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Topology

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Topology emerged as a separate branch of mathematics at the end of the nineteenth and the beginning of the twentieth century as a result of efforts to deal with convergence problems, to lay the foundations of real analysis and functional analysis, and to understand the geometric aspects of complex analysis. The main contributions were made in Western Europe, especially by Bernhard Riemann, Georg Cantor, Jules Henri Poincaré and Felix Hausdorff.

Hungarian research in topology started with nine articles published by Frigyes Riesz between 1904 and 1908 (*Qeuvres Complètes,* pp. 27–161). The first of these papers considers the Schoenflies theorem (the converse of the Jordan theorem on simple closed curves). In 1905 he published a paper on Borel's covering theorem for line segments in which he showed that Borel's assumption of countability of the covering can be omitted. Also in 1905, he published a paper on a theorem of L. Zoretti and a paper in which he attempted to give a description of R^n using a collection of n relations of order.

F. Riesz published two papers in Hungarian in 1906 and 1907, then their translation in German (Die Genesis des Raumbegriffes) also in 1907, in which he made an attempt to define something similar to the later concept (due to Hausdorff) of a topological space. Riesz's concept is based on the notion of an accumulation point. It appeared approximately at the same time as the investigations of Maurice Fréchet. A short description of the ideas of Riesz was contained in his talk at the International Congress of Mathematicians, held in Rome in 1908 (Stetigkeitsbegriff und abstrakte Mengenlehre).

Also in this talk he introduced a new structure, the so called "Verkettungstypus" (chaining type). The starting point is the observation that not every property of continuity can be described with the help of accumulation points. Such properties are for example the Cantor connectedness or the property of a planar domain to be bounded by a simple closed curve.

For that reason Riesz introduced this new structure based on the notion that any two subsets of the underlying set of the structure are either near to or far from each other (Are they "verkettet" or not?). Also he began to develop a theory of these structures.

The ideas of Riesz with respect to the "Verkettungstypus" were almost totally forgotten. As far as we are aware, before 1950 the only reference to this structure can be found in a paper of Tibor Radó and Paul Reichelderfer {24} where they write:

"The earliest suggestion of this kind of axiomatic treatment seems to be due to F. Riesz, who proposed to use (essentially) the concept of a pair of not mutually separated sets as a primitive concept (Stetigkeitsbegriff und abstrakte Mengenlehre)" (*Oeuvres Complètes I,* pp. 155–161).

The whole paper on this talk had little impact on the further development of topology before 1950.

It should be mentioned that a few decades later in 1951, independently of Riesz, V. A. Efremovich introduced the so called proximity spaces, which are in some respect similar to the "Verkettungstypus" of Riesz.

After these early papers Riesz abandoned publishing papers on topology except for a proof of the Jordan curve theorem (1938 in Hungarian and 1939 in French). He wrote exclusively on analysis. But his inspirations, letters and also some oral communications on topology were presented in the works of some other mathematicians.

First of all we should mention the only paper of Károly Kaluzsay which was published in Hungarian in 1915. Kaluzsay proved a three dimensional analogue of Schoenflies's theorem. This paper was quoted by Raymond Louis Wilder {28}. Here Wilder writes:

"I have recently come across an early attempt at the converse for the case of an H_2 in E_3 in a paper by K. Kaluzsay, A felületre vonatkozó Jordan tétel megfordítása, *Matematikai és Physikai Lapok,* vol. 24 (1915), pp. 101–141. Upon obtaining a translation of Kaluzsay's results, I have found the interesting fact, that these conditions (No. 5 apparently gives the uniform connectedness *im kleinen* of the complementary domains) closely approximate those which I gave in the Transaction paper just cited, except that instead of the condition on the Betti number, which I used, he assumed that (condition 3) any closed polygon in a complementary domain can be continuously deformed into a point, thus yielding only a special case."

According to Kaluzsay the inspiration of the problem treated in this paper came from F. Riesz.

Also to be mentioned is Dénes Kőnig's proof of the Helly–Radon theorem. D. Kőnig considers first a finite number of bounded closed convex sets and for passing from the finite case to the infinite one, he refers to an oral communication of a proof by F. Riesz. Thus in fact Riesz indirectly published a topological proof in 1921.

On the other hand, Béla Kerékjártó refers in his book [87] to a letter of Riesz's which contains a simple proof of Tietze's extension theorem. This proof of Riesz's figures in the book of Kerékjártó.

Beside Riesz there were a few other Hungarian mathematicians, widely known researchers on other chapters of mathematics than topology, also publishing in topology in this period.

György Pólya was a well-known analyst. However, in 1913 he published a construction of a Peano curve such that each point belongs at most to three values of the parameter (the existence of such a curve was claimed earlier).

Dénes Kőnig is known for his work in graph theory. But in 1918 he published a small book (in Hungarian) on the elements of analysis situs. The book contains a proof of the classification of closed and of bounded surfaces. For its clarity and suggestivity this book was very popular in Hungary for several years.

Kőnig published between 1911 and 1924 five papers which deal with the genus of systems of lines, with the combinatorial properties of surfaces and with one- and two-sidedness of manifolds of dimension higher than 2. Also he published a paper in 1922 in Acta Szeged, where he gave a generalization of a theorem of Borel.

It should be mentioned that Kőnig's investigations in graph theory are closely related to combinatorial topology as is referred in the introduction of his book [94].

Tibor Radó mainly worked in analysis, but he wrote an early paper in which he proved the triangulability of two-dimensional manifolds with a countable base in {22}:

"LEMMA 2. *If R is a two-dimensional manifold for which the countability axiom holds, then it can be triangulated.*" (Translation from German, p. 111).

Radó's proof is the first of this fundamental theorem of topology.

Later in 1936 he wrote two papers in connection with the topological index of a point under a continuous complex-valued function with respect to a directed continuous closed curve. In the second paper {23} he proves a lemma about the topological index with respect to some sequences of closed continuous curves and shows some corollaries of this lemma.

In a paper written together with J. W. T. Youngs and published in Acta Szeged in 1940, referring to the works of Moore, Alexandroff and Kerékjártó, the authors consider upper semicontinuous collections. The upper semicontinuity is the subject also of a paper co-authored with E. J. Mickle and published in the Proceedings of the *Amer. Math. Soc.* in 1950. The Borel transformations, i.e. the mappings, where the preimages of Borel sets are Borel sets, play an important role in this paper.

Tibor Radó also published a paper on semicontinuity of functions and functionals in the *Amer. Math. Monthly* in 1942.

Chiefly interesting is the paper of T. Radó and P. Reichelderfer "On cyclic transitivity" mentioned above. The authors suggest the desirability of a full axiomatic treatment of the theory of the structure of a general space using the notion of a "connected" set as an undefined concept. They refer also to section 2.1 of their paper, however this part of the paper was burned by the Germans. The editors of the *Fundamenta Mathematicae* announced on page 14:

"The manuscript of the work of Messrs. T. Radó and P. Reichelderfer was burned by the Germans and a large part of the already typeset pages destroyed. We were left by chance with these few sheets which had printed. ..." (Translation from French.)

We should mention that a condensed version of this paper was published in Duke Math. Journal but this version ends with section 1.23 and thus there is no reference to 2.1.

Between 1943 and 1949 T. Radó published three papers on Peano spaces and after 1950 five papers on algebraic topology: on singular homology, on chain homotopy and on general cohomology theory. It should be mentioned that in the monograph of T. Radó and P. Reichelderfeld [145], published in 1955, the subject of almost half of the book is topology, mainly cohomology theory.

The first paper of T. Radó, where the concept of the Čech cohomology group with integral coefficients occurs, is a paper co-authored with P. Reichelderfer written in 1949 (see {25}). The authors use this concept for the definition of the index in Euclidean n-space.

Two lemmas on metric spaces occur in a paper published in 1958 (E. J. Mickle and T. Radó {17}). One of them says that if A is a subset of a separable metric space M and $\emptyset \neq X$ is covering of A by proper closed spheres such that the diameters of these spheres have a finite upper bound, then there is a sequence $C_n = \gamma(a_n, r_n)$ of pairwise disjoint members of X such that $A \subset \bigcup_n \gamma(a_n, 5r_n)$, where $\gamma(a_n, s)$ is the closed sphere in M with centre a_n and radius s.

Gyula Pál mainly worked in analysis. However he also published some papers on topology. Three papers concern the construction of Jordan curves with given projections. In the third one {19} Gy. Pál proves that each locally connected planar continuum with at least two points is the orthogonal projection of a closed Jordan curve of the Euclidean 3-space.

Two papers are devoted to plane topology. In the second one {20} he describes the topic of a projected volume on topology.

Gyula Szőkefalvi Nagy mainly worked in geometry. However among his 150 papers there are also some papers belonging to topology. He published a paper in Hungarian and then in German {26} proving a theorem of Gauss on the double points of closed planar curves. The theorem says that under a successive numeration of the points of self intersection of a closed planar curve without singularities the serial number of each such point is once an even and once an odd number. The proof is based on the fact that two closed planar curves in relative general position have an even number of common points. Gy. Sz. Nagy's proof is also quoted in the book of Hans Rademacher and Otto Toeplitz {21}.

Gy. Sz. Nagy introduced in the paper mentioned above a new topological symbol connected with the cycles of the curves in question. He showed that planar curves with at most four double points can be fully characterized with these symbols.

Three papers of Gy. Sz. Nagy deal with closed curves on the sphere and on surfaces. In the first one {27} he considered cuts of self intersections of oriented closed curves on surfaces and proves the equality $h+k=n+2-2r$ where n is the number of self intersection points of the curve C in question, r is a nonnegative integer, h is the number of components of the figure obtained from C after cutting it at each point of self intersection, and k is the number of components of the figure obtained from C making the complementary cut at each point of self intersection.

Pál Erdős was one of the most prolific and most travelled mathematicians ever. He worked in number theory and combinatorial analysis, and is considered the founder of discrete mathematics. Among his more than 1500 papers, there are also papers belonging to topology.

In his paper written in 1940 he states and proves the astonishing fact that the Menger–Urysohn dimension of the rational points of Hilbert space is 1 (*Ann. of Math.,* **41** (1940), 734–736).

One of the papers of Erdős published in 1944 concerns connected and biconnected sets. Some problems and questions related to these concepts are included there.

Also in 1945 and 1946 he published two papers on the Hausdorff dimension of some sets in Euclidean spaces.

In a paper co-authored with George Piranian (*Michigan Math. J.,* **5** (1958), 139–148) the authors have shown how the convergence field of certain regular Toeplitz matrices can serve as neighborhoods in the topologization of a quotient space of bounded sequences.

Erdős investigated Borel sets together with Arthur Stone (Proceedings of the *Amer. Mat. Soc.,* **25** (1970), 304–306). Erdős and Stone proved that the linear sum of two Borel subsets of the real line need not be Borel, even if one of them is compact and the other is a G_δ. However, after publication the authors have been informed of the earlier work of B. S. Sodnomov who proved in 1951 a theorem equivalent to the first theorem of Erdős and Stone.

Some papers of Erdős on topology are closely related to set theory.

In a paper co-authored with A. Tarski and published in 1943 the authors give a set theoretical equivalent of the statement that in every topological space of power $\leq 2^{\aleph_0}$ there exists a family of mutually disjoint open sets with a maximal power.

Two papers co-authored with András Hajnal and published in 1961 consider propositions of the form: Any product of $\aleph_\mu$ discrete λ-compact spaces is κ-compact, where μ, λ, κ are ordinals and for $\mu > 0$ $\aleph_\mu$ is the least cardinal number larger than all cardinals $\aleph_\delta$ with $\delta < \mu$, moreover a topological space is said to be μ-compact if each open cover has a subcover with cardinality less than $\aleph_\mu$. Using the generalized continuum hypothesis the authors show that the topological product of $\aleph_\kappa$ 1-compact (i.e. Lindelöf) discrete spaces is not necessarily κ-compact for any finite κ.

The subject of a paper of Erdős and Mary Ellen Rudin is the box product (1973). Erdős proves there that it is consistent with the usual axioms of set theory that the box product $\omega_k \times (\omega_0 + 1) \times (\omega_0 + 1) \times \ldots$ is ether normal or not normal for all integers $k > 1$, where ω_m is the initial ordinal number with the cardinality $\aleph_m$ for all nonnegative integers m.

Finally, one of the main results of an abstract and of a paper co-authorized with F. S. Cater and Fred Galvin (1976 and 1978) is that if κ and λ are infinite cardinals with $\lambda \leq \kappa^+$, where κ^+ is the least cardinal larger than κ, and X is a topological space with density $d(X) \geq 2$, where the density $d(Y)$ of a topological space Y is the minimum cardinality of a dense subset of Y, then the cofinality cfd $(X^\kappa)_{(\lambda)} \geq \operatorname{cf} \lambda$, where $(X^\kappa)_{(\lambda)}$ is the product of κ copies of X with λ-box topology, i.e. the basic open sets of this product are of the form $\prod_{\xi \in \kappa} U_\xi$, where U_ξ is open in X and the cardinality of the set $\{\xi;\, U_\xi \neq X\}$ is less than λ, moreover for any infinite cardinal number μ the cofinality $\operatorname{cf} \mu$ of μ is the least cardinal α such that μ is the sum of α cardinals less then μ. In a certain sense this result generalizes the theorem of Gyula Kőnig that $\operatorname{cf} 2^\kappa > \kappa$.

György Alexits worked mainly in the theory of orthogonal series. However between 1932 and 1942 he was primarily interested in topology and wrote around 15 papers related to it. A part of them is devoted to Menger's theory of curves, another to locally connected continua.

One of his results is that each homogeneous rational curve is homeomorphic to the circle. This result was published only in Hungarian in {1}. It is closely related to a theorem of Mazurkiewicz which says that each locally connected homogeneous plane curve is homeomorphic to the circle.

His last paper on topology concerns spaces which are the union of a countable set of hereditarily locally connected continua. Among others, generalizing certain theorems of Gordon Thomas Whyburn, Alexits shows that a subset of such a space is totally disconnected if and only if it is zero dimensional, or if and only if each quasicomponent of the subset is a singleton, see {2}.

Alexits was also interested in defining some concepts of differential geometry (curvature, torsion) in metric and semimetric spaces using topological methods. One of the papers in this direction was co-authored with Jenő Egerváry.

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The only Hungarian researcher of the first half of the century who devoted his activity mainly to topology was Béla Kerékjártó who wrote between 1919 and 1944 more than 70 scientific papers. Furthermore he is the author of the monograph [87]. He also planned to write the second volume of this books, but it has never been done.

The book itself is divided into two parts. The first deals with the topology of the plane, while the subject of the second part is the topology of surfaces. In the introductory chapter, which gives a view of the development of topology, also the notion of an n-manifold is formulated as follows:

An n-manifold is a connected Hausdorff space having a countable basis such that each member of this basis is homeomorphic to the interior of an n-ball.

Nowadays the existence of a countable basis is usually not assumed, but otherwise in general the formulation of Kerékjártó or some formulation equivalent to it is used for the notion of an n-manifold.

The main subject of the first part of the monograph are Jordan's theorem and numerous assertions closely related to this theorem. Here one can find the so called Kerékjártó theorem, which says that if $J_1, \ldots, J_k$ are simple closed curves in the plane such that each pair of them has at least two common points, then the boundary of each complementary domain of $J_1 \cup \cdots \cup J_k$ is a simple close curve (p. 87). This theorem is quoted in the book of M. H. A. Newman {18}.

The classification theorem of open surfaces, i.e. of the non-compact 2-manifolds, can be found in the second part. Kerékjártó's result in this relation was published earlier in the Jahresbericht der Deutschen Matematischen Vereinigung in 1922 (Bd. 31) with the heading: Haupsatz der Flächentopologie bei unendlich hohen Zusammenhang.

There is a fundamental difference between the open and the closed surfaces. In fact there is only a countable set of pairwise non-homeomorphic closed (i.e. compact) surfaces and they can be characterized in both the orientable and non-orientable cases with only one integer, e.g. with their genus. On the other hand there is a set of pairwise non-homeomorphic open surfaces of the cardinality of the continuum. This is true also in the case where we consider only simple surfaces, i.e. surfaces which are divided into two parts by each of their simple closed curves. One of the theorems of Kerékjártó says that each simple open surface is homomorphic to a domain of the plane, where the complementary set of the domain is totally disconnected. Moreover two open surfaces are homeomorphic if and only if these complementary sets are homeomorphic. But in the plane there can be found pairwise non-homeomorphic totally disconnected closed subsets of the cardinality of the continuum and the complement of each such subset is a domain in the plane, i.e. a simple open surface.

Kerékjártó formulated his scientific programme in his Privatdocent lecture held in Szeged on 15. December 1921 with the title "Az analízis és a geometria topológiai alapjairól" (On the topological fundaments of analysis and geometry). Here the intention formulated by Brouwer in his Privatdocent lecture in 1910 is emphasized, namely that complex analysis should be built with instruments of topology without metric elements such as length and area.

The principal direction of Kerékjártó's investigation is attached to this intention. He mainly investigates the properties of the topological transformations and groups of transformations of surfaces with special emphasis on continuous transformation groups.

One of the important periods of his investigations is the first half of the thirties. At that time he introduced the notion of regular map. He calls a bijective map τ of the plane onto itself regular at a point P if to each positive ε there is a positive δ such that if the distance of P and Q is less than δ then for each integer n the distance of $\tau^n(P)$ and $\tau^n(Q)$ is less than ε, where the distance is obtained from the spherical distance by a stereographic projection. The map τ itself is said to be regular if it is regular at each point P of the plane.

The notion of regularity can be defined in the same way in arbitrary metric spaces, and it can be transported to topological spaces as it is shown in a paper of Kerékjártó.

Now a fundamental statement of Kerékjártó says that the orientation preserving regular transformations without fixed points of the plane are equivalent to translations.

Several papers of Kerékjártó deal with regular topological transformations of various surfaces. One of them states that if $p > 1$ then each regular map of an orientable closed surface with genus p onto itself is periodic.

It should be mentioned that a somewhat strengthened version of the denial of Kerékjártó's regularity, the so called expensiveness, had a fundamental role in several branches of dynamical systems.

In the early forties Kerékjártó also investigates compact topological transformation groups. For example he examines the question of which closed surfaces with or without boundary have infinite compact transformation groups. He shows that there are only seven surfaces of this kind. These are: the sphere, the closed disc, the annulus, the torus, the projective plane, the Moebius strip and the Klein bottle.

Kerékjártó was the author also of the monographs "A geometria alapjairól" (Foundation of geometry) I. and II. (see [88]). The first volume was printed in 1934 and the second in 1944. The topological view appears in both volumes. For example in the introductory chapter of the second volume Kerékjártó mentions that each 2-times transitive continuous group of the topological transformations of the plane is equivalent to the group of similarity transformations of the Euclidean plane, where a group of topological transformations of a topological space is said to be n-times transitive if for any two n-tuples $(A_1, \ldots, A_n)$ and $(A'_1, \ldots, A'_n)$ of pairwise distinct points of the space there is precisely one element of the group which takes for $i = 1, \ldots, n$ the point A_i into A'_i. Also the topological concepts play an important role at the end of the second volume at the common characterization of the real and complex projective geometry.

We have to mention the simple proofs of Kerékjártó to some well known topological theorems of fundamental importance, e.g. the proof of the theorem of Schoenflies for the invariance of domains in the plane. The theorem says that if M and N are homeomorphic subsets of the plane then either both of them are open, or neither of them is open. Kerékjártó published his proof first in Hungarian, but the same proof appears in his monograph "Vorlesungen über Topologie". The proof is based on the fact that if Q is a square, and we consider a topological map of Q into the plane, then the image of the midsegments of Q cannot lie in the external residual domain of the image of the boundary of Q.

The proof of Kerékjártó, as well as the original proof of Schoenflies, requires an extension of the Jordan curve theorem, formulated by Schoenflies, however Kerékjártó's proof is quite different from that of Schoenflies.

To the Jordan curve theorem itself Kerékjártó gave two totally different proofs. One of them appeared also in his book "Vorlesungen über Topologie". More interesting is the second one which was published in 1930 in the Acta Szeged. The fundamental idea of this second proof is one of the simplest among the proofs of the theorem.

In the introductory chapter of the "Vorlesungen" Kerékjártó describes also the content of the second volume planned. However his early death prevented the implementing of his plans.

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At present Hungarian research in topology is mainly inspired by the concept of syntopogeneous spaces due to Ákos Császár. They are a common generalization of topological spaces, uniform spaces introduced by André

Weil, and proximity spaces introduced by V. A. Efremovich. The first monograph about syntopogeneous spaces appeared in 1960 (Ákos Császár, [23]).

The productivity of two members of the present Hungarian school of topology is already closed by their early deaths.

János Czipszer as a publisher's reader of the first monograph of Ákos Császár gave the author valuable aid in preparing this edition. Also a basic theorem of this volume is due to J. Czipszer (Theorem 12.35, 12.37 in the second English edition). The second edition of the monograph (Ákos Császár, [24]) contains several results of Czipszer. The author writes in the preface of this edition:

"I have to express my warmest thanks to my collaborator J. Czipszer who, after having lent me valuable aid in the preparing the first edition, kindly communicated a series of his unpublished results which constitute an essential supplement to the theory of syntopogeneous structures. The material of Chapters 17, 18 and 19 is almost entirely due to him, and his ideas greatly contributed also to the subjects dealt with in the other chapters."

Among the 12 published papers of János Czipszer there are three dealing with topological questions. Two of them are co-authored with Ákos Császár. In the first paper {4} the authors consider curves without ramification points. Among others, they generalize one of Menger's theorem as follows: If X is a nondegenerate, connected, compact Hausdorff space which contains no point of infinite order and at most a finite number of points of order greater than 2, then X is the union of a finite collection of generalized arcs no one of which contains a non-end point of another.

In the second paper {5} the authors consider generalizations of the Stone–Weierstrass theorem. One of the main results says that to a family of bounded real-valued functions γ defined on a set E such that γ coincides with the set of all uniformly continuous (with respect to U) real-valued bounded functions if and only if γ is nonempty, it is closed with respect to uniform convergence and it satisfies the following conditions:

(a) $f \in \gamma \Longrightarrow f + \alpha \in \gamma$ for each $\alpha \in \mathbb{R}$,

(b) $f \in \gamma \Longrightarrow \alpha f \in \gamma$ for each $\alpha \geq 0$,

(c) $f, g \in \gamma \Longrightarrow \min(f, g) \in \gamma$ and $\max(f, g) \in \gamma$.

The third paper of J. Czipszer {3} is connected with the following problem proposed by G. Alexits:

Given a convergent sequence of real-valued functions defined over a complete separable metric space R, does there exist a dense subset of R on which the sequence converges locally uniformly?

A demonstration is given and produces an affirmative answer to the above question. Moreover it is shown that if the continuum hypothesis is assumed, then there exists a convergent sequence of real-valued functions defined over the real line $\mathbb{R}$ which does not converge locally uniformly on any subset of $\mathbb{R}$ having the power of the continuum.

Jenő Deák first published a few papers on analysis, then during the period from 1975 to his early death more than fifty papers on general topology, mostly devoted to the study of various topological structures.

His first papers on topology address the theory of directional structures, initiated in 1964 by Ervin Deák; the purpose of this theory was a topological characterization of the topological spaces homeomorphic to a subspace of the Euclidean space $\mathbb{R}^n$ $(n \in \mathbb{N})$. In order to reach this aim, let us say that a pseudo-direction $\mathcal{R}$ in a topological space X is a set of ordered pairs (G, F), where G is open and F is closed in X, and some natural conditions are satisfied. A pseudo-directional structure in X is a collection $\mathfrak{R}$ of pseudo-directions; $\mathfrak{R}$ is compatible with X if the sets G and $X \setminus F$ such that $(G, F) \in \mathcal{R} \in \mathfrak{R}$ constitute a subbase for the topology of X.

Now the main result of J. Deák in this theory is a stronger version of a theorem of E. Deák and says that a separable metrizable space X can be topologically embedded into $\mathbb{R}^n$ if and only if X admits a compatible pseudo-directional structure $\mathfrak{R}$ satisfying $|\mathfrak{R}| \leq n$, see {8}.

E. Deák has introduced a concept of dimension equal to the minimal cardinality of a compatible directional structure. J. Deák has presented, in a series of papers published between 1976 and 1980, a thorough discussion of this kind of dimension and of other similar concepts, partly introduced by him, gave a series of counter-examples and, in particular, he proved in {7} a generalization of a theorem of J. de Groot in dimension theory.

In {9} J. Deák develops further the investigations in which completely regular spaces are characterized by the existence of subbases possessing some special property. Continuing the studies of Péter Hamburger in this subject, he has presented several increasingly stronger results.

In {10} J. Deák gives an analysis of the ideas of Riesz and, in particular, he points out that the concept of a proximity, introduced in 1951 by V. A. Efremovich, can be brought in fact into a near connection with the chainings

(Verkettung) of Riesz, but the relation of the two concepts needs a careful discussion.

Extraordinarily intensive was the research work of J. Deák in the field of the theory of quasi-uniform spaces and bitopological spaces. As it is well-known, quasi-uniformities constitute a non-symmetric version of uniformities; i.e. a quasi-uniformity on a set X is a filter $\mathcal{U}$ on the product set $X \times X$ with the property that $(x, x) \in U$ for $x \in X$, $U \in \mathcal{U}$ and, for a given $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V^2 = V \circ V \subset U$. A bitopology on a set X is simply an ordered pair $(\mathcal{T}_1, \mathcal{T}_2)$ of topologies on X. The two kinds of structures are closely related because each quasi-uniformity $\mathcal{U}$ on X induces a topology $\mathcal{U}^{tp}$ for which the neighbourhoods of $x \in X$ are the sets $U(x)$ with $U \in \mathcal{U}$ so that the pair $(\mathcal{U}^{-tp}, \mathcal{U}^{tp})$ is a bitopology on X (of course, $\mathcal{U}^{-tp}$ denotes the topology induced by the quasi-uniformity $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$).

As to bitopological spaces, a series of three papers (see {12}) presents (as a kind of a little monograph) a thorough discussion of separation axioms for bitopological spaces, then develops further results of R. E. Smithson on the relation of completely regular bitopological spaces and multifunctions, finally applies (pseudo-)directional structures to the characterization of completely regular bitopological spaces and their compactifications.

The problems belonging to the theory of extensions stand in the centre of the investigations of J. Deák on quasi-uniformities and bitopologies. In 1990 and 1991, he published several papers on such subjects; the most important is perhaps {11}. In these two papers the fundamental question is the following: we are given a quasi-uniform space $(X, \mathcal{U})$ inducing a bitopology $(\mathcal{U}^{-tp}, \mathcal{U}^{tp})$, further a set $Y \supset X$ and, on Y, a bitopology $(\mathcal{T}_{-1}, \mathcal{T}_1)$ such that $\mathcal{T}_{-1}|X = \mathcal{U}^{-tp}$ and $\mathcal{T}_1|X = \mathcal{U}^{tp}$; is there a quasi-uniformity $\mathcal{V}$ on Y such that $\mathcal{V}|(X \times X) = \mathcal{U}$ (i.e. $\mathcal{V}$ is an extension of $\mathcal{U}$) and $\mathcal{T}_{-1} = \mathcal{V}^{-tp}$, $\mathcal{T}_1 = \mathcal{V}^{tp}$ (i.e. $\mathcal{V}$ is compatible with the given bitopology)?

J. Deák presents a careful discussion of possible questions in connection with this problem: necessary or sufficient conditions for the existence of a compatible extension in general or satisfying further (e.g. cardinality) conditions, constructions for obtaining such extensions, etc. He also discusses a quite similar problem where, instead of quasi-uniformities, we consider syntopogenous structures in the sense of Császár.

A delicate question belonging to the same group of problems is the search for complete extensions, because the concept of completeness of a quasi-uniform space may be defined, and was in fact defined in the literature, in several more or less natural ways, and it is difficult to choose which is the most natural of them; probably the answer depends on the point of view the author wishes to emphasize. In any case, J. Deák's methods are useful for simplifying the situation in this rather complicated field of research (see {13}).

J. Deák published in 1993 a grandiose survey paper {14}. The paper gives a collection of results on extensions both of uniformities and of quasi-uniformities, in the latter case involving topologies or bitopologies, and contains not only a survey of the results in the literature but also a long list of new results and of open problems.

Still the questions related to extension problems are the subject of J. Deák's paper {15}; a quasi-metric on X is a function $d : X \times X \to [0, +\infty)$ satisfying $d(x, y) = 0 \Leftrightarrow x = y$ and $d(x, z) \le d(x, y) + d(y, z)$. A quasi-metric induces a quasi-uniformity $\mathcal{U}$ if we set $U_{d,\varepsilon} = \{(x, y) : d(x, y) < \varepsilon\}$ for $\varepsilon > 0$ and then the topology $\mathcal{U}^{tp}$. The paper investigates the problem of the existence of a quasi-metric inducing a given topology on X and coinciding with a given quasi-metric on a subset of X.

A long series of further papers concerns questions on special properties of quasi-uniformities: weak symmetry properties, uniform local symmetry, doubly uniformly strict extensions, co-regularity, Cauchy-type properties, properties preserved by extensions, co-stability (the latter with co-author S. Romaguera). Through all this research work, J. Deák has deserved the rank of a leading researcher in the theory of quasi-uniform spaces.

While quasi-uniformities and topologies are particular cases of the general concept of a syntopogenous structure in the sense of Császár, other types of structures were considered in a further series of papers; these are particular cases of the general concept of a merotopy in the sense of Miroslav Katětov introduced in 1965. In an equivalent form due to Horst Herrlich, a merotopy on a set X is a set $\mathfrak{M}$ of covers of X with the property that a cover $\mathfrak{c}$ belongs to $\mathfrak{M}$ as soon as one of its refinements belongs to $\mathfrak{M}$ and, if $\mathfrak{c}_1$ and $\mathfrak{c}_2$ belong to $\mathfrak{M}$ then some common refinement of them also belongs to $\mathfrak{M}$; $\mathfrak{c}'$ is a refinement of $\mathfrak{c}$ if $C' \in \mathfrak{c}'$ implies the existence of $C \in \mathfrak{c}$ satisfying $C' \subset C$.

Special cases of merotopies are contiguities introduced by V. M. Ivanova and A. A. Ivanov in 1959 (sets of finite covers), Čech proximities introduced in the monograph of Eduard Čech in 1966 (two-element covers); less simple to explain but further special cases are filter merotopies (M. Katětov 1965) and in particular Cauchy structures (H. Keller 1968), Čech closures (E. Čech

1966), limitations (H. J. Kowalsky 1954). All these kinds of structures are connected by the fact that in the series

$$\text{merotopy} \to \text{filter merotopy} \to \text{contiguity} \to \text{Čech proximity} \\ \to \text{Čech closure} \to \text{limitation}$$

each kind of structure induces in a natural way a structure belonging to one of the later types in the series; we say that a richer structure induces another one. Also if $\mathcal{D}$ is a structure belonging to one type of the above list, given on a set X and $A \subset X$, there is a natural restriction $\mathcal{D}|A$ of $\mathcal{D}$ to A.

Now a very general problem that first appeared in a series of papers due to Császár and J. Deák is the following: we are given a structure $\mathcal{D}$ on a set X and subsets X_i of X ($i \in I$, where I is arbitrary, in general infinite, but possibly $I = \emptyset$) together with richer structures $\mathcal{C}_i$ given on each X_i. We look for a structure $\mathcal{C}$ of the same type inducing $\mathcal{D}$ and such that $\mathcal{C}|X_i = \mathcal{C}_i$; this is the problem of simultaneous extension of the type $(\mathcal{C}, \mathcal{D})$.

The series of papers co-authored with Császár (see {6}) contains investigations in this direction. In each case, necessary and/or sufficient conditions are given for the existence of a simultaneous extension and it is also discussed whether there is a coarsest or finest simultaneous extension. In later papers, J. Deák investigates other cases in the same spirit.

J. Deák also wrote a brilliant survey paper on the problem of simultaneous extensions (see {16}); his aim was to present the problem for mathematicians in general and not only for specialists in general topology.

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