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Differential Equations: Hungary, the Extended First Half of the 20th Century

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Introduction. It is well known that the theory of differential equations does not belong to the most important chapters in the history of Hungarian mathematics. Yet, when making preparations for this paper, both of us were astonished to realize how many prominent Hungarian scholars had been concerned with the theory of differential equations, even if marginally, and how much they had been aware of the relations of their primary fields to our topic.

Summary. We give a detailed account on

- the relation between Fejér's summation theorem and Dirichlet's problem on the unit disc;
- Fejér's work in stability theory (in connection to his habilitation lecture);
- F. Riesz' subharmonic functions;
- Haar's inequality for partial differential equations of the first order;
- the Haar–Radó results in the calculus of variations (with a particular emphasis on the minimal surface problem)
- what is called 'von Neumann's stability analysis' and the underlying Lax equivalence "consistency & stability $\Leftrightarrow$ convergence";
- Lax's contribution to the theory of conservation laws (a field of research he entered under the influence of Neumann's interest in shock waves);
- M. Riesz' theory of fractional potentials;
- the work of Pólya and Szegő on isoperimetric inequalities.

The concluding pages are devoted to differential equations in Hungary after the second world war. As before, the emphasis is placed on

- results providing a significant contribution to classical problems (like Bihari's 1956 inequality, the first nonlinear version of Gronwall's Lemma);
- results which pave the way to modern theories (like the 1950–60 contributions by Rényi and Barna to the emerging theory of interval maps).

In connection to some statistical data from the

- first decade of the twentieth century;
- years 1928 and 1953 (representing the period before and after the second world war)

a couple of general remarks are also made.

At the turn of the century. At the beginning of the twentieth century the general theory of differential equations was closer to physics, and within this, to mechanics than it is nowadays. Let us mention here that even the famous paper on linear inequalities by Gyula Farkas had its roots in the one-sided constraints of mechanics. In the volumes of Mathematische Annalen between 1900 and 1910 34 articles (among approx. 450) had Hungarian authors. Out of them 16 are concerned with differential equations in a wider sense, including 8 papers on physical applications, namely: 4 papers by Mór Réthy discuss the variational principles of mechanics, 1 paper by Győző Zemplén treats hydrodynamics, another is on electrodynamics, 1 paper by Gyula Farkas is on shock waves and, finally, 1 paper by Lipót Fejér is concerned with the variational principles of mechanics. The last one will be discussed in detail later. Three papers on linear ordinary differential equations by Lajos Schlesinger come under pure mathematics, definitely. One paper by Károly Goldziher and 4 papers by József Kürschák, who is known mainly as an algebraist, are on partial differential equations. E.g. Kürschák provides a new proof to a theorem of Lie according to which, under certain compatibility conditions, Monge–Ampère equations are transferred into Monge–Ampère equations by contact transformations {44}.

Although the authors received their most important professional stimulations from abroad, even, while staying abroad, these results could not have been achieved without certain Hungarian scholarly antecedents. Gyula König taught courses on differential equations regularly at the Budapest University of Technology, so did Gyula Vályi at the University of Kolozsvár. Their most important results obtained in the field of partial differential equations were highly appreciated by the international scientific community. The studies in partial equations of second order by Vályi, König and József Kürschák, a follower of König, belong to the theory of formal integrability. Their main aims were to set up compatibility criteria on the reduction of equations (i.e. to a system of partial equations of the first order or that of ordinary ones). At the Budapest University of Technology Gusztáv Kondor, who belonged to an earlier generation and had much more modest abilities than Vályi and König, also read lectures on differential equations. Among the predecessors, the name of József Petzval, a pioneer of photography, who left the chair of the University of Budapest for that of Vienna University in 1837, should also be mentioned. His two-volume work on differential equations was highly appreciated all over the Austro–Hungarian Monarchy.

The papers mentioned, as well as the preliminaries discussed, represent demonstratively the state of the theory of differential equations in the early twentieth century: it had a very close relation to various branches of physics and there was a great effort to solve equations in a closed form. Even in our days it seems to be surprising how many of them could be solved by integral representations and series expansions in terms of elementary and special (higher transcendental) functions. By 1900 the development of the general theory of linear ordinary differential equations was already at a fairly high level. This can be exemplified, primarily, by the generalization of Galois theory, widely known in the field of algebraic equations, to linear ordinary differential equations (whose coefficients are meromorphic functions). On the other hand, at that time only the germs of a general theory of nonlinear ordinary differential equations, namely, the existence theorems of Peano and Picard, and elements of the qualitative theory began to emerge. But the number of known nonlinear types of equation which could be solved by quadrature became several dozens. As far as partial differential equations are concerned, we cannot speak about a general theory at all, except for the equations of first order. Here emphasis was laid on the concrete solutions of initial-boundary value problems for linear and, to a much lesser extent, for nonlinear equations closely connected with physical applications.

Lajos Schlesinger and his work. In the history of differential equations in Hungary Lajos Schlesinger holds a special and distinguished place for two reasons: he was the first mathematician in Hungary whose prime field of activity was the study of abstract differential equations, namely that of the linear ordinary differential equations, during most of his life and who gained an international fame and reputation for this. His major work, a two-volume monograph on linear ordinary differential equations {73} was published again by the Johnson Reprint Corporation in 1968. While the activities and lives of Gyula König and Gyula Vályi are discussed by Barna Szénássy in his excellent book on the history of Hungarian mathematics of early times, (i.e. before the twentieth century) in detail, the life and work of Lajos Schlesinger are not treated at all since most of his activity took place after 1900.

Lajos Schlesinger was born in Nagyszombat in 1864. He started studying mathematics at Heidelberg, finished his studies in Berlin, and took his doctor's degree with Lazarus Fuchs whose daughter he married (and, thus, he confirmed a peculiar law of genetics according to which the genes carrying mathematical abilities are passed from a father–in–law to a son–in–law. Another example of this is Aurél Wintner born and educated in Hungary, who married Otto Hölder's daughter). Lajos Schlesinger spent most of his life in Germany and died as Professor of the University of Giessen in 1933. His activities are attached closely to his father–in–law's. Similar to the latter, he used methods of the theory of analytic functions to a greater extent and those of group theory to a lesser extent in the theory of ordinary differential equations. Further on, he wrote a great number of papers on Fuchsian equations and Fuchsian groups and published the correspondence of Fuchs and Weierstrass. By the evidence of his references, based on studying and processing almost 2000 professional articles, he wrote his survey {74} on the development of the theory of linear ordinary differential equations from 1865 on, the publication year of his father-in-law's famous paper. Together with Abraham Plessner he wrote a remarkable book on Lebesgue integrals and Fourier series and took part in processing Gauss's unpublished manuscripts. Undoubtedly, he had a very rich career in mathematics. In a late paper related to the solution of the differential equation $\dot{z} = C(t)z$ where $z \in \mathbb{C}^N$ and C is a complex-valued matrix function, he became the forerunner of the theory of product integrals (i.e. of generalizations of Lie's matrix formula $e^{A+B} = \lim_{n\to\infty} (e^{A/n}e^{B/n})^n$). This activity of Schlesinger is mentioned with admiration by Felix Browder, too, in the preface to the Mathematical Encyclopedia volume 'J. D. Dollard & C. N. Friedman, *Product Integration with Applications to Differential Equations*, Addison–Wesley, Reading, Mass., 1979'.

Lajos Schlesinger was Professor at the University of Kolozsvár between 1897 and 1911. Among Hungarian mathematicians he stimulated Manó Beke and his impact can be felt on a paper by Lipót Fejér. Beke investigated the irreducibility of homogeneous linear ordinary differential equations the coefficients of which are rational functions. Also, for the equations with such coefficients the concept of irreducibility itself was introduced by him {3}. (An equation is irreducible if it has no solution common with a homogeneous linear ordinary differential equation which is of a lower order and has coefficients of the same class.) With a highly effective application of Cauchy's majorant method, Fejér {21} gave a new proof to Fuchs's theorem on the singularities of the solutions of homogeneous linear differential equations.

For Hungarians, Schlesinger's reputation is indicated by the fact that he became the subject of the well-known anecdote attached to Henri Poincaré's visit to Budapest. In 1905 Poincaré came to Budapest to receive the Bolyai Prize. At his arrival his reply to the greetings of the notables who were meeting him was as follows: 'Thank you! But where is Fejér?' Similar to the legends about the lives of medieval saints, this story has become extant in another version in which the name of Schlesinger replaces that of Fejér's. Indeed, it is conceivable that Poincaré wanted to meet Schlesinger since he himself had been concerned with homogeneous linear ordinary differential equations with rational coefficients. It was he who gave the name of Fuchsian functions, in honour of Immanuel Lazarus Fuchs, to a certain class of automorphic functions which played an important role in the integration of the aforementioned equations.

Fejér summation theorem and the Dirichlet problem on the unit disc. In the first decade of the twentieth century Lipót Fejér was concerned with the theory of ordinary differential equations in a wider sense in several papers. Pál Turán mentions in the Introduction to Lipót Fejér's Collected Works that the discovery of the famous Fejér's summation theorem is closely related to the Dirichlet problem on the unit disc $B = \{ (x, y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1 \}$. Given a continuous function $g : \partial B \to \mathbb{R}$, does there exist a continuous function $u : B \to \mathbb{R}$ with the properties that u is harmonic on $B \setminus \partial B$ (i.e. u is twice continuously differentiable on $B \setminus \partial B$ and satisfies $\Delta u = 0$ on $B \setminus \partial B$) and $u|_{\partial B} = g$? The positive answer, together with the form of the solution function

$$(1) \quad u(r\cos\varphi, r\sin\varphi) = \begin{cases} a_0/2 + \sum_{k=1}^{\infty}(a_k \cos k\varphi + b_k \sin k\varphi)r^k & \text{if } r < 1 \\ g(\varphi) & \text{if } r = 1 \end{cases}$$

(where $a_0, a_1, b_1, a_2, b_2, \ldots$ are the corresponding Fourier coefficients) had already been guessed earlier but only H. A. Schwarz succeeded in proving it with the help of the so-called Poisson integral representation of the series expansion

$$a_0/2+\sum_{k=1}^{\infty}(a_k \cos k\varphi+b_k \sin k\varphi)r^k = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos\,(\varphi-\psi)+r^2} g(\psi)\, d\psi$$

valid for $r < 1$.

Fejér spent the academic year 1899/1900 in Berlin where he was influenced greatly by the lectures of H. A. Schwarz. When discussing the Dirichlet problem on the unit disc Schwarz stated that it would be expedient to give an existence proof by the theory of series exclusively, moreover he spoke about the unsuccessful attempts made in that direction. The problem was that due to the possible divergence of the Fourier series of function g the Abel summation theorem could not be applied. The way out of the situation was that — at the points of ∂B — continuity of the solution function defined by formula (1) followed from a new summation procedure. Fejér was given the decisive impetus to frame a new summation procedure by the theorem of Frobenius according to which Abel's convergence assumption on $\sum c_k$ for the existence of $\lim_{r\to 1^-} \sum c_k r^k$ can be weakened to assuming the convergence of $\sum s_n$ where $s_n = (c_0+c_1+\cdots+c_n)/(n+1)$, $n \in \mathbb{N}$. All these considerations led Fejér to prove that the arithmetic means of the partial sums of the Fourier series of continuous 2π-periodic functions are uniformly convergent. Compared to his revolutionary discovery the original question posed, i.e. to prove the solvability of the Dirichlet problem on the unit disc by the theory of series exclusively, remained entirely in the background.

In Fejér's famous Comptes Rendus note {13} there is only one sentence which indicates that his summation theorem is also applicable to the theory of Poisson integrals. He worked out the details in two separate papers published in Hungary. The first one {14} discusses the Dirichlet problem on the unit disc. The second one {15} treats the heat equation $u_t = u_{xx}$ equipped with the initial condition $u(0,x) = g(x)$ where $g : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic continuous function. The results of these two short papers are included in Section 3/a of an extensive paper on his summation theorem he published in Mathematische Annalen in 1904 {16}.

The work of Fejér in mechanics and his habilitation lecture. Between 1905 and 1911 Lipót Fejér worked at the Department of Mathematics and Physics of the University of Kolozsvár. This must explain the fact that he chose the topic of his "habilitation" lecture not from the theory of Fourier series but from the stability theory of ordinary differential equations. First, he gave an outline of the definitions of stability used in those days (strangely enough, including the recurrence property as well): 'The concept of stability carries highly different contents even within the framework of mass point systems. It is no use arguing which one of them is the best since, except for some inherent features of it, stability as a popular concept is so indefinite and so relative that, owing to the variety of existing relations, stability definitions highly differing from one another may be formulated without getting into contradiction with the popular concept'. Among the definitions of stability listed by him we can find the one accepted in general nowadays but it is considered too narrow by Fejér, joining Felix Klein's opinion. Then, he discusses some simpler aspects of the three-body problem and finishes his habilitation lecture with the discussion of the Lagrange–Dirichlet theorem.

In connection with his habilitation lecture {19} Fejér published a finding of his own in the problem of equilibrium instability {18}, {20}. (Ref. {20} is a word–for–word German translation of the Hungarian original {18}.) Slightly changing notations, Fejér investigated equation

$$\ddot{\underline{r}} = \operatorname{grad} \pi(\underline{r}) - \dot{\underline{r}} f\left(|\dot{\underline{r}}|\right)/|\dot{\underline{r}}|, \quad \underline{r} = (x, y, z) \in \mathbb{R}^3 \tag{2}$$

under the conditions below: The potential function $\pi : \mathbb{R}^3 \to \mathbb{R}$ is analytic in a small vicinity of the origin $\underline{0}$ and its Taylor series about $\underline{0}$ begins with a negative definite 3-variable homogeneous polynomial of order $2n$ for some integer $n \geq 1$ (in particular, the potential function π has an isolated maximum at $\underline{0}$), $f : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function, $f(0) = 0$, $f(v) > 0$ for $v > 0$ and

$$\limsup_{v \to 0^+} f(v)/v < \infty. \tag{3}$$

Starting from Jacobi's identity

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{2}|\underline{r}|^2\right) = |\dot{\underline{r}}|^2 + \langle \operatorname{grad} \pi(\underline{r}), \underline{r} \rangle - \langle \dot{\underline{r}}, \underline{r} \rangle f\left(|\dot{\underline{r}}|\right)/|\dot{\underline{r}}|$$

Fejér proved via elementary ad hoc inequalities that the equilibrium state $\underline{0}$ is unstable. He raised the question of whether the result of instability will remain true without the condition (3), e.g. for the function $f(v) = v^\alpha$, $\alpha \in (0, 1)$. The positive answer is a consequence of La Salle's invariance principle elaborated more than 50 years later: A 1968 theorem of Luigi Salvadori (see Thm. III. 5.8 as well as the accompanying discussion) in 'N. Rouche, P. Habets & M. Laloy, *Stability Theory by Liapunov's Second Method*, Springer, Berlin, 1977' can be directly applied.

Another paper of Fejér's {17} published in 1906 is related to the so-called Ostwald principle. In his work entitled 'Lehrbuch der allgemeinen Chemie' published in 1893 Ostwald stated that 'von allen möglichen Energieumwandlungen wird diejenige eintreten, welche in gegebener Zeit den grösstmöglichen Umsatz ergibt' (from among all possible conversions of energy the one which produces the greatest increment in the given time will happen). For a given planar potential field $U(\underline{r})$ and starting point $\underline{r}_0$ Fejér studied such equations of motion

$$\ddot{\underline{r}} = -\operatorname{grad} U(\underline{r})$$

for which *Case 1* $U(\underline{r}_0) - U\big(\underline{r}(t)\big)$ is maximal for a single and fixed $t = T_0 > 0$ or, alternatively, *Case 2* $U(\underline{r}_0) - U\big(\underline{r}(t)\big)$ is maximal for all $t \in [0, T_0]$. He stated that in *Case 1* the initial velocity $\dot{\underline{r}}(0)$ has to meet some conditions of compatibility and the motion itself will be brachistochronal, and in *Case 2* the orthogonal semitrajectory through $\underline{r}_0$ should be a half-line and $\dot{\underline{r}}(0)$ should be parallel to the direction vector of this half-line. Otherwise, Ostwald wanted to demonstrate the general validity of his principle by the motions along half-lines obtained in *Case 2*. In spite of the great prestige and merits of Ostwald — he was awarded a Nobel Prize in chemistry in 1909 — this energy principle had already been highly criticized. All parties in the debate emphasized that the original phrase 'von allen…' provided several opportunities for different mathematical interpretations. Most of them, including Mór Réthy {66}, investigated which modifications and improved versions of the Ostwald principle were in harmony with the general laws of mechanics and other physical sciences.

On partial differential equations, more precisely on the equation of the form

$$\sum_{i,k=1}^{n} a_{ik}(x) u_{x_i x_k} + \sum_{r=1}^{n} b_r(x) u_{x_r} + c(x) u = 0, \quad a_{ik} = a_{ki} \tag{4}$$

Lipót Fejér wrote only a single article {22}. He assumed that the coefficients were analytic functions and $c(\underline{0}) < 0$. By using the Cauchy–Kowalevskaya theorem he proved that (4) had a solution having a positive maximum at

$\underline{0}$ if and only if matrix $\left\{a_{ik}(\underline{0})\right\}_{i,k=1}^{n}$ is not positive semidefinite. This is a contribution to the theory of maximum principles.

F. Riesz' subharmonic functions. Out of the papers of Frigyes Riesz those on subharmonic functions are related to the theory of differential equations directly. The concept of subharmonic functions was introduced by Riesz in a lecture in Stockholm in 1924 {68}. The definition is as follows: Let $\Omega \subset \mathbb{R}^n$ be open and connected, and $f : \Omega \to [-\infty, \infty)$ an upper semicontinuous function, $f(x) \neq -\infty$ for at least one $x \in \Omega$. The function f is subharmonic if for any pair (Ω', u), the inequality $f|_{\partial\Omega'} \leq u|_{\partial\Omega'}$ implies that $f|_{\Omega'} \leq u$. Here Ω' is a bounded domain, $\overline{\Omega}' = \mathrm{cl}\,(\Omega') \subset \Omega$, $u : \overline{\Omega}' \to \mathbb{R}$ is a continuous function which is harmonic on Ω'. The definition of subharmonic functions was inspired, partly, by some characteristics of analytic functions discovered by Hardy and Landau and, partly, by Perron's method, which was elaborated for proving the existence of classical $C(\overline{\Omega}) \cap C^2(\Omega)$ solutions to the Dirichlet boundary value problem $\Delta u = 0$, $u|_{\partial\Omega} = g$ (where $g : \partial\Omega \to \mathbb{R}$ is a given continuous function and $\Omega \subset \mathbb{R}^n$ is a bounded domain having a 'nice' boundary.) Prior to his lecture in Stockholm Riesz had generalized {67} the results of Hardy and Landau, and in a joint paper with Tibor Radó, he gave a simplification of Perron's method {72}. The core of Perron's method as conceived by Riesz and Radó is the following assertion: If $\Omega \subset \mathbb{R}^n$ is a bounded domain and $g : \partial\Omega \to \mathbb{R}$ is a continuous function, then

$$u(x) = \sup\left\{v(x) \mid v \in C(\overline{\Omega}), v|_{\partial\Omega} \leq g \text{ and } v|_{\Omega} \text{ is subharmonic}\right\}$$

defines a harmonic function on Ω. (If $\partial\Omega$ is nice e.g. if it is of class C^2 or satisfies the outer sphere condition, then $u \in C(\overline{\Omega})$ and $u|_{\partial\Omega} = g$.) Connections to the celebrated Fejér–Riesz proof of Riemann's conformal mapping theorem (published as part of the Introduction in a paper by Radó' (*Acta Sci. Math. (Szeged),* **1** (1922/23), 240–251. (reprinted as pp. 841–843 of the Appendix to Fejér's 'Gesammelte Arbeiten')) are transparent.

The basic characteristics of subharmonic functions and their relation to potential theory was clarified by Frigyes Riesz in two consecutive papers {69}, {70}. A function $f \in C^2(\Omega)$ is subharmonic, if and only if $\Delta f \geq 0$ on Ω. The main result is that any subharmonic function can be represented as a potential plus a harmonic function. If $f : \Omega \to [-\infty, \infty)$ is a subharmonic function and, as before, Ω' is a bounded domain with $\overline{\Omega}' \subset \Omega$, then

$$f(x) = \int_{\Omega'} \Gamma(x,y)\, d\mu_y + h(x), \quad x \in \Omega' \tag{5}$$

where Γ is the fundamental solution of Laplace's equation $\Delta u = 0$ on $\mathbb{R}^n$, μ_y is a (uniquely defined) Borel measure and $h : \Omega' \to \mathbb{R}$ is a harmonic function. It is worth comparing (5) with Green's classical representation formula

$$w(x) = \int_{\Omega'} \Gamma(x,y)\Delta w(y)\, dy + \int_{\partial\Omega'} \left(w(y)\frac{\partial \Gamma(x,y)}{\partial \nu_y} - \Gamma(x,y)\frac{\partial w(y)}{\partial \nu} \right) dS_y,$$

$x \in \Omega'$ for $C^2(\Omega)$ functions. (Note also that the function defined by the boundary integral is harmonic.) If $f = w \in C^2(\Omega)$ then $d\mu_y = \Delta w(y)\, dy$. In the book written by Tibor Radó [143] several variations of formula (5) can be found. Some of them originate from Frigyes Riesz. (Radó reviewed the development of the theory of subharmonic functions till 1937 but he was not concerned with its applications to partial differential equations.)

The late period of Frigyes Riesz had little relevance to differential equations. Techniques applied in his 1938–48 series of papers on ergodic theory belong to general functional analysis and the theory of measure and integration. However, he considers ergodic theory as part of a dynamical system theory for measure–preserving transformations and interprets, occasionally, his own results from this view–point. It is amazing how much the old Riesz felt the importance of functional iterations! He asked (himself and) each reader of Matematikai Lapok (**3** (1952), Problem No. 54) to "determine the set of those complex initial points z_1 for which, given a complex parameter a, the iteration $z_{n+1} = (z_n^2 + a)/2$ is converging." The Editor of the Problem Session had waited two years for the solution in vain when he stated (**5** (1954), p. 283) that Problem No. 54 was entirely open and promised to publish the solution any time it arrives. (Actually, Problem No. 54 concerns the structure (of the complement) of Julia sets for the quadratic family, a basic problem in fractal geometry... Unfortunately, Mandelbrot was apparently unaware of the existence of Matematikai Lapok and published his results somewhere else ...)

The works of F. Riesz and Haar on linear integral equations. It is important to mention that integral equations played a fundamental role in the formulation of the papers on compact linear operators by Frigyes

Riesz, as well as in the emergence of the whole functional analysis. Alfréd Haar, a colleague of Frigyes Riesz at Kolozsvár and, later, at Szeged was also concerned with integral equations. As a student in Göttingen, while applying Hilbert's method elaborated for the problem $\Delta u = 0$, $u|_{\partial\Omega} = g$ in his first paper, Haar {30} discusses the reducibility of the boundary value problem

$$\Delta^2 u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0,\ u|_{\partial\Omega} = g_0,\ \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = g_1$$

to integral equations and proves Fredholm type alternative theorems.

Haar's inequality for partial equations of the first order. The most important results in differential equations of Hungarian mathematics in the first half of the twentieth century are attached to the name of Alfréd Haar. He made essential contributions to both the theory of general equations of the first order and of quasilinear elliptic equations of the second order. His results in the latter topic are related to his pioneering work in the field of the calculus of variations, to put it more precisely, in the study of the minimal surface problem and of the minimal surface equation

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0.$$

Here we sum up Alfréd Haar's papers on partial differential equations of the first order based on his lecture delivered in Bologna in 1928 {39}.

Consider a finite interval $[x_1, x_2] \subset \mathbb{R}$. Given a parameter $a > 0$, define the triangle

$$\mathcal{T} = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \le y \le (2a)^{-1}(x_2 - x_1) \text{ and } x_1 + ay \le x \le x_2 - ay \right\}.$$

Finally, let $z : \mathcal{T} \to \mathbb{R}$ be a C^1 function and set $M = \max\left\{ \left|z(x,0)\right| \mid x_1 \ge x \ge x_2 \right\}$. Assume that, for some constants $b > 0$ and $c > 0$,

$$\left|z_y(x,y)\right| \le a\left|z_x(x,y)\right| + b\left|z(x,y)\right| + c \quad \text{whenever} \quad (x,y) \in \mathcal{T}.$$

Then

$$\left|z(x,y)\right| \le Me^{by} + cb^{-1}(e^{by} - 1) \quad \text{whenever} \quad (x,y) \in \mathcal{T}. \tag{6}$$

Inequality (6) — just like its one-variable counterpart, the famous Gronwall lemma in the theory of ordinary differential equations — has far-reaching consequences. As a preliminary, we may state that the problem

$\mathcal{F}(u_y, u_x, u, x, y) = 0$, $u|_S = h$ ($h : S \to \mathbb{R}$ is a continuous function, $S \subset \partial\Omega$, $\Omega \subset \mathbb{R}^2$) can be transformed to the form (7) under very general conditions.

Let $g : [x_1, x_2] \to \mathbb{R}$ be a continuous function and let $F : \mathbb{R}^2 \times \mathcal{T} \to \mathbb{R}$ be a continuous function with the property that

$$\left| F(p, z, x, y) - F(\tilde{p}, \tilde{z}, x, y) \right| \leq a|p - \tilde{p}| + b|z - \tilde{z}|$$

whenever $(p, z, x, y), (\tilde{p}, \tilde{z}, x, y) \in \mathbb{R}^2 \times \mathcal{T}$. Applying inequality (6) to the difference of two possible solutions, it is immediate that the first order problem

$$u_y = F(u_x, u, x, y) \quad \text{and} \quad u|_{[x_1,x_2]\times\{0\}} = g \tag{7}$$

has at most one C^1 solution on $\mathcal{T}$ {36}. Formerly, uniqueness results were known only for C^2 solutions and proved within the framework of the theory of characteristics. Inequality (6) has some consequences to characteristics in return {37}. As has been mentioned by Hadamard in his additional comment published jointly with Haar's paper {36}, Haar's inequality leads not only to a result on uniqueness but also to the assertion that the C^1 solutions of equation (7) on the triangle $\mathcal{T}$ depend continuously on function g.

Haar finished his lecture delivered at Bologna with the extension of inequality (6) to systems of partial differential equations with a simple structure. His promise to devote another paper to the topic could not be fulfilled due to his early death. Incidentally, Alfréd Haar was already concerned with systems of first order partial differential equations in one of his early articles {41} the coauthor of which was Tódor Kármán, one of his fellow students in Göttingen.

Haar's existence and uniqueness theorem in the calculus of variations. We are going to sum up Alfréd Haar's work on calculus of variations based on his own lecture held in Hamburg in 1930 {40} as well as on the monograph of Tibor Radó [142] who himself achieved fundamental results in this field. The focus of the discussion will be placed on Alfréd Haar's existence and uniqueness theorem {35}. In the proof a lemma originating from Tibor Radó {63} plays an important part which, by its external form, is a geometric statement on saddle surfaces but in essence is an a priori estimate for the gradient of solutions to quasilinear elliptic equations which will be reviewed separately here. Naturally, the abstract existence and uniqueness theorem has important consequences for the classical minimal surface problem {34} which motivates it. Finally, we are going to discuss these consequences. Emphasis will be laid on the regularity properties of the solution, more exactly, on the analytic feature of the minimal surface.

Consider a bounded domain $\Omega \subset \mathbb{R}^2$ with a convex Jordan curve $\Gamma = \partial\Omega$ as its boundary, a C^2 function $F : \mathbb{R}^2 \to \mathbb{R}$, and a continuous function $\varphi : \Gamma \to \mathbb{R}$. The variational problem

$$I(u) = \iint_\Omega F(p,q)\,dx\,dy \longrightarrow \min_{u|_\Gamma = \varphi} \tag{8}$$

is called regular if the Hessian matrix of F is positive definite for all $p, q \in \mathbb{R}^2$. Of course, $p = u_x$ and $q = u_y$. The Euler–Lagrange equation of problem (8) is

$$F_{pp}u_{xx} + 2F_{pq}u_{xy} + F_{qq}u_{yy} = 0. \tag{9}$$

In the most important special case, i.e. in the minimal surface problem we have $F(p,q) = (1 + p^2 + q^2)^{1/2}$, and thus equation (9) simplifies to

$$(1 + q^2)u_{xx} - 2pqu_{xy} + (1 + p^2)u_{yy} = 0. \tag{10}$$

Using geometrical terms equation (10) expresses that the mean curvature of a minimal surface equals zero.

As a preliminary, observe that the functional I can be defined for all elements of the function class

$$\mathcal{L} = \{u : \overline{\Omega} \to \mathbb{R} \mid u \text{ is a Lipschitz function}\}$$

and the double integral in (8) is understood in the sense of Lebesgue. The validity of the formula

$$\operatorname{Area}(u,\Omega) = \iint_\Omega (1 + u_x^2 + u_y^2)^{1/2}\,dx\,dy, \quad u \in \mathcal{L}$$

was proven first by Zoárd Geőcze {25} where $\operatorname{Area}(u,\Omega)$ stands for the area of surface $S = \{(x, y, u(x,y)) \in \mathbb{R}^3 \mid (x,y) \in \overline{\Omega}\}$ as defined by Lebesgue for continuous surfaces in his famous doctoral thesis. Also, observe that both (9) and (10) are quasilinear elliptic equations. The eigenvalues of the coefficient matrix of equation (10) are $\lambda_1(p,q) = 1 + p^2 + q^2$ and $\lambda_2(p,q) = 1$. Since the ratio λ_1/λ_2 is unbounded, (10) is nonuniformly elliptic. It is well-known that the solvability of a Dirichlet boundary value problem for nonuniformly elliptic quasilinear equations depends crucially on the geometric assumptions on the pair (Γ, φ). The crucial assumption of Haar's existence and uniqueness theorem is that the pair (Γ, φ) satisfies Hilbert's so-called three–point condition. In other words, it is assumed that, for some constant $K > 0$, every set of three distinct points on the curve $\left\{\left(x, y, \varphi(x, y)\right) \in \mathbb{R}^3 \mid (x, y) \in \Gamma\right\}$ lies in a plane of slope $\leq K$. The three–point condition is the assumption which implies the property $\mathcal{L} \neq \emptyset$. It implies also the strict convexity of Γ.

After such preparations we are in a position to formulate Haar's existence and uniqueness theorem {35}. Consider the variational problem (8). Assume that (8) is regular and that the three–point condition with constant K is satisfied. Then (8) has a unique solution in the function class $\mathcal{L}$ and the Lipschitz constant of this solution is $\leq K$.

T. Radó's regularity Lemma. Uniqueness is a rather elementary consequence of the regularity assumption on (8). The starting point of the existence proof is the observation that the functional $I : \mathcal{L} \to \mathbb{R}$ is lower semicontinuous (with respect to uniform convergence). The difficulty is that minimizing sequences are not precompact. A nice geometric property of saddle functions, which was conjectured by Haar and proved by Radó {63}, helps, instead. A continuous function $u : \overline{\Omega} \to \mathbb{R}$ is a saddle function if, given arbitrarily three constants $\alpha, \beta, \gamma \in \mathbb{R}$ and an open set $\emptyset \neq \Omega' \subset \Omega$, the maximum-minimum principle

$$\min_{(x,y)\in\overline{\Omega'}} \left(u(x,y) - (\alpha x + \beta y + \gamma)\right) = \min_{(x,y)\in\partial\Omega'} \left(u(x,y) - (\alpha x + \beta y + \gamma)\right),$$

$$\max_{(x,y)\in\overline{\Omega'}} \left(u(x,y) - (\alpha x + \beta y + \gamma)\right) = \max_{(x,y)\in\partial\Omega'} \left(u(x,y) - (\alpha x + \beta y + \gamma)\right)$$

is satisfied. Radó's lemma concerns saddle functions $u : \overline{\Omega} \to \mathbb{R}$ for which the pair $\left(\Gamma, u|_{\Gamma}\right)$ is subject to the three–point condition with constant K and states that such functions are Lipschitz continuous on $\overline{\Omega}$ and the Lipschitz constant is $\leq K$.

Radó's lemma was given a new and a much simpler proof by János Neumann {51}. Both the original method of proving and the one given by Neumann lead, automatically, to Lemma 12.6 in 'D. Gilbarg & N. S. Trudinger, *Elliptic Partial Differential Equations of the Second Order*, Springer, Berlin, 1983': Let Ω be a bounded domain in $\mathbb{R}^2$, and let $\varphi : \partial\Omega \to \mathbb{R}$ be a continuous function satisfying the three-point condition with constant K. Suppose

$u \in C(\overline{\Omega}) \cap C^2(\Omega)$ with $u|_{\partial\Omega} = \varphi$ satisfies a quasilinear elliptic equation of the form $au_{xx} + 2bu_{xy} + cu_{yy} = 0$ where a, b, c are continuous functions in the variables (x, y, u, p, q). Then $|u_x|, |u_y| \le K$ on Ω. (Besides the Lax–Milgram lemma, which is considered 'half-Hungarian' due to Péter Lax, this is the only result in the well-known book of Gilbarg and Trudinger which originates from a Hungarian mathematician. It is worth mentioning that Lemma 12.6 and, in relation to this, the name of Tibor Radó are mentioned in the preface of this monograph.

Haar's Lemma on the variation of double integrals. Applying the existence and uniqueness theorem for the variational problem (8) in the special case $F(p,q) = (1 + p^2 + q^2)^{1/2}$ we obtain that there exists a $u^* \in \mathcal{L}$ for which

$$\text{Area}\,(u^*, \Omega) < \text{Area}\,(u, \Omega) \quad \text{for all} \quad u^* \neq u \in \mathcal{L}.$$

Actually, $u^*|_\Omega$ is analytic. The argumentation leading to this will be outlined below. The starting point is one of Alfréd Haar's earlier results {32}, namely the two-variable counterpart of Du Bois–Reymond's fundamental lemma of the calculus of variations. In order that the novelty of Haar's result should be emphasized, we would like to present Du Bois–Reymond's lemma: If $f : [a, b] \to \mathbb{R}$ is a continuous function such that

$$\int_a^b f(x)\varphi'(x)\,dx = 0 \ \text{ for each } \ \varphi \in C[a,b] \cap C^1(a,b) \ \text{ with } \varphi(a) = \varphi(b) = 0,$$

then f is a constant function.

Using the previously introduced notation, let $v, w : \overline{\Omega} \to \mathbb{R}$ be continuous functions and assume that

$$\iint_\Omega (v\xi_x + w\xi_y)\,dx\,dy = 0 \tag{11}$$

for each continuous function $\xi : \overline{\Omega} \to \mathbb{R}$ satisfying $\xi|_\Gamma = 0$ and for which the partial derivatives derivatives $\xi_x, \xi_y : \Omega \to \mathbb{R}$ are continuous and bounded. Then there exists a C^1 function $\omega : \Omega \to \mathbb{R}$ with the properties that

$$v = \omega_y \quad \text{and} \quad w = -\omega_x \quad \text{on} \quad \Omega. \tag{12}$$

If we assume merely that that functions v and w are measurable and bounded, then function ω satisfies only the Lipschitz condition and (12) is valid almost everywhere. Now let u denote the solution of the variational problem (8). Since

$$\delta I(u;\xi) = \iint_\Omega (F_p(u_x,u_y)\xi_x + F_q(u_x,u_y)\xi_y)\,dx\,dy = 0$$

for all ξ (which have the above mentioned properties), (12) goes over into the system of equations

$$F_p(u_x,u_y) = \omega_y \quad \text{and} \quad F_q(u_x,u_y) = -\omega_x. \tag{13}$$

We arrived at a system of first order partial equations which replaces the Euler–Lagrange equation and can be used excellently for clarifying the differentiability properties of the solutions of (8). Under adequate differentiability conditions, of course, (9) can be derived from (13) easily. The hard fact is that we cannot foresee whether these differentiability conditions will be met by u, moreover, examples are also known when the unique solution to (8) is not twice differentiable.

Fortunately, Lipschitz solutions of the system

$$\frac{u_x}{(1+u_x^2+u_y^2)^{1/2}} = \omega_x \quad \text{and} \quad \frac{u_y}{(1+u_x^2+u_y^2)^{1/2}} = -\omega_y \tag{14}$$

— this is to what (13) is simplified in the special case $F(p,q) = (1+p^2+q^2)^{1/2}$ — are analytic functions. Eventually, the latter assertion depends on the fact proven by Rademacher according to which Lipschitz solutions of the Cauchy–Riemann system are analytic functions. Analyticity of C^1 solutions of the system (14) was shown by Tibor Radó {62} while Haar had been working on the proof to the existence and uniqueness theorem. (Analyticity of C^2 solutions of equation (10) had been known much earlier.) Radó's argumentation works for Lipschitz solutions as well. Thus, the proof for the existence and uniqueness theorem was also a justification for the analytic nature of function $u^*|_\Omega$, at the same time. As has been shown by Tibor Radó, $\text{Area}\,(u^*,\Omega) \le \text{Area}\,(u,\Omega)$ for all $u \in \mathcal{C}$, where

$$\mathcal{C} = \left\{ u : \overline{\Omega} \to \mathbb{R} \mid u \text{ is a continuous function and } u|_\Gamma = \varphi \right\}.$$

From a later result of McShane it also follows that

$$\text{Area}\,(u^*,\Omega) < \text{Area}\,(u,\Omega) \quad \text{for all} \quad u^* \ne u \in \mathcal{C}.$$

The implication (11) ⇒ (12) is called Haar's Lemma in the literature. Incidentally, this was Haar's first result in the field of the calculus of variations which he published in Hungarian in 1917 {31} and in German in 1919 {32}. The converse of Haar's Lemma, the implication (12) $\Rightarrow$ (11) is also true. The latter was proven by Schauder (*Acta Sci. Math (Szeged),* **4** (1929), 38–50). For the fulfilment of the equivalence between (11) and (12) the convexity assumption on Ω is too restrictive. Even Haar and Schauder themselves worked with Jordan domains. Definitely, Haar was aware of the fact that his lemma could be generalized for multiple integrals on n-dimensional domains and it was in close relation to the n-dimensional Stokes theorem but in those directions he worked out very few or rather no details at all. His work had been continued by others, amongst them, by Jenő Gergely {26}, Antal Sólyi {76} and Adolf Szücs {80} in Hungary. It is important to mention that the concept of the adjoint variational problem was introduced also by Alfréd Haar {38}. The pattern to that was provided by adjoint minimal surfaces as well as by an earlier result {33} of his generalizing an observation of Zermelo on the Du Bois–Reymond lemma in one dimension.

In the development of the minimal surface problem it turned out that the three–point condition for the pair (Γ, φ) is superfluous. Keeping only the convexity assumption on $\partial\Omega = \Gamma$, Tibor Radó [142] proved that equation (10) has a solution in the function class $\mathcal{C} \cap C^2(\Omega)$. (The proof depends, essentially, upon the existence theorem for conformal mappings of approximating polyhedra and it does not indicate at all that the three–point condition can be omitted from the general conditions of Haar's existence and uniqueness theorem.) This $\mathcal{C} \cap C^2(\Omega)$ function is analytic on Ω and constitutes the unique solution of the variational problem $J : \mathcal{C} \to \mathbb{R}$, $J(u) = \text{Area}\,(u, \Omega) \longrightarrow min$ as well. Extensions of the minimal surface problem allow surfaces which cannot be parametrized globally in the form $\big\{\big(x, y, u(x, y)\big) \in \mathbb{R}^3 \mid (x, y) \in \overline{\Omega}\big\}$, whose boundary is disconnected or may be just a knotted Jordan curve, etc. If the problem is studied in this generalized sense, great difficulties arise both for existence and uniqueness and, eventually, neither of them holds true. However, the relevant methods (in the elaboration of which Tibor Radó himself participated according to his monograph [142]) are quite far from those in differential equations.

An early paper on billiards. There is a paper of Adolf Szücs, jointly with Dénes König {43} which is worth to be discussed in detail. They considered the motion of a single, dimensionless particle included in an immobile cube. The impacts on the walls follow the laws of elastic reflection. By using elementary geometry and elementary number theory, they classified the orbits as closed, dense in a polyhedral surface, and dense within the whole cube. The three possibilities do not depend on the initial state but only on the three components of the initial velocity. This is one of the earliest results in the theory of rational billiards.

Neumann's method of stability analysis. In connection with the argumentation leading to Haar's existence and uniqueness theorem we have already mentioned the name of János Neumann who gave a new proof for Radó's lemma on saddle surfaces which was the most important part of the aforementioned argumentation from the aspect of the general theory of partial differential equations. Neumann need not be introduced here. He was one of the 'Martians' who came out from the leading secondary schools of Budapest at the turn of the XIX and XX centuries. Once Jenő Wigner was asked how it happened that so many geniuses had been born in Budapest a century earlier. Wigner replied that he did not understand the question because in those days, as he had said it before, just one genius was born and that person was called János Neumann. In the early phase of planning this volume of studies the idea cropped up that a special chapter should be written about Neumann, while keeping the division by the broad mathematical themes. Although this proposal was not realized, it demonstrates well what a great personality Neumann was and what a prominent role he played in the history of mathematics.

Neumann exerted a great influence on the general development of differential equations. This was not achieved by his articles but by his consulting activities with which he kept track of the operation of the first digital computer. The first genuine tasks solved by eletronic computers were as follows: initial-boundary value problems of thermonuclear reactions, neutron diffusion and transport, radiation flows, and fluid dynamics. The respective partial differential equations were solved by numerical stepsize integration procedures, namely, by the so-called method of finite differences. At each point of the integration net, partial derivatives, i.e. differential quotients were replaced by (finite) difference quotients. That procedure led to large-scale systems of algebraic equations. The number of unknowns was the same as that of the points of the integration net. If the original partial differential equation was linear, then the system of approximating algebraic equations became also linear. The computer was used for the solution of that system of algebraic equations. Actually, in most of the cases the computing task to be performed was to solve a system of linear equations, or to put it in another way, the inversion of a large-scale matrix. Needless to say, the original initial-boundary value problems emerged during the development of wartime technology. All related research results were considered top military secrets and their publication was quite out of question. What Neumann was allowed to publish as a 'by-product' of the numerical solution of the initial-boundary value problems was only an expository article in 1947 on the numerical inversion of matrices of high order with the co–authorship of H. H. Goldstine {27} (in today's terminology, they established the stability of the LU and the Choleski factorizations) and a six-page paper with R. D. Richtmyer {53} on the numerical calculation of hydrodynamic shocks in 1950. (Some of) his earlier reports to various divisions of the National Defence Research Committee on shocks — actually, the generals were interested in detonation/blast waves — remained unpublished for nearly two decades, see pp. 178–347 in Volume 6 of the Collected Works of John von Neumann (and others may remain unpublished still). The fact that the discretization of partial differential equations might lead to matrices of high order could be found out by any reader of that time, therefore it was mentioned in the introduction of the first paper briefly. However, it was not mentioned at all whether partial differential equations and, mainly which ones, had been discretized concretely, and whether their solutions had been computed. Moreover, no mention was made of the capacity and character of the computing devices used. (With the pride of a father and the precision of a book-keeper, however, he lists technical and finantial data of the first high-speed 'fully automatic electronic computing machines' in his confidential reports.) Yet, the aim was just to compute the concrete solutions of the concrete initial-boundary value problems with the aid of the first electronic/digital computers which had been built by the mid-forties. The second paper with Goldstine {28}, a 1951 continuation of {27} contains a probabilistic analysis of rounding errors.

Neumann kept track of the numerical solution of those concrete initial-boundary value problems closely and, in particular, he was concerned with the computational stability of the discretization methods applied. The main difficulty was that truncation errors (which emerged from replacing partial derivatives by the respective finite differences) could be amplified considerably in the course of the numerical-computational procedure. Thus, the establishment of stability criteria, the fulfilment of which would prevent elementary errors (truncation errors, rounding errors, or (local) errors of any kind) from becoming so amplified as to make gibberish the whole calculation, was of vital importance. Neumann acquainted his colleagues with his ideas on the concept of computational stability in his lectures held at the Los Alamos laboratories and other military research institutes. The 'von Neumann's method of stability analysis' has become known since the early l950s, primarily, owing to the paper written by G. C. O'Brien, M. A. Hyman & S. Kaplan (*J. Math. Phys.,* **29** (1950), 223–239) who incorporated and presented Neumann's heuristic techniques of stability with his consent. (The 1947 interior report had been remained unknown for a long time.) Now let us quote A. H. Taub's words from his Editorial Note on page 664 of Volume 5 of the Collected Works: '... his procedure was, as he always emphasized, at all times heuristic. It was used in various "practical" situations for practical purposes but, as he once wrote (letter of November 10, 1950 to Werner Leutert, "as far as any evidence that is known to me (J. v. N.) goes, ... it has not led to false results so far".' Robert Richtmyer (*Difference Methods for Initial-Value Problems,* Interscience, New York, 1957), a further close witness of the emergence of 'von Neumann's method of stability analysis' declares his opinion in the very same sense: '... the new development has been based more on empiricism and intuition and less on a mathematical basis than the classical development. One should not blame the new development for this, if we were to wait for convergence proofs and error estimates for the new methods, most of the computers now in use in technology and industry would come grinding to a halt.'

For Neumann the pattern was provided by R. Courant, K. Friedrichs H. Lewy (*Math. Ann.,* **100** (1928), 32–74), who studied the convergence of the solutions of the approximating system of difference equations and, through this, proved existence theorems for partial differential equations of various types. Courant, Friedrichs and Lewy observed for the first time that restrictions on the permissible size of Δt in terms of the size of the other increment (e.g. inequality $\Delta t \le \Delta x/c$ for the wave equation $u_{tt} = c^2 u_{xx}$ expressing that the triangle-shaped domain of dependence on local data cannot become larger under discretization) are necessary for the convergence of the finite difference approximations.

However, as it is shown by the quotations above, Neumann's interest in analysing difference approximations was focused on error amplification and not on convergence. The essence of his approach can be easily extracted from his 1947/48 interim "First/Second Report on the Numerical Calculation of Flow Problems" reprinted on pages 652–750 of Volume 5 of the Collected Works: Consider, for simplicity, the one-dimensional heat equation $u_t = u_{xx}$, $t \ge 0$, $u(0,x) = f(x)$, $x \in \mathbb{R}$. The simplest discretization procedure leads to a system of linear equations of the form

$$(15) \qquad \frac{u_{i+1,j} - u_{i,j}}{\Delta t} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta x)^2} \quad i \in \mathbb{N},\ j \in \mathbb{Z}.$$

Neumann's basic idea was to investigate how errors (existing at some initial or intermediate time level) evolve in the course of solving system (15). Substituting $\varepsilon(t,x) = a(t)\sin kx$ into the error formula

$$\varepsilon(t+\Delta t, x) - \varepsilon(t,x) = \mu\big[\varepsilon(t, x+\Delta x) - 2\varepsilon(t,x) + \varepsilon(t, x-\Delta x)\big]$$

where $\mu = \Delta t/(\Delta x)^2$ (cf. (15) but note that $\varepsilon(t,x)$ is different from $e(t,x) = u(t,x) - \overline{u}(t,x)$, the difference between the computed and the exact values of the solution), it is readily checked that the coefficient $a(t)$ is amplified at each integration step by a factor of $1 + 2\mu(\cos k\Delta x - 1)$, see p. 653. If the interval of variability of x is from 0 to π and $u_t = u_{xx}$ is equipped with Dirichlet boundary conditions, then $\varepsilon_{i,j}$ can be represented as $\varepsilon_{i,j} = \sum_{k=1}^{N-1} a_{i,k} \sin k(j\Delta x)$, and the amplification factor is the same, i.e.

$$a_{i,k+1} = a_{i,k}\big(1+2\mu(\cos k\Delta x - 1)\big) \quad \text{for} \quad k = 1, 2, \ldots, N-1 \text{ with } N = \pi/\Delta x,$$

see p. 691. Observe that $\big\{1 + 2\mu(\cos k\Delta x - 1) \mid k = 1, 2, \ldots, N-1\big\}$ is the set of eigenvalues of the tridiagonal error amplification matrix (cf. (15))

$$A = \{\alpha_{i,j}\}_{i,j=1}^{N-1}, \quad \alpha_{i,j} = 1 - 2\mu \text{ if } i = j \text{ and } \mu \text{ if } |i-j| = 1 \text{ (and 0 otherwise).}$$

Hence, for two different reasons,

$$\big|1 + 2\mu(\cos\beta - 1)\big| \leq 1 \quad \text{for all} \quad \beta \in (0,\pi) \;\Leftrightarrow\; \mu \leq 1/2$$

is a necessary condition for preventing error amplification. He also proved that an implicit version of (15) is computationally stable, independently of the fact how much the concrete value of the mesh ratio $\Delta t/(\Delta x)^2$ is, see p. 707. (It is worth mentioning here that the inequality $\mu \leq 1/2 \Leftrightarrow \Delta t \leq (\Delta x)^2/2$ plays no role in the Courant, Friedrichs & Lewy paper.) As for the linear system (15) equipped with initial conditions $u_{0j} = f(j\Delta x)$ ($j \in \mathbb{Z}$, f sufficiently nice), they only prove that, in the special case $\Delta t/(\Delta x)^2 = 1/2$, the solutions of (15) converge to the exact solution $\overline{u}$ of the original initial value problem. In fact, they point out by a direct computation that

$$u_{i,j} = \sum_{k=0}^{i} \frac{1}{2^i}\binom{i}{k} f\big((j+i-2k)\Delta x\big) \quad \longrightarrow$$

$$\longrightarrow \quad \overline{u}(t,x) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(\xi - x)^2}{4t}\right) f(\xi)\, d\xi$$

whenever $i, j \to \infty$ with $i\Delta t \to t$ and $j\Delta x \to x$ for any $t > 0$ and $x \in \mathbb{R}$ fixed. (Actually, they consider equation $2u_t = u_{xx}$ and — in order to make the combinatorics for $u_{i,j}$ simpler — they take $\Delta t = (\Delta x)^2$ (and mention that similar convergence results hold true for general parabolic equations as well).) In the Courant, Friedrichs & Lewy paper the emphasis is put on hyperbolic equations. On the other hand, though the affirmative answer is implicitely contained in his considerations, Neumann does not seem to pay any attention to the question if, at least in some technical sense and for f and $\overline{u}$ sufficiently nice, inequality $\mu \leq 1/2$ is enough to imply that $e(t, x) \to 0$ as $\Delta t \to 0$.)

The model results collected in the previous paragraph illustrate how Neumann applied his Fourier series and eigenvalue techniques to studying the computational stability of finite difference approximations. Real-life examples can be found in his weather forecast paper {8} and in the aforementioned 1947/48 confidential reports intended for military and industrial use. Thus, Neumann can be regarded, rightly, as the founder of the stability theory of the numerical methods of differential equations even if he always used the concept of stability in an empirical sense. Several applications of the Fourier approach and the eigenvalue one initiated by him can be found in Arieh Iserles' remarkable textbook. With good reason, 'A. Iserles, *A First Course in the Numerical Analysis of Differential Equations,* Cambridge University Press, Cambridge, 1996' criticizes that the bulk of the literature (in particular, the older literature) terms each of these approaches as 'von Neumann's method of stability analysis', without any distinction and in a rather confusing way. A more descriptive and less ambiguous terminology is preferred.

Lax equivalence theorem, the theoretical result behind. Neumann's 'naive' argumentations concerning the computational stability of difference approximations were raised to the level of an abstract theory in linear functional analysis by Péter Lax in the early fifties. Together with his parents, the 15 year old Lax left Hungary for the US in 1941. Letters of recommendation from Dénes König and Rózsa Péter to Neumann accompanied him. As a student and young researcher, he was very much influenced by Neumann, Friedrichs (his PhD adviser), and Courant. Thirty years after the death of Neumann, he remembered his mentor in an interview by saying: 'Von Neumann, who was the central figure of the mid-century, firmly believed that computing was central not only to the numerical side of applied mathematics but also to progress in theory. That is why he invented computers and pushed for their development. He foresaw that computations are essential to discover basic phenomena in nonlinear systems.'

Drafted for the war, Lax ended up at Los Alamos and remained there until May 1946. His Los Alamos stay shaped his general attitude to mathematics as well the choice of his research subjects considerably. He belongs to a minority of mathematicians who consider themselves both pure and applied.

For Banach space problems $u_t = Au$, $u(0) = u_0$ generating C^0 operator semigroups, e.g. for well-posed linear evolutionary partial differential equations, Lax gave a precise definition of the stability of approximating linear procedures. The core of the definition is, on each time interval $[0, T]$, the uniform boundedness of all products of the approximating small-stepsize linear operators. Simultaneously, Lax opened up the series of equivalence theorems

$$\text{consistency} \quad \& \quad \text{stability} \;\Leftrightarrow\; \text{convergence}$$

which played a vital role in the theory of discretizations. For details, see e.g. 'R. D. Richtmyer & K. W. Morton, *Difference Methods for Initial-Value Problems,* Wiley, New York, 1967'. In the background of the later equivalence theorems, too, there is the Neumann idea according to which the convergence of numerical procedures can be proven through an argumentation of the following type: 'small local errors' plus 'no (significant) error amplification' imply 'small global error'.

The work of Lax on a single conservation law. In what follows we give a brief description of Lax's work on shock waves which had grown out of his Los Alamos experiences. 'The existing literature on this question is unsatisfactory' — summarized Neumann his opinion in 1943. Thanks to the development that followed, in particular to Lax's contribution {45}, {46}, Neumann's statement had lost much of its validity by the time of his death in 1957. We follow the respective chapters in 'L. C. Evans *Partial Differential Equations,* AMS, Providence, R.I., 1998' and 'J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer, Berlin, 1982' very closely.

Consider the scalar conservation law in a single space variable

$$u_t + \big(f(u)\big)_x = 0, \quad x \in \mathbb{R}, \; t \geq 0 \tag{16}$$

with initial data $u(x, 0) = u_0(x)$, $x \in \mathbb{R}$. Characteristics are straight lines of the form $\Big\{\Big(f'\big(u_0(x_0)\big)\,t + x_0, t\Big) \;\Big|\; t \geq 0\Big\}$, $x_0 \in \mathbb{R}$. If $x_0 < \tilde{x}_0$ and $f'\big(u_0(x_0)\big) < f'\big(u_0(\tilde{x}_0)\big)$ for some $x_0, \tilde{x}_0 \in \mathbb{R}$, the characteristic lines through $(x_0,0)$ and $(\tilde{x}_0,0)$ meet at some point in $t>0$. Since classical solutions are constant along characteristics, one has either to accept that solutions cannot be defined for all time or, alternatively, to look for a more general concept of a solution that allows the appearance of discontinuities (even in solutions with continuous initial data). A function $u\in L_\infty(\mathbb{R}\times\mathbb{R}^+,\mathbb{R})$ is called a weak solution of (16) with initial data $u_0\in L_\infty(\mathbb{R},\mathbb{R})$ if

$$\int_0^\infty\int_{-\infty}^\infty \big(\varphi_t u+\varphi_x f(u)\big)\,dx\,dt+\int_{-\infty}^\infty \varphi(\cdot,0)u_0\,dx=0$$

for any C^∞ function $\varphi:\mathbb{R}^2\to\mathbb{R}$ with compact support in $\big\{(x,t)\mid x\in\mathbb{R},\ t\geq 0\big\}$. In general, unfortunately, as it is demonstrated by the example of Burger's equation $u_t+uu_x=0$ (i.e. $f(u)=u^2/2$) equipped with Riemann initial data $u_0(x)=u_\ell\in\mathbb{R}$ if $x<0$ and $u_r\in\mathbb{R}$ if $x>0$, there exists an abundance of weak solutions defined for all $t\geq 0$. Now equations of the form (16) arise in the physical sciences and so one must have some mechanism to pick out the 'physically relavant' one. Mathematically, the basic question is to impose an a priori condition on weak solutions that ensures existence and uniqueness. This a priori assumption is O. Oleinik's entropy condition (we present as inequality (18) below) found in 1957 (*Usp. Mat. Nauk.,* **12** (1957), 3–73. (in Russian) (*AMS Transl. Ser. 2,* **26** (1957), 95–172.). Until very recently, no such results were known for systems of conservation laws of general type.) In the very same year, Lax {46} also proved an existence and uniqueness theorem. He considered a subclass of systems of conservation laws and proved existence and uniqueness within a class of piecewise continuous functions with a finite number of certain shock and contact discontinuities. His abstract result applies to Riemann's classical tube problem in gas dynamics and gives a rigorous proof for the earlier results.

The key is to look at the Hamilton–Jacobi equation

$$(17)\qquad w_t+f(w_x)=0\quad\text{with initial data}\quad w_0(x)=\int_0^x u_0(y)\,dy.$$

(Formal differentiation shows that u can be taken for w_x.) From now on, assume that $f\in C^2$, $f(0)=0$, $\inf\big\{f''(u)\mid u\in\mathbb{R}\big\}>0$ and let L denote the Legendre transform of f. The uniform convexity assumption on f implies that f' is a C^1 self-diffeomorphism of $\mathbb{R}$. For later purposes, set $g=(f')^{-1}$, the inverse of f'. The Hopf–Lax formula

$$w(x,t) = \inf\left\{ \int_0^t L\big(\dot{q}(s)\big)\,ds + w_0(y) \,\Big|\, q \in C^1\big([0,t],\mathbb{R}\big),\right.$$
$$\left. q(0) = y,\ q(t) = x \right\}$$
$$= \min\left\{ tL\left(\frac{x-y}{t}\right) + w_0(y) \,\Big|\, y \in \mathbb{R} \right\},$$

a basic result in the calculus of variation defines a weak solution to (17) in the sense that $w : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ is Lipschitz continuous, (and hence, by the Rademacher theorem, w is almost everywhere differentiable), it satisfies $w_t(x,t) + f\big(w_x(x,t)\big) = 0$ for almost every $(x,t) \in \mathbb{R}\times\mathbb{R}^+$, and $w(x,0) = w_0(x)$ for each $x \in \mathbb{R}$. Actually, given $t > 0$ arbitrarily, the mapping $x \to w(t,x)$ is differentiable for almost every $x \in \mathbb{R}$. In addition, there exists for all but at most countably many values of $x \in \mathbb{R}$ a unique $y(x,t) \in \mathbb{R}$ such that

$$w(x,t) = tL\left(\frac{x - y(x,t)}{t}\right) + w_0\big(y(x,t)\big),$$

the mapping $x \to y(x,t)$ is nondecreasing, and $\frac{\partial}{\partial x} w(x,t) = g\big(\frac{x-y(x,t)}{t}\big)$ holds true for almost every $x \in \mathbb{R}$. The final result is that the Lax–Oleinik formula

$$u(x,t) = g\left(\frac{x - y(x,t)}{t}\right)$$

defines a weak solution for (16) satisfying, with some absolute constant $C = C(u_0)$, for each $t > 0$, the one-sided Lipschitz estimate

$$u(x+z,t) - u(x,t) \le Cz/t \tag{18}$$

for almost every $(x,z) \in \mathbb{R}\times\mathbb{R}^+$. As a reformulation of (18), the function $x \to u(x,t) - Cx/t$ is nondecreasing for each $t > 0$. Thus, even though the initial data u_0 is merely an $L_\infty(\mathbb{R},\mathbb{R})$ function, the Lax–Oleinik solution u immediately becomes fairly regular in $t > 0$. It is a crucial result of Oleinik that (in some technical sense) u depends continuously on u_0 and — up to a set of measure zero — no other weak solution of (16) with initial data u_0 satisfies (18).

Consider a C^1 curve $\Gamma = \big\{\big(x(t),t\big)\big\}$ of discontinuties in a weak solution u and assume that u extends continuously from either side of Γ to Γ. By choosing the test function to concentrate at the discontinuity, one arrives easily at the Rankine–Hugoniot jump identity

$$s(u_+ - u_-) = f(u_+) - f(u_-). \tag{19}$$

Here $s = \dot{x}(t)$ is the speed of the discontinuity in $\big(x(t),t\big)$ and $u_\pm = u\big(x(t)\pm 0,t\big)$ are the states at the jump. The motion along the discontinuity curve is called a shock wave. In a rough analogy to the thermodynamic principle that physical entropy cannot decrease as time goes forward, Lax introduced the entropy condition

$$f'(u_+) < s < f'(u_-) \tag{20}$$

required at each point of Γ. A more direct analogy for requiring (20) is that information may vanish at the shock but may not be created at a shock — geometrically, (20) means that characteristic lines may enter a shock but may not leave it. Armoured with (19) and (20), it is not hard to single out the 'physically relevant' solution in a great number of cases. For conservation laws with uniformly convex f, (20) is an easy consequence of (18).

The work of Lax on systems of conservation laws. The modern theory of systems of conservation laws $\mathbf{u}_t + \big(\mathbf{f}(\mathbf{u})\big)_x = 0$, $(x \in \mathbb{R},\ t \geq 0)$ started with Lax's fundamental paper {46}. It is there where one first encounters the basic ideas in the subject: the shock inequalities (that replace Lax's entropy condition (20) for systems), the notion of genuine nonlinearity, the one-parameter families of shock- and rarefaction-wave curves, as well as the solution to the general Riemann problem. We do not enter the details here but, indicating the complexity of {46}, describe the solution of Riemann's classical tube problem in gas dynamics instead. Consider a long, thin, cylindrical tube containing gas separated at $x = 0$ by a thin membrane. It is assumed that the gas is at rest on both sides of the membrane, but it is of different constant pressures and densities on each side. At time $t = 0$, the membrane is broken, and the problem is to determine the ensuing motion of the gas. This leads to a system of conservation laws with dependent variable $\mathbf{u} = (v,\rho,p) = (\text{velocity}, \text{density}, \text{pressure})$ and initial data $(v_\ell,\rho_\ell,p_\ell) \in \mathbb{R}^3$ for $x < 0$ and $(v_r,\rho_r,p_r) \in \mathbb{R}^3$ for $x > 0$. Note that $v_\ell = v_r = 0$ and consider the case $\rho_\ell > \rho_r$, $p_\ell > p_r$. By symmetry,

$v(x,t) = \alpha(s)$, $\rho(x_t) = \beta(s)$, $p(x,t) = \gamma(s)$ for some real functions α, β, γ where $s = x/t$, $x \in \mathbb{R}$, $t > 0$. The solution $\mathbf{u}(x,t)$ can be described as follows. The initial discontinuity breaks up into two discontinuities, the shock wave and a contact discontinuity with constant speed $s_4 > 0$ and $s_3 \in (0, s_4)$, respectively. In addition, there exist constants $s_1 < 0$ and $s_2 \in (s_1, 0)$ such that

$$u(s) = \begin{cases} 0 \\ mif(s) \\ u_0 \\ u_0 \\ 0 \end{cases}, \quad \rho(s) = \begin{cases} \rho_\ell \\ mdf(s) \\ \rho_1 \\ \rho_2 \\ \rho_r \end{cases}, \quad p(s) = \begin{cases} p_\ell & \text{if } s < s_1 \\ mdf(s) & \text{if } s_1 < s < s_2 \\ p_0 & \text{if } s_2 < s < s_3 \\ p_0 & \text{if } s_3 < s < s_4 \\ p_r & \text{if } s_4 < s \end{cases}$$

(i.e. gas in original high pressure state, rarefaction wave, rarified gas, compressed gas, gas in original low pressure state) where $u_0 > 0$, $p_0 \in (p_r, p_\ell)$, $\rho_1 \in (0, \rho_\ell)$, $\rho_2 > \min\{\rho_1, \rho_r\}$ are constants and mdf and mif stand for certain decreasing and increasing functions, respectively.

The Lax–Milgram Lemma. It is a must to discuss the Lax–Milgram Lemma. Consider, for simplicity, the Dirichlet problem for the Laplacian

$$\Delta u = f \in L_2(\Omega), \quad u|_{\partial\Omega} = 0$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$ nice. Function $u \in H_0^1(\Omega)$ is called a weak solution if

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega f v \, dx = 0$$

for each C^∞ function v with compact support in Ω. The Lax–Milgram Lemma is an abstract result in linear functional analysis. (Actually, a simple consequence of Riesz' theorem on the dual space of a Hilbert space: Given a bounded, coercive bilinear form b on a Hilbert space H with scalar product $\langle ., . \rangle$, there exists a uniquely defined linear self-homeomorphism S of H such that $b(Sf, v) + \langle f, v \rangle = 0$ whenever $f, v \in H$. Coercivity of the not necessarily symmetric bilinear form b means that $b(v,v) \geq \beta \langle v, v \rangle$ for some $\beta > 0$ and all $v \in H$.) The Lax–Milgram Lemma guarantees existence and uniqueness for the weak solution. It also works in the case when the Laplacian is replaced by a more general elliptic operator. Thus the Lax–Milgram Lemma reduces the existence proof for a solution in $H^2(\Omega) \cap H_0^1(\Omega)$

to a regularity lemma for the weak solution. Here $H^2(\Omega)$ and $H_0^1(\Omega)$ are Sobolev spaces. As it is suggested by the observation that follows (18), note that the natural space for weak solutions of a conservation law is some bounded variation space.

In line with the editorial principles, we do not pursue the scientific career of Lax any further but restrict ourselves to call the attention of the reader to [107], a survey on systems of conservation laws. The *MathSciNet Full Search* option gives evidence that his name — in expressions like Lax difference operator, Lax equations, Lax–Friedrichs scheme, Lax integrability, Lax monoids, Lax pairs, Lax–Phillips scattering theory, Lax representation, Lax–Wendroff scheme etc. — appears in the respective titles of more than 600 mathematical research papers.

A cross-section in 1928. The state of art. Neumann must have known the R. Courant, K. Friedrichs & H. Lewy (*Math. Ann.,* **100** (1928), 32–74) paper very early because an article of his own, the one in which he proved the minimax theorem of game theory, was published in the same volume of Mathematische Annalen. If we have started our study with mentioning how many papers of Hungarian mathematicians were published in Mathematische Annalen between 1900 and 1910, let us cite similar statistics here, based on Volumes 98–99–100 which were issued in 1928. Out of about 100 papers 17 were written by Hungarian authors or coauthors. Since Gyula Szőkefalvi-Nagy, János Neumann and Gábor Szegő are represented by several papers, the number of Hungarian authors is 13. Out of them Pólya lived in Zürich, Switzerland, Neumann, Szász and Szegő lived in Germany, and Szőkefalvi-Nagy was a resident in Romania. Later Szőkefalvi-Nagy moved to Szeged and the other four, threatened by the worsening of their working conditions in a continental Europe under German influence and foreseeing the dimensions of a racial persecution that culminated in the *Endlösung*, emigrated to the USA. Tibor Radó, after the failure of his 1929 application for a professorship at Debrecen University, joined them in the self-chosen/from-outside-enforced exile. Though supported by a commission of the four most respected Hungarian professors who recommended him *primo et unico loco,* Radó was surpassed by a protègè of a clique of local potentates with some political support in Budapest. (Also Riesz' and Haar's applications were refused in their days. They applied for a professorship at Budapest University but neither of them was appointed.)

Returning to the Hungarian contribution to Volumes 98–99–100 of the Mathematische Annalen, we note that only two of the 17 papers treat differential equations. These are Alfréd Haar's aforementioned paper on adjoint problems of the calculus of variations {38} and an additional one by Aurél Wintner {82}. During his U.S. years Wintner did not consider himself a Hungarian mathematician, therefore in this report, in compliance with the editorial principles (which exclude discussing the oeuvre of Arthur Erdélyi and that of Paul Halmos, for example) we are going to discuss only his earliest career. The cited paper is concerned with analytic solutions of differential equations in Hilbert spaces: Wintner provides an infinite–dimensional generalization of the Cauchy–Kowalewskaya theorem. By the way, in one of his papers, the coauthor of which was S. Bochner, Neumann treats ordinary differential equations in Hilbert spaces, most precisely, with almost periodic solutions of one type of these equations {6}. It is worth mentioning another paper of his, written jointly with G. W. Brown {7}, in which they give a new proof using Liapunov functions of game dynamics for the existence of good strategies for zero–sum two-person games. Similarly to his works on partial differential equations, here also, practical aspects are emphasized: "The proof is 'constructive' in a sense that lends itself to utilization when, actually, computing the solutions of specific games." Our report on the relation of Neumann's oeuvre to the theory of differential equations will be complete if we mention his contribution of great impact to ergodic theory — his 1940/41 Princeton lectures on invariant measures were published quite recently {52} — as we did in the case of Frigyes Riesz.

Methods of differential equations appeared occasionally in the works of Károly Jordan. We restrict ourselves to quoting his monograph on difference calculus (actually, on basic combinatorial enumeration from a probabilist's view but including a long chapter on linear difference equations and a short one on linear equations of partial differences) {42}. On the other hand, differential equation methods infiltrated the works of István Grynaeus, whose illness and untimely death in 1936 deprived the circle of Hungarian differential geometers of its most talented member, to a much deeper extent, e.g. in {29} which is an application of the Ricci calculus to a Pfaffian system.

Pólya and Szegő on isoperimetric inequalities. Several works of György Pólya and Gábor Szegő can be considered to be about differential equations; primarily the ones in which they proved certain isoperimetric inequalities with the aid of the methods of potential theory and calculus of variations. Pólya and Szegő were led to isoperimetric inequalities, partly, by their notorious problem-solving attitude and, partly, by their profound knowledge of complex function theory (and, within complex function theory, by the fact that potential theory in dimension two is essentially equivalent to the theory of conformal mapping). Among the antecedents it should be mentioned that Gábor Szegő translated Webster's partial differential equations textbook in the late 1920s. Since Szegő added some mathematical details to the original text of the physicist Webster, who neglected, or elaborated only roughly, certain parts of it, the translation became a revision and, thus, the translator became a co-author. The German edition of the work was published under both of their names [195]. Also, it is worth mentioning that in the encyclopedic work of 'P. Frank and R. von Mises *Die Differential- und Integralgleichungen der Mechanik und Physik*, Vieweg, Braunschweig, 1925' the chapter on potential theory was written by Szegő whose own first result in that field was on a relationship between Green functions and the transfinite diameter of plane curves. This latter concept was introduced by Mihály Fekete in his famous work on generalizing Chebishev polynomials (which arise in the case of a line segment) {23}. Later, Pólya joined Szegő's research of this type. By isoperimetric inequalities we mean statements on extremal properties of set functions which have obvious geometric or physical interpretations. The model statement (which was due, originally, to Pólya in 1920) can be taken from Pólya and Szegő [129, Problem IX. I. 2]: Consider a corn hill the base of which is a unit disc in a horizontal plane. Then $V/S \geq \pi/3$ where V and S stand for the volume and the maximal slope, respectively. Equality is attained for circular cones. Based on the machinery necessary to their formulation and proof, isoperimetric inequalities can be classified as belonging to the relevant branches of mathematics.

In what follows let $V_1 \subset V \subset V_0$ denote a nested triplet of closed solids in $\mathbb{R}^3$ with closed regular surface boundaries. It is assumed that $\partial V_1 \subset V \setminus \partial V$ and $\partial V \subset V_0 \setminus \partial V_0$. Let u denote the uniquely defined solution to the Dirichlet problem

$$(21) \qquad \Delta u = 0 \quad \text{on} \quad V_0 \setminus (V_1 \cup \partial V_0), \quad \text{and} \quad u|_{\partial V_1} = 1, \quad u|_{\partial V_0} = 0.$$

The capacity of the nested pair $(\partial V_1, \partial V_0)$ is defined as

$$C = -\frac{1}{4\pi} \int_{\partial V} \frac{\partial u}{\partial \nu} \, dS \qquad \text{(the normal vector } \nu \text{ points outwards)}$$

— the integral does not depend on the particular choice of V. The nested pair $(\partial V_1, \partial V_0)$ itself is termed a condenser. The terms 'capacity' and 'condenser' refer to the meaning of the Dirichlet problem (21) in electrostatics. (Of course, the function u can be interpreted as an equilibrium solution of the heat equation also.) The capacity of ∂V_1 is defined via the Dirichlet problem

$$\Delta u = 0 \quad \text{or} \quad \mathbb{R}^3 \setminus V_1, \quad \text{and} \quad u|_{\partial V_1} = 1, \quad u(\infty) = 0$$

(that corresponds to the limiting process $\partial V_0 \to \infty$).

Szegő's main result {78} is as follows: Among all nested pairs $(\partial V_1, \partial V_0)$ with volume (V_1) and volume (V_0) given, the capacity is minimal if and only if V_1 and V_0 are concentric balls. The limiting process $\partial V_0 \to \infty$ leads to the proof of a conjecture due to Poincaré: Of all surfaces ∂V_1 with volume (V_1) given, the sphere has the smallest capacity. Similarly {78}, of all surfaces $\partial(V_1)$ with area (V_1) given, the sphere has the largest capacity. Naturally, the abovementioned statements, the planar versions of which had been known before, could be reformulated in the form of inequalities as well. In his latter work {79} Szegő verified Maxwell's conjecture $C \le d/2$, too. Here d stands for the usual diameter and the equality holds only for spheres. It should be mentioned that exact indications to a satisfactory proof of Poincaré's conjecture can be found in an earlier paper by G. Faber (*Sitzungsber. Bayr. Acad. Wiss.* (**1923**), 169–172).

The main finding of Georg Faber's aforesaid paper is the proof of one of Rayleigh's important conjectures: of all vibrating membranes, the closed disc emits the gravest fundamental tone. The mathematical task is to minimize $\lambda_1(D)$ where D is a closed regular domain in $\mathbb{R}^2$ with area (D) given, say π, and $\lambda_1(D)$ stands for the principal eigenvalue of the negative Laplacian equipped with the Dirichlet boundary condition $u|_{\partial D} = 0$. Recall that

$$\lambda_1(D) = \min\left\{\frac{\iint_D (u_x^2 + u_y^2)\,dx\,dy}{\iint_D u^2\,dx\,dy} \;\middle|\; u \in C^1(D \setminus \partial D) \cap C(D) \text{ and } u|_{\partial D} = 0\right\},$$

the minimum is attained for the principal eigenfunction e_1, the level sets of e_1 are (except for one point) simple closed curves, and $e_1(x,y) > 0$ for $(x,y) \in D \setminus \partial D$. In essence, the major observation of Faber and of Edgar Krahn (*Math. Ann.,* **94** (1924), 97–100) (the latter obtained the same result nearly simultaneously but independently of the former) is that

$$\iint_D \left((e_1)_x^2 + (e_1)_y^2\right)\,dx\,dy \ge \iint_{B_1} (v_x^2 + v_y^2)\,dx\,dy$$

where B_1 is the closed unit disc and the function $v = v(e_1, D) : B_1 \to \mathbb{R}^+$ is (uniquely) defined by the property as follows: For any $\kappa \in \big[0, \max\big\{ e_1(x,y) \mid (x,y) \in D \big\}\big]$,

(22)
$$v(x,y) = \kappa \text{ if and only if } x^2 + y^2 = \pi^{-1} \cdot \text{area}\Big(\big\{ (x,y) \in D \mid e_1(x,y) \geq \kappa \big\}\Big).$$

From this the proof of Rayleigh's conjecture can be derived easily. The name of the procedure applied in formula (22) is symmetrization with respect to a point. Faber remarks that the very same method leads to the proof of Poincaré's conjecture on minimal capacities and, that is indeed the case.

Szegő {78} followed a totally different, simpler and ad hoc way, but the family of symmetrization methods some elements of which had already been known by Jacob Steiner and Hermann Amandus Schwarz in the nineteenth century proved to be much more successful in the long run. At least, a dozen quantities in geometry and physics increase or decrease under a certain symmetrization procedure. Pólya and Szegő, jointly and individually, proved several assertions of this type and, through them, isoperimetric inequalities {61}. With the help of the symmetrization methods Pólya {57} proved de Saint-Venant's conjecture of 1856 (which de Saint-Venant supported by convincing physical considerations and several particular cases, but did not prove in a mathematical sense): Of all cross-sections with a given area, the circular cross-section has the largest torsional rigidity. The torsional rigidity or stiffness $P(D)$ of a cross-section D (i.e. of an infinite beam with a given plane domain D as cross-section) can be defined as

$$P(D) = 4 \cdot \max\left\{ \frac{\big(\iint_D u\,dx\,dy\big)^2}{\iint_D (u_x^2 + u_y^2)\,dx\,dy} \;\Big|\; u \in C^1(D \setminus \partial D) \cap C(D) \text{ and } u|_{\partial D} = 0 \right\}.$$

Note that the maximum is attained if and only if $u = cv$ where $c \neq 0$ is a real constant and v solves the boundary value problem

$$v_{xx} + v_{yy} + 2 = 0 \quad \text{on} \quad D \setminus \partial D, \quad \text{and} \quad v|_{\partial D} = 0.$$

In their 1951 book Pólya and Szegő [130] presented the 'state of the art' of the questions concerning isoperimetric inequalities of that age. The influence of this so-called 'smaller Pólya–Szegő' can be felt even nowadays and this work of theirs continues to be the source of inspirations. At least half the book discusses Pólya and Szegő's own results. They formulated, improved and optimized in it the inequalities about various set functions. They treated the cases of nearly circular and nearly spherical domains as well as several techniques for handling parameters. After the publication of their book the study of the topic was continued by both of them. Among their co-authors Menahem Schiffer's name should be mentioned: With him, Pólya proved {59} an old conjecture of his according to which the transfinite diameter of a convex plane curve is no less than one-eighth of the perimeter.

A special mention should be made of Pólya and Szegő's joint paper {60} on qualitative properties of the one-dimensional heat equation. Applying Descartes's generalized rule of signs and Sturm's oscillation theorem they state that the number of roots and/or the extrema of each individual solution is a decreasing function of time. In one of his papers {57} Pólya treats similar questions again but the longer study intended has never been written. If it had been written, it might have accelerated the recognition how important a role is played by the number of sign changes in the qualitative theory of linear and nonlinear parabolic equations of one dimension. The 1952 paper of Pólya {57} on combining finite differences with the Rayleigh–Ritz method is frequently interpreted as a preparatory step towards the discovery of finite element methods.

M. Riesz' fractional potentials. Now we are going to discuss the contribution to differential equations of the younger Riesz brother. Marcel Riesz was concerned with differential equations only in a rather late period of his career, from the early 1930's on. His most important results were in the field of potential theory and wave propagation. His interest was motivated, partly, by the application to the theory of relativity. All his work on partial differential equations until that time was summarized by Marcel Riesz himself in a book-size paper written in a book style, published in 1949 {71}. We are going to discuss this monumental work below.

Marcel Riesz worked out several basic techniques in multidimensional fractional integration and generalized the concept of the classical Riemann–Liouville integral

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1}\,dt, \qquad \alpha > 0$$

in different directions. The one associated with the m-dimensional Laplacian Δ is

$$(23) \qquad (I_\Delta^\alpha f)(x) = \frac{1}{H_{\Delta,m}(\alpha)} \int_{\mathbb{R}^m} |x-y|^{\alpha-m} f(y)\, dy$$

where $H_{\Delta,m}(\alpha) = \pi^{m/2} 2^\alpha \Gamma(\alpha/2)/\Gamma\big((m-\alpha)/2\big)$. Case $m=3$, $\alpha=2$ simplifies to the standard Newtonian potential. Patterned on the simple identity

$$(I^\alpha f)(x) = \frac{f(a)}{\Gamma(\alpha+1)}(x-a)^\alpha + (I^{\alpha+1} f')(x) = (I^{\alpha+1} f')(x)$$

valid for $f \in C^1(\mathbb{R})$ with $f(a)=0$ and $\alpha > -1$, Green's formula applies for $f : \mathbb{R}^m \to \mathbb{R}$ sufficiently nice, extends the operator $\alpha \to (I_\Delta^\alpha f)$ by analytic continuation and leads to the properties $\Delta(I_\Delta^{\alpha+2} f) = -I_\Delta^\alpha f$ and $\Delta(I_\Delta^2 f) = -f$. If $f\,dy$ is a mass distribution with a finite total mass in $\mathbb{R}^m$, then the integral in (23) makes sense for $0<\alpha<m$ and $I_\Delta^\alpha f$ is called the fractional potential of order α of $f\,dy$. By passing to the limit, $I_\Delta^0 f = f$. A further fundamental fact established by Riesz is that $I_\Delta^{\alpha+\beta} f = I_\Delta^\alpha (I_\Delta^\beta f)$ whenever $\alpha>0$, $\beta>0$ and $\alpha+\beta<m$. The very same semigroup properties hold true if $f\,dy$ is replaced by $d\mu(y)$ where μ is a general mass distribution in $\mathbb{R}^m$. In this setting $(I_\Delta^\alpha f)$ is the fractional potential of the mass while the energy of μ with respect to the fractional potential is defined by $\int (I_\Delta^\alpha \mu)(x)\, d\mu(x)$. Existence, uniqueness and basic properties of the equilibrium distribution in a compact set $F \subset \mathbb{R}^m$ (i.e. of a distribution having minimal energy in the class of mass distributions supported by F and having a given total mass) were proven rigorously in the 1935 PhD thesis of Otto Frostman, a famous disciple of Riesz. In fact, Frostman's very general approach and method of proving the existence of the equilibrium measure is considered the foundation of modern general potential theory. Riesz' functional potentials thus generated a far reaching development including weighted potentials as well as the Wiener theory of Brownian motion.

The fractional integral associated with the D'Alembertian operator $\Box = \partial_1^2 - \partial_2^2 - \cdots - \partial_m^2$, $m \ge 2$ is

$$(I_\Box^\alpha f)(x) = \frac{1}{H_{\Box,m}(\alpha)} \int_{x-C} \big(r(x-y)\big)^{\alpha-m} f(y)\, dy.$$

Here $1/H_{\Box,m}(\alpha)$ is a suitable 'Γ–factor' — suitable to imply $\Box(I_\Box^{\alpha+2} f) = I_\Box^\alpha f$ and $\Box(I_\Box^2 f) = f$ for $f : \mathbb{R}^m \to \mathbb{R}$ sufficiently nice —, $r^2(x-y) =$

$(x_1 - y_1)^2 - (x_2 - y_2)^2 - \cdots - (x_m - y_m)^2$ is the square of the Lorentzian distance and

$$x - C = \left\{ x - y \in \mathbb{R}^m \mid r^2(y) \geq 0 \text{ and } y_1 > 0 \right\},$$

the retrograde light-cone with its vertex at x. Semigroup properties for $\alpha, \beta \geq 0$ are also established. The last chapter of {71} contains a similar theory for the wave operator in arbitrary Riemannian spaces.

A major part of {71} is devoted to the Cauchy problem for the wave equation $\Box u = f$ with initial data on a codimension one surface of the form $S = \left\{ x \in \mathbb{R}^m \mid x_1 = s(x_2, \ldots, x_m) \right\}$. Riesz establishes integral representations for the solution involving certain divergent integrals which obtain a meaning by analytic continuation methods. This is more elegant than the parallel theory of Hadamard on 'finite parts' of divergent integrals because it does not distinguish between even and odd numbers of dimensions. On the basis of his formula, Riesz clarifies that Huygens phenomenon is a consequence of the fact that, for $m > 2$ even, function $H_{\Box,m}$ has a simple pole at $\alpha = 2$. He gives a purely geometric interpretation of the solution for the physically most important case, namely $m = 4$, with a disussion of certain line congruences and caustics. The entire discussion is important with respect to the Lorentz group. Then he applies his method to the Maxwell and Dirac equations and analyses the Liénard-Wiechert potential of a moving electron, too. Similar to Hadamard, Riesz also extends his solution representation formula for the wave equation with variable coefficients and initial data on S.

Marcel Riesz's work {71} reflects the state of differential equations which preceded the introduction of Schwartz distributions and Sobolev spaces. Since that time one of his main goals, the proper interpretation of divergent integrals, has been attained in a much larger framework through the theory of distributions. Although he must have been rather distant from defining the appropriate function spaces, his results in potential theory pointed towards the introduction of fractional powers of the Laplacian. In the later development of linear partial differential equations from among his disciples two of them, Lars Gårding and Lars Hörmander played a basically important role. Apart from his work on spinors and Clifford algebras in the late period of his life Riesz himself contributed relatively little to his earlier differential equation results.

The work of Egerváry. The first result obtained in the post–war period in Hungary we present is due to Jenő Egerváry and Pál Turán {11} and devoted to the memory of D. König and A. Szücs who did not survive the tragic days of 1944/45. Combined with hard analytic tools which go back to H. Weyl, Egerváry and Turán used the geometric ideas of D. König and A. Szücs {43} in proving a weak, somewhat artificial form of the Boltzmannian Hypothesis in the kinetic theory of gases. They considered an oversimplified differential equation model (which is very carefully chosen but not a differential equation model any more — nevertheless, we feel that the differential equation chapter is the right place to discuss it) of n particles: the n particles are included in an immobile cube $C = \{(x_1, x_2, x_3) \mid 0 \le x_1, x_2, x_3 \le \pi\}$, they are dimensionless, of equal mass, no attractive or exterior forces acting, the impacts on the walls according to the laws of elastic reflection, collisions between three or more particles excluded, collisions between two particles according to the law of elastic impact, the initial conditions of the n particles at time $t_0 = 0$ are arbitrary and, with $\vartheta_1 = 1$, $\vartheta_2 = 2^{1/2}$, $\vartheta_3 = 3^{1/2}$, the initial velocities satisfy

$$v_k^i \in n^{2/5}\left(1 + \frac{k}{n^{101/100}}\right) \cdot \left(\vartheta_i - \frac{1}{n^{10}}, \vartheta_i + \frac{1}{n^{10}}\right)$$

$$i = 1, 2, 3 \quad \text{and} \quad k = 1, 2, \ldots, n.$$

For simplicity, Egerváry and Turán assumed that the n particles are *equidistributed* at time t if for any rectangular body R in C, the number of particles $N(R, t)$ in R at t satisfies

$$\left|\frac{N(R,t)}{n} - \frac{\mathrm{vol}\,(R)}{\pi^3}\right| \le \frac{1}{n^{1/10}}.$$

They prove that the particles are equidistributed for the time interval $0 \le t \le n^{1/4}$ except time intervals whose total length does not exceed $c_0 n^{-1/10} \log^4 n$ where c_0 stands for a moderate numerical constant. If n is of the order 10^{23}, then $n^{1/4}$ is on the order of several days, and $c_0 n^{-1/10} \log^4 n$ is on the order of several seconds long. Estimates which are slightly better and work for more realistic initial velocities can be found in {12} which is a technically improved version of {11}. In both papers, the intention of the authors is to support the opinion that (some reasonable variant of) the Boltzmannian hypothesis can be derived as a consequence of the basic laws of mechanics.

Jenő Egerváry, a professor at the Budapest University of Technology, is one of the very few Hungarian mathematicians whose entire career is closely related to applied mathematics. Starting from his 1913 PhD Thesis (dedicated to a single linear Fredholm integral equation {9}) to his latest results (including his 1956 paper on a large system of fourth–order linear differential equations modelling suspension bridges {10}) he wrote several articles on the convergence of the method of finite differences. He had papers on the three–body problem, on heat conduction, and on the motion of the electron as well.

A cross-section in 1953. The state of art. As far as the application of mathematics is concerned, the decade preceding 1956 played a role of special importance in Hungary. Obviously, in the age of reconstruction of war–time damages (with the priority of rebuilding the bridges destroyed on the Danube) and during a period of an unprecedented development of heavy industry most of the applications were closely related to differential equations. (Motivations of mathematicians to take part in this work were diverse: with genuine enthusiasm some supported the efforts to establish the new society which was called people's democracy officially; others did the same out of fear of the Communist Party under external and internal pressures; there were still others who just wanted to earn money.) In the meantime, central industrial research institutes were set up and even the Research Institute of Applied Mathematics organized by Egerváry and Rényi, which was the legal predecessor of today's Rényi Institute (Research Institute of Mathematics of the Hungarian Academy of Sciences), there was a Department of Chemical Industry, a Department of Mechanics and Statics as well as a group on Electrotechnic (precisely, an Independent Group on Electronics and Function Approximation). In compliance with the above mentioned administrative structure of mathematical research dozens of papers of practical importance were born in the field of the application of differential equations. From the mid- and late fifties researchers of mathematical analysis in a broader sense turned to more abstract research topics.

From 1960 to 1970 the Department of Differential Equations of the Research Institute of Mathematics — the attribute 'Applied' was taken away after the fifties — was led by Károly Szilárd, the brother of Leó Szilárd. Károly Szilárd left Hungary in 1919 and returned in 1960. He spent 14 years in Germany (PhD in Göttingen, 1925) and 27 years in the USSR (Stalin Prize in 1953, after several years in a 'prison-research-institute'). A further emblematic figure of applied mathematical analysis was Samu Borbély. He worked in a research laboratory of the German aviation industry in the thirties, then returned to Hungary for reasons of conscience, and fled the Gestapo in 1944. While in a USSR 'prison-research-institute' after the war, he could conceale his expertize in aviation matters and worked for the artillary. Being a member of the Hungarian Academy of Sciences from 1949 onward, he taught mathematics in Miskolc and, later, at the Budapest University of Technology. Like Szilárd, he has a very limited number of publications (in the literature with general availability).

Having discussed the data between 1900 and 1910 as well as those in 1928, let us have a look at the state of differential equations in light of the statistical figures found in the 1953 volumes of the old Acta Scientiarum Mathematicarum (Szeged) **1/32** and the recently founded new journals Acta Mathematica Hungarica (Budapest) **1/23**, Publicationes Mathematicae (Debrecen) **2/33** — papers only in English, French, German, and Russian; MTA III. Osztály Közleményei (Proceedings of the Third Branch (Mathematics and Physics) of the Hungarian Academy of Sciences) **1/21**, MTA Alkalmazott Matematika Intézetének Közleményei (Proceedings of the Research Institute of Applied Mathematics of the Hungarian Academy of Sciences) **10/36** — papers only in Hungarian. The name of each journal is followed by a fraction. The denominator is the number of papers in the journal written by Hungarian authors whereas the numerator is the number of papers that may be ranked among differential equations in a broader sense. Since in 1953 mathematicians in Hungary could hardly think to publish their work abroad, actually, the number of their papers in the five periodicals mentioned were almost identical with the total number of their publications of that year.

Bihari inequality. The 1956 paper of Imre Bihari {4} is probably the most frequently cited ordinary differential equation paper ever written by a Hungarian mathematician. It contains what we call today the Bihari inequality, the first nonlinear version of the classical Gronwall lemma. Let $u,v : [a,b) \to \mathbb{R}^+$, $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous functions. Assume that ω is increasing and $\omega(u) > 0$ whenever $u > 0$. In addition, let K be a nonnegative constant and assume that

$$u(t) \le K + \int_a^t v(s)\omega\big(u(s)\big)\,ds \quad \text{whenever} \quad t \in [a,b).$$

Then

(24)

$$u(t) \le \begin{cases} \Omega^{-1}\left(\Omega(K) + \displaystyle\int_a^t v(s)\,ds\right) & \text{if } \Omega(K) > -\infty \\ 0 & \text{if } \Omega(K) = -\infty \end{cases} \quad \text{whenever } t \in [a,c)$$

where, with some fixed positive u_0,

$$\Omega(u) = \int_{u_0}^{u} \frac{1}{\omega(t)}\,dt, \quad u \geq 0,$$

$$c = \begin{cases} \min\left\{b, \sup\left\{t \geq a \,\middle|\, \Omega(K) + \int_a^t v(s)\,ds < \lim_{u\to\infty} \Omega(u)\right\}\right\} & \\ & \text{if } \Omega(K) > -\infty \\ b & \text{if } \Omega(K) = -\infty \end{cases}$$

and Ω^{-1} stands for the inverse function of Ω. Note that

$$\Omega(K) = -\infty \quad \text{if and only if} \quad K = 0 \text{ and } \int_{u_0}^{0} \frac{1}{\omega(t)}\,dt = -\infty.$$

(The result in the degenerate case follows from the inequality in the nondegenerate case simply by choosing $K = k^{-1}$, $k = 1, 2, \ldots$ and letting $k \to \infty$. Bihari did not specify the domains of his functions.) Neither c (the case $c = b = \infty$ is not excluded) nor the right–hand side of inequality (24) depends on the particular choice of u_0. If $\omega(u) = u$ for each $u \geq 0$, then (24) simplifies to

$$u(t) \leq K \exp\left(\int_a^t v(s)\,ds\right) \qquad \text{whenever} \quad t \in [a, b),$$

i.e. to Bellman's version of the classical Gronwall lemma. Bihari's inequality (24) has direct implications on questions of uniqueness and continuous dependence. The relationship between (24) and the Alexeev–Gröbner nonlinear variation-of-constants formula is more or less the same as the relationship between the classical Gronwall lemma and the standard, linear variation-of-constants formula. Bihari {4} himself discusses the uniqueness criteria of Osgood, Perron, and Nagumo as well as the nonuniqueness criterion of Tamarkine in the light of his inequality and presents an application to continuous dependence on initial conditions. In an accompanying paper {5}, he applies inequality (24) to problems of stability and boundedness. Inequality (24) has been generalized in various directions, by a great number of authors.

In the sixties the interest of Bihari was focused on establishing a Sturm–Liouville theory for certain types of second-order nonlinear ordinary differential equations he called half-linear. The one-dimensional p-Laplacian

$$\big(q(x)\Phi(y')\big)' + r(x)\Phi(y) = 0$$

(where $q, r > 0$ and $\Phi(s) = |s|^{p-2}s$ ($s \in \mathbb{R}$) with some $p \in (1, \infty)$) provides an example. (We have to admit Bihari's terminology was not always consistent. One can venture to state the more an equation is subject to Sturm–Liouville theory the more this equation is half-linear.)

The contributions of Makai. Three years after Bihari's inequality, a paper by Endre Makai {47} attracted international interest, too. He proved that the principal eigenvalue $\lambda_1(D)$ in Reyleigh's conjecture and the torsional rigidity $P(D)$ in Saint-Vernant's conjecture (we discussed in connection with the work of Pólya and Szegő on isoperimetric inequalities) satisfy

$$(25) \qquad \lambda_1(D)\frac{\text{area}^2(D)}{\text{length}^2(\partial D)} \le 3 \quad \text{and} \quad P(D)\frac{\text{area}^3(D)}{\text{length}^2(\partial D)} \le 1,$$

respectively. An important ingredient of Makai's proof is the observation that, with $D(\varepsilon)$ denoting the Euclidean ε-neighborhood of D in $\mathbb{R}^2$, $\text{length}\big(\partial D(\varepsilon)\big)$ is an increasing function in ε. Makai proved this observation in a generality which suited his purposes — the difficulty is of course related to the existence of the length (the exceptional ε-set in $\mathbb{R}^+$ where $\text{length}\big(\partial D(\varepsilon)\big)$ does not make sense is countable) — nevertheless, in the last version of his paper finally published he refers to the more general geometric inequalities of Béla Szőkefalvi-Nagy {50} obtained in the meantime. The 'method of interior parallels' of Makai and Szőkefalvi-Nagy helped Pólya {58} to find the sharp upper bounds $\pi^2/4$ and $3/4$ in (25) later — the equalities are approached as D approaches an infinite strip.

A further interesting result of Makai {49} concerns the eigenfunctions of the Laplacian for the Dirichlet and the Neuman problem on the m-dimensional symplex

$$S_m = \big\{ (x_1, x_2, \dots, x_m) \mid 0 \le x_1 \le x_2 \le \dots \le x_m \le \pi \big\}.$$

The eigenfunctions are

$$\text{determinant}\, [\sin n_i x_j]_{i,j=1}^m \quad \text{whenever} \quad 0 < n_1 < \dots < n_m, \quad \text{integers}$$

and

$$\text{permanent}\, [\cos n_i x_j]_{i,j=1}^m \quad \text{whenever} \quad 0 \le n_1 \le \dots \le n_m, \quad \text{integers}$$

with eigenvalues $\sum n_i^2$, respectively. Related results for the isosceles rectangular triangle S_2 as well as for the equilateral triangle were obtained by Makai {48} a couple of years earlier.

The papers of Rényi and Barna on interval maps. In a 1957 paper {65}, Alfréd Rényi elaborated a method for proving that certain interval maps admit absolutely continuous ergodic measures. His examples include what is called today Rényi transformation

$$R_\beta : [0,1] \to [0,1], \quad x \to \beta x \pmod 1$$

where $\beta > 1$ is a real parameter and the absolutely continuous ergodic measure ν_β is equivalent to the standard Lebesgue measure λ on $[0,1]$. (Note that an absolutely continuous ergodic measure is necessarily unique.) Rényi's interest comes from number theory: If $\beta = 10$, then $\nu_\beta = \lambda$ and his result simplifies to Borel's Normal Number Theorem stating that, for almost every $x \in [0,1]$, the frequency of any digit in the decimal expansion of x is $1/10$. He reproves the corresponding result for continued fraction expansions and has also a similar application to an 1832 algorithm of Farkas Bolyai.

A further class of transformations with absolutely continuous ergodic measures Rényi investigates are mappings of the form

$$S : [0,1] \to [0,1], \quad x \to \tau(x) \pmod 1$$

where $\tau : [0,1] \to \mathbb{R}^+$ is a C^1 function with $\tau(0) = 0$, $\tau(1) \in \{2, 3, \dots\}$, and satisfying the expanding condition $\tau'(x) > 1$ for each $x \in [0,1]$ as well as a technical condition (C). While keeping condition (C), Rényi points out that the remaining set of assumptions can be replaced by three alternative sets of conditions under which the existence of an absolutely continuous ergodic measure can be established. Condition (C) itself is a so-called distortion inequality, a uniform bound for the build-up of nonlinearities under the iterates of S. Though condition (C) involves an infinite number of iterates of S, it can be checked in a number of various circumstances. As it was observed by Adler in the afterword to a posthumous paper by R. Bowen (*Comm. Math. Phys.,* **69** (1979), 1–17), condition (C) is automatically satisfied if $\tau : [0,1] \to \mathbb{R}^+$ is a C^2 function with $\tau(0) = 0$, $\tau(1) \in \{2, 3, \dots\}$, and the expanding condition $\tau'(x) > 1$ for each $x \in [0,1]$. Condition (C) and other distortion inequalities have remained extremely useful in the later development of the subject. The number of contributors in the sixties and the seventies became so large that, following Adler, the collection of Rényi-type results on the existence of absolutely continuous ergodic measures for general Markov maps of the interval is termed usually as The Folklore Theorem.

The theory of invariant measures for interval maps began with the 1947 result of Stanislaw Ulam & János Neumann {81} who pointed out that

$d\lambda/(\pi\big(x(1-x)\big)^{1/2})$ defines an absolutely continuous ergodic measure for the logistic map $[0,1] \to [0,1]$, $x \to 4x(1-x)$.

The most interesting Hungarian contribution to the early theory of interval maps is the work of Béla Barna on divergence properties of Newton's method when applied for approximating real roots of real polynomials. His results remained unnoticed for about two decades. In an 1985 survey paper, however, S. Smale (*Bull. Amer. Math. Soc.,* **13** (1985), 87–121) mentions his name, together with those of Fatou and Julia, as one of the pioneers of the iteration theory of rational functions.

The work of Barna originates in two questions of Rényi posed at the end of his 1950 half-scientific, half-educational paper {64} on Newton's method. Rényi's interest is mainly qualitative. He describes in detail the results of Cauchy and Fourier on convergence criteria but does not mention that the order of convergence is, in fact, quadratic. He turns his attention to "bad initial points" instead and gives a sufficient condition for a particularly strong form of divergence.

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function. For $x \in \big\{y \in \mathbb{R} \mid f'(y) \neq 0\big\}$, set $N_f(x) = x - f(x)/f'(x)$. A point $x_0 \in \mathbb{R}$ is *convergent* if the infinite orbit sequence $x_0, x_1 = N_f(x_0), x_2 = N_f(x_1), \ldots$ is (defined and) convergent (and then, necessarily, $\lim_{n\to\infty} x_n$ is a zero of f). Otherwise x_0 is *divergent.* For an arbitrary C^2 function with the properties that f'' is strictly increasing and f has exactly three simple roots say A_1, A_2, A_3, Rényi {64} proves that the set of divergent points is countable, there exists a unique period–two orbit $x_0^*, x_1^*, x_0^*, \ldots$ and, last but not least, for $i = 1, 2, 3$, any neighborhood of x_0^* contains a point whose orbit converges to A_i, a strikingly sensible dependence on initial values near x_0^*. Rényi asks 1.) if for real polynomials without complex roots the set of divergent points is always countable and 2.) if there is a real polynomial with the properties that not all roots are complex and the set of divergent points contains an interval.

Answering the first question of Rényi in the negative, Barna {1} shows that, given a fourth–degree real polynomial with four simple real roots, the set of divergent points is a compact set of the form $C \cup F$ where C is a Cantor set, $C \cap F = \emptyset$, and

$$F = \{x \in \mathbb{R} \mid \text{the iteration } x_0, x_1, \ldots \text{ breaks up in a finite number of steps}\}$$

is a countable set of isolated points. Moreover, the set

$$S = \{x \in C \mid \text{ the infinite orbit } x_0, x_1, \ldots \text{ is eventually periodic}\}$$

is also countable and, for each $k \geq 2$, contains periodic orbits with minimal period k. The number of such periodic orbits is $2k^{-1}(2^{k-1}-1)$ if $k \neq 2$ is a prime number and 3 if $k = 2$. (If k is not a prime number, Barna asserts that the number of periodic orbits with minimal period k can be computed via a complicated recursion but gives no details at all.) Given $x_0 \in C \setminus S$ arbitrarily, Barna — in today's terminology — shows that the omega–limit set $\omega(x_0) = \cap_{k=0}^{\infty} \mathrm{cl}\left(\{x_k, x_{k+1}, \dots\}\right)$ is not finite.

The consecutive four papers of Barna {2a}, {2b}, {2c}, {2d} are devoted to real polynomials of degree m, $m \geq 4$. He proves that all the $m = 4$ cardinality and topological results on C, F, S, and the structure of the set of divergent points remain valid under the condition that the roots of the polynomial are real and simple say $A_1, A_2, \dots, A_m$. In addition, he proves that, given $x_0 \in C$ and $i \in \{1, 2, \dots, m\}$ arbitrarily, any neighborhood of x_0 in $\mathbb{R}$ contains a point whose orbit converges to A_i. He provides two different proofs to this latter result. The first one {2a} is based on the general, complex–variable theory of Fatou and Julia on iterating rational functions whereas the second one {2c}, like the whole approach of Barna, is completely elementary. No $m \geq 5$ version of the "$2k^{-1}(2^{k-1}-1)$ if $k \neq 2$" combinatorial result is given. In the last paper of the series {2d}, Barna proves that his Cantor set C is a Lebesgue null set.

The answer to Rényi's second question is, in contrast to the conjecture in {64}, affirmative. Barna's example in {2b} is $f(x) = 11x^6 - 34x^4 + 39x^2$ for which $N_f(1) = -1$, $N_f(-1) = 1$ and $N_f'(1) = N_f'(-1) = 0$. Thus $1, -1, 1, \dots$ is an asymptotically stable period–two orbit of N_f and sufficiently small intervals about $x_0 = 1$ consist entirely of divergent points (attracted by the period–two orbit $1, -1, 1, \dots$).

A further early contribution to the modern theory of dynamical systems is due to György Szekeres, a childhood friend of Pál Erdős. He presents a detailed study of the one-dimensional conjugacy equation $\mathcal{H}\big(f(x)\big) = \mu\mathcal{H}(x)$, $\mu \neq 0, 1$ in 1958 {77}. Here the real function f is strictly increasing, defined on some finite or infinite interval $[0, c) \subset \mathbb{R}^n$, and satisfies $f(0) = 0$, $f(x) < x$ for $x \neq x_0 = 0$. He looks for strictly monotone solutions $\mathcal{H}$ representable as limits of iterations like $\mathcal{H}(x) = \eta \lim_{n\to\infty} \mu^{-n} f^n(x)$, $x \in [0, c)$ in the regular case $f'(0) = \mu \in (0, 1)$, ($f \in C^r$, $r \geq 1$; $\eta \neq 0$ is a real parameter) on some interval $[0, b)$ or $(0, b)$. In the singular cases $\mu \neq f'(0) = 0$ or $\mu \neq f'(0) = 1$, the existence of such solutions is pointed out under certain asymptotic conditions on f at the fixed point $x_0 = 0$. Similar results are proved for Abel's functional equation $\alpha\big(f(x)\big) = \alpha(x) + c$ ($c \neq 0$) as well as for the embeddability of $f = \Phi(1, \cdot)$ in a local iteration group (i.e. in a continuous-time local dynamical system satisfying) $\Phi(t+\tau,\cdot) = \Phi\big(t,\Phi(\tau,\cdot)\big)$, $t,\tau \in \mathbb{R}$. Szekeres' results generalize, complete, and unify those of Koenigs and Schröder who worked with f analytic, and can be considered as 'variations and fugue' on the 1959/60 Grobman–Hartman Lemma in one dimension.

Further workers, further works. Finally, we mention the names of several mathematicians who started their careers in the late fourties and whose research topics were, to a lesser or greater extent, connected to the theory of differential equations. These are György Alexits (actually, he belonged to an earlier generation but his scientific activities could not freely develop in the pre-war period — he worked mainly in approximation theory but his 1924 PhD Thesis was devoted to the Laplace equation), István Fenyő (his main area was, as it is demonstrated by the title of his major work with H. W. Stolle {24}, the theory and praxis of linear integral equations), Géza Freud (he is well known as an expert on orthogonal polynomials — but published several papers on partial differential equations in the early years of his career), Miklós Mikolás (his fractional calculus papers contain several applications to ordinary differential equations with fractional derivatives), and György Targonski (who combined the theory of iterations with those of functional equations). As for representatives of the ten years younger generation, the names of Tamás Fényes and of his blind friend, Pál Kosik, are mentioned (a great part of their joint papers is devoted to the Mikusinski operational calculus).

Epilogue. And this ends our report on the history of differential equations: Hungary, the extended first half of the 20th century, a terrific place and time.

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Addendum. *Árpád Elbert,* a devoted researcher in the fields of half-linear and delay equations as well as in the theory of special functions, died during the preparation of this paper. He intended this report to be a tribute to the memory of his masters, *Imre Bihari* and *Kató Rényi.*

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