BOLYAI SOCIETY
MATHEMATICAL STUDIES, 14

A Panorama of Hungarian Mathematics in the Twentieth Century, pp. 537–548.

Contribution of Hungarian Mathematicians to Game Theory

FERENC FORGÓ

Game theory, when defined in the broadest sense, is a collection of mathematical models formulated to study situations of conflict and cooperation. It is mainly concerned with finding the best actions for the individual decision makers in these situations and/or recognizing stable outcomes. Game theory attempts to provide a normative guide to rational behavior for individuals pursuing more or less different goals and make predictions about the outcomes thus realized.

The mathematical complexity of the models poses a real challenge to mathematicians who, if they want to be really successful, must possess not only the technical skills but have a deep understanding of the problems in a very wide variety of applications ranging from biology and human behavior to economics and warfare. János (John) von Neumann and János (John) Harsányi have left irremovable marks by their contributions to the very foundation of cooperative and noncooperative game theory. Jenő Szép, as an educator and textbook writer has helped generations of economists and decision scientists to keep abreast of the latest developments in this rapidly growing field. Their most important results, together with the context in which they are interpreted are briefly outlined below.

János (John) von Neumann, the last renaissance scientist of our time, was not only a brilliant mathematician but he also took interest in other sciences as well. His model of economic equilibrium has been a subject of study and a source of inspiration for generations of economists. While classical and neoclassical economic equilibrium models work with economic agents (producers, consumers) whose individual contribution in forming prices and producing goods is assumed to be negligible, there are several industries where concentration of capital creates situations in which a few major players decide on production volumes and/or prices. Strategic aspects of interaction on the market became an issue and raised questions that classical economic theory could not answer. So the need for a theory to study strategic behavior of participants in a conflict situation and the presence of the open mind of a genius came together in the thirties and forties of the 20^{th} century to give birth to a brand new body of knowledge commonly known as game theory.

Von Neumann did not publish too much in game theory as far as the number of papers and books is concerned. His two major works, however, are landmarks in game theory. One is a paper in which the first, complete minimax theorem is stated and proved, the other is a book [117] in which the theoretical foundation of cooperative game theory is laid down. For the latter he found an economist, Oskar Morgenstern to help him reinforce the economic relevance of the model and make the monumental work the starting point of any research in cooperative game theory.

Though Emile Borel (1924) was the first who defined pure and mixed strategies for symmetric two-person games, he was unable to prove the existence of equilibrium in mixed strategies. In fact, he was in doubt about the validity of the minimax theorem, an equivalent to the existence theorem in case of two-person zero-sum games. It was von Neumann (1928) who first gave a complete proof of a minimax theorem which covers the special case of the mixed extension of finite, two-person zero-sum games.

Theorem 1 (von Neumann). *Let C and U be unit simplexes of finite dimensional Euclidean spaces, and f be a jointly continuous real valued function defined on $X \times Y$. Suppose that f is quasiconcave on X, that is to say, for all $y \in Y$ the upper level sets of f are convex, and f is quasiconvex on Y, that is to say, for all $x \in X$ the lower level sets of f are convex. Then*

$$\min_Y \max_X f = \max_X \min_Y f.$$

This result was later extended by von Neumann (1937) himself by replacing unit simplexes with nonempty compact, convex subsets of Euclidean spaces. In both cases, von Neumann used topological and fixed-point arguments. It turned out later on that fixed-point theorems are not necessary for the proof, convex analysis, in particular linear separation is enough to prove even more general results where the continuity of f is replaced by partial upper and lower semicontinuity in the respective variables, Sion (1958). Theorem 1 has opened a whole avenue of research about minimax theorems and their various generalizations and has been applied in many fields inside and outside of mathematics. For a good overview of minimax theorems we recommend Simons (1995).

Von Neumann was not only concerned with the existence of equilibria for two-person zero-sum games but also proposed a unique method for computing one in the case of symmetric matrix games.

The mixed extension of a finite, two-person zero-sum game can be completely defined by a matrix $\mathbf{A}$ of m rows and n columns with general element $a_{i,j}$ which represents the payoff player 1 gets from player 2 if player 1 plays his i^{th} and player 2 his j^{th} pure strategy. The players are allowed to mix their pure strategies by choosing probability vectors $\mathbf{x}$ and $\mathbf{y}$, respectively, and are concerned with expected payoffs $\mathbf{xAy}$ player 1 gets on the average from player 2. It is easily seen that Theorem 1 ensures the existence of an equilibrium pair of strategies $\mathbf{x}^*$, $\mathbf{y}^*$ to satisfy

$$\mathbf{xA\ y}^* \leq \mathbf{x}^*\mathbf{A\ y}^* \leq \mathbf{x}^*\mathbf{A\ y}$$

for all probability vectors $\mathbf{x}$ and $\mathbf{y}$ of dimension m and n, respectively. We will briefly refer to the mixed extension of a finite two-person zero-sum game as a matrix game and define it by its payoff matrix $\mathbf{A}$.

A matrix game is said to be symmetric if $\mathbf{A} = -\mathbf{A}^T$. The value (i.e., the payoff at equilibrium) of symmetric games is 0 and both players have the same equilibrium strategies. Von Neumann proposed the following method to find an equilibrium strategy of player 2 (which is also an equilibrium strategy of player 1).

Player 2's strategy $\mathbf{y}(t)$ is assumed to be a function of a continuous parameter t (time) and we suppose $t \geq 0$. Define the following functions:

$$u_i : \mathbb{R}^n \to \mathbb{R}, \quad u_i(\mathbf{y}) = \mathbf{e}_i\mathbf{Ay}, \quad (i = 1, \ldots, n)$$

$$\phi : \mathbb{R} \to \mathbb{R}, \quad \phi(a) = \max\{0, a\},$$

$$\Phi : \mathbb{R}^n \to \mathbb{R}, \quad \Phi(\mathbf{y}) = \sum \phi\big(u_i(\mathbf{y})\big)$$

where $\mathbf{e}_i$ denotes the i^{th} unit vector.

Let $\mathbf{y}^0$ be any strategy of player 2. Consider the following system of differential equations:

$$\begin{aligned} &y_j'(t) = \phi\big(u_j\big(\mathbf{y}(t)\big)\big) - \Phi\big(\mathbf{y}(t)\big)\, y_j(t) \\ &y_j(0) = y_j^0 \end{aligned} \qquad (j = 1, \ldots, n).$$

Since the right-hand side of the above system is continuous, it has at least one solution. Let $\mathbf{y}(t)$ be a solution and $t_1, t_2, \dots$ be a positive monotone increasing sequence tending to ∞. Von Neumann's proved the following theorem:

Theorem 2. *Any limit point of the sequence* $\{\mathbf{y}(t_k)\}$, $(k = 1, 2, \dots)$ *is an equilibrium strategy of player* 2 *in the symmetric matrix game* $\mathbf{A}$. *Furthermore, there exists a constant* C, *such that*

$$\mathbf{e}_i \mathbf{A} \mathbf{y}(t_k) \leq \sqrt{n}/(C + t_k), \qquad (i = 1, \dots, n).$$

Numerically solving the differential equation system provides a good approximation of an equilibrium strategy of either player.

Using linear programming is a more efficient way of solving matrix games, but von Neumann's method remains an elegant, alternative approach which has also been an inspirational source for various kinds of learning processes.

Von Neumann's minimax theorem in the two-person zero-sum case was the precursor to the equilibrium concept developed by John Nash in 1951 for general-sum, n-person noncooperative games. A game G in normal (strategic) form is given as the $2n$-tuple

$$G = \{S_1, \dots, S_n;\, f_1, \dots, f_n\},$$

where for each $i = 1, \dots, n$, S_i is the strategy set of player i and $f_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ is his payoff function. Given the $(n-1)$-tuple of strategies $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ of all players but i, a strategy $t \in S_i$ is said to be a best reply (to the strategy profile of the rest of the players), if

$$f_i(s_1, \dots, s_{i-1}, t, s_{i+1}, \dots, s_n) \geq f_i(s_1, \dots, s_{i-1}, u, s_{i+1}, \dots, s_n)$$

holds for all $u \in S_i$.

A strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is called a *Nash equilibrium point* of game G if

$$f_i(s_1^*, \dots, s_n^*) \geq f_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*)$$

holds for all $s_i \in S_i$ and $i = 1, \dots, n$, or equivalently, s_i^* is a best reply to $(s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_n^*)$.

Not only noncooperative game theory received the initiating impetus from von Neumann but the cooperative theory as well. In the seminal book written together with Oskar Morgenstern [117] he sets up the model of a cooperative game most commonly used for analysis ever since. Given a finite set of players, $N = \{1, \dots, n\}$, a pair $G = (N, v)$ is defined an n-person cooperative game in characteristic function form (with side payments) where v is a real valued function defined on the subsets (coalitions) of N. The function v assigns a real number $v(S)$ to every coalition, with the convention $v(\emptyset) = 0$. The value $v(S)$ represents the transferable utility coalition S can achieve on its own when its members fully cooperate. The theory is mostly concerned with how the utility $v(N)$ achievable by the grand coalition N can be distributed taking into account the power, as expressed by the characteristic function, the coalitions have. Von Neumann and Morgenstern consider only essential, constant-sum games in their book. A game $G = (N, v)$ is essential if

$$\sum_{i \in N} v\big(\{i\}\big) < v(N)$$

and it is constant-sum if

$$v(S) + v(N - S) = v(N)$$

holds for any coalition S. We call the game $G = (N, v)$ superadditive if

$$v(S) + v(T) \leq v(S \cup T)$$

for all disjoint coalitions S, T.

For a given game $G = (N, v)$, let S be a coalition and $\mathbf{x} = (x_1, \dots, x_n)$ a real n-vector.

Define

$$x(S) = \sum_{i \in S} x_i.$$

An n-vector $\mathbf{x} = (x_1, \dots, x_n)$ is called an imputation if it is individually rational, i.e.,

$$x_i \geq v\big(\{i\}\big) \quad \text{for all} \quad i \in N$$

and Pareto optimal or efficient, i.e.,

$$x(N) = v(N).$$

An imputation represents a distribution of $v(N)$ among players in such a way that no player will get less than his own value $v\big(\{i\}\big)$. We say that imputation $\mathbf{x}$ dominates imputation $\mathbf{y}$, written $\mathbf{x}\,\mathrm{dom}\,\mathbf{y}$, if there is a coalition S such that $x_i > y_i$ for all $i \in S$ and $x(S) \leq v(S)$. If $\mathbf{x}\,\mathrm{dom}\,\mathbf{y}$, then coalition S can block the imputation $\mathbf{y}$ since it can give more to its members and also has the power to achieve this.

Von Neumann and Morgenstern define a "solution" to a game $G = (N, v)$ as a subset V of the set all imputations which is both internally and externally stable, i.e.

(i) there is no $\mathbf{x}, \mathbf{y} \in V$ such that $\mathbf{x}\,\mathrm{dom}\,\mathbf{y}$,

(ii) if $\mathbf{y} \notin V$, then there is an $\mathbf{x} \in V$ such that $\mathbf{x}\,\mathrm{dom}\,\mathbf{y}$.

To distinguish from other solution concepts that have emerged since the ground breaking work of von Neumann and Morgenstern the above "solution" is usually referred to as *stable set* or *von Neumann–Morgenstern solution.* Von Neumann and Morgenstern interpreted stable sets as a standard of behavior within a society. Distribution of the commonly gained wealth is accepted and can be maintained if it belongs to a stable set. Within this set no distribution is both favorable and achievable by any segment of the society while any distribution outside of this set can be prevented from becoming socially acceptable by certain social groups which have both the motivation and power to do so. Von Neumann and Morgenstern left open the question of which imputation in a stable set V will actually realize. This is assumed to be determined by the bargaining ability of the players, outside forces, chance etc. They were not disturbed at all by the fact that in many games there is a multitude of different stable sets. They considered each as a standard of behavior and did not consider part of their model which one of these will realize. They were, however, deeply concerned with the existence of stable sets. They could prove in their book the following theorem.

Theorem 3 [117]**.** *Every superadditive, essential, three-person game has at least one stable set.*

Although some special classes of games were shown by von Neumann and Morgenstern to have stable sets, they were unable to prove a general existence theorem. To settle the existence of stable sets has proved to be a hard problem over the years. Lucas constructed a 10-person, nonconstant-sum game in 1969 which has no stable sets and Bondareva et al. (1974) proved that all 4-person games have stable sets. The question of existence for general games is unsettled for 5 to 9-persons. No one has proved or disproved the original conjecture of von Neumann and Morgenstern that every constant-sum game has at least one stable set. It is also conjectured that games with no solutions are "rare", known examples are unstable in the sense that minor changes in the characteristic function value causes them to have stable sets.

Stable sets have a surprisingly rich mathematical structure and give rise to extremely difficult problems. An excellent overview of the developments from von Neumann and Morgenstern to the early nineties is given by Lucas (1992).

One of the basic assumptions of classical game theory is complete information and common knowledge: rules of the game, the available strategies, the payoff functions are assumed to be known by each player together with the infinite hierarchy: "all players know it, each one knows that everyone knows it, and so on". If we want to come closer to reality, we have to find ways for analyzing conflict situations where players have only partial information about certain constituents of the game.

János (John) Harsányi, the Economics Nobel Laureate of 1994, was the first to provide a consistent model for games with incomplete information which became the most commonly applied approach to treat informational disparities of agents not only in game theory but in the new, rapidly developing field of Economics of Information as well. The very heart of the concept is usually referred to as the *Harsányi Doctrine* or *The Common Prior Assumption:* If C denotes the possible states of the world with generic element c which is a specification of all parameters that may be the object of uncertainty in a game G, then all the players share the same prior probability distribution on C which is common knowledge among them. This does not imply that all players have the same subjective probabilities since they may have different information about the true state of nature. The subjective probability of a player is his posterior (in the Bayesian sense) given his information. Posteriors may well be different but differences in probability estimates of distinct individuals should be explained by differences in information and experience.

We will demonstrate the power of the *Harsányi Doctrine* by a brief outline of his original model published in a series of articles in 1967. When asked, he himself thought that these articles had brought him the Nobel prize. It took thirty years for the scientific community to really appreciate the contribution of the original model and the underlying idea (The Harsányi Doctrine) to help better understand strategic behavior in conflicts with incomplete information.

Following Harsányi we will call a game of incomplete information in normal (strategic) form an *I-game* which is given as

$$G := \{S_1, \dots, S_n;\, C_1, \dots, C_n;\, R_1, \dots, R_n;\, V_1, \dots, V_n\}$$

where for player i, S_i is his strategy set, C_i is the (finite) set of the information vectors available to him, R_i is a function that assigns to every information vector $c_i \in C_i$ a (subjective) probability vector $R_i(c_{-i} \mid c_i)$ over the set $C_{-i} = C_1 \times \cdots \times C_{i-1} \times C_{i+1} \times \cdots \times C_n$, $c_{-i} = (c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n)$, V_i is his expected payoff function which assigns a real number (utility) to any n-tuple of strategies and information vectors using the probability distribution R_i. We can view the information vector c_i as representing certain physical, social, and psychological attributes of player i himself, in that it summarizes some crucial parameters of player i's own payoff function, as well as the main parameters of his beliefs about his social and physical environment including his beliefs about the beliefs of the other players. From this point of view, vector c_i may be called player i's *attribute vector* or *type.* R_i is player i's subjective probability distribution over the types of the other players conditioned on his own type.

Applying the Harsányi Doctrine, we assume that there is an objective probability distribution R^* over the product of the information sets (types) whose conditional probabilities coincide with the subjective conditional probabilities of the players, that is

$$R_i(c_{-i} \mid c_i) = R^*(c_{-i} \mid c_i)$$

holds for all attribute vectors and all $i = 1, \dots, n$. Then the I-game with the common prior R^* is called the *Bayesian game associated with* G and is denoted by

$$G^* = \{S_1, \dots, S_n;\, C_1, \dots, C_n;\, R^*;\, V_1, \dots, V_n\}.$$

With the help of the distribution R^*, we can now define the normal form $N(G^*)$ of an I-game G (and G^*) as

$$N(G^*) = \{S^*_{1,} \dots, S^*_n;\, W_1, \dots, W_n\}.$$

The strategy sets in $N(G^*)$ are the sets of the so-called *normalized strategies* which are functions from the range set of the information sets to the original strategy sets. In other words, a normalized strategy $s^*_i \in S^*_i$ tells player i what strategy to use from S_i for each possible value of his own information vector c_i. The payoff functions, $W_1, \ldots, W_n$, are expected payoffs with respect to the information vector c using the objective probability distribution R^*. Notice that $N(G^*)$ does not include the $c_i's$ and R^* any more.

If in the same game, instead of the distribution R^*, we use the conditional subjective distribution functions $R_i(c_{-i} \mid c_i)$ when defining the payoff functions, then we get the *semi-normal form* $SN(G^*)$ of G^* as

$$SN(G^*) = \{S_1^*, \ldots, S_n^*; C_1, \ldots, C_n; R^*; Z_1, \ldots, Z_n\}$$

where the Z_i's are conditional expected payoffs obtained by using the conditional probabilities $R^*(c_{-i} \mid c_i)$ which are equal to $R_i(c_{-i} \mid c_i)$ by the Harsányi doctrine.

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ of game $SN(G^*)$ is said to be a *Bayesian equilibrium point* of the I-game G, if, for all $i = 1, \ldots, n, s_i^*$ is a best reply to the strategy profile $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$ of the rest of the players for all possible values of c_i (with the possible exception of a set of measure 0 in C_i).

Theorem 4 (Harsányi 1967). *Let G be an I-game and G^* the Bayesian game associated with G. A normalized strategy profile s^* is a Bayesian equilibrium point of G if and only if s^* is a Nash equilibrium point of the normal form $N(G^*)$.*

Harsányi's Bayesian approach is based on beliefs of players concerning certain parameters of the *I-game.* But it also involves beliefs about other players' beliefs and so on, leading to an infinite hierarchy of beliefs. The introduction of *type* has proved to be a useful tool to get around this difficulty operationally, but leaving a void for the formal mathematical treatment. Mertens and Zamir (1985) justified the validity of the Harsányi model with the strictest mathematical rigor.

Harsányi's model of games of incomplete information and the Harsányi doctrine has been so incorporated into modern economics that it is impossible to give a complete list of its applications. Harsányi himself used it in addressing other fundamental questions in game theory. We just mention two without going into any details.

One of the basic problems in noncooperative game theory is how to select a particular Nash equilibrium point of the mixed extension of a finite n-person noncooperative game if there are many equilibria. Additional assumptions and requirements have been proposed, as they say, to *refine* the equilibrium concept. The refinement due to Harsányi has become famous under the name: *tracing procedure,* Harsányi (1975). In this model, it is assumed that each player starts his analysis of the game situation by assigning a subjective prior probability distribution to the set of all pure strategies available to each other player. The Harsányi doctrine is also assumed: these distributions are conditionals of a common prior. Then the players will modify their subjective probability distributions in a continuous manner until all of these probability distributions converge to a specific equilibrium point of the game.

Another controversial issue is how to interpret mixed strategies for finite games. This is a problem when theory is confronted with practice: hardly anyone would randomize in the way it is assumed in classical models of game theory, that is, before each play of the game a lottery is performed and the players will do whatever is dictated by its outcome. In Harsányi's model, Harsányi (1973), a game with incomplete information is associated with the finite game under study. This game is obtained by a small perturbation of the payoffs. Then, as is proved by Harsányi, if the variations in payoffs are small, almost any mixed equilibrium of the finite game is close to a pure strategy equilibrium of the associated Bayesian game and vice versa. Thus even if no player makes any effort to use his pure strategies with the required probabilities, the random variations in the payoffs make every player choose his pure strategies with the right frequencies. The assumption of small variations is not very restrictive since payoffs are usually utilities of players whose estimations are never exact.

János von Neumann's and János Harsányi's contributions to modern game theory have had a great impact on the directions in which game theory has developed and their work is still a significant marker in contemporary research in the field. Hungarian mathematics, and science in general, must be very proud of their accomplishments and cherish the fame they have brought to Hungary. Though they got their first university degrees in Hungary, due to several reasons and special historical circumstances they lived their active life mostly abroad, thereby sharing the fame and publicity they earned between the homeland and the country they lived in.

Until the early sixties, any economic theory or methodology other than the Marxian was taboo in Hungary and completely missing from university curricula. A few professors at the University of Economics in Budapest realized in the early sixties that part of the methodology of modern economics, such as activity analysis with the mathematical programming underpinning and game theory when stripped of any ideology and used for the analysis of economic problems existing in modern societies, whether it be a free market economy or a (partially) planned and government controlled socialist economy, could be introduced in the curriculum of the university. They even selected a small, special group of students who were given a special curriculum heavily loaded with mathematics and "western-style" economics.

Professor **Jenő Szép,** head of the Mathematics Department at the University of Economics at that time and the late professor Béla Krekó were the main driving forces in this endeavor. Jenő Szép designed the first course in game theory which was based on Burger's book [20] and included both noncooperative and cooperative games. Out of these lectures, grew the first game theory book in Hungarian [176] co-authored by one of his former students, Ferenc Forgó. In addition to covering the standard topics, the introduction of the Nash equilibrium concept was embedded into a more general equilibrium model invented and studied by Jenő Szép (1970) himself.

The success of this book led to the German and the English version [176]. The English version served as text for game theory courses in several universities world wide. Later another author, Ferenc Szidarovszky of the University of Arizona joined Szép and Forgó to write an extended and improved text [45]. The specialty of this book is that it enhances the nonconvexities arising in various models of conflict situations.

Jenő Szép, as an educator, text-book author and inspirational source for generations of scientists and practitioners has done a lot to further the cause of meaningful and creative application of mathematics in economics in general and in game theory in particular.

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