences" {32, p. 26}.) The popular and valuable book {27}, written with M. Biot, significantly advanced this program. It contained elementary treatments of ordinary differential equations, linear algebra, Bessel functions, Fourier methods, and finite differences in the setting of classical and structural mechanics.

Aeronautics and astronautics. Whereas liquids like water are virtually incompressible, gases are not, and the effects of compressibility in gases become pronounced when they move at speeds exceeding about a fifth of the speed of sound. The type of the governing partial differential equations depends crucially upon whether the fluid is viscous, whether it is compressible, and the local speed at which it moves. The most striking effect of compressibility is the appearance of shocks (strictly speaking for an inviscid compressible fluid), which are discontinuities in the derivatives of the velocity field and in the pressure field. Von Kármán had published some early papers on gas dynamics. In the 1930's, well before high-speed flight became a reality, he advocated the creation of a comprehensive theory and began his fundamental work on it with {20}. In the 1940's, he began serious work on rockets and jet propulsion, which would be the main focus of his activities for the rest of his life. To handle the practical complexities of high-speed flight both near and away from the earth, he promoted the development of aerothermochemistry in which fluid-dynamical, thermal, and chemical effects are coupled, as for example in combustion. It was not long before many of these ideas formed the heart of graduate teaching in aeronautics. The most accessible scientific treatment of his work in this area is in his own posthumous tract {33}.

Summary. Von Kármán's work on fluid dynamics was immediately assimilated into the main stream of the general theory and forms an extensive contribution of permanent value. Accounts of much of this work can be found in standard references on fluid dynamics. Von Kármán's work on solid mechanics, on the other hand, represented pioneering attacks on nonlinear problems of great theoretical and practical importance. His analyses continue to challenge his successors, but they cannot be said to represent permanent contributions, partly because the nonlinear problems he grappled with had not yet been subsumed under a cohesive and mature theory like that of fluid dynamics.

Appreciations of von Kármán's scientific contributions are given in numerous obituaries and memorials, among which are {9, 12, 40, 42, 44} all by fluid-dynamicists. The best place to start to learn of the personal side of von Kármán is his autobiography {32}, which is valuable also for his discussion of his research.

References

[84] Kármán, Tódor, *Collected Works of Theodore von Kármán,* Volumes 1–4, Butterworths Scientific Publications (London, 1956); Volume 5, Von Kármán Institute for Fluid Dynamics, Rhode-St. Genese (Belgium, 1975).

{1} S. S. Antman, The influence of elasticity on analysis: Modern developments, *Bull. Amer. Math. Soc.* (New Series), **9** (1983), 267–291.

{2} S. S. Antman, *Nonlinear Problems of Elasticity,* Springer, 1995.

{3} I. Babuška and L. Li, The problem of plate modeling: Theoretical and computational results, *Comp. Meths. in Appl. Mech. Engg.,* **100** (1992), 249–273.

{4} G. K. Batchelor, *The Theory of Homogeneous Turbulence,* Cambridge Univ. Pr. (1953).

{5} P. G. Ciarlet, A justification of the von Kármán equations, *Arch. Rational Mech. Anal.,* **73** (1980), 349–389.

{6} P. G. Ciarlet, *Mathematical Elasticity, Volume II: Theory of Plates,* North-Holland (1997).

{7} P. G. Ciarlet & P. Rabier, *Les Equations de von Kármán,* Springer (1980).

{8} J.-L. Davet, Justification de modèles de plaques nonlinéaires pour des lois de comportment générales, *Mod. Math. Anal. Num.,* **20** (1986), 147–192.

{9} H. Dryden, Theodore von Kármán, 1881–1963, *Biog. Mem. Nat. Acad. Sci.,* **38** (1966), 345–384.

{10} K. O. Friedrichs and J. J. Stoker, The nonlinear boundary value problem of the buckled plate, *Amer. J. Math.,* **63** (1941), 839–888.

{11} G. Friesecke, R. D. James, and S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity, *Comm. Pure Appl. Math.,* **55** (2002), 1461–1506.

{12} S. Goldstein, Theodore von Kármán, 1881–1963, *Biog. Mem. Fellows Roy. Soc.,* **12** (1966), 335–365.

{13} M. H. Gorn, *The Universal Man, Theodore von Kármán's Life in Aeronautics,* Smithsonian Inst. (1992).

{14} J. Hutchinson, Plastic buckling, *Advances in Applied Mechanics,* Vol. 14, Academic Press (1974), 67–144.

{15} Th. von Kármán, *Festigkeitsprobleme in Maschinenbau,* in *Encyklopädie der mathematischen Wissenschaften,* Vol. IV/4, (1910), 311–385, edited by F. Klein and C. Müller, Teubner, reprinted in [84, Vol. 1, 141–207].

{16} Th. von Kármán, Über den Mechanismus des Widerstandes, den ein bewegter Körper in einer Flüssigkeit erfährt, *Göttinger Nachr.* (1911), 509–517, (1912), 547–556, reprinted in [84, Vol. 1, 324–338].

{17} Th. von Kármán, Über laminäre und turbulente Reibung, *Z. angew. Math. Mech.,* **1** (1921), 233–252, reprinted in [84, Vol. 2, 70–97].

{18} Th. von Kármán, Mathematik und technische Wissenschaften, *Naturwiss.,* **18** (1930), 12–16, reprinted in [84, Vol. 2, 314–321].

{19} Th. von Kármán, Mechanische Ähnlichkeit und Turbulenz, *Göttinger Nachr.* (1930), 58–76, reprinted in [84, Vol. 2, 322–336].

{20} Th. von Kármán and N. B. Moore, Resistance of slender bodies moving with supersonic velocities, with special reference to projectiles, *Trans. Amer. Soc. Mech. Engrs.* (1932), 303–310, reprinted in [84, Vol. 2, 376–393].

{21} Th. von Kármán, On the statistical theory of turbulence, *Proc. Nat. Acad. Sci.,* **23** (1937), 98–105, reprinted in [84, Vol. 3, 222–227].

{22} Th. von Kármán, The fundamentals of the statistical theory of turbulence, *J. Aero. Sci.,* **4** (1937), 131–138, reprinted in [84, Vol. 3, 228–244].

{23} Th. von Kármán and L. Howarth, On the statistical theory of isotropic turbulence, *Proc. Roy. Soc. A,* **164** (1938), 192–215, reprinted in [84, Vol. 3, 280–300].

{24} Th. von Kármán and H.-S. Tsien, The buckling of spherical shells by external pressure, *J. Aero. Sci.,* **7** (1939), 43–50, reprinted in [84, Vol. 3, 368–380].

{25} Th. von Kármán, Some remarks on mathematics from the engineer's viewpoint, *Mechanical Engrg.* (1940), 308–310, reprinted in [84, Vol. 4, 1–6].

{26} Th. von Kármán, L. G. Dunn, and H.-S. Tsien, The influence of curvature on the buckling characteristics of structures, *J. Aero. Sci.,* **7** (1940), 276–289, reprinted in [84, Vol. 4, 7–31].

{27} Th. von Kármán and M. Biot, *Mathematical Methods in Engineering,* McGraw-Hill (1940).

{28} Th. von Kármán, The engineer grapples with nonlinear problems, *Bull. Amer. Math. Soc.,* **46** (1940), 615–683, reprinted in [84, Vol. 4, 34–93].

{29} Th. von Kármán and H.-S. Tsien, The buckling of thin cylindrical shells under axial compression, *J. Aero. Sci.,* **8** (1941), 303–312, reprinted in [84, Vol. 4, 107–126].

{30} Th. von Kármán, Tooling up mathematics for engineering, *Quart. Appl. Math.,* **1** (1943), 2–6, reprinted in [84, Vol. 4, 189–192].

{31} Th. von Kármán, L'aérodynamique dans l'art de l'ingénieur, *Mém. Soc. Ingénieurs de France.* (1948), 155–178, reprinted in [84, Vol. 4, 372–393].

{32} Th. von Kármán, *The Wind and Beyond,* Little–Brown (1967).

{33} Th. von Kármán, *From Low-Speed Aerodynamics to Astronautics,* Pergamon (1963).

{34} L. M. Milne-Thomson, *Theoretical Hydrodynamics,* 5th edition, Macmillan (1968).

{35} P. M. Naghdi, Theory of Shells, in: *Handbuch der Physik,* Vol. VIa/2, edited by C. Truesdell, Springer (1972), pp. 425–640.

{36} L. Prandtl, *Essentials of Fluid Dynamics,* Blackie (1952).

{37} P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.,* **7** (1971), 487–513.

{38} C. Reid, *Courant,* Springer (1976).

{39} W. R. Sears, Recollections of Theodore von Kármán, *J. Soc. Indust. Appl. Math.,* **13** (1965), 175–183.

{40} W. R. Sears, The scientific contributions of Theodore von Kármán, 1881–1963, *Phys. Fluids,* **7** (1964), v–viii.

{41} A. Sommerfeld, *Mechanics of Deformable Bodies,* Academic Press (1964).

{42} G. I. Taylor, Memories of von Kármán, 1881–1963, *J. Fluid Mech.,* **16** (1963), 478–480.

{43} I. I. Vorovich, *Nonlinear Theory of Shallow Shells,* Springer (1999).

{44} F. L. Wattendorf and F. J. Malina, Theodore von Kármán, 1881–1963, *Astronautica Acta,* **10** (1964), 81–92.

Stuart S. Antman

Department of Mathematics
Institute for Physical Science and Technology
and Institute for Systems Research
University of Maryland
College Park, MD 20742-4015, U.S.A.

ssa@math.umd.edu