Extremal Properties of Polynomials

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This article focuses on those problems about extremal properties of polynomials that were considered by the Hungarian mathematicians Lipót Fejér, Mihály Fekete, Marcel Riesz, Alfréd Rényi, György Pólya, Gábor Szegő, Pál Erdős, Pál Turán, Géza Freud, Gábor Somorjai, and their associates, who lived and died mostly in the twentieth century. It reflects my personal taste and is far from complete even within the subdomains we focus on most, namely inequalities for polynomials with constraints, Müntz polynomials, and the geometry of polynomials. There are separate chapters of this book devoted to orthogonal polynomials, interpolation, and function series, so here we touch these issues only marginally.

1. Markov- and Bernstein-Type Inequalities

Let $\|f\|_A$ denote the supremum norm of a function f on A. The Markov inequality asserts that

$$\|p'\|_{[-1,1]} \leq n^2 \|p\|_{[-1,1]}$$

holds for every polynomial p of degree at most n with complex coefficients. The inequality

$$\left| p'(y) \right| \leq \frac{n}{\sqrt{1 - y^2}} \|p\|_{[-1,1]}$$

holds for every polynomial p of degree at most n with complex coefficients and for every $y \in (-1, 1)$, and is known as Bernstein inequality. Various analogues of the above two inequalities are known in which the underlying intervals, the maximum norms, and the family of functions are replaced by more general sets, norms, and families of functions, respectively. These

inequalities are called Markov- and Bernstein-type inequalities. If the norms are the same on both sides, the inequality is called Markov-type, otherwise it is called Bernstein-type (this distinction is not completely standard). Markov- and Bernstein-type inequalities are known on various regions of the complex plane and n-dimensional Euclidean space, for various norms such as weighted L_p norms, and for many classes of functions such as polynomials with various constraints and exponential sums of n terms, just to mention a couple. Markov- and Bernstein-type inequalities have their own intrinsic interest. In addition, they play a fundamental role in approximation theory.

The inequality

$$\left\| p^{(m)} \right\|_{[-1,1]} \le T_n^{(m)}(1) \cdot \|p\|_{[-1,1]}$$

for every (algebraic) polynomial p of degree at most n with complex coefficients was first proved by V.A. Markov in 1892. Here, and in the sequel T_n denotes the Čebyshov polynomial of degree n defined by

$$T_n(x) := \frac{1}{2} \left(\left(x + \sqrt{x^2 - 1} \right)^n + \left(x - \sqrt{x^2 - 1} \right)^n \right)$$

(equivalently, $T_n(\cos\theta) := \cos(n\theta)$, $\theta \in \mathbb{R}$). V. A. Markov was the brother of the more famous A. A. Markov who proved the above inequality for $m = 1$ in 1889 by answering a question raised by the prominent Russian chemist, D. I. Mendeleev. Sergei N. Bernstein presented a shorter variational proof of V. A. Markov's inequality in 1938. The simplest known proof of Markov's inequality for higher derivatives is due to Duffin and Schaeffer {15}, who gave various extensions as well.

Let $\mathbb{T} := \mathbb{R} \pmod{2\pi}$. The inequality

$$\|t'\|_{\mathbb{T}} \le n\|t\|_{\mathbb{T}}$$

for all (real or complex) trigonometric polynomials of order n is also called the Bernstein inequality. It was proved by Bernstein in 1912 with $2n$ in place of n. See also {41}. The sharp inequality appears first in a paper of Fekete in 1916 who attributes the proof to Fejér. Bernstein attributes the proof to Edmund Landau. A clever proof based on zero-counting may be found in many books dealing with approximation theory. In books Markov's inequality for the first derivative is then deduced as a combination of Bernstein's inequality and an inequality due to Issai Schur:

$$\|p\|_{[-1,1]} \le (n+1) \max_{x \in [-1,1]} \left| p(x)\sqrt{1 - x^2} \right|$$

for every polynomial p of degree at most n with real coefficients.

Bernstein used his inequality to prove inverse theorems of approximation. Bernstein's method is presented in the proof of the next theorem, which is one of the simplest cases. However, several other inverse theorems of approximation can be proved by straightforward modifications of the proof of this result. That is why Bernstein- and Markov-type inequalities play a significant role in approximation theory. Direct and inverse theorems of approximation and related matters may be found in many books on approximation theory, including [111], {14}, and {56}.

Let $\mathcal{T}_n$ be the collection of all trigonometric polynomials of order at most n with real coefficients. Let Lip_α, $\alpha \in (0, 1]$, denote the family of all real-valued functions g defined on $\mathbb{T}$ satisfying

$$\sup\left\{\frac{|g(x) - g(y)|}{|x - y|^\alpha} \; : \; x \neq y \in \mathbb{T}\right\} < \infty.$$

For $f \in C(\mathbb{T})$, let

$$E_n(f) := \inf\left\{\|t - f\|_{\mathbb{T}} \; : \; t \in \mathcal{T}_n\right\}.$$

An example for a direct theorem of approximation is the following. *Suppose f is m times differentiable on $\mathbb{T}$ and $f^{(m)} \in \mathrm{Lip}_\alpha$ for some $\alpha \in (0, 1]$. Then there is a constant C depending only on f so that*

$$E_n(f) \leq C n^{-(m+\alpha)}, \qquad n = 1, 2, \ldots .$$

A proof may be found in [111], for example. The inverse theorem of the above result can be formulated as follows. *Suppose $m \geq 1$ is an integer, $\alpha \in (0, 1)$, and $f \in C(\mathbb{T})$. Suppose there is a constant $C > 0$ depending only on f such that*

$$E_n(f) \leq C n^{-(m+\alpha)}, \qquad n = 1, 2, \ldots .$$

Then f is m times continuously differentiable on $\mathbb{T}$ and $f^{(m)} \in \mathrm{Lip}_\alpha$.

We outline the proof of the above inverse theorem. We show only that f is m times continuously differentiable on $\mathbb{T}$. The rest can be proved similarly, but its proof requires more technical details. See, for example, George G. Lorentz's book [111]. For each $k \in \mathbb{N}$, let $Q_{2^k} \in \mathcal{T}_{2^k}$ be chosen so that

$$\|Q_{2^k} - f\|_{\mathbb{T}} \leq C \, 2^{-k(m+\alpha)}.$$

Then

$$\|Q_{2^{k+1}} - Q_{2^k}\|_{\mathbb{T}} \leq 2C \, 2^{-k(m+\alpha)}.$$

Now

$$f(\theta) = Q_{2^0}(\theta) + \sum_{k=1}^{\infty}(Q_{2^{k+1}} - Q_{2^k})(\theta), \qquad \theta \in \mathbb{T},$$

and by Bernstein's inequality

$$\left|Q_{2^0}^{(j)}(\theta)\right| + \sum_{k=0}^{\infty}\left|(Q_{2^{k+1}} - Q_{2^k})^{(j)}(\theta)\right|$$

$$\leq \|Q_1\|_{\mathbb{T}} + \sum_{k=0}^{\infty}\left(2^{k+1}\right)^{j}\left\|Q_{2^{k+1}} - Q_{2^k}\right\|_{\mathbb{T}}$$

$$\leq \|Q_1\|_{\mathbb{T}} + \sum_{k=0}^{\infty}\left(2^{k+1}\right)^{j}2C \, 2^{-k(m+\alpha)}$$

$$\leq \|Q_1\|_{\mathbb{T}} + 2^{j+1}C\sum_{k=0}^{\infty}\left(2^{j-m-\alpha}\right)^{k} < \infty$$

for every $\theta \in \mathbb{T}$ and $j = 0, 1, \ldots, m$, since $\alpha > 0$. Now we can conclude that $f^{(j)}(\theta)$ exists and

$$f^{(j)}(\theta) = Q_1^{(j)}(\theta) + \sum_{k=0}^{\infty}(Q_{2^{k+1}} - Q_{2^k})^{(j)}(\theta)$$

for every $\theta \in \mathbb{T}$ and $j = 0, 1, \ldots, m$. The fact that $f^{(m)} \in C(\mathbb{T})$ can be seen by the Weierstrass M-test. This finishes the proof.

For Erdős, Markov- and Bernstein-type inequalities had their own intrinsic interest and he explored what happens when the polynomials are restricted in certain ways. It had been observed by Bernstein that Markov's inequality for monotone polynomials is not essentially better than for arbitrary polynomials. Bernstein proved that *if n is odd, then*

$$\sup_{p} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} = \left(\frac{n+1}{2}\right)^{2},$$

where the supremum is taken over all polynomials p of degree at most n with real coefficients which are monotone on $[-1, 1]$. This is surprising, since one

would expect that if a polynomial is this far away from the "equioscillating" property of the Čebyshov polynomial T_n, then there should be a more significant improvement in the Markov inequality. In the short paper {20}, Erdős gave a class of restricted polynomials for which the Markov factor n^2 improves to cn. He proved that there is an absolute constant c such that

$$\left| p'(y) \right| \leq \min \left\{ \frac{c\sqrt{n}}{(1-y^2)^2}, \; \frac{en}{2} \right\} \|p\|_{[-1,1]}, \qquad y \in (-1,1),$$

for every polynomial of degree at most n that has all its zeros in $\mathbb{R} \setminus (-1,1)$. This result motivated several people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. Generalizations of the above Markov- and Bernstein-type inequality of Erdős have been extended in many directions by many people including G. G. Lorentz, John T. Scheick, József Szabados, Arun Kumar Varma, Attila Máté, Quazi Ibadur Rahman, and Narendra K. Govil. Many of these results are contained in the following essentially sharp result, due to Peter Borwein and Tamás Erdélyi {7}: *there is an absolute constant c such that*

$$\left| p'(y) \right| \leq c \min \left\{ \sqrt{\frac{n(k+1)}{1-y^2}}, \; n(k+1) \right\} \|p\|_{[-1,1]}, \qquad y \in (-1,1),$$

for every polynomial p of degree at most n with real coefficients that has at most k zeros in the open unit disk.

Let K_α be the open diamond of the complex plane with diagonals $[-1,1]$ and $[-ia, ia]$ such that the angle between $[ia, 1]$ and $[1, -ia]$ is $\alpha\pi$. A challenging question of Erdős, that Gábor Halász {45} answered in 1996, is: how large can the quantity

$$\frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}}$$

be, assuming that p is a polynomial of degree at most n which has no zeros in a diamond K_α, $\alpha \in [0, 2)$? He proved that *if $\alpha \in [0,1)$ then there are constants $c_1 > 0$ and $c_2 > 0$ depending only on α such that*

$$c_1 n^{2-\alpha} \leq \sup_p \frac{\left| p'(1) \right|}{\|p\|_{[-1,1]}} \leq \sup_p \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_2 n^{2-\alpha},$$

where the supremum is taken for all polynomials p of degree at most n (with either real or complex coefficients) having no zeros in K_α. He also showed that there is an absolute constant $c_2 > 0$ such that

$$\|p'\|_{[-1,1]} \leq c_2 n \log n \|p\|_{[-1,1]}$$

holds for all polynomials p of degree at most n with complex coefficients having no zeros in K_1, while for every $\alpha \in (1,2)$ there is a constant c_2 depending on α such that

$$\|p'\|_{[-1,1]} \leq c_2 n \|p\|_{[-1,1]}$$

for all polynomials p of degree at most n with complex coefficients having no zeros in K_α.

Halász reduced the proof to the following result of Szegő {82}: *the inequality*

$$\left|p'(0)\right| \leq c_\alpha n^{2\alpha} \|p\|_{D_\alpha}$$

holds for every polynomial p of degree at most n with complex coefficients, where c_α is a constant depending only on α, and

$$D_\alpha := \left\{ z \in \mathbb{C} \: : \: |z| \leq 1, \; \left|\arg(z)\right| \leq \pi(1-\alpha) \right\}, \qquad \alpha \in (0,1].$$

The identity

$$t'(\theta) = \sum_{\nu=1}^{2n} (-1)^{\nu+1} \lambda_\nu t(\theta + \theta_\nu)$$

with

$$\lambda_\nu := \frac{1}{n} \frac{1}{\left(2\sin\left(\frac{1}{2}\theta_\nu\right)\right)^2}, \qquad \theta_\nu := \frac{2\nu-1}{2n}\pi, \qquad \nu = 1, 2, \ldots, 2n,$$

for all trigonometric polynomials t of order at most n was established by M. Riesz {70} and it is called as the Riesz Interpolation Formula. Here, choosing $t(\theta) := \sin(n\theta)$ and the point $\theta = 0$, we obtain that $\sum_{\nu=1}^{2n} \lambda_\nu = n$. The above identity can be used to prove not only Bernstein's inequality, but an L_p version of it for all $p \geq 1$. Namely, combining the triangle inequality and Hölder's inequality in the Riesz Interpolation Formula, and then integrating both sides, we obtain

$$\int_0^{2\pi} \left|t'(\theta)\right|^p d\theta \leq n^p \int_0^{2\pi} \left|t(\theta)\right|^p d\theta$$

for all trigonometric polynomials t of order at most n. It is interesting to note that the sharp L_p version of Bernstein's inequality with Bernstein factor n for all $0 < p < 1$ was established only much later, in 1981, by Vitali V. Arestov {1}. It followed the paper {61} by Máté and Nevai, where the $11^{1/p}n$ Bernstein factor was proved. A short and elegant proof of Arestov's result due to Manfred Golitschek and G. G. Lorenz is presented in {14, pages 104–109}.

For *real* trigonometric polynomials t the inequality

$$t'(\theta)^2 + n^2 t(\theta)^2 \le n^2 \|t\|_{\mathbb{R}}^2, \qquad \theta \in \mathbb{R},$$

holds and is known as the Bernstein–Szegő inequality. Various extensions and generalizations of this have been established throughout the century.

There is a Bernstein inequality on the unit circle ∂D of the complex plane. It states that

$$\|p'\|_{\partial D} \le n \, \|p\|_{\partial D}$$

for all polynomial p of degree at most n with complex coefficients.

It was conjectured by Erdős and proved by Péter Lax {53} in 1944 that

$$\|p'\|_{\partial D} \le \frac{n}{2} \, \|p\|_{\partial D}$$

for every polynomial p of degree at most n with complex coefficients having no zeros in D.

The question about the right Bernstein factor on the unit circle (between $n/2$ and n) is unsettled in the case when we know that there are k zeros inside the open unit disk and $n - k$ zeros are outside it.

A technical detail related to the proof of the Bernstein–Szegő inequality is known as the Riesz Lemma after Marcel Riesz, see {8}, for instance. It states that *if t is a real trigonometric polynomial of order n and for an $\alpha \in \mathbb{R}$ $t(\alpha) = \|t\|_{\mathbb{R}} = 1$, then*

$$t(\theta) \ge \cos\big(n(\theta - \alpha)\big), \qquad \theta \in \left[\alpha - \frac{\pi}{2n}, \alpha + \frac{\pi}{2n}\right].$$

In particular t does not vanish in $\big(\alpha - \frac{\pi}{2n}, \alpha + \frac{\pi}{2n}\big)$.

Géza Freud paid serious attention to Markov–Bernstein-type inequalities on the real line associated with $w_\alpha(x) := \exp\big(-|x|^\alpha\big)$, $\alpha > 0$ in L_p norm. After Freud the name *Freud weight* has become common to refer to the weights w_α and their generalizations. Freud handled the Hermite weight,

case $\alpha = 2$, coupled with the assumption $1 \leq p \leq \infty$. However, it was his student, Paul Nevai, together with Eli Levin, Doron S. Lubinsky, and Vilmos Totik who put the right pieces together to obtain

$$\|Q'w_\alpha\|_p \leq Cn^{1-1/\alpha}\|Qw_\alpha\|_p, \qquad \alpha > 1,$$

$$\|Q'w_\alpha\|_p \leq C\log n\|Qw_\alpha\|_p, \qquad \alpha = 1,$$

and $\qquad \|Q'w_\alpha\|_p \leq C\|Qw_\alpha\|_p, \qquad 0 < \alpha < 1,$

for every polynomial Q of degree at most n with real coefficients and for every $0 < p \leq \infty$, where $C = C(\alpha, p)$ is a constant depending only on α and p, and

$$\|f\|_p^p := \int_{\mathbb{R}} |f(t)|^p \, dt \qquad \text{and} \qquad \|f\|_\infty := \|f\|_{\mathbb{R}}.$$

In their proof the idea of an infinite-finite range inequality played a significant role. This also goes back to Freud who observed that

$$\big\|Q(x)\exp(-x^2)\big\|_{\mathbb{R}} \leq Cn^{1/2}\big\|Q(x)\exp(-x^2)\big\|_{I_n}$$

for every polynomial Q of degree at most n with real coefficients, where

$$I_n := \big[-(3/2)n^{1/2}, (3/2)n^{1/2}\big],$$

and C is an absolute constant.

There is an elementary paper of Szegő {84} dealing with weighted Markov- and Bernstein-type inequalities on $[0, \infty)$ with respect to the Laguerre weight e^{-x} on $[0, \infty)$, which proves

$$\big\|p'(x)e^{-x}\big\|_{[0,\infty)} \leq (8n+2)\big\|p(x)e^{-x}\big\|_{[0,\infty)}$$

for every polynomial p of degree at most n with real coefficients. A sharp L_2 version of the above inequality was the topic of Turán's paper {87}. Turán has also found the extremal polynomials in this L_2 case.

An interesting inequality of Turán {86}) states that

$$\frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} > \frac{1}{6}\sqrt{n}$$

for every polynomial p of degree n having all its zeros in $[-1, 1]$. He also showed that

$$\|p'\|_{\overline{D}} \geq \frac{n}{2}\|p\|_{\overline{D}}$$

if p has each of its zeros in the closed unit disk $\overline{D}$ of the complex plane.

Turán posed some problems about bounding the derivative of a polynomial if its modulus is bounded by a certain curve. One of his problems asked for the right upper bound for $\|p'\|_{[-1,1]}$ for polynomials p of degree at most n with real coefficients satisfying $|p(x)| \leq (1 - x^2)^{1/2}$ for every $x \in [-1, 1]$. This problem has been solved by Q. I. Rahman {69} who established the inequality $\|p'\|_{[-1,1]} \leq 2(n - 1)$ for all such polynomials.

2. MÜNTZ POLYNOMIALS AND EXPONENTIAL SUMS

James A. Clarkson and Erdős wrote a seminal paper on the density of Müntz polynomials. C. Herman Müntz's classical theorem characterizes sequences $\Lambda := (\lambda_j)_{j=0}^{\infty}$ with

$$(1) \qquad\qquad 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

for which the Müntz space $M(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}$ is dense in $C[0, 1]$. Here the span denotes the collection of all finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \ldots$ with real coefficients, and $C(A)$ is the space of all real-valued continuous functions on $A \subset [0, \infty)$ equipped with the uniform norm. If $A := [a, b]$ is a finite closed interval, then the notation $C[a, b] := C([a, b])$ is used.

Müntz's Theorem. *Suppose* $\Lambda := (\lambda_j)_{i=0}^{\infty}$ *is a sequence satisfying* (1). *Then* $M(\Lambda)$ *is dense in* $C[0, 1]$ *if and only if* $\sum_{j=1}^{\infty} 1/\lambda_j = \infty$.

The original Müntz Theorem proved by C. Müntz {63} in 1914, by Ottó Szász {76} in 1916, and anticipated by Bernstein was only for sequences of exponents tending to infinity. Szász proved more than Müntz. He did not assume (1). He assumed only that the numbers λ_k are arbitrary complex with $\lambda_0 = 1$, Re $\lambda_k > 0$, $k = 1, 2, \ldots$, and the exponents λ_k are distinct. He gave a necessary and sufficient conditions which characterize denseness in the case when Re $\lambda_k \geq c > 0$, $k = 1, 2, \ldots$.

The point 0 is special in the study of Müntz spaces. Even replacing $[0,1]$ by an interval $[a,b] \subset [0,\infty)$ in Müntz's Theorem is a non-trivial issue. Such an extension is, in large measure, due to James A. Clarkson and Erdős {13} and Laurent Schwartz [164]. In {13}, Clarkson and Erdős showed that Müntz's Theorem holds on any interval $[a,b]$ with $a \geq 0$. That is, *for any increasing positive sequence $\Lambda := (\lambda_j)_{j=0}^{\infty}$ and any $0 < a < b$, $M(\Lambda)$ is dense in $C[a,b]$ if and only if $\sum_{j=1}^{\infty} 1/\lambda_j = \infty$.* Moreover, they showed that under the assumptions $\sum_{j=1}^{\infty} 1/\lambda_j < \infty$ and

$$\inf\left\{\lambda_{j+1} - \lambda_j \,:\, j = 0, 1, 2, \ldots\right\} > 0$$

every function $f \in C[a,b]$ from the uniform closure of $M(\Lambda)$ on $[a,b]$ is of the form

$$(2) \qquad f(x) = \sum_{j=0}^{\infty} a_j x^{\lambda_j}, \qquad x \in [a,b).$$

In particular, f can be extended analytically throughout the open disk centered at 0 with radius b, cut by $(-\infty, 0]$.

Erdős considered this result his best contribution to complex analysis. Later, by different methods, L. Schwartz extended some of the Clarkson–Erdős results to the case when the exponents λ_i are arbitrary distinct positive real numbers. For example, in that case, under the assumption $\sum_{j=1}^{\infty} 1/\lambda_j < \infty$ every function $f \in C[a,b]$ from the uniform closure of $M(\Lambda)$ on $[a,b]$ can still be extended analytically throughout the region

$$\left\{z \in \mathbb{C} \setminus (-\infty, 0] \,:\, |z| < b\right\},$$

although such an analytic extension does not necessarily have a representation given by (2). The Clarkson–Erdős results were further extended by P. Borwein and T. Erdélyi {10} from the interval $[0,1]$ to compact subsets of $[0,\infty)$ with positive Lebesgue measure. That is, *if $\Lambda := (\lambda_j)_{j=0}^{\infty}$ is an increasing sequence of positive real numbers with $\lambda_0 = 0$ and $A \subset [0,\infty)$ is a compact set with positive Lebesgue measure, then $M(\Lambda)$ is dense in $C(A)$ if and only if $\sum_{j=1}^{\infty} 1/\lambda_j = \infty$.* This result had been expected by Erdős and others for a long time.

Somorjai {74} and Joseph Bak and Donald J. Newman {2} proved that

$$R(\Lambda) := \left\{p/q \,:\, p, q \in M(\Lambda)\right\}$$

is always dense in $C[0,1]$. This surprising result says that while the set $M(\Lambda)$ of Müntz polynomials may be far from dense, the set $R(\Lambda)$ of Müntz

rationals is always dense in $C[0,1]$ no matter what the underlying sequence Λ is. Newman was truly impressed by Somorjai's result. In the light of Somorjai's theorem, Newman, in 1978 [123, p. 50] raises "the very sane, if very prosaic question": are the functions

$$\prod_{j=1}^{k}\left(\sum_{i=0}^{n_j} a_{i,j} x^{i^2}\right), \qquad a_{i,j} \in \mathbb{R}, \quad n_j \in \mathbb{N},$$

dense in $C[0,1]$ for some fixed $k \geq 2$? In other words does the "extra multiplication" have the same power that the "extra division" has in the Bak–Newman–Somorjai result? Newman speculated that it did not. P. Borwein and T. Erdélyi proved this conjecture in {10} in a generalized form.

Müntz–Jackson type theorems via interpolation have been considered in {60} by László Márki, Somorjai, and Szabados.

The main results of {8, Section 4.2} and {16} are the following.

Full Müntz Theorem in $C[0,1]$**.** *Suppose* $(\lambda_j)_{j=1}^{\infty}$ *is a sequence of distinct real numbers greater than 0. Then* span $\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *is dense in* $C[0,1]$ *if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} = \infty.$$

Moreover, if

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} < \infty,$$

then every function from the $C[0,1]$ *closure of* span $\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *is infinitely many times differentiable on* $(0,1)$.

Full Müntz Theorem in $L_p(A)$**.** *Let* $A \subset [0,1]$ *be a compact set with positive lower density at 0. Let* $p \in (0,\infty)$. *Suppose* $(\lambda_j)_{j=1}^{\infty}$ *is a sequence of distinct real numbers greater than* $-(1/p)$. *Then* span $\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *is dense in* $L_p(A)$ *if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{\left(\lambda_j + (1/p)\right)^2 + 1} = \infty.$$

Moreover, if

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{\left(\lambda_j + (1/p)\right)^2 + 1} < \infty,$$

then every function from the $L_p(A)$ closure of span $\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ can be represented as an analytic function on $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}$ restricted to $A \cap (0, r_A)$, where

$$r_A := \sup \left\{ y \in \mathbb{R} : m(A \cap [y, \infty)) > 0 \right\}$$

($m(\cdot)$ denotes the one-dimensional Lebesgue measure).

These improve and extend earlier results of Müntz {63}, Szász {76}, and Clarkson and Erdős {13}. Related issues about the denseness of span $\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ are also considered.

Based on the work of Edmond Laguerre {52} and Pólya {66} the following is known.

Let $\mathcal{A}_1$ denote the class of entire functions f of the form

$$f(z) = C z^m e^{-c^2 z^2 + az} \prod_{k=1}^{\infty} (1 + \alpha_k z) e^{-\alpha_k z}, \qquad z \in \mathbb{C},$$

where $C, c, a, \alpha_k \in \mathbb{R}$, m is a nonnegative integer, and $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. Then $\mathcal{A}_1$ is the collection of the analytic extensions of those functions defined on $\mathbb{R}$ which may be obtained as the uniform limit, on every compact subset of $\mathbb{R}$, of polynomials having only real zeros.

Let $\mathcal{A}_2$ denote the class of entire functions f of the form

$$f(z) = C z^m e^{-az} \prod_{k=1}^{\infty} (1 - \alpha_k z), \qquad z \in \mathbb{C},$$

where $C \in \mathbb{R}$, $a > 0$, m is a nonnegative integer, $\alpha_k \geq 0$, and $\sum_{k=1}^{\infty} \alpha_k < \infty$. Then $\mathcal{A}_2$ is the collection of the analytic extensions of those functions defined on $\mathbb{R}$ which may be obtained as the uniform limit, on every compact subset of $\mathbb{R}$, of polynomials having only positive zeros.

The functions of the classes $\mathcal{A}_1$ and $\mathcal{A}_2$ are sometimes called the Pólya–Laguerre functions.

In {68} Pólya posed the question: for which sequences $0 < \beta_1 < \beta_2 < \cdots$ are the linear combinations of the functions

$$\cos(\beta_k t), \quad \sin(\beta_k t), \qquad k = 1, 2, \ldots,$$

complete in $C[0, 2\pi]$? Pólya himself conjectured that

$$\limsup_{k \to \infty} \frac{\beta_k}{k} < 1$$

is sufficient. Szász {77} proved Pólya's conjecture.

The following pretty results of Fejér may be found in {8} (see also {40}):
Let

$$p(z) := \sum_{k=0}^{n} a_k z^{\lambda_k}, \qquad a_k \in \mathbb{C}, \quad a_0 a_1 \neq 0.$$

Then p has at least one zero $z_0 \in \mathbb{C}$ so that

$$|z_0| \leq \left(\frac{\lambda_2 \lambda_3 \cdots \lambda_n}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \cdots (\lambda_n - \lambda_1)} \right)^{1/\lambda_1} \left| \frac{a_0}{a_1} \right|^{1/\lambda_1}.$$

From the above result the following beautiful consequence follows easily:
Suppose

$$f(z) = \sum_{k=0}^{\infty} a_k z^{\lambda_k}, \qquad a_k \in \mathbb{C}$$

is an entire function so that $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$, that is, the entire function f satisfies the Fejér gap condition. Then there is a $z_0 \in \mathbb{C}$ so that $f(z_0) = 0$. Hence for every $a \in \mathbb{C}$ there is a z_0 such that $f(z_0) = a$, that is f has no Picard exceptional value.

Important results of Turán are based on the following observations: *Let*

$$g(\nu) := \sum_{j=1}^{n} b_j z_j^{\nu}, \qquad b_j, z_j \in \mathbb{C}.$$

Suppose

$$\min_{1 \leq j \leq n} |z_j| \geq 1, \qquad j = 1, 2, \dots, n.$$

Then

$$\max_{\nu = m+1, \dots, m+n} |g(\nu)| \geq \left(\frac{n}{2e(m+n)} \right)^n |b_1 + b_2 + \cdots + b_n|$$

for every positive integer m.

A consequence of the preceding is the famous Turán Lemma: *if*

$$f(t) := \sum_{j=1}^{n} b_j e^{\lambda_j t}, \qquad b_j, \lambda_j \in \mathbb{C}.$$

and

$$\min_{1 \leq j \leq n} \operatorname{Re}(\lambda_j) \geq 0,$$

then

$$\left| f(0) \right| \leq \left(\frac{2e(a+d)}{d} \right)^n \|f\|_{[a,a+d]}$$

for every $a > 0$ *and* $d > 0$.

Another consequence of this is the fact that *if*

$$p(z) := \sum_{j=1}^{n} b_j z^{\lambda_j}, \qquad b_j \in \mathbb{C}, \quad \lambda_j \in \mathbb{R}, \quad z = e^{i\theta},$$

then

$$\max_{|z|=1} \left| p(z) \right| \leq \left(\frac{4e\pi}{\delta} \right)^n \max_{\substack{|z|=1 \\ \alpha \leq \arg(z) \leq \alpha+\delta}} \left| p(z) \right|$$

for every $0 \leq \alpha < \alpha + \delta \leq 2\pi$.

Turán's inequalities above and their variants play a central role in the book of Turán {88}, where many applications are also presented. The main point in these inequalities is that the exponent on the right-hand side is only the number of terms n, and so it is independent of the numbers λ_j. An inequality of type

$$\max_{|z|=1} \left| p(z) \right| \leq c(\delta)^{\lambda_n} \max_{\substack{|z|=1 \\ \alpha \leq \arg(z) \leq \alpha+\delta}} \left| p(z) \right|,$$

where $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n$ are integers and $c(\delta)$ depends only on δ, could be obtained by a simple direct argument, but it is much less useful than Turán's inequality. Fedor Nazarov has a paper {64} devoted to Turán-type inequalities for exponential sums, and their applications to various uniqueness theorems in harmonic analysis of the uncertainty principle type. The author derives an estimate for the maximum modulus of an exponential sum

$$\sum_{k=1}^{n} c_k e^{\lambda_k t}, \qquad c_k, \lambda_k \in \mathbb{C},$$

on an interval $I \subset \mathbb{R}$ in terms of its maximum modulus on a measurable set $E \subset I$ of positive Lebesgue measure:

$$\sup_{t \in I} \left| p(t) \right| \leq e^{\max |\operatorname{Re}\lambda_k| m(I)} \left(\frac{Am(I)}{m(E)} \right)^{n-1} \sup_{t \in E} \left| p(t) \right|.$$

In {9} a subtle Bernstein-type extremal problem related to Turán's result is solved by establishing the equality

$$\sup_{0 \neq f \in \widetilde{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} = 2n - 1,$$

where

$$\widetilde{E}_{2n} := \left\{ f \ : \ f(t) = a_0 + \sum_{j=1}^{n} \left(a_j e^{\lambda_j t} + b_j e^{-\lambda_j t} \right), \quad a_j, b_j, \lambda_j \in \mathbb{R} \right\}.$$

This settles a conjecture of G. G. Lorentz and others and it is surprising to be able to provide a sharp solution. It follows fairly simply from the above that

$$\frac{1}{e-1} \frac{n-1}{\min\{y-a, b-y\}} \leq \sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a, b-y\}}$$

for every $y \in (a, b)$, where

$$E_n := \left\{ f \ : \ f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j, \lambda_j \in \mathbb{R} \right\}.$$

3. Geometric Properties of Polynomials

There are a number of contributions by Hungarian mathematicians exploring the relation between the coefficients of a polynomial and the number of its zeros in certain regions of the complex domain. I find the following result of Erdős and Turán {38} especially attractive. It states that *if* $p(z) = \sum_{j=0}^{n} a_j z^j$ *has* m *positive real zeros, then*

$$m^2 \leq 2n \log \left(\frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}} \right).$$

This result was originally due to I. Schur. Erdős and Turán rediscovered it with a short proof that we outline now:

Step 1. We utilize the following observation due to Szász, see page 173 of {8}. Let $\gamma, \lambda_0, \lambda_1, \ldots, \lambda_n$ be distinct real numbers greater than $-1/2$. Then the $L_2[0,1]$ distance d_n from x^γ to span $\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$ is given by

$$d_n = \frac{1}{\sqrt{2\gamma+1}} \prod_{j=0}^{n} \left| \frac{\gamma - \lambda_j}{\gamma + \lambda_j + 1} \right|.$$

Step 2. Let

$$p(z) = a_n \prod_{k=1}^{n} (z - r_k e^{i\theta_k})$$

and

$$q(z) := \prod_{k=1}^{n} (z - e^{i\theta_k}).$$

Note that for $|z| = 1$,

$$\frac{|z - re^{i\theta}|^2}{r} \geq |z - e^{i\theta}|^2.$$

Use this to deduce that

$$\left| q(z) \right|^2 \leq \frac{|p(z)|^2}{|a_0 a_n|} \leq \left(\frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}} \right)^2$$

whenever $|z| = 1$.

Step 3. Let ∂D be the unit circle of the complex plane. Since p has m positive real roots, q has m roots at 1. Use the change of variables $x := z + z^{-1}$ applied to $z^n q(z^{-1}) q(z)$ to show that

$$\|q\|_{\partial D}^2 \geq \min_{\{b_k\}} \left\| (z-1)^m (z^{n-m} + b_{n-m-1} z^{n-m-1} + \cdots + b_1 z + b_0) \right\|_{\partial D}^2$$

$$\geq \min_{\{c_k\}} \left\| x^m (x^{n-m} + c_{n-m-1} x^{n-m-1} + \cdots + c_1 x + c_0) \right\|_{[0,4]}$$

$$= 4^n \min_{\{d_k\}} \left\| x^m (x^{n-m} + d_{n-m-1} x^{n-m-1} + \cdots + d_1 x + d_0) \right\|_{[0,1]}$$

$$\geq \frac{4^n}{\sqrt{2n+1}\binom{2n}{n+m}},$$

where the last inequality follows from Step 1.

Step 4. Now one can easily show that

$$\log\left(\frac{4^n}{\sqrt{2n+1}\binom{2n}{n+m}}\right) \geq m^2/n$$

for $1 < m < n$, and the proof of Erdős and Turán is finished.

Another beautiful result of Erdős and Turán $\{38\}$ states that if the zeros of $p(z) = \sum_{j=0}^n a_j z^j$ are denoted by

$$z_k = r_k e^{i\varphi_k}, \qquad k = 1, 2, \ldots, n,$$

then for every $0 < \alpha < \beta \leq 2\pi$ we have

$$\left| \sum_{k \in I(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n \right| \leq 16\sqrt{n \log R},$$

where

$$R := \frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}}$$

and

$$I(\alpha, \beta) := \left\{ k \in \{1, 2, \ldots, n\} : \varphi_k \in [\alpha, \beta] \right\}.$$

André Bloch and Pólya $\{68\}$ proved that the average number of real zeros of a polynomial from

$$\mathcal{F}_n := \left\{ p : p(z) = \sum_{k=0}^n a_k z^k, \ a_k \in \{-1, 0, 1\} \right\}.$$

is at most $c\sqrt{n}$. They also proved that a polynomial from $\mathcal{F}_n$ cannot have more than

$$\frac{cn \log \log n}{\log n}$$

real zeros. This quite weak result appears to be the first on this subject. Schur $\{73\}$ and by different methods Szegő $\{83\}$ and Erdős and Turán $\{38\}$ improve this to $c\sqrt{n \log n}$ (see also $\{8\}$). Their results are more general, but in this specialization not sharp. In $\{12\}$ the right upper bound $c\sqrt{n}$ is found for the number of real zeros of polynomials from a much larger class, the class of all polynomials of the form

$$p(x) = \sum_{j=0}^n a_j x^j, \qquad |a_j| \leq 1, \quad |a_0| = |a_n| = 1, \quad a_j \in \mathbb{C}.$$

In fact our method is able to give $c\sqrt{n}$ as an upper bound for the number of zeros of a polynomial p of degree at most n with $|a_0| = 1$, $|a_j| \leq 1$, inside any polygon with vertices on the unit circle (of course, c depends on the polygon). This is discussed in {11}.

Bloch and Pólya {4} also prove that there are polynomials $p \in \mathcal{F}_n$ with

$$\frac{cn^{1/4}}{\sqrt{\log n}}$$

distinct real zeros of odd multiplicity. (Schur {73} claims they do it for polynomials with coefficients only from $\{-1, 1\}$, but this appears to be incorrect.)

A surprising theorem of Szegő {80} states that *if*

$$f(x) := \sum_{k=0}^{n} a_k \binom{n}{k} x^k, \qquad\qquad a_n \neq 0,$$

$$g(x) := \sum_{k=0}^{n} b_k \binom{n}{k} x^k, \qquad\qquad b_n \neq 0,$$

and

$$h(x) := \sum_{k=0}^{n} a_k b_k \binom{n}{k} x^k,$$

f has all its zeros in a closed disk $\overline{D}$, and g has zeros $\beta_1, \ldots, \beta_n$, then all the zeros of h are of the form $-\beta_j \gamma_j$ with $\gamma_j \in \overline{D}$. An interesting consequence of this is the fact that *if a polynomial p of degree n has all its zeros in $\overline{D}_1 := \left\{ z \in \mathbb{C} : |z| \leq 1 \right\}$, then the polynomial q defined by $q(x) := \int_0^x p(t)\, dt$ has all its zeros in $\overline{D}_2 := \left\{ z \in \mathbb{C} : |z| \leq 2 \right\}$.*

In {18}, Erdős proved that the arc length from 0 to 2π of a real trigonometric polynomial f of order at most n satisfying $\|f\|_{\mathbb{R}} \leq 1$ is maximal for $\cos n\theta$. An interesting question he posed quite often is the following: *Let $0 < a < b < 2\pi$. Is it still true that the variation and arc-length of a real trigonometric polynomial with $\|f\|_{\mathbb{R}} \leq 1$ in $[a, b]$ is maximal for $\cos(n\theta + \alpha)$ for a suitable α?* The following related conjecture of Erdős was open for quite a long time: *Is it true that the arc length from -1 to 1 of a real algebraic polynomial p of degree at most n with $\|p\|_{[-1,1]}$ is maximal for the*

Čebyshov polynomial T_n? This was proved independently by Gundorph K. Kristiansen {51} and by Borislav Bojanov {5}.

A well-known theorem of Piotr L. Čebyshov states that *if p is a real algebraic polynomial of degree at most n and $z_0 \in \mathbb{R} \setminus [-1,1]$, then $|p(z_0)| \leq |T_n(z_0)| \cdot \|p\|_{[-1,1]}$, where T_n is the Čebyshov polynomial of degree n.* The standard proof of this is based on zero counting which can no longer be applied if z_0 is not real. By letting $z_0 \in \mathbb{C}$ tend to a point in $(-1,1)$, it is fairly obvious that this result cannot be extended to all $z_0 \in \mathbb{C}$. However, a surprising result of Erdős {23} shows that Čebyshov's inequality can be extended to all $z_0 \in \mathbb{C}$ outside the open unit disk.

Erdős and Turán were probably the first to discover the power and applicability of an almost forgotten result of Evgenii Ja. Remez. The so-called Remez inequality is not only attractive and interesting in its own right, but it also plays a fundamental role in proving various other things about polynomials. Let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n with real coefficients. For a fixed $s \in (0,2)$, let

$$\mathcal{P}_n(s) := \left\{ p \in \mathcal{P}_n \,:\, m\Big(\big\{x \in [-1,1] \,:\, |p(x)| \leq 1\big\}\Big) \geq 2 - s \right\},$$

where $m(\cdot)$ denotes linear Lebesgue measure. The Remez inequality concerns the problem of bounding the uniform norm of a polynomial p of degree n on $[-1,1]$ given that its modulus is bounded by 1 on a subset of $[-1,1]$ of Lebesgue measure at least $2-s$. That is, *how large can $\|p\|_{[-1,1]}$ (the uniform norm of p on $[-1,1]$) be if $p \in \mathcal{P}_n(s)$?* The answer is given in terms of the Čebyshov polynomials T_n. We have

$$\|p\|_{[-1,1]} \leq T_n\left(\frac{2-s}{2+s}\right)$$

for all $p \in \mathcal{P}_n(s)$, and the extremal polynomials for the above problem are the Čebyshov polynomials $\pm T_n\big(h(x)\big)$, where h is a linear function which maps $[-1, 1-s]$ or $[-1+s, 1]$ onto $[-1,1]$. See page 228 of {8}.

One of the applications of the Remez inequality by Erdős and Turán {37} deals with orthogonal polynomials. Let w be an integrable weight function on $[-1,1]$ that is strictly positive almost everywhere. Denote the sequence of the associated orthonormal polynomials by $(p_n)_{n=0}^{\infty}$. The leading coefficient of p_n is denoted by $\gamma_n > 0$. Then a theorem of Erdős and Turán {37} states that

$$\lim_{n \to \infty} \big[p_n(z)\big]^{1/n} = z + \sqrt{z^2 - 1}$$

holds uniformly on every closed subset of $\mathbb{C} \setminus [-1, 1]$.

Erdős and Turán {35} established a number of results on the spacing of zeros of orthogonal polynomials. One of these is the following. *Let w be an integrable weight function on* $[-1, 1]$ *with* $\int_{-1}^{1} \left(w(x) \right)^{-1} dx =: M < \infty$, *and let*

$$(1 >) x_{1,n} > x_{2,n} > \cdots > x_{n,n} (> -1)$$

be the zeros of the associated orthonormal polynomials p_n in decreasing order. Let

$$x_{\nu,n} = \cos \theta_{\nu,n}, \qquad 0 < \theta_{\nu,n} < \pi, \qquad \nu = 1, 2, \ldots, n.$$

Let $\theta_{0,n} := 0$ and $\theta_{n+1,n} := \pi$. Then there is a constant K depending only on M such that

$$\theta_{\nu+1,n} - \theta_{\nu,n} < \frac{K \log n}{n}, \qquad \nu = 0, 1, \ldots, n.$$

Pólya {67} proved that if

$$p(x) = \sum_{k=0}^{n} a_k x^k,$$

then

$$|a_n| \le \frac{1}{2} \left(\frac{4}{m(E)} \right)^n \sup_{x \in E} |p(x)|$$

for every measurable set $E \subset \mathbb{R}$, $0 < m(E) < \infty$. Equality holds if and only if E is an interval $[a, a + \lambda]$ and $P(x) = AT_n\big(2(x - a)/\lambda - 1\big)$, where $a, A \in \mathbb{R}$ and $\lambda > 0$. Here, as before, T_n denotes the Čebyshov polynomial of degree n.

Let $H \subset \mathbb{C}$ be a compact set. Let

$$\mu_n = \inf \left(\max_{z \in H} |p(z)| \right),$$

where the infimum is taken for all monic polynomials p of degree n with complex coefficients. Let

$$\widetilde{\mu}_n = \inf \left(\max_{z \in H} |p(z)| \right),$$

where the infimum is taken for all monic polynomials p of degree n with complex coefficients and with all zeros in H. The numbers

$$\mu := \mu(H) = \lim_{n \to \infty} \mu_n^{1/n} \qquad \text{and} \qquad \widetilde{\mu} := \widetilde{\mu}(H) = \lim_{n \to \infty} \widetilde{\mu}_n^{1/n}$$

exist and are called the Čebyshov constant and modified Čebyshov constant, respectively.

Let

$$d(z_1, z_2, \ldots, z_n) = \prod_{1 \leq k < j \leq n} |z_k - z_j|$$

and

$$d_n := \sup \left(d(z_1, z_2, \ldots, z_n) \right)^{\frac{2}{n(n-1)}},$$

where the supremum is taken for all $z_1, z_2, \ldots, z_n \in H$. The points $z_1, z_2, \ldots, z_n$ for which the above supremum is achieved are called nth *Fekete points*. Then the value

$$d(H) := \lim_{n \to \infty} d_n$$

exists and is called the transfinite diameter (Fekete constant) of H.

The logarithmic energy $I(\mu)$ of a $\mu \in \mathcal{M}(H)$ is defined as

$$I(\mu) := \int_H \int_H \log \frac{1}{|z - t|} \, d\mu(z) \, d\mu(t),$$

and the energy V of H by

$$V := \inf \left\{ I(\mu) \, : \, \mu \in \mathcal{M}(H) \right\},$$

where $\mathcal{M}(H)$ is the collection of all positive Borel measures with $\mu(H) = 1$ and with support in H. Then V turns out to be finite or $+\infty$ and in the finite case there is a unique measure $\mu = \mu_H \in \mathcal{M}(H)$ for which the infimum defining V is attained. This μ_H is called the equilibrium distribution or measure of a compact set H. The quantity $\text{cap}\,(H) := e^{-V}$ is called the logarithmic capacity of H.

Fekete $\{42\}$ and Szegő $\{81\}$ proved that the the Čebyshov constants, the transfinite diameter and the logarithmic capacity of a compact set $H \subset \mathbb{C}$ are equal, that is,

$$\mu(H) = \widetilde{\mu}(H) = d(H) = \text{cap}\,(H)$$

for every compact subset H of the complex plane.

Tamás Kövári has two papers, {48} (written jointly with Pommerenke) and {47}, on the distribution of Fekete points.

Approximation of functions $f \in C(H)$ by polynomials with integer coefficients on a compact set $H \subset \mathbb{C}$ has been considered by Fekete. Several papers appeared on the subject, the two most important of them are {42} and {44}. His typical results include that a function $f \in C[a,b]$, $b - a \geq 4$, is approximable from the collection of polynomials with integer coefficients if and only if f is a polynomial with integer coefficients. Also, a function $f \in C[0,1]$ is approximable from the collection of polynomials with integer coefficients if and only if $f(0)$ and $f(1)$ are both integers.

Another important result of Fekete {42} is an extension of a sresult of David Hilbert related to the so-called Integer Chebyshev Problem. An integer Chebyshev polynomial Q_n for a compact subset $E \subset \mathbb{C}$ is a polynomial Q_n of degree at most n with integer coefficients such that $\|Q_n\|_E = \inf_{P_n} \|P_n\|_E$, where the infimum is taken for all not identically zero polynomials P_n of degree at most n with integer coefficients (it is easy to see that at least one such Q_n exists). The integer Chebyshev constant for a compact subset $E \subset \mathbb{C}$ is defined by $t_{\mathbb{Z}}(E) := \lim_{n \to \infty} \|Q_n\|_E^{1/n}$ (it is a routine argument to show that the limit exists). In the above notation Fekete's result simply reads as

$$t_{\mathbb{Z}}(E) \leq \sqrt{\operatorname{cap}(E)},$$

and contains Hilbert's result as a special case when the set E is a closed interval of the real line. The Integer Chebyshev Problem has continued to attract a large number of well known mathematicians later in the century. Yet, there are many unanswered questions about it posed in papers appearing in general mathematics journals in the XXI-st century.

For a prime p, the polynomials

$$f_p(z) := \sum_{k=1}^{p-1} \left(\frac{k}{p} \right) z^k$$

are named after Fekete and have a variety of remarkable properties (the coefficients are Legendre symbols).

Erdős and Freud {30} worked together on orthogonal polynomials with regularly distributed zeros. Let α be a positive measure on $(-\infty, \infty)$ for which all the moments

$$\mu_m := \int_{-\infty}^{\infty} x^m \, d\alpha(x), \qquad m = 0, 1, \dots,$$

exist and are finite. Denote the sequence of the associated orthonormal polynomials by $(p_n)_{n=0}^{\infty}$. Let $x_{1,n} > x_{2,n} > \cdots > x_{n,n}$ be the zeros of of p_n in decreasing order. Let $N_n(\alpha, t)$ denote the number of positive integers k for which

$$x_{k,n} - x_{n,n} \geq t(x_{1,n} - x_{n,n}).$$

The distribution function β of the zeros is defined, when it exists, as

$$\beta(t) = \lim_{n \to \infty} n^{-1} N_n(\alpha, t), \qquad 0 \leq t \leq 1.$$

Let

$$\beta_0(t) = \frac{1}{2} - \frac{1}{\pi} \arcsin(2t - 1).$$

A positive measure α for which the array $x_{k,n}$ has the distribution function $\beta_0(t)$ is called an arc-sine measure. If $d\alpha(x) = w(x)\,dx$ is absolutely continuous and α is an arc-sine measure, then w is called an arc-sine weight. One of the theorems of Erdős and Freud {30} states that *the condition*

$$\limsup_{n \to \infty} (\gamma_{n-1})^{1/(n-1)} (x_{1,n} - x_{n,n}) \leq 4$$

implies that α is arc-sine and

(3) $$\lim_{n \to \infty} (\gamma_{n-1})^{1/(n-1)} (x_{1,n} - x_{n,n}) = 4,$$

where, as before, γ_{n-1} is the leading coefficient of p_{n-1}.

They also show that the weights $w_\alpha(x) := \exp\left(-|x|^\alpha\right)$, $\alpha > 0$, are not arc-sine. It is further proved by a counter-example that even the stronger sufficient condition (3) in the above-quoted result is not necessary in general to characterize arc-sine measures. As the next result of their paper shows, the case is different if w has compact support. Namely they show that *a weight w, the support of which is contained in $[-1, 1]$, is arc-sine on $[-1, 1]$ if and only if*

$$\limsup_{n \to \infty} (\gamma_n)^{1/n} \leq 2.$$

A set $A \subset [-1, 1]$ is called a determining set if all weights w with support contained in $[-1, 1]$, the restricted support $\{x : w(x) > 0\}$ of which contain A, are arc-sine on $[-1, 1]$. A set $A \subset [-1, 1]$ is said to have minimal capacity c if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for every $B \subset [-1, 1]$ having Lebesgue measure less than $\delta(\varepsilon)$ we have cap $(A \setminus B) > c - \varepsilon$. Another remarkable result of this paper by Erdős and Freud is that *a measurable set $A \subset [-1, 1]$ is a determining set if and only if it has minimal capacity $1/2$.*

Erdős' paper {32} with Fritz Herzog and George Piranian on the geometry of polynomials is seminal. In this paper, they proved a number of interesting results and raised many challenging questions. Although quite a few of these have been solved by Christian Pommerenke and others, many of them are still open. Erdős liked this paper very much. In his talks about polynomials, he often revisited these topics and mentioned the unsolved problems again and again. A taste of this paper is given by the following results and still unsolved problems from it. As before, associated with a monic polynomial

$$(4) \qquad f(z) = \prod_{j=1}^{n} (z - z_j), \qquad z_j \in \mathbb{C},$$

let

$$E = E(f) = E_n(f) := \left\{ z \in \mathbb{C} : \left| f(z) \right| \leq 1 \right\}.$$

One of the results of Erdős, Herzog, and Piranian tells us that *the infimum of $m\big(E(f)\big)$ is 0, where the infimum is taken over all polynomials f of the form (4) with all their zeros in the closed unit disk (n varies and m denotes the two-dimensional Lebesgue measure).* Another result is the following. *Let F be a closed set of transfinite diameter less than 1. Then there exists a positive number $\rho(F)$ such that, for every polynomial of the form (4) whose zeros lie in F, the set $E(f)$ contains a disk of radius $\rho(F)$.* There are results on the number of components of E, the sum of the diameters of the components of E, some implications of the connectedness of E, some necessary assumptions that imply the convexity of E. An interesting conjecture of Erdős states that *the length of the boundary of $E_n(f)$ for a polynomial f of the form (4) is $2n + O(1)$.* This problem seems almost impossible to settle. The best result in this direction is $O(n)$ by P. Borwein {6} that improves an earlier upper bound $74n^2$ given by Pommerenke.

One of the papers where Erdős revisits this topic is {33} written jointly with Elisha Netanyahu. The result of this paper states that *if the zeros $z_j \in \mathbb{C}$ are in a bounded, closed, and connected set whose transfinite diameter is $1 - c$ ($0 < c < 1$), then $E(f)$ contains a disk of positive radius ρ depending only on c.*

I have a postcard from Erdős asking for a proof of the fact that the diameter of $E(f)$ is always at least 2 for monic polynomials f (of the form (4)). "This ought to be trivial, but I do not have a proof" is commented by Erdős on the card. Based on a result of Pommerenke, a proof is presented in {8, p. 354}.

János Erőd {39} attributes the following interesting result to Erdős and Turán and presents its proof in his paper. *If*

$$(5) \qquad f(x) = \pm \prod_{j=1}^{n} (x - x_j), \qquad -1 \le x_1 \le x_2 \le \cdots \le x_n \le 1,$$

and f is convex between x_{k-1} and x_k for an index k, then

$$x_k - x_{k-1} \le \frac{16}{\sqrt{n}}.$$

It is not clear to me whether or not Erdős and Turán published this result. In any case, Erőd proves more, namely he shows that in the Erdős–Turán inequality the constant 16 can be replaced by

$$c_n = 2\sqrt{\frac{n}{2n-3}} \quad \text{if } n \text{ is even, and} \qquad c_n = \frac{2n}{n-1}\sqrt{\frac{n-2}{2n-3}} \quad \text{if } n \text{ is odd}$$

(so $\lim_{n\to\infty} c_n = \sqrt{2}$ in both cases).

An elementary paper of Erdős and Tibor Grünwald (Gallai) {31} deals with some geometric properties of polynomials with only real zeros. One of their results states that *if f is a polynomial of the form (5), then*

$$\int_{x_k}^{x_{k+1}} |f(x)| \, dx \le \frac{2}{3} (x_{k+1} - x_k) \max_{x \in [x_k, x_{k+1}]} |f(x)|.$$

Some extensions of the above are proved in {19}. In this paper Erdős raised a number of questions. For example, he conjectured that if t is a real trigonometric polynomial with only real zeros and with $\|t\|_{\mathbb{R}} \le 1$, then

$$\int_0^{2\pi} |t(\theta)| \, d\theta \le 4.$$

Concerning polynomials p of degree at most n with all their zeros in $(-1, 1)$ and with $\|p\|_{[-1,1]} = 1$, Erdős conjectured that *if $x_k < x_{k+1}$ are two consecutive zeros of p, then*

$$\int_{x_k}^{x_{k+1}} |p(x)| \, dx \le d_n(x_{k+1} - x_k),$$

where

$$d_n := \frac{1}{y_{k+1} - y_k} \int_{y_k}^{y_{k+1}} |T_n(y)| \, dy,$$

T_n is the usual Čebyshov polynomial of degree n, and $y_k < y_{k+1}$ are two consecutive zeros of T_n. (Note that d_n is independent of k and that $\lim_{n \to \infty} d_n = 2/\pi$.) These conjectures and more have all been proved in 1974, see Edward B. Saff and T. Sheil-Small {71} and Kristiansen {49}. {49} contains an error. This was corrected in {50}. See also {75}.

In {36} Erdős and Turán proved the following. Let $\{\zeta_\nu^{(n)}\}$ be a triangular sequence of numbers such that

$$1 \geq \zeta_1^{(n)} > \zeta_2^{(n)} > \cdots > \zeta_n^{(n)} \geq -1.$$

Let

$$\zeta_\nu^{(n)} = \cos \phi_\nu^{(n)}, \qquad 0 \leq \phi_\nu^{(n)} \leq \pi, \qquad \text{and} \qquad \omega_n(\zeta) = \prod_{\nu=1}^{n} (\zeta - \zeta_\nu^{(n)}).$$

For $(\alpha, \beta) \subset (0, \pi)$, let $N_n(\alpha, \beta)$ denote the number of $\phi_\nu^{(n)}$ in (α, β). Suppose $\left| \omega_n(\zeta) \right| < 2^{-n} A(n)$ on $(-1, 1)$ for every n. Then for every subinterval (α, β) of $(0, \pi)$ one has

$$\left| N_n(\alpha, \beta) - \frac{\beta - \alpha}{\pi} n \right| < \frac{8}{\log 3} \left(n \log A(n) \right)^{1/2}.$$

Extending the results of his paper {36} with Turán, Erdős {21} proved that *if there are absolute constants* $c_1, c_2 > 0$ *and a function* f *such that*

$$\frac{c_2 f(n)}{2^n} \leq \max_{\zeta_{\nu+1}^{(n)} \leq \zeta \leq \zeta_\nu^{(n)}} \left| \omega_n(\zeta) \right| \leq \frac{c_1 f(n)}{2^n}, \qquad \nu = 0, 1, \ldots, n,$$

then for $(\alpha, \beta) \subset (0, \pi)$,

$$N_n(\alpha, \beta) = \frac{\beta - \alpha}{\pi} n + O\big((\log n)\big(\log f(n) \big) \big).$$

This result has been extended by various people in many directions. See, for example, Totik {85}.

Erdős {28} gives an extension of some results of Bernstein and Antoni Zygmund. Bernstein had asked the question whether one can deduce boundedness of $\left| P_n(x) \right|$ on $[-1, 1]$ for polynomials p_n of degree at most n if one knows that $\left| P_n(x) \right| \leq 1$ for $m > (1 + c)n$ values of x with some $c > 0$. His answer was affirmative. He showed that *if* $\left| P_n\big(x_i^{(m)}\big) \right| \leq 1$ *for*

all zeros $x_i^{(m)}$ of the mth Čebyshov polynomial T_m with $m > (1+c)n$, then $\left| P_n(x) \right| \le A(c)$ for all $x \in [-1, 1]$, with $A(c)$ depending only on c. Zygmund had shown that the same conclusion is valid if T_m is replaced by the mth Legendre polynomial L_m. Erdős established a necessary and sufficient condition to characterize the system of nodes

$$-1 \le x_1^{(m)} < x_2^{(m)} < \cdots < x_n^{(m)} \le 1$$

for which

$$\left| P_n\big(x_i^{(m)}\big) \right| \le 1, \qquad i = 1, 2, \ldots, m, \qquad m > (1+c)n,$$

imply $\left| P_n(x) \right| \le A(c)$ for all polynomials P_n of degree at most n and for all $x \in [-1, 1]$, with $A(c)$ depending only on c. His result contains both that of Bernstein and of Zygmund as special cases. Note that such an implication is impossible if $m \le n + 1$, by a well-known result of Georg Faber.

Erdős wrote a paper {22} on the coefficients of the cyclotomic polynomials. The cyclotomic polynomial F_n is defined as the monic polynomial whose zeros are the primitive nth roots of unity. It is well known that

$$F_n(x) = \prod_{d \mid n} (x^{n/d} - 1)^{\mu(d)},$$

where μ is the Möbius function. For $n < 105$, all coefficients of F_n are ± 1 or 0. For $n = 105$, the coefficient 2 occurs for the first time. Denote by A_n the maximum over the absolute values of the coefficients of F_n. Schur proved that $\limsup A_n = \infty$. Emma Lehmer proved that $A_n > cn^{1/3}$ for infinitely many n. In his paper {22}, Erdős proved that *for every k, $A_n > n^k$ for infinitely many n.* This is implied by his even sharper theorem saying that

$$A_n > \exp\left[c \, (\log n)^{4/3} \right]$$

for $n = 2 \cdot 3 \cdot 5 \cdot \cdots \cdot p_k$ *with k sufficiently large, where p_k denotes the kth prime number.* Recent improvements and generalizations of this can be explored in {57}, {58}, and {59}.

Erdős {25} has a note on the number of terms in the square of a polynomial. Let

$$f_k(x) = a_0 + a_1 x^{n_1} + \cdots + a_{k-1} x^{n_{k-1}}, \qquad 0 \ne a_i \in \mathbb{R},$$

be a polynomial with k terms. Denote by $Q(f_k)$ the number of terms of f_k^2. Let $Q_k := \min Q(f_k)$, where the minimum is taken over all f_k of the above

form. László Rédei posed the problem whether $Q_k < k$ is possible. Rényi, László Kalmár, and Rédei proved that, in fact, $\liminf_{k\to\infty} Q_k/k = 0$, and also that $Q(29) \leq 28$. Rényi further proved that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{Q_k}{k} = 0.$$

He also conjectured that $\lim_{k\to\infty} Q_k/k = 0$. In his short note {25}, Erdős proves this conjecture. In fact, he shows that *there are absolute constants $c_1 > 0$ and $0 < c_2 < 1$ such that $Q_k < c_2 k^{1-c_1}$*. Rényi conjectured that $\lim_{k\to\infty} Q_k = \infty$. He also asked whether or not Q_k remains the same if the coefficients are complex. These questions remained open at the time of the writing of this paper.

Erdős {27} proved a significant result related to his conjecture about polynomials with ± 1 coefficients. He showed that if

$$f_n(\theta) := \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta)$$

is a trigonometric polynomial with real coefficients,

$$\max_{1\leq k\leq n} \left\{ \max \left\{ |a_k|, |b_k| \right\} \right\} = 1 \qquad and \qquad \sum_{k=1}^{n} (a_k^2 + b_k^2) = An,$$

then there exists a $c = c(A) > 0$ depending only on A for which $\lim_{A\to 0} c(A) = 0$ and

$$\max_{0\leq\theta\leq 2\pi} \left| f_n(\theta) \right| \geq \frac{1 + c(A)}{\sqrt{2}} \left(\sum_{k=1}^{n} (a_k^2 + b_k^2) \right)^{1/2}.$$

Closely related to this is a problem for which Erdős offered \$100 and which has become one of my favorite Erdős problems (see also Problem 22 in {26}): *Is there an absolute constant $\varepsilon > 0$ such that the maximum modulus on the unit circle of any polynomial $p(x) = \sum_{j=0}^{n} a_j x^j$ with each $a_j \in \{-1, 1\}$ is at least $(1 + \varepsilon)\sqrt{n}$?* Erdős conjectured that there is such an $\varepsilon > 0$. Even the weaker version of the above, with $(1 + \varepsilon)\sqrt{n}$ replaced by $\sqrt{n} + \varepsilon$ with an absolute constant $\varepsilon > 0$, looks really difficult. (The lower bound $\sqrt{n+1}$ is obvious by the Parseval formula.) These problems are unsettled to this date.

Let D be the open unit disk of the complex plane. Let ∂D be the unit circle of the complex plane. Let

$$\mathcal{K}_n := \left\{ p \,:\, p(z) = \sum_{k=0}^{n} a_k z^k, \ a_k \in \mathbb{C}, \ |a_k| = 1 \right\}.$$

The class $\mathcal{K}_n$ is often called the collection of all (complex) unimodular polynomials of degree n. Given a sequence (ε_{n_k}) of strictly positive numbers tending to 0, we say that a sequence (P_{n_k}) of polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k})-ultraflat if

$$(1 - \varepsilon_{n_k})\sqrt{n_k + 1} \le \left| P_{n_k}(z) \right| \le (1 + \varepsilon_{n_k})\sqrt{n_k + 1}, \qquad z \in \partial D,$$

or equivalently

$$\max_{z \in \partial D} \left| \left| P_{n_k}(z) \right| - \sqrt{n_k + 1} \right| \le \varepsilon_{n_k}\sqrt{n_k + 1}.$$

The existence of an ultraflat sequence of unimodular polynomials (for some sequence (ε_{n_k}) of positive real numbers tending to 0) seemed very unlikely, in view of an extended version of the above mentioned Erdős conjecture to polynomials $P \in \mathcal{K}_n$ with $n \ge 1$.

Yet, refining a method of Thomas W. Körner, Jean-Pierre Kahane {46} proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where

$$\varepsilon_n = O\!\left(n^{-1/17}\sqrt{\log n}\,\right).$$

Thus this extended version of the above mentioned Erdős conjecture was disproved for the classes $\mathcal{K}_n$.

The structure of ultraflat sequences of unimodular polynomials is beautiful. The following uniform distribution theorem for the angular speed, conjectured by Saffari {72}, is proved in {17}. *Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. We write*

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}, \qquad R_n(t) = \left| P_n(e^{it}) \right|, \qquad t \in \mathbb{R}.$$

It is a simple exercise to show that α_n can be chosen so that it is differentiable in t on $\mathbb{R}$. Then in the interval $[0, 2\pi]$, the distribution of the normalized angular speed $\alpha_n'(t)/n$ converges to the uniform distribution as $n \to \infty$. That is, we have

$$m\!\left(\left\{\, t \in [0, 2\pi] \,:\, 0 \le \alpha_n'(t) \le nx \right\}\right) = 2\pi x + \gamma_n(x)$$

for every $x \in [0,1]$, where $\lim_{n\to\infty} \gamma_n(x) = 0$ for every $x \in [0,1]$. As a consequence, $\left|P_n'(e^{it})\right|/n^{3/2}$ also converges to the uniform distribution as $n \to \infty$. That is, we have

$$m\left(\left\{t \in [0, 2\pi] : 0 \le \left|P_n'(e^{it})\right| \le n^{3/2}x\right\}\right) = 2\pi x + \gamma_n(x)$$

for every $x \in [0,1]$, where $\lim_{n\to\infty} \gamma_n(x) = 0$ for every $x \in [0,1]$ ($m(\cdot)$ denotes the one-dimensional Lebesgue measure). In both statements the convergence of $\gamma_n(x)$ is uniform on $[0,1]$.

For higher derivatives, the following result is proved in {17}. Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then

$$\left(\frac{\left|P_n^{(m)}(e^{it})\right|}{n^{m+1/2}}\right)^{1/m}$$

converges to the uniform distribution as $n \to \infty$. More precisely, we have

$$m\left(\left\{t \in [0, 2\pi] : 0 \le \left|P_n^{(m)}(e^{it})\right| \le n^{m+1/2}x^m\right\}\right) = 2\pi x + \gamma_{m,n}(x)$$

for every $x \in [0,1]$, where $\lim_{n\to\infty} \gamma_{m,n}(x) = 0$ for every fixed $m = 1, 2, \ldots$ and $x \in [0,1]$. For every fixed $m = 1, 2, \ldots$, the convergence of $\gamma_{n,m}(x)$ is uniform on $[0,1]$.

Several topics from Erdős's problem paper {29} have already been discussed before. Here is one more interesting group of problems. Let $(z_k)_{k=1}^{\infty}$ be a sequence of complex numbers of modulus 1. Let

$$A_n := \max_{|z|=1} \prod_{k=1}^{n} |z - z_k|.$$

What can one say about the growth of A_n? Erdős conjectured that $\limsup A_n = \infty$. In my copy of {29} that Erdős gave me, there are some handwritten notes (in Hungarian) saying the following. "Wagner proved that $\limsup A_n = \infty$. It is still open whether or not $A_n > n^c$ or $\sum_{k=1}^{n} A_k > n^{1+c}$ happens for infinitely many n (with an absolute constant $c > 0$). These are probably difficult to answer." See {89}.

Erdős was famous for anticipating the "right" results. "This is obviously true; only a proof is needed" he used to say quite often. Most of the times, his conjectures turned out to be true. Some of his conjectures failed for the more or less trivial reason that he was not always completely

precise with the formulation of the problem. However, it happened only very rarely that he was essentially wrong with his conjectures. If someone proved something that was in contrast with Erdős' anticipation, he or she could really boast to have proved a really surprising result. Erdős was always honest with his conjectures. If he did not have a sense about which way to go, he formulated the problem "prove or disprove". Erdős turned even his "ill fated" conjectures into challenging open problems. The following quotation is a typical example of how Erdős treated the rare cases when a conjecture of his was disproved. It is from his problem paper {29} entitled "Extremal problems on polynomials". For this quotation, we need to recall the following notation. Associated with a monic polynomial $f(z) = \prod_{j=1}^{n}(z - z_j)$, where z_j are complex numbers, let $E_n(f) := \{z \in \mathbb{C} : |f(z)| \leq 1\}$. In his problem paper Erdős writes (in terms of the notation employed here): "In [7] we made the ill fated conjecture that the number of components of $E_n(f)$ with diameter greater than $1 + c$ $(c > 0)$ is less than δ_c, δ_c bounded. Pommerenke [14] showed that nothing could be farther from the truth, in fact he showed that for every $\varepsilon > 0$ and $k \in \mathbb{N}$, there is an $E_n(f)$ which has more than k components of diameter greater than $4 - \varepsilon$. Our conjecture can probably be saved as follows: Denote by $\Phi_n(c)$ the largest number of components of diameter greater than $1 + c$ $(c > 0)$ which $E_n(f)$ can have. Surely, for every $c > 0$, $\Phi_n(c) = o(n)$, and hopefully $\Phi_n(c) = o(n^\varepsilon)$ for every $\varepsilon > 0$. I have no guess about a lower bound for $\Phi_n(c)$, also I am not sure whether the growth of $\Phi_n(c)$, $(1 < c < 4)$ depends on c very much."

If $p(x) = \sum_{j=0}^{n} a_j x^j$, then we introduce $l_1(p) := \sum_{j=0}^{n} |a_j|$. An interesting problem of Erdős and György Szekeres is to minimize $l_1(p)$ over all polynomials

$$p(x) = \prod_{k=1}^{N}(1 - x^{\alpha_k}),$$

where N is a fixed positive integer, while the positive integers $\alpha_1, \alpha_2, \ldots, \alpha_N$ vary. Let $E_N^* := \min l_1(p)$, where the minimum is taken for all p of the above form. It is conjectured by Erdős and Szekeres that $E_N^* \geq N^K$ for any fixed K and sufficiently large N. Erdős and Szekeres {34} proved a sub-exponential upper bound for E_N^*. The best known upper bound for E_N^* today is $\exp\left(O(\log n)^4\right)$ given by A. S. Belov and Sergei V. Konyagin {3}, that improves weaker bounds given earlier by Andrew Odlyzko and Mihail N. Kolountzakis.

For a fixed $z \in \mathbb{C}$ and a Borel measure μ on $[-\pi, \pi)$ the Christoffel function $\omega_n(\mu, z)$ is defined to be the minimum of

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| p(e^{i\theta}) \right|^2 d\mu(\theta)$$

taken over all polynomials p of degree less than n with $p(z) = 1$. Szegő {78} studied $\omega_n(\mu, 0)$ for absolutely continuous measures μ in 1915. Later, in 1922, Szegő {79} showed that

$$\lim_{n \to \infty} n \omega_n(\mu, e^{it}) = \mu'(t), \qquad t \in (-\pi, \pi),$$

assuming that μ is absolutely continuous and $\mu' > 0$ is twice continuously differentiable. This result is very important for applications in orthogonal polynomials, probability theory, and statistics (linear prediction), and other areas. It gives a useful and numerically adaptable method of computing the weight function for orthogonal polynomials. In their paper {62}, Máté, Nevai, and Totik show that

$$\lim_{n \to \infty} n \omega_n(\mu, e^{it}) = \mu'(t)$$

holds almost everywhere on every interval $I \subset [-\pi, \pi)$ for which

$$\int_I \log \mu'(\theta) \, d\theta > -\infty.$$

In an earlier paper {61} Máté and Nevai showed that

$$\int_{-\pi}^{\pi} \log \mu'(\theta) \, d\theta > -\infty$$

implies

$$2e^{-1} \mu'(t) \leq \liminf_{n \to \infty} n \omega_n(\mu, e^{it}) \leq \limsup_{n \to \infty} n \omega_n(\mu, e^{it}) = \mu'(t)$$

for almost every real t.

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