BOLYAI SOCIETY
MATHEMATICAL STUDIES, 14

A Panorama of Hungarian Mathematics in the Twentieth Century, pp. 427–454.

Discrete and Convex Geometry

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1. Introduction

Mathematical research in Hungary started with geometry: with the work of the two Bolyais early in the 19th century. The father, Farkas Bolyai, showed that equal area polygons are equidecomposable. The son, János Bolyai, laid down the foundations of non-Euclidean geometry. The study of geometric objects has been continuing ever since. The present chapter of this book is devoted to describing what investigations took place in Hungary in the 20th century in the field of convex and combinatorial geometry. This includes incidence geometries, finite geometries, and stochastic geometry as well. The selection of the material is, of course, a personal one, and some omissions are inevitable (though most likely unjustified). The choice is made difficult by the wide variety of topics that were to be included.

Besides mathematics, or discrete and convex geometry (to be more precise), this survey is about people, is about mathematicians. Whenever appropriate, I have tried to say a few words not only about the mathematics but the person as well. There are quite a lot of them, but the heroes of the story are two giants who stand out head and shoulder above the rest. They are László Fejes Tóth and Pál Erdős. Both of them helped to create the school of Hungarian discrete geometry, and both of them were extremely successful problem solvers and exceptionally prolific problem raisers. Yet they were of different taste, style, and character. Their questions, their results, and, in general, their mathematics have, to a large extent, determined what discrete and convex geometry in Hungary means, and how and

*Partially supported by Hungarian Science Foundation Grant T 032452 and T 037846.

in what directions the Hungarian school of geometry has developed. I have tried to group geometric research in Hungary around topics, and not so much around people. Sometimes this has been difficult and occasionally I have had to take the liberty of proceeding differently.

2. The beginnings

Geometric research in Hungary, after the work of the two Bolyais, besides containing many scattered results, is concentrated around two topics. One is geometric constructions, and the other one is equidecomposability of equal area polygons.

Geometric constructions has always been a popular topic in high school mathematics in Hungary, and the theory of geometric constructions had been popular subject in geometric research as well. The starting point is a theorem of Hilbert's from Grundlagen der Geometrie (Teubner, 1899) saying that usual geometric constructions (with ruler and compass) can be accomplished by a ruler alone if a circle, together with its centre, is drawn in the plane. (He also showed that the centre is necessary.) Hilbert proved, further, that the compass is needed only to copy segments on a line, that is, the ruler and the ability of copying segments suffice for the usual geometric constructions. Continuing Hilbert's work, József Kürschák proved (Math. Ann., **55** (1902), 597–598) that whatever can be constructed using a ruler and the ability of copying segments, can be constructed by a ruler and the ability of copying a fixed segment, say the unit segment. The algebraist Mihály Bauer proved (Ungar. Ber., **20** (1905), 43–47), that in Kürschák's "unit segment" theorem the unit segment cannot be replaced by any fixed, unmovable segment, or even by any fixed, unmovable polygon. Richárd Obláth continues these investigations. He simplifies Kürschák's proof (Math. Phys. Lapok, **18** (1909), 174–176). In Obláth (Monatsh. Math., **26** (1915), 295–298) it is proved that in Hilbert's theorem the circle can be replaced by an arbitrary arc of the circle (its centre is, of course, necessary). This was also shown by Gyula Szőkefalvi-Nagy (Tohoku J. Math., **40** (1934), 76–78). Many decades later, Obláth proved (Matematikai Lapok, **2** (1951), 219–221) that, the circle and its centre can be replaced by an arbitrary arc together with the two points that split the arc into three arcs of equal length. Gyula Szőkefalvi-Nagy wrote a book on geometric constructions: A geometriai szerkesztések elmélete (Kolozsvár, 1943). Pál Szász and Gyula Strommer had also worked in this area.

Farkas Bolyai's equidecomposablity theorem has been a popular subject as well. Mór Réthy (Ann. Math., **38** (1891), 405–428) extends the result from polygons to some other planar regions, and Zsigmond Spiegl (Math. Phys. Lapok, **2** (1893), 17–30) gives a new proof of the result. 60 years later Tamás Varga found a short and transparent proof of Farkas Bolyai's theorem (Mathematikai Lapok, **5** (1954, 101–114).

In (Ungar. Ber., **15** (1898), 196–197) Kürschák gave a purely geometric proof of the fact that the area of the regular twelvegon P is exactly three times the area of the square Q whose side length is the radius of circle, circumscribed about P. The proof is accomplished by decomposing P and three copies of Q into finitely many congruent pieces. W. Csillag (Ungar. Ber., **19** (1903), 70–73) gives an alternative proof, based on a remark of Kürschák.

Concerning best approximation problems, József Kürschák gave the first elementary proof (see [41], page 6) of the following inequalities. Let R resp. r the circumradius and inradius of a convex n-gon K_n with area A and perimeter L. Then

$$nr^2 \tan\frac{\pi}{n} \le A \le \frac{1}{2} nR^2 \sin\frac{2\pi}{n},$$

and

$$2nr\tan\frac{\pi}{n} \le L \le 2nR\sin\frac{\pi}{n},$$

and equality holds in all inequalities if and only if K_n is the regular n-gon.

There are further geometric results from the turn of the century. For instance, Lipót Klug (Monatsh. Math., **10** (1889), 84–87) proves the following interesting generalization of Pythagoras's theorem. Denote by $[x_1, \dots, x_k]$ the $(k-1)$-dimensional volume of the simplex with vertices $x_1, \dots, x_k \in \mathbb{R}^n$, where $k \le n$. If $v_1, \dots, v_n$ is a system of n pairwise orthogonal vectors in $\mathbb{R}^n$, then

$$\sum \left[v_{i_1}, \dots, v_{i_k}\right]^2 = (n-k+1) \sum \left[0, v_{j_1}, \dots, v_{j_{k-1}}\right]^2,$$

where the summation is taken over all k, resp. $(k-1)$ membered subsets of $\{1, \dots, n\}$. It is readily seen that the case $k = 2$ is a simple consequence of the Pythagoras theorem.

Gusztáv Rados (Ungar. Ber., **22** (1907), 1–12) considers regular star-n-gons inscribed in the unit circle. (A star-n-gon is obtained by connecting two vertices of a reguler n-gon when there are exactly $k-1$ vertices between them; k must be relative prime to n.) Their number is clearly $\frac{1}{2}\varphi(n)$ (where $\varphi(n)$ is Euler's totient function). The star-n-gon is subdivided by its edges into cells, the cell containing the centre is called the "kernel". Denote the sum of the areas of the $\frac{1}{2}\varphi(n)$ kernels by I_n, and the area of the regular convex n-gon, circumscribed about the unit circle by C_n. Then the ratio I_n/C_n is a rational number which is equal to $\frac{1}{4}\big(\varphi(n)+\mu(n)\big)$ where $\mu(n)$ is the Möbius function. As expected, the proof uses geometry and number theory. It is shown further that if n is a prime, then $C_{2n}I_n = n^2/2$.

Gyula Vályi (1855–1913) was a respected professor in Kolozsvár, who lectured on various subjects. He was almost blind. He did some work in geometry. For instance, (Math. Phys. Lapok, **10** (1901), 309–321), he considers the foot-triangle, $A_1B_1C_1$, of the triangle ABC where A_1 is the foot of the altitude starting at A and B_1 and C_1 are defined similarly. The foot triangle has its own foot triangle A_2, B_2, C_2, and so on. Can the nth foot triangle be similar to the original triangle? Vályi shows that there are exactly $2^n(2^n-1)$ (non-similar) triangles that are similar to their nth foot-triangle. Vályi together with Gyula Kőnig was also interested in perspective triangles and tetrahedra.

Dénes Kőnig's (1884–1944) main interest was graph theory, but had a few nice results in geometry as well. For instance, his joint paper with Adolf Szűcs (Mat. Természettud. Ért., **31** (1913), 545–558) investigates the orbit of a point in the 3-dimensional cube when it starts moving in direction v, and is reflected like light whenever it meets the boundary of the cube. They show that the orbit is periodic if and only if the ratio of any two components of v is rational (v is a rational vector, for short), the orbit is everywhere dense if and only if v is not orthogonal to any rational vector, and if v is orthogonal to exactly one rational vector, then the orbit lies on the boundary of a polyhedron and is everywhere dense there. He proves Helly's theorem (Math. Zeitschrift, **14** (1922), 208–210); the proof is identical with Helly's original proof that only appeared in 1923. (Helly found his famous theorem in 1913 but could not publish it because of the First World War. The first proof, by Radon, appeared in 1921.)

In this book, there are two long chapters about Lipót Fejér and his work in analysis. In his student years, Fejér had been attracted to geometry where he surprised his colleagues by beautiful elementary proofs. One survives, see [140] or [40], Vol. II, pp. 844–847: Assume ABC is an acute triangle.

Then its foot-triangle has the the smallest perimeter among all triangles XYZ where the point X comes from the line through B, C, the point Y from the line through C, A, and Z from the one through A, B.

Of course, János Bolyai's ground-breaking result on non-Euclidean geometry, whose real importance was understood quite late, was a central topic in the mathematical life of the time. In 1897, János Bolyai's Appendix appeared in Hungarian translation for the first time. Even more significantly, there was to be a volume on the achievements of mathematics, edited by Poincaré, which was to contain a chapter on "Géométrie de Lobatschewsky". The title of this chapter finally became "Géométrie de Bolyai et Lobatschewsky", thanks to the work of a committee consisting of G. Rados, B. Tőttössy, J. Kürschák, and L. Kopp. In another development, under the auspices of Gyula Kőnig and Mór Réthy, the Tentamen of Farkas Bolyai appeared a second time in two volumes, the first in 1897, the second in 1904.

3. Packings and coverings by circles

László Fejes Tóth has been working in geometry since 1939. His interests are very broad: packing and covering, approximation, isoperimetric inequalities for polytopes, and much more. We start by describing his ground-breaking research in the theory of packings and coverings.

One of László Fejes Tóth's early results is a new proof (the first correct proof, according to C. A. Rogers) of a theorem of Thue from 1882:

Thue's theorem ([41], page 58). *The density of any packing of congruent circles in the plane is at most* $\pi/\sqrt{12}$.

Here and in what follows packing means a collection of pairwise (internally) disjoint sets, while a covering is a collection of sets whose union contains the set which is to be covered. Density has just the usual definition: on a bounded set D it is the total area of the circles divided by $\operatorname{Area} D$, and one takes the limit if the set in question is not bounded. Dual to Thue's theorem is that of Kershner.

Kershner's theorem ([41], page 58). *The density of any covering of the plane with congruent circles is at least* $2\pi/\sqrt{27}$.

László Fejes Tóth has proved many extensions and generalizations of these results. The theory of "packings and coverings" he developed in 2 and 3-dimensions is the content of his book Lagerungen in der Ebene, auf dem Kugel und im Raum [41]. The extension of the theory to higher dimension is carried out in the book by C. A. Rogers, Packing and covering, Cambridge, 1964. This extension is rather restricted since the higher dimensional packing and covering problems are much more difficult, and as a consequence, there are only few results about them.

We describe now some generalizations of Thue's and Kershner's theorem that are due to László Fejes Tóth, see [41], page 67.

Theorem. *Every packing of (at least two) congruent circles in a convex domain has density at most* $\pi/\sqrt{12}$.

Theorem. *Every covering of a convex domain by (at least two) congruent circles has density at least* $2\pi/\sqrt{27}$.

When one is only interested in the asymptotic behaviour of an extremal system of circles, the shape of the domain does not matter much. So it is quite natural to consider hexagons instead of general convex domains. The dual problems of packing and covering can be unified in the following way. How to place congruent circles in the plane, when the density is given a priori, and we want to maximize the area covered by the circles. The answer is in the following theorem, see [41], page 80.

Theorem. *Given a hexagon of area* H *and a system of congruent circles of total area* T*, let* A *denote the area of that part of the hexagon that is covered by the circles. Then* $A \leq A^*$ *where* A^* *is the area of the intersection of the circle of area* T *and a concentric regular hexagon of area* H.

This result is a special case of the so-called Moment Theorem (see [41], page 81 and Section 5 below for this particular application) which was invented in connection with isoperimetric problems for polyhedra (see later). The Moment Theorem has found several further extensions and applications in the works of P. M. Gruber and Gábor Fejes Tóth.

4. Packings and coverings by incongruent circles

The problems become more involved when incongruent circles are used. In a joint work of L. Fejes Tóth and J. Molnár (Math. Nachr., **18** (1958), 235–243), any kind of circle of radius r from the interval $[a, b]$ can be used, and the question is how to choose and arrange such circles to obtain a densest possible packing, or a thinnest possible covering. Upper and lower bounds are given for the densities in question. József Molnár constructed examples of packings and coverings, using only circles of radius a and b, that are almost optimal. An interesting remark from [41], page 79 says that if the ratio b/a is close to one, then the density is maximal if the system is the densest lattice packing of congruent circles. This line of research has been continued by Károly Böröczky, Aladár Heppes, András Bezdek, Károly Bezdek, József Molnár, Gábor Fejes Tóth, and others.

Another result concerning packings with incongruent circles (see [41], page 75) says that if a hexagon H contains n non-overlapping circles with radii $r_1, \ldots, r_n$, then $(r_1 + \ldots r_n)^2 \le n \operatorname{Area} H \sqrt{12}$. This means, roughly speaking, that the total area of circles, packed in a hexagon, is maximal if they are congruent and each is touched by six others.

It is perhaps worth mentioning here that in this field there is still much to be discovered.

5. Packings and coverings by convex sets in the plane

In connection with packings and coverings it is quite natural to consider not only circles but other convex bodies as well. The following far-reaching generalization of Thue's theorem is due to László Fejes Tóth ([41], page 85).

Theorem. *Let $K \subset \mathbb{R}^2$ be a convex body, and let P_6 be a hexagon, of minimum area, circumscribed about K. If n congruent (non-overlapping) copies of K are packed in a convex hexagon H, then $n \operatorname{Area} P_6 \le \operatorname{Area} H$.*

In the proof one grows the congruent copies, K_i, of K into (non-overlapping) convex polygons R_i with $K_i \subset R_i$. Next, Euler's theorem on planar graphs implies that the total number of sides of the R_is is at most $6n$. Then Dowker's theorem (see a little later) and an application of the Jensen inequality finishes the proof. For details see [41]. When K is a circle, then $\operatorname{Area} P_6 = \frac{\pi}{\sqrt{12}} \operatorname{Area} K$, which gives Thue's theorem. A very general corollary of the previous theorem says the following:

Corollary. *No packing of congruent, centrally symmetric convex bodies in the plane can have density larger then that of the densest lattice packing of the same convex body.*

Rogers proved that the above theorem remains valid for non-symmetric convex bodies K as well provided only translated copies of K are allowed to form the packing. A beautiful and short proof of this fact was given by L. Fejes Tóth (Mathematika, **30** (1983), 1–3). In the covering version of the previous theorem an extra (and most likely only technical) condition is needed, namely, that the covering is "non-crossing". Two copies K_i and K_j are said to be crossing if removing their intersection from their union, both sets split into disjoint parts.

Theorem ([41], page 86). *Let $K \subset \mathbb{R}^2$ be a convex body, and let p_6 be a hexagon, of maximum area, inscribed in K. If n congruent and pairwise non-crossing copies of K cover a convex hexagon H, then $n \,\mathrm{Area}\, p_6 \geq \mathrm{Area}\, H$.*

An interesting generalization of the last three theorems is due to Gábor Fejes Tóth (Acta Math. Acad. Sci. Hungar., **23** (1972), 263–270). It goes as follows.

For a convex body $K \subset R^2$ of unit area let $f_K(x)$ denote the maximum area of the intersection of K and a hexagon of area x. Further, let $\bar{f}_K(x)$ be the least concave function greater than or equal to $f(x)$. Given a convex hexagon of area H and a system of congruent non-crossing copies of K with total area T, let A denote the area of that part of the hexagon which is covered by the copies of K. Then

$$A \leq T \bar{f}_K(H/T).$$

This bound is sharp if K is centrally symmetric. In this case, for given density T/H an arrangement of a large number of copies of K for which A is arbitrarily close to the upper bound is generally not lattice-like, but is given by an appropriate combination of two lattice arrangements. This phenomenon shows a remarkable analogy with the phase transition of crystals, as Gábor Fejes Tóth and László Fejes Tóth remark (Computers Math. Applic., **17** (1989), 251–254).

This is the point where a result of István Fáry (1922–1984) should be mentioned. Besicovitch proved that every convex body K in the plane contains a centrally symmetric hexagon whose area is at least $2/3 \,\mathrm{Area}\, K$. Fáry gave an alternative proof of this and characterized the case of equality

(Bull. Soc. Math. France, **78** (1950), 152–161). Moreover, he used it in the following result on packings and coverings in the plane (see [40], page 100).

Theorem. *Assume $K \subset \mathbb{R}^2$ is a convex body. Let $\delta_L(K)$ and $\theta_L(K)$ denote the density of the densest lattice packing and the density of the thinnest lattice covering of K (by translates of K) in the plane. Then*

$$\delta_L(K) \geq \frac{2}{3} \quad \text{and} \quad \theta_L(K) \leq \frac{3}{2},$$

with equality if and only if K is a triangle.

We mention, however, that the question whether $\theta(K) \leq 3/2$ for non-lattice coverings by translates of a triangle K is still wide open. There are plenty of such questions in this area.

6. Packings and coverings on the sphere

Packing and covering of the 2-dimensional sphere by spherical caps is a problem, analogous to the previous. For instance, Thammes's question asks for the densest packing of n circles (caps) on the sphere. The dual problem is that of the thinnest covering by n circles (caps) of the sphere. L. Fejes Tóth gives upper resp. lower bounds for the densities in question. Set $\omega_n = \frac{n}{n-2}\frac{\pi}{6}$.

Theorem ([41], page 114). *When $n \geq 3$ congruent caps are packed on the sphere, then their density is at most*

$$\frac{n}{2}\left(1 - \frac{1}{2\sin\omega_n}\right).$$

When the sphere is covered by $n \geq 3$ congruent caps, then their density is at least

$$\frac{n}{2}\left(1 - \frac{1}{\sqrt{3}\tan\omega_n}\right).$$

These inequalities solve the question of densest packing, resp. thinnest covering of the sphere for $n = 3, 4, 6, 12$ (see [41]). The proof shows at the same time, that the extremal systems are formed by the circles, inscribed in, resp. circumscribed about, the faces of the regular mosaics with symbols $\{k, 3\}$ with $k = 2, 3, 4, 5$. (These mosaics, for $k > 2$, correspond to a regular polytope in 3-space whose facets are regular k-gons.) Also, the above result gives an alternative proof of the theorems of Thue and Kershner (when n goes to infinity), and shows that the extremal configuration corresponds to the regular mosaic $\{6, 3\}$ which is the usual tiling of $\mathbb{R}^2$ by regular hexagons.

These questions are closely related to isoperimetric problems about polytopes. For instance, assume that V is the volume, F is the surface area of a convex polytope $P \subset \mathbb{R}^3$ with f faces. What's the smallest value of the so-called isoperimetric quotient F^3/V^2? Steiner conjectured that if the polytope is combinatorially equivalent to a regular polytope, then the isoperimetric quotient is minimal for the corresponding regular polytope. (Steinitz had doubts about this conjecture.) László Fejes Tóth proved the validity of the conjecture for the case of the tetrahedron, cube, and dodecahedron ($f = 4, 6, 12$) with the following theorem (see [41], page 135, and [42], page 283).

Theorem. *Under the above conditions*

$$\frac{F^3}{V^2} \geq 54(f-2)\tan\omega_f(4\sin^2\omega_f - 1).$$

The proof of Steiner's conjecture goes via Lindelöf's theorem (stating that any polytope, extremal to the isoperimetric quotient is circumscribed about the sphere), and the so-called Moment Theorem (see [42], page 219). The Moment Theorem has various forms, here we give the one used in the plane ([41], page 81). We assume that $g : R^+ \to R^+$ is an increasing function, $H \subset \mathbb{R}^2$ is a convex hexagon and $a_1, \ldots, a_n$ are points in H. Set finally $d(x) = \min\big\{ |x - a_1|, \ldots, |x - a_n| \big\}$.

Moment Theorem. *Under these conditions*

$$\int_H g\big(d(x)\big)\, dx \leq n \int_h g\big(|x|\big)\, dx,$$

where h is a regular hexagon, of area Area H/n, *centered at the origin.*

There are several other isoperimetric problems concerning packings and coverings. For instance, in a given a hexagon $H \subset \mathbb{R}^2$ n non-overlapping convex bodies are placed. What is the minimum of the total perimeter of these n convex bodies, if each has area at least a? This is the *perimeter problem.* Or what is the maximum of the total area of the n convex bodies, if each has perimeter at most p? This is the so called *area problem,* which was solved by László Fejes Tóth, [42], page 175. He and Aladár Heppes solved the perimeter problem, see [42], page 174. The extremal configuration consists, in both cases, either of arbitrarily arranged circles, or of certain "smooth polygons" inscribed in the faces of (a bounded piece) of the regular mosaic $\{6, 3\}$. Heppes proved further (Publ. Math. Inst. Hungar. Acad. Sci., **8** (1964), 365–371) that in the area problem the same conclusion holds without assuming the convexity of the pieces.

7. Packings and coverings in the hyperbolic plane

Packing and covering problems arise naturally in hyperbolic spaces as well. However, it is impossible to define the density of a packing or covering that would satisfy the most natural and simple requirements. This was pointed out by an ingenious example due to Károly Böröczky. László Fejes Tóth gives an alternative notion of densest packing and thinnest covering, that of "solidity'. A packing of convex bodies is called *solid* if altering the positions of finitely many of the bodies, and leaving the remaining ones unmoved, the new packing obtained this way is always congruent with the original one. The definition is analogous for coverings. A solid packing (and covering) on the sphere and on the Euclidean plane is automatically the densest (thinnest, resp.). So the following theorems, due to Margit Imre (Acta Math. Hung., **15** (1964), 115–121), generalize the corresponding spherical and Euclidean results. We mention that the regular mosaic $\{n, 3\}$ forms a tessellation of the sphere for $n = 2, 3, 4, 5$, of the Euclidean plane for $n = 6$, and of the hyperbolic plane for $n > 6$.

Theorem. *For every $n \geq 2$, the circles inscribed in the faces of the regular mosaic with symbol $\{n, 3\}$ form a solid packing.*

For every $n \geq 3$, the circles circumscribed about the faces of the regular mosaic with symbol $\{n, 3\}$ form a solid covering.

We mention in passing that András Bezdek extended the first statement from the above theorem by showing that for $n > 7$ the packing is "super-solid", which means that removing n of the circles and putting back only $n - 1$ so as to form a packing, one gets back the original system minus one circle. The question is still open for $n = 6, 7$.

The Hungarian school of discrete geometry was founded by László Fejes Tóth and was strongly influenced by his results, insight and inspiring questions. László Fejes Tóth had the exceptional ability of addressing the right question to the right people. Several young mathematicians-to-be started working in discrete geometry because of an intriguing problem of László Fejes Tóth, and became discrete geometers under his guidance and influence. The Hungarian school of discrete geometry has been incessantly working on the "Fejes Tóth" theory of packings and coverings. The following names must be mentioned here: Imre Bárány, András Bezdek, Károly Bezdek, Károly Böröczky, Károly Böröczky Jr., Gábor Fejes Tóth, Zoltán Füredi, Aladár Heppes, Jenő Horváth, Gábor Kertész, Endre Makai Jr., Emil Molnár, József Molnár, and János Pach.

8. Packings and coverings in higher dimensions

In higher dimensions the problem of densest packing and thinnest covering (of congruent balls, say) is much harder. The densest sphere packing problem in 3-dimensional space goes back to Kepler and is part of problem 18 of Hilbert's famous set of unsolved problems. In the early 50's László Fejes Tóth made a significant step toward the solution of this problem. He proposed a strategy which, if carried out succesfully, solves the problem by reducing it to a finite optimization problem. A decade later, he even foresaw the possibility of using computers in the solution. In [42], page 300, László Fejes Tóth writes that "...this problem can be reduced to the determination of the minimum of a function of a finite number of variables, providing a programme realizable in principle. In view of the intricacy of this function we are far from attempting to determine the exact minimum. But, mindful of the rapid development of our computers, it is imaginable that the minimum may be approximated with great exactitude."

We close this section by recalling two general covering theorem of Erdős and Rogers. One is from (J. London Math. Soc., **28** (1953), 287–293) and gives the first nontrivial lower bound on the density of any covering, by congruent balls, of $\mathbb{R}^d$. The second result (Acta Arithmetica, **7** (1962), 281–285) is very general and has been used often. Its proof is a powerful combination of random methods and maximal lattice packing.

Theorem. *For every convex body $K \subset \mathbb{R}^d$ there exists a covering of $\mathbb{R}^d$ by translates of K whose density is less than $d(\log d + \log\ \log d + 4)$ and so that no point is covered more than $ed(\log d + \log\ \log d + 4)$ times.*

9. Approximation

Approximation problems of convex bodies by special classes of convex bodies, usually by polytopes with few vertices or faces belong to the theory of convex sets. László Fejes Tóth has made pioneering research in this direction and used his results in the theory of packings and coverings.

Assume $K \subset \mathbb{R}^2$ is a convex body. Denote by T_n (resp. t_n) the area of the maximal (minimal) area convex n-gon inscribed in (circumscribed about) K. Answering a question of Kershner, Dowker (Bull. AMS., **50** (1944), 120–122) proved that the sequence $T_3, T_4, \dots$ is concave, and the sequence $t_3.t_4, \dots$ is convex. In other words,

$$T_{n-1} + T_{n+1} \leq 2T_n \quad \text{and} \quad t_{n-1} + t_{n+1} \geq 2t_n.$$

László Fejes Tóth and József Molnár extended this result to inscribed (circumscribed) polygons with maximal (minimal) perimeter: (Molnár, Matematikai Lapok, **26** (1955), 210–218; Fejes Tóth, Math. Phys. Semesterber., **6** (1958/59), 253–261).

It had been generally known or assumed that, among all d-dimensional convex bodies, the Euclidean ball can be approximated worst. When we measure approximation by an affinely invariant quantity (like missed area) the extreme case should be the set of ellipses.

An early example of this phenomenon is given by a theorem of Ernő Sas. Answering a question of L. Fejes Tóth, he proved the following, (see E. Sas, Compositio Math., **6** (1939), 468–470), and [41], page 36, as well. For a convex body $K \subset \mathbb{R}^2$, T_n still denotes the maximal area of an inscribed convex n-gon. Then

$$T_n \geq \operatorname{Area} K \frac{n}{2\pi} \sin \frac{2\pi}{n},$$

with equality if and only if K is an ellipse.

Dezső Lázár, a promising young mathematician (who became a victim of holocaust in 1942) proved the following result, (Lázár, Acta Univ. Szeged, **11** (1947), 129–132) and [41], page 40, as well. We keep the previous notation t_n and T_n. Then

$$\frac{T_n}{t_n} \geq \cos^2 \frac{\pi}{n},$$

with equality if and only if K is an ellipse. The analogous statement for the best approximation in perimeter was proved by László Fejes Tóth ([41], page 30).

There is a strong relation between best approximation of a convex body $K \subset \mathbb{R}^2$ and its affine perimeter, when approximation is measured by an affinely equivariant quantity. This was pointed out by László Fejes Tóth. The affine perimeter was defined by Blaschke as $\int \kappa^{1/3} ds$ where the integration goes on the boundary of K according to arc-length. Fejes Tóth proves, among other similar results, the following one. Again, K and t_n have the same meaning as above.

Theorem. *Assume K has twice differentiable boundary, and its affine perimeter is λ. Then $\lambda^3 \leq 8t_n n^2 \sin^2(\pi/n)$.*

This result implies Blaschke's affine isoperimetric inequality, which says, that $\lambda^3 \leq 8\pi^2 \operatorname{Area} K$. Fejes Tóth proves a further remarkable property of the affine perimeter: in a given triangle ABC, among all curves connecting A to B within the triangle, and bounding, together with the side AB, a fixed area, the largest affine arc-length goes with the cone-section that touches the side AC at A, and BC at B.

L. Fejes Tóth proves [41], page 89, the following result concerning affine perimeter and packings by convex bodies:

Theorem. *Assume n convex bodies are packed in a hexagon H, and let Λ denote the sum of the affine perimeters of the convex bodies. Then*

$$\Lambda^3 \leq 72n^2 \operatorname{Area} H.$$

10. The Erdős–Szekeres theorem

Discrete geometry in Hungary was born when Erdős and his friends (Tibor Gallai, György Szekeres, Pál Turán, Eszter Klein, and many others) were very young and became interested in all kinds of combinatorial questions. A good example of this is the so-called Erdős–Szekeres theorem, which grew out of the following observation of Eszter Klein from 1934. From every set of five points in general position in the plane one can choose four that are in convex position, where k points are said to be in convex position if none of them is contained in the convex hull of the other $k-1$. Erdős immediately generalized the question and, together with Szekeres, proved the following result.

Erdős–Szekeres theorem. *For every $k \geq 4$ there exists a finite number $N(k)$ such that every set $X \subset \mathbb{R}^2$ in general position with $|X| \geq N(k)$ contains k points that are in convex position.*

This appeared in Erdős, Szekeres (Compositio Math., **2** (1935), 463–470), in the proof they rediscover Ramsey's theorem. They published a second paper on this problem 26 years later (Ann. Univ. Budapest, Sectio Math., **3-4** (1961), 53–62). They prove, denoting by $N(k)$ the smallest integer the theorem is valid for, the following bounds

$$2^{k-2} + 1 \leq N(k) \leq \binom{2k-4}{k-2}.$$

Earlier Endre Makai Sr. and Pál Turán showed that $N(5) = 9$, and Eszter Klein's observation gives $N(4) = 5$. This is in accordance with the conjecture that $N(k) = 2^{k-2} + 1$ which has become known as the Happy End Problem because Eszter Klein and Szekeres got married, escaped from Hungary to Australia via Shanghai (because of the holocaust) and have been living happily ever since. (Actually, there is no other evidence than $N(4) = 5$ and $N(5) = 9$ for the conjecture.)

Later, Erdős asked whether among sufficiently many points (in general position) in the plane one can always find the vertices of an *empty k-gon,* that is, k points in convex position such that their convex hull contains no more points from the original set. This turned out to be true for $k = 3, 4, 5$, false for $k > 6$. Very annoyingly, the problem is still open for $k = 6$.

There are several further extensions, generalizations, and applications of the Erdős–Szekeres theorem that are beyond the scope of this survey. For instance, the theory of order types (started by Goodman and Pollack) grew out of an attempt to prove the Happy End Conjecture. The recent overview of these developments by W. Morris and V. Soltan (Bull. AMS., **37** (2000), 437–458) lists more than 200 references. The Hungarian school of discrete geometers, namely Imre Bárány, Tibor Bisztriczky, Gábor Fejes Tóth, Zoltán Füredi, Gyula Károlyi, János Pach, József Solymosi, Géza Tóth, have been actively pursuing Erdős–Szekeres type phenomena.

11. Repeated distances, distinct distances in the plane

Erdős was interested in all kinds mathematics, he knew very well that mathematics develops by asking questions, as they constitute the raw material mathematicians can work on. He himself was a prolific problem raiser, often more proud of a good question he asked than a theorem he proved. He once said that he had never been jealous of a result of someone else, but he had often been jealous of a good problem someone else asked. He raised several questions a day, some based on new insight or new theorems, some in the hope of getting closer to the solution of the some old problem, sometimes the question came just out of curiosity. With the following two questions (Erdős, Amer. Math. Monthly, **53** (1946), 248–250), he struck gold:

At most how many times can a given distance occur among a set of n points in the plane?

What is the minimum number of distinct distances determined by a set of n points in the plane?

To be more formal, let X be a set of n points in the plane, and let $f(X)$ denote the number of pairs $x, y \in X$ such that their distance $|x-y|$ is equal to one, and let $g(X)$ denote the number of distinct distances $|x-y|$, $x, y \in X$. Define

$$f(n) = \max f(X), \quad \text{and} \quad g(n) = \min g(X).$$

With this notation, Erdős's question is to find, or at least estimate, $f(n)$ and $g(n)$. These two questions have turned out both extremely hard and extremely influential.

Erdős proves, in the same paper, that $f(n) \leq cn^{3/2}$. In the proof Erdős uses a simple geometric argument to show that the graph of unit distances (with vertex set X) does not contain the complete bipartite graph $K_{2,3}$. Since such a graph cannot have more than $cn^{3/2}$ edges, the upper bound on $f(n)$ follows immediately. This is the first application of extremal graph theory in combinatorial geometry, that has been followed by many others. The effect is mutual and mutually beneficial: a question in combinatorial geometry often leads to a problem in extremal graph or hypergraph theory. Erdős did pioneering work in this direction. The best upper bound to date is $f(n) \leq cn^{4/3}$ (due to Spencer, Szemerédi, Trotter). Here is another formulation of the "unit distances" question: given n points in the plane and the n unit circles centred at these points, how many point-circle incidences can occur among them? In this form, the question immediately leads to incidence problems to be discussed in Section 12.

Again in the same paper, Erdős gives the lower bound, (which is conjectured to be the proper order of magnitude of $f(n)$):

$$f(n) > n^{1+c/\log\log n}.$$

The construction is just the $\sqrt{n} \times \sqrt{n}$ grid; the proof uses a little number theory. The same construction gives, for the number of distinct distances, that

$$g(n) \le \frac{cn}{\sqrt{\log n}}.$$

This is again the conjectured value of $g(n)$. Moser gave the lower bound

$$g(n) > cn^{2/3}$$

which has been improved several times by methods combining geometry and combinatorics. The current best lower bound (due to Katz and G. Tardos, based on earlier work of Solymosi and Cs. Tóth) is $cn^{.864\ldots}$. (A recent result of Imre Ruzsa shows that the current techniques cannot give anything of the form $n^{8/9}$.)

The problem changes if one strengthens the non-collinearity condition on X by assuming, say, that the points are in convex position, or that X is in general position. The convexity condition gave rise to the theory of *forbidden submatrices.* For the general position case, Erdős, Füredi, Pach, Ruzsa (Discrete Math., **111** (1993), 189–196) show that

$$g^{\text{gen}}(n) \le ne^{\sqrt{c\log n}},$$

while the lower bound $(n-1)/3$ is due to Szemerédi. In the same paper Erdős et al. show that, if X contains no three points on a line and no four on a circle, then the inequality $g(X) \le C|X|$ does not hold for any constant C. The proof uses a celebrated result of Freiman from additive number theory.

Erdős also asked, in his 1946 Monthly paper, how often the maximal, minimal distance can occur among pairs of points of a set $X \subset \mathbb{R}^2$. The minimal distance problem has been completely solved in $\mathbb{R}^2$, but not in higher dimensions. The maximal distance can occur n times in $\mathbb{R}^2$, and $2n-2$ times in $\mathbb{R}^3$ (the latter result is due to Heppes and Grünbaum). For higher dimensions, the Lenz construction (see in the next chapter) gives asymptotically optimal point sets. A more general question concerns the distribution of distances. Erdős, Lovász, Vesztergombi (Discrete Comp. Geom., **4** (1989), 341–349) investigate the graph determined by the k largest distances.

Concerning the possible distribution of distances, Ilona Palásti constructed examples of point sets $X \subset \mathbb{R}^2$ with $|X| = k$ for $k = 4, 5, 6, 7, 8$ where the $k(k-1)/2$ distances occur with very special distribution: one distance occurs once, another twice, a third three times, etc. See for instance Palásti (Discrete Math., **76** (1989), 155-156). In general, she was working on geometric problems proposed by Erdős, we will encounter another result of hers in Section 14.

12. Repeated and distinct distances elsewhere

Of course the same questions can be asked in any dimension. Denoting the corresponding functions by $f_d(n)$ and $g_d(n)$, Erdős proved (Publ. Math. Inst. Hung., **5** (1960), 165–169) that

$$cn^{4/3} \leq f_3(n) \leq cn^{5/3}.$$

By now there are better estimates for $f_3(n)$. The behaviour of $f_d(n)$ for $d > 3$ is simple, because of the so-called Lenz construction, (see the same paper of Erdős): half of the points are on the circle $(x, y, 0, 0)$ with $x^2 + y^2 = 1/2$, the other half on the circle $(0, 0, u, v)$ with $u^2 + v^2 = 1/2$. This gives that $f_4(n)$ is asymptotically $n^2/4$. Even more precise information on $f_d(n)$ is available. The question of distinct distances does not, however, become simpler. Here Erdős proved, still in the 1946 Monthly paper, that

$$cn^{3/(3d-2)} \leq g_d(n) \leq cn^{2/d}.$$

Many of these results have been improved since, and many by the Hungarian school of combinatorial geometry: József Beck, Zoltán Füredi, Endre Makai Jr., János Pach, Imre Ruzsa, László Székely, Endre Szemerédi, Csaba Tóth, Gábor Tardos.

Erdős, together with Hickerson and Pach (Amer. Math. Monthly, **96** (1989), 569–577) consider the same problem on the 2-dimensional unit sphere S^2 and show that every distance $d \in (0, 2)$ can occur $cn \log^* n$ times, and the special distance $\sqrt{2}$, surprisingly, occurs $cn^{4/3}$ times; this bound is optimal. (Here $\log^* n$ is the number one has to take logarithm from n to get below 2.)

A minor modification of the Lenz construction shows, further, that the maximal distance in $\mathbb{R}^d$, $d \geq 4$ can occur asymptotically

$$\frac{n^2}{2}\left(1-\frac{1}{\lfloor d/2\rfloor}\right)$$

times. The maximal distance question is related to the famous Borsuk conjecture stating that every set $S \subset \mathbb{R}^d$ can be partitioned into $d+1$ sets of smaller diameter. So the modified Lenz construction was an indication that the Borsuk conjecture might be false. This turned out to be the case later, from dimension 1000 onwards (but with a different example).

It is natural to ask the same questions about angles, directions instead of distances, and Erdős, of course, was asking, popularizing, and answering such questions. For details, see the survey by Erdős, Purdy: Extremal problems in combinatorial geometry (Handbook of Combinatorics, North Holland, (1995)). The following intriguing problem of Erdős is again of a similar kind: How many similar copies of a regular pentagon can an n element planar point set contain? The answer, by Erdős and Elekes (Intuitive Geometry, Colloq. Math. Soc. János Bolyai **63**, 85–104, North-Holland, 1994) is surprising: the construction of a *pentagonal lattice* in $\mathbb{R}^2$ contains cn^2 regular pentagons. Far reaching generalizations of this construction were given by Miklós Laczkovich and Imre Ruzsa.

The two questions asked by Erdős in 1946 started a novel and exciting research field in discrete geometry that has given rise to many beautiful results and hundreds of new problems. Erdős himself writes in his 80th birthday volume: "My most striking contribution to geometry is, no doubt, my problem on distinct distances".

13. Incidences

In the Educational Times in 1893, J. J. Sylvester raised the following question. Assume n points are given in the plane, not all of them on a line. Is it true then that they determine an ordinary line, that is, a line containing exactly two of the given n points. It seems that the problem lay dormant until Erdős revived it some 40 years later. Soon after that Tibor Gallai (1912–1992) found a beautiful proof which appeared (Amer. Math. Monthly, **51** (1944), 169–171) as a solution to a question posed by Erdős.

The following Euclidean Ramsey theorem, probably the first of its kind, is also due to Gallai: Given a finite set $P \subset \mathbb{R}^d$, and a colouring of $\mathbb{R}^d$ by r colours, there always exists a monochromatic and homothetic copy of P. Gallai never published this result which appeared first in R. Rado (Sitzungsber. Preuss. Akad. Wiss., Phys.-Math., **16/17** (1933), 589–596).

Now back to the Sylvester–Gallai theorem, which clearly implies that n points (not all of them on a line) determine at least n lines. A far reaching combinatorial generalization of this fact (including the case of finite projective planes) was proved by Erdős, de Bruijn (Indag. Math., **10** (1948), 421–423): Suppose $\{A_1, \dots, A_m\}$ are proper subsets of a ground set $\{a_1, \dots, a_n\}$ Suppose also that each pair a_i, a_j occurs in one and only one A. Then $m \geq n$.

Motivated, among others, by the Sylvester–Gallai theorem, Erdős conjectured that given n points in the plane, the number of lines containing at least $\sqrt{n}$ of the points is at most $c\sqrt{n}$ (where c is some positive constant). This was proved by Szemerédi and Trotter, and independently and about the same time by József Beck. In fact, Szemerédi and Trotter proved a much stronger conjecture of Erdős which says that the number of incidences between n points and m lines in the plane cannot exceed $O\big(m^{2/3}n^{2/3}+m+n\big)$. A minor modification of Erdős's construction for the upper bound for $f(n)$ shows that this bound is best possible (apart from the implied constant). This conjecture of Erdős, which is now called Szemerédi–Trotter theorem, has turned out to be a central result in the theory of complexity of line arrangements. It is not only point-line incidences that are important, but point-curve incidences as well. The curves here should by defined by fixed degree polynomials. This type of problems have been considered by Szemerédi, Beck, Pach, Székely, Tóth. We have seen above that the "unit distance" problem of Erdős can be formulated as a question on incidences between points and unit circles.

Incidence problems are closely related to the complexity of geometric objects. For instance, a set of n lines dissects the plane into cells. The complexity of a cell is the number of lines incident to the cell. In computational geometry, interest is frequently focused on the complexity of a cell, or the total complexity of some cells, or the sum of the complexities of all cells. The smaller this complexity is, the simpler the description of the system. Miraculously, or maybe not so miraculously, the complexity bounds are often close to the corresponding incidence bounds. Here is a sample theorem (due to Clarkson et al. (1990)):

Theorem. *Given a system of n lines in the plane, and some m distinct cells they determine, the total number of edges bounding one of these m cells is at most $c\big(m^{2/3}n^{2/3}+n\big)$.*

This estimate is best possible. This is shown, again, by a small modification of Erdős's $\sqrt{m}\times\sqrt{m}$ grid construction.

This is perhaps the point where the problem of halving lines should be mentioned. Given a set $X\subset\mathbb{R}^2$ of n points in general position (with n even), how many pairs $x,y\in X$ determine a halving line? That is, a line that has $(n-2)/2$ points of X on both sides. Denote this number by $h(X)$ and define $h(n)=\min h(X)$. What's the value of $h(n)$?

This innocent looking question is still unsolved. László Lovász proved in 1972, that the number of halving lines is at most $(2n)^{3/2}$, the lower bound $cn\log n$ is due to Erdős et al. (Proc. Internat. Symp., Fort Collins, Colo. (1973), 139–149, North-Holland). The best bounds, currently known are $O\big(n^{4/3}\big)$ (upper bound, by Tamal Dey) and $\Omega\big(ne^{\sqrt{\log n}}\big)$ (lower bound, by Géza Tóth). The dual to the halving lines problem is that of the complexity of the mid-level of an arrangement of n lines. This turned out to be important in computational geometry. Higher dimensional variants and analogous questions have been intensively investigated by the Hungarian school of discrete geometry, namely by Bárány, Füredi, Lovász, Pach, Szemerédi, Tardos, Tóth.

The following theorem, due to Erdős and Péter Komjáth (Discrete Comp. Geom., **5** (199)), 325–331), is just a sample of similar results from an interesting mixture of discrete geometry, combinatorics, and set theory.

Theorem. *The continuum hypothesis is equivalent to the existence of a colouring of the plane, with countably many colours, with no monochromatic right angled triangles.*

14. Miscellaneous results in combinatorial geometry

We have mentioned Tibor Gallai's result on ordinary lines. Gallai mainly worked in combinatorics, graph theory and was extremely modest, and had not published much. (But, according to Erdős, he should have published a theorem that he had proved which later became known as Dilworth's theorem.) However, a question of Gallai which appeared first in Fejes Tóth's book [41], page 97, motivated by combinatorial analogues, has proved to be very important and has become the starting point of a whole theory. This question is related to Helly's theorem: Assume that a system of unit circles in the plane has the property that any two of them have a point in common. Does this condition imply the existence of a set $F \subset \mathbb{R}^2$ of at most k points such that F intersects every circle in the family. (The answer is yes: Danzer proved that $k = 4$ always works, and cannot be improved, earlier Ungár and Szekeres showed $k \leq 7$, and L. Sztachó proved $k \leq 5$.)

József Molnár was mainly working in the theory of packings and coverings. He has an interesting Helly-type result as well. The question is the incidence structure of a finite family of convex sets in $\mathbb{R}^n$, which is only solved for $n = 1$. Molnár proves (Matematikai Lapok, **8** (1957), 108–117) the following generalization of Helly's topological theorem.

Theorem. *Let $\mathcal{C}$ be a finite family of connected compact sets in $\mathbb{R}^2$, $|\mathcal{C}| \geq 3$. Assume any two of the sets have connected intersection, and any three have nonempty intersection. Then there is a point common to all sets in $\mathcal{C}$.*

Danzer and Grünbaum proved that if every angle spanned by three points of a set $X \subset \mathbb{R}^d$ is at most $\pi/2$, then X has at most 2^d elements. (The cube shows that this bound is sharp.) They conjectured that, for $n \geq 3$, the size of X is at most $2n - 1$ if all angles spanned by three points of X are strictly smaller than $\pi/2$. This conjecture turned out to be absolutely wrong: Erdős and Füredi (Combinatorial Mathematics, North Holland Math. Studies **75** (1983), 275–283) constructed a set, X, of $n = 1.15^d$ points in $\mathbb{R}^d$ such that all angles spanned are acute. The construction is a random subset of the vertices of the unit cube, with a few unsuitable vertices deleted. A similar construction (in the same paper) gives a set X of size $(1 + \delta)^d$ with all distances within X are almost all equal: any two of them are at distance $\left(1 + O\left(\sqrt{\delta}\,\right)\right)$.

Ákos Császár has been working mainly in measure theory and topology. In 1949 he constructed a "polyhedron without diagonals", that is, a 3-dimensional polyhedron P with triangular faces and straight edges such that each pair of vertices is connected by an edge. P has seven vertices and is homeomorphic to the torus (see Császár, Acta Sci. Math. Szeged, **13** (1949), 140–142). This beautiful construction has become known as Császár's torus in the literature.

In (Acta Sci. Math. Szeged, **11** (1948), 229–233) István Fáry proves that every every planar graph can be drawn in the plane so that its edges are non-crossing straight line segments. (Actually, this follows from a remarkable theorem of Koebe from 1936, but the connection was not known at the time.) Erdős considered the problem of straight line planar representation of graphs with few crossing edges. For instance, Alon and Erdős show (Discrete Comp. Geom., **4**, (1989), 287–290) that any straight line planar drawing of a graph with n vertices and $6n-5$ edges contains three pairwise disjoint edges. This type of problems about *geometric graphs* was initiated by Erdős and Perles. By now, due to the work of János Pach and his students, the theory of geometric graphs is an exciting new field on the boundary of geometry and graph theory, rich with beautiful results and intriguing questions.

In connection with Sylvester's Orchard problem (Educational Times, **59** 1893) Ilona Palásti, together with Füredi (Proc. AMS., **92** (1984), 561–566) constructs a set of n lines, A_n, such that the number of triangles determined by the cell decomposition defined by A_n is $\frac{1}{3}n(n-3)$. A_n is a simple arrangement (no three lines concur), and it is known that the number of triangles determined by a simple arrangement of n lines is at most $\frac{1}{3}n^2 + O(n)$. So A_n is an asymptotically optimal arrangement.

15. Finite geometries

The outstanding Hungarian number theorist and algebraist, László Rédei, had made several interesting excursions to geometry. The first is closer to algebra than to geometry and is, in fact, about polynomials and finite geometries. Let p be a prime and U a subset of p elements of the affine plane over $GF(p)$. What Rédei (together with Megyesi) proves in [149] is that U determines at least $(p+3)/2$ directions unless it is a line. Further research in this direction is due to Blokhuis, Szőnyi, Lovász, and Schrijver.

We mention in passing that the analogous question (due to Erdős) for the Euclidean plane was solved by Péter Ungár (J. Comb. Theory Ser. A., **20** (1967)). His result says that $2n$ non-collinear points in the plane determine at least $2n$ distinct directions. The proof uses allowable sequences, or order types, if you like.

Rédei gave a new proof (J. London Math. Soc., **34** (1959), 205–207) of a result of Delone stating that, given a 2-dimensional lattice L, there always exists a lattice parallelogram P, such that $L \cap P$ consists of the vertices of P and these four vertices lie in four different quadrants of the plane. (The origin need not belong to L.) The "book-proof" of this theorem was found by János Surányi (Acta Sci. Math. Szeged, **22** (1961), 85–90), together with several applications.

János Surányi has been working mainly in number theory, and in geometry of numbers, in particular. He gave beautiful combinatorial geometric proofs of Wilson's theorem and Fermat's little theorem (Matematikai Lapok, **23** (1972), 25–29; joint work with K. Härtig).

We have encountered the name of Endre Makai Sr., in connection with the Erdős–Szekeres theorem. In (Mat. Fiz. Lapok, **50** (1943), 47–50) he gave an elementary proof of the fact that an empty lattice triangle has area 1/2.

Ferenc Kárteszi's field of interest was projective and later finite geometries. He ran a popular seminar on this subject. He and his disciples (G. Korchmáros, E. Boros, G. Kiss, M. H. Nguyen, T. Szőnyi and others) extended the notion of affine regular n-gon to finite geometries, see for instance, G. Kiss (Pure Math. Appl. Ser. A, **2** (1991), 59–66). An interesting result of Kárteszi (Publ. Math. Debrecen, **4** (1955), 16–27) says that, given n points in the plane, no three on a line, no point can be contained in more than $n^3/24$ of the triangles, spanned by the points.

16. Stochastic geometry

Crofton defined the mass of a set of lines in $\mathbb{R}^2$ as $\int dp d\phi$ where p and ϕ are the polar coordinates of the projection of the origin onto the line. Pólya was an analyst whose interests were very broad. For instance, he shows in (J. Leipz. Ber., **69** (1917), 457–458) that, if a mass distribution on lines is positive, additive, and independent of the position, then it is, apart from a constant factor, necessarily the one defined by Crofton. This fact has obvious implication on how to define a natural probability distribution on a (compact) subset of lines in the plane.

Alfréd Rényi (1920–1971) was a probabilist with broad interests in mathematics. He was a very influential mathematician and an able organizer. He is the founding father, and first director, of the Mathematical Institute of the Hungarian Academy of Sciences which now carries his name. He is the author of severals short popular books on mathematics, including Dialogues on Mathematics that has been translated into seven languages.

He wrote two papers on stochastic geometry: the motivation came from the so-called four-point problem of J. J. Sylvester (1863) who asked the probability that four points randomly chosen on the plane form the vertices of a convex quadrilateral. Rényi, together with Sulanke (Z. Wahrscheinlichkeitstheorie **2** (1963), 75–84, and **3** (1964), 138–147) modifies the question: drop n uniform, random, independent points $x_1, \dots, x_n$ in a convex body $K \subset \mathbb{R}^2$, let K_n be their convex hull. What's the expectation of the number of vertices, area, and perimeter, of K_n? They determine these expectations for smooth enough convex bodies and for polygons. For instance, when K is a polygon with k vertices, then the expected number of vertices of K_n is equal to

$$\frac{2}{3} k \log n \bigl(1 + o(1).$$

When K is smooth with curvature κ, then the expected number of vertices is

$$\left(\frac{2}{3}\right)^{2/3} \Gamma\left(\frac{5}{3}\right) \int_{bdK} \kappa^{1/3} n^{1/3} \bigl(1 + o(1)\bigr).$$

These two papers initiated a new direction that have resulted in hundreds of papers on the study of the so-called *random polytopes.*

The second paper contains the following interesting, and purely geometric, result: Let P be a convex polygon with vertices $v_1, \dots, v_k$. Write $\triangle_i$ for the triangle with vertices v_{i-1}, v_i, v_{i+1}. Then the product

$$\prod_1^k \frac{\operatorname{Area} \triangle_i}{\operatorname{Area} P}$$

is the largest when P is an affinely regular k-gon. Actually, László Fejes Tóth theorem from Section 8 (or rather its proof, see L. Fejes Tóth (Matematikai Lapok, **29** (1977/81), 33–38)) gives the stronger inequality that

$$\sum_1^k \left(\frac{\operatorname{Area} \triangle_i}{\operatorname{Area} P}\right)^{1/3}$$

is the largest when P is an affinely regular k-gon.

17. Miscellaneous results in convex geometry

Gyula Pál was working mainly in convex geometry. He was born in Hungary and later moved to Denmark. In an often cited paper J. Pál (Kgl. Danske

Videnskab. Selskab Med. **3** (1920), 1–35) he proves two interesting results. The first is that for every compact set $S \subset \mathbb{R}^2$ there is a convex set, $K \subset \mathbb{R}^2$, of constant width with $S \subset K$ and having the same diameter as S. The other result is about universal covers: every set $S \subset \mathbb{R}^2$ of diameter at most one is contained in a regular hexagon of width 1. This shows that the regular hexagon of width 1 is a universal cover for sets of diameter one. (This universal cover theorem can be used to show the validity of the Borsuk conjecture in the plane.) In the same paper, Pál constructs another universal cover with slightly smaller area than the hexagon.

The following nice result on universal covers is due to Károly Bezdek (Amer. Math. Monthly, **96** (1989), 789–806, joint work with R. Connelly). Let $\mathcal{C}$ be the class of closed planar curves of length one; a set $K \subset \mathbb{R}^2$ is *universal translation cover* for $\mathcal{C}$, if every curve in $\mathcal{C}$ is contained in a translated copy of K. Now the cited result says that every convex body of constant width $\frac{1}{2}$ is a universal translation cover for $\mathcal{C}$. Moreover, every universal translation cover for $\mathcal{C}$ which is convex and has minimal perimeter is of constant width $\frac{1}{2}$.

We mention here that Jenő Egerváry (1891–1958), who mainly worked in algebra and matrix theory, proved an isoperimetric result on curves in $\mathbb{R}^3$ (Publ. Math. Debrecen, **1** (1949), 65–70): he finds, among such curves of length one that have at most three coplanar points, the one whose convex hull has minimal volume.

In connection with geometric constructions, we encountered the name of Gyula Szőkefalvi-Nagy (1982–1959). He worked in various fields of mathematics. He considered the minimal ring containing a convex curve in the plane in (Acta Sci. Math. Szeged, **10** (1943), 174–184). In another paper (Acta Math. Hung., **5** (1954), 165–167) he proves that, given finitely many planes (not all parallel with a line) in 3-space, the set of points with sum of distances to the planes equal to $d > d_0$ form the boundary of a convex polytope. Here $d_0 > 0$ is a constant that depends only on the set of given planes.

Béla Szőkefalvi-Nagy (1914–1998) was an analyst whose research field was Hilbert spaces and operators on Hilbert spaces. He liked geometry and had written about 6 papers in geometry. (One of them is mentioned below, together with his coauthor Rédei.) In a paper (Bull. Soc. Math. France, **69** (1941), 3–4) he constructs, in dimension 4 and higher, convex polytopes, different from the simplex, that have no diagonals. This is an early example of the so-called neighbourly polytopes. Szőkefalvi-Nagy's most famous result in convex geometry states that the Helly number of axis parallel boxes (in $\mathbb{R}^d$) is 2. That is, if in a family of axis parallel boxes in $\mathbb{R}^d$, every two boxes have a point in common, then there is a point common to every box in the family. See Szőkefalvi-Nagy (Acta Sci. Math. Szeged, **14** (1954), 169–177). This paper turned out to be very influential, and the Helly number of various families of convex sets has been thoroughly investigated, for instance in the work of V. Boltjanski and János Kincses.

László Rédei and Béla Szőkefalvi-Nagy proved an interesting result in convex geometry. It is a Heron-type formula which expresses the product of the areas of two convex polygons as a polynomial of the distances between the vertices of the two polygons. For details see Rédei, Szőkefalvi-Nagy (Publ. Math. Debrecen, **1** (1949), 42–50).

Another result, again from convex geometry, of Rédei is joint with István Fáry and is about the maximal volume of a centrally symmetric convex set contained in a fixed convex body $K \subset \mathbb{R}^d$ (see Fáry, Rédei, Math. Ann., **122** (1950), 205–220). If the centrepoint is $x \in K$, then this maximal body is exactly $K \cap (2x - K)$. Fáry and Rédei show that the level sets of the function $x \to \mathrm{Vol}\big(K \cap (2x - K)\big)$ are convex, the function has a unique maximum, and compute it when K is the d-dimensional simplex.

György Hajós (1912–1972) was a very influential person in Hungarian mathematical life. He is the author of the textbook "Introduction to Geometry" that was used at Eötvös University for teaching geometry to several generations of mathematicians and high-school teachers of mathematics. On his famous Monday evening seminar one could learn clarity of ideas, precision in proofs, and rigour in presentation. He published surprisingly few papers, but there is one among them that made Hajós world-famous. It contains the solution of a long-standing conjecture of Minkowski (Hajós, Math. Z., **47** (1941), 427–467). The conjecture which is now Hajós's theorem states that in every lattice tiling of $\mathbb{R}^d$ by congruent d-dimensional cubes, there always exists a "stack" of cubes in which each two adjacent cubes meet along a full facet. The theorem has several equivalent forms and Hajós's proof is algebraic.

Hajós and Heppes construct a three-dimensional (non-convex) polyhedron P whose supporting planes intersect exactly at the vertices of the polyhedron, (see Hajós, Heppes, Acta Math. Hung., **21** (1970), 101–103). Here a supporting plane is a plane that contains at least one point of P and P is contained in the one of the halfspaces bounded by the plane.

István Vincze was a statistician who was interested in convex geometry. In 1939, motivated by a sharpening of the planar isoperimetric inequality due to Bonnesen and Fenchel, he considered the following question. Given a convex body $K \subset \mathbb{R}^2$, and a point $x \in K$, let $R(x)$ denote the radius of the smallest disk centered at x which contains K. Similarly, let $r(x)$ denote the radius of the largest disk, centered at x, which is contained in K. The function $x \to R(x) - r(x)$ attains its minimal value at a unique point $x_0 \in K$, and the circular ring about x_0 with radii $R(x_0)$ and $r(x_0)$ is called the minimal ring containing the boundary of K. Vincze (Acta Sci. Math. Szeged, **11** (1947), 133–138) proved that

$$\min\left\{ R(x) : x \in K \right\} \geq \frac{\sqrt{3}}{2} R(x_0), \quad \text{and} \quad \max\left\{ r(x) : x \in K \right\} < 2r(x_0).$$

Both inequalities are best possible.

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