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Holomorphic Functions

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1. Prologue

It is quite well known that, as Jean-Pierre Kahane tells us elsewhere in this volume, Hungarian mathematics started the twentieth century with a bang when in October 1900 a twenty year old student, who had just returned from Berlin after a year there, proved that the Fourier series of a continuous function is uniformly Cesàro-summable to the function. It is, however, less well known that Lipót Fejér has already previously published an article containing some simple theorems concerning power series (Mat. Fiz. Lapok **9** (1900), 405–410; [40], No. 1, p. 29). A typical result is the following: if g is a positive integer and f an entire function of genus $\leq g-1$, then the radius of convergence of $\sum_{n=0}^{\infty} c_n f(n) x^{n^g}$ is not smaller than the the radius of convergence of $\sum_{n=0}^{\infty} c_n x^{n^g}$. An application is the fact that a power series and the series obtained from it by termwise differentiation have the same radius of convergence.

2. The Jensen formula

Fejér got to know Constantin Carathéodory during the academic year he spent in Berlin. At that time Carathéodory was an engineer for the Suez Canal Company. His family, as many Greeks, was in the diplomatic service of the Ottoman Empire: his father was ambassador in Brussels, his uncle, who was also his father-in-law, ambassador in Berlin. Carathéodory was always interested in mathematics, so while visiting his uncle he went to see what is going on at the seminar of Hermann Amandus Schwarz. According to the legend, he found a man seven years his junior who presented a proof of the characterization of the triangle with shortest perimeter inscribed into a triangle ([40], Appendix III, vol. II, p. 847; [140]). Carathéodory was so impressed that then and there he decided to abandon his career as an engineer and to become a mathematician.

The Comptes Rendus note of Carathéodory and Fejér from 1907 (**145**, 163–165; [40], No. 19) is based on the geometric theorem according to which if P is a point inside a circle with center O and radius R, and Q is its polar, i.e. OPQ are collinear and $OP \cdot OQ = R^2$, then the ratio $PM : QM$ of the distances from P and from Q to a point M on the circumference equals the constant OP/R.

Consider now a circle in the complex plane $\mathbb{C}$ with radius R and center at the origin. If $a \in \mathbb{C}$ is such that $|a| < 1$, then its polar with respect to the circle is

$$\hat{a} = \frac{R^2}{\bar{a}} = \frac{R^2 a}{|a|^2}$$

and so

$$\left| \frac{z-a}{z-\hat{a}} \right| = \frac{|a|}{R}$$

for all $z \in \mathbb{C}$ with $|z| = R$. Carathéodory and Fejér consider all functions f holomorphic in $\{z \in \mathbb{C} : |z| < R\}$ with $f(0) = A$, which vanish at points $a_1, \ldots, a_n$ inside the circle, and ask for which such function is $M = \max_{|z|<1} \left| f(z) \right|$ the smallest. They divide f by the product

$$Q(z) = \frac{z-a_1}{z-\hat{a}_1} \cdot \frac{z-a_2}{z-\hat{a}_2} \cdots \frac{z-a_n}{z-\hat{a}_n}$$

and obtain from the fact that $\left| Q(z) \right|$ is constant on $|z| = R$ that

$$M \geq \frac{|A| R^n}{|a_1|\,|a_2| \ldots |a_n|},$$

and that the only function for which M attains this lower bound is

$$\frac{A R^{2n}}{|a_1^2|\,|a_2^2| \ldots |a_n^2|} Q(z).$$

If we replace $\hat{a}_i$by $R^2/\bar{a}_i$ in $Q(z)$ and multiply it by $R^n / \prod_{j=1}^n |a_j|$, then we obtain the rational function

$$R^n \frac{z-a_1}{R^2 - \bar{a}_1 z} \cdot \frac{z-a_2}{R^2 - \bar{a}_2 z} \cdots \frac{z-a_n}{R^2 - \bar{a}_n z}$$

which has zeros at $a_1, a_2, \dots, a_n$ and absolute value 1 on $|z| = R$. It has entered complex analysis under the name of "Blaschke product" after Wilhelm Blaschke who considered it, however, only in 1915.

3. Polynomials

Edmund Landau considered in 1906 the trinomial equation

$$a_0 + a_1 z + a_n z^n = 0$$

and proved that it has a root in the disk

$$|z| \leq 2 \left| \frac{a_0}{a_1} \right| .$$

He also considered the quadrinomial equation

$$a_0 + a_1 z + a_m z^m + a_n z^n = 0$$

and proved that it has a root in the disk

$$|z| \leq 8 \left| \frac{a_0}{a_1} \right| . \tag{1}$$

The remarkable fact about these two estimates is that the bounds depend only on a_0, a_1 and not on m, n, a_m, a_n. Landau asked the question whether a similar result holds for a general polynomial equation of the form

$$a_0 + a_1 z + a_2 z^{n_2} + a_3 z^{n_3} + \cdots + a_k z^{n_k} = 0, \tag{2}$$

where $a_1 \neq 0$, $1 < n_2 < n_3 < \cdots < n_k$. Lipót Fejér gave an answer to this question (Comptes Rendus, Paris **145** (1907), 459–461, Math. Ann., **65** (1908), 413–423, Mat. Fiz. Lapok **17** (1908), 308–324; [40], Nos. 21, 23, 24). He starts out from a then almost forgotten theorem of C. F. Gauss (also called the Gauss–Lucas theorem) according to which if $f(z)$ is a polynomial, then each root of $f'(z) = 0$ is inside the convex hull of the roots of $f(z) = 0$ unless it coincides with a root of $f(z) = 0$. In the third article quoted he gives a proof of the theorem based on the fact that if

$$f(z) = (z - z_1)^{\alpha_1} (z - z_2)^{\alpha_2} \dots (z - z_n)^{\alpha_n},$$

where $z_1, z_2, \dots, z_n$ are the distinct zeros of $f(z)$, then

$$\frac{f'(z)}{f(z)} = \frac{\alpha_1}{z - z_1} + \frac{\alpha_2}{z - z_2} + \cdots + \frac{\alpha_n}{z - z_n}.$$

This implies that if $f'(\zeta) = 0$, then ζ is the center of gravity of positive masses placed at the points z_j, from where Gauss' theorem follows immediately. Alternatively one can say that a unit mass placed at ζ is in equilibrium under forces of attraction inversely proportional to the distance from z_j to ζ due to masses placed at the points z_j. The theorem of Gauss follows also from this interpretation.

Fejér uses a consequence of Gauss' theorem, namely that the largest (in absolute value, of course) root of $f(z) = 0$ is not smaller than the largest root of $f'(z) = 0$. His main theorem is the following:

The polynomial equation

$$a_0 + a_1 z^{n_1} + a_2 z^{n_2} + \cdots + a_k z^{n_k} = 0, \tag{3}$$

where $a_1 \neq 0$, $1 \leq n_1 < n_2 < \cdots < n_k$, has a root in the disk

$$|z| \leq \left(\frac{n_1 n_2 \dots n_k}{(n_2 - n_1)(n_3 - n_1) \dots (n_k - n_1)} \right)^{\frac{1}{n_1}} \left| \frac{a_0}{a_1} \right|^{\frac{1}{n_1}}. \tag{4}$$

So the disk does not depend on the coefficients $a_2, a_3, \dots a_k$.

A corollary of the theorem is that equation (3) has at least one solution in the disk

$$|z| \leq k \left| \frac{a_0}{a_1} \right|^{\frac{1}{n_1}}. \tag{5}$$

Here the right hand side is independent also of $n_2, n_3, \dots, n_k$. Another corollary states that (3) has at least one root in

$$|z| \leq k \max \left(\left| \frac{a_0}{a_1} \right|, 1 \right), \tag{6}$$

where now the bound does not even depend on n_1. Fejér points out that it is very easy to obtain an estimate where in (6) the "length" k is replaced by the possibly much larger degree $n = n_k$.

Fejér's results were generalized by Paul Montel and Mieczysław Biernacki to the case when not only the first two but the first p coefficients are fixed, see Pál Turán's note in [40], p. 333 and Chapter IV of Dieudonné's expository account [29] which discusses also the results concerning polynomials of several other Hungarian mathematicians: Elemér Bálint, Jenő Egerváry, Mihály Fekete, István Lipka, Gábor Szegő, Gyula Szőkefalvi Nagy. Thus Fekete improved the estimate (4) to

$$|z| \leq \min_{1\leq r\leq k-1} \left(\frac{n_{r+1}\dots n_k}{(n_{r+1}-n_r)\dots(n_k-n_r)}\right)^{\frac{1}{n_r}} \left|\frac{a_0}{a_1}\right|^{\frac{1}{n_r}}$$

and deduced from it the following result related to the theorem of J. H. Grace: If the coefficients of the polynomial (3) satisfy a linear relation

$$\lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k = 0,$$

where $\lambda_0 \neq 0$, then (3) has a root in the disk

$$|z| \leq 2k \max_{1\leq i\leq k} \left(\left|\frac{\lambda_i}{\lambda_0}\right|, 1\right).$$

From (5) with $n_1 = 1$ we see that the equation (2) originally considered by Landau has a root in the disk

$$|z| \leq k \left|\frac{a_0}{a_1}\right|, \tag{7}$$

thus in (1) the factor 8 can be replaced by 3. The factor k is sharp as the equation

$$a_0\left(1+\frac{a_1 z}{k a_0}\right)^k = a_0 + a_1 z + \cdots = 0$$

shows. Nevertheless Fejér proved (Jahresber. Deutsch. Math.-Verein., **24** (1917), 114–128; [40], No. 56) that a root of (2) can always be found in a region whose area is one-fourth of the area of the disk given by (7), namely the disk which has a diameter joining the points 0 and $-k\frac{a_0}{a_1}$ of $\mathbb{C}$.

Fejér obtains analogous results in the more general case of equation (3). Then a root can be found in each of n_1 disks contained in the disk (5). It follows that if $n_1 \geq 2$, then (3) has at least two distinct roots in (5). Using Fejér's method Fekete proved that for the equation

$$a_0 + a_1 x + \cdots + a_\nu x^\nu + b_1 x^{n_1} + b_2 x^{n_2} + \cdots + b_k x^{n_k} = 0, \tag{8}$$

where $a_\nu \neq 0$, $\nu < n_1 < n_2 < \cdots < n_k$, there exists a bound B, depending only on $a_0, a_1, \ldots, a_\nu$ and k, such that (8) has at least ν roots in the disk $|z| \leq B$.

A result, somewhat similar to Fejér's, where a subset can be excluded from a region in which the roots a priori lie, is due to István Lipka. It is well known that all the roots of

$$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0$$

lie in the disk with center 0 and radius ξ, where ξ is the unique positive root of

$$z^n - |a_1| z^{n-1} - |a_2| z^{n-2} - \cdots - |a_{n-1}| z - |a_n| = 0$$

([129], III. 17). The unique positive root η of

$$z^{n-1} - |a_1| z^{n-2} - |a_2| z^{n-3} - \cdots - |a_{n-1}| = 0$$

satisfies $\eta < \xi$. Lipka proves that if $a_n > 0$, then all the roots of $f(z) = 0$ lie in the cogwheel-shaped region obtained by excluding from the disk $|z| \leq \xi$ the annular sectors described, using the polar representation $z = \rho e^{i\varphi}$, by the inequalities

$$\eta < \rho \leq \xi, \quad \frac{4k-1}{2n} < \varphi < \frac{4k+1}{2n}, \quad k = 0, 1, \ldots, n-1.$$

A significant number of Lipka's publications deal with the rule of signs of Descartes.

The article (Math. Ann., **85** (1922), 41–48; [40], No. 60) of Fejér, written in honor of Hilbert's 60^{th} birthday, uses the observation that one can approach simultaneously all the trees of a forest only if one is outside the convex hull of the forest. Within the convex hull whenever one gets closer to some trees, the distance from others increases.

Let $n \geq 1$ be an integer, denote by $\mathcal{P}_n$ the set of all polynomials of the form

$$g(z) = z^n + c_1 z^{n-1} + \cdots + c_n$$

($c_j \in \mathbb{C}$), and let S be a closed set in $\mathbb{C}$ which has at least n points. Associate with each $g \in \mathcal{P}_n$ a number $A(g) \geq 0$ which is increasing in the following sense: if

$$\big|g(z)\big| < \big|h(z)\big| \quad \text{for} \quad h(z) \neq 0, \quad z \in S,$$

$$\big|g(z)\big| = \big|h(z)\big| \quad \text{for}\, h(z) = 0, \quad z \in S,$$

then $A(g) < A(h)$. Fejér's theorem states that if the polynomial $t \in \mathcal{P}_n$ is such that $A(t) \leq A(g)$ for all $g \in \mathcal{P}_n$, then the zeros of t lie in the convex hull of S.

Examples of functionals A are

$$M_\infty(g) = \max_{z \in S} \left| g(z) \right| \quad \text{and} \quad \int_S \left| g(x) \right|^p dz.$$

When S is the interval $[-1, 1]$, then the polynomial minimizing M_∞ is 2^{1-n} times the usual Čebishov polynomial $T_n(x) = \cos(n \arccos x)$, so Fejér's theorem yields the classical result that all the zeros are in $[-1, 1]$.

Proof. Write $t(z) = (z - z_1)(z - z_2) \dots (z - z_n)$, and assume that z_1 is not in the convex hull of S. There exists $\zeta \in \mathbb{C}$ such that $|s - \zeta| < |s - z_1|$ for every "tree" s in the "forest" S. Set $g(z) = (z - \zeta)(z - z_2) \dots (z - z_n)$. For every $s \in S$ such that $(s - z_2) \dots (s - z_n) \neq 0$ (and such exist by hypothesis) one has $\left| g(s) \right| < \left| t(s) \right|$, hence $A(g) < A(t)$, which contradicts the minimality of t. ■

Fejér's article inspired a large amount of research. When Neumann János (a.k.a. John von Neumann) was still a student in the Lutheran "Gymnasium" of Budapest, his parents engaged Fekete as a tutor. Not that Neumann needed a tutor, he has read on his own Carathéodory's "Reelle Funktionen" at the age of fifteen, but Fekete needed an income. For political reasons he was not only dismissed from his job as a secondary school teacher but his title of "Privatdozent" was taken from him and he was expelled from the Mathematical Society. It is difficult to imagine Fekete, the meekest of men, as a dangerous revolutionary.

Together they wrote Neumann's first paper (Jahresber. Deutsch. Math.-Verein., **31** (1922), 125–128; [118]) which apeared when he was nineteen years old. They prove Gauss' theorem using Fejér's result by considering a polynomial $p(z) = \frac{z^n}{n} + p_1 z^{n-1} + \cdots + p_n$, taking for S the set of its roots, and for the functional A the expression

$$A(g) = \max_{z \in S} \frac{\left| g(z) \right|}{\left| p'(z) \right|}.$$

They prove that the polynomial $g(z) \in \mathcal{P}_{n-1}$, for which $A(g)$ is the smallest, is $p'(z)$ which has thus its zeros in the convex hull of S.

Another celebrated theorem concerning the location of the zeros of the derivative of a polynomial was stated by J. L. W. V. Jensen ([129], III.35), and was first proved by Gyula Szőkefalvi Nagy (Jahresber. Deutsch. Math.-Verein., **27** (1918), 44–48): Let $f(z)$ be a polynomial with real coefficients so that its zeros are either real or occur in pairs of conjugate complex numbers. Then the zeros of $f'(z)$ are either real or are contained in the union of disks with a diameter that has two conjugate zeros of $f(z)$ as endpoints.

Fekete and von Neumann base a proof of Jensen's theorem on the following analogue of Fejér's principle: Let S be a closed subset of $\mathbb{C}$ which is symmetric to the real axis. If z is a point in $\mathbb{C}$ lying outside the union of the disks with diameters having $s \in S$ and $\bar{s} \in S$ as endpoints, then there exists a point $\zeta \in \mathbb{C}$ such that $|\zeta - s| \cdot |\zeta - \bar{s}| < |z - s| \cdot |z - \bar{s}|$ To prove this, they first consider the case when S consists of just two points $s \neq \bar{s}$. Let z be outside the disk D having the segment $[s, \bar{s}]$ as diameter. If z is real, then the assertion is obvious. If z is complex, then $\bar{z}$ is also outside D, and so are some points ζ of the open segment $]z, \bar{z}[$. The required inequality follows from the area formula $A = \frac{1}{2}ab\sin\gamma$ we learned in trigonometry. The proof is similar when S consists of a finite number of pairs of points, and the general case follows by a continuity argument.

Towards the end of his life Fekete returned to the principles of Fejér and of Fekete–von Neumann alone (Proc. Nat. Acad. Sci. U.S.A., **37** (1951), 95–103, Bull. Res. Council, Israel **5A** (1955), 11–19) and with Joseph L. Walsh (J. Analyse Math., **4** (1954/55), 49–87, J. Analyse Math., **5** (1956), 47–76, Pacific J. Math., **7** (1956), 1037–1064; [194], 163, 178, 181). Then Walsh continued alone ([194], 145, 172, 187, 188, 201, 202), and in collaboration with T. S. Motzkin ([194], 151, 166, 174, 179, 189, 193), M. Zedek ([194], 170) and Oved Shisha ([194], 209, 222, 226, 227).

In the case when the functional A is a weighted maximum

$$A(g) = \max_{z \in S} w(z)\big|g(z)\big|,$$

where $w > 0$ is continuous, Gyula Szőkefalvi Nagy brings precisions to the theorems of Fejér and Fekete–von Neumann (Acta Sci. Math. Szeged **6** (1932), 49–58). Let $g^*(z) \in \mathcal{P}_n$ be the polynomial which minimizes $A(g)$ and let $S^* \subset S$ be the set of points where $w(z)|g^*|$ achieves its maximum. Then:

a) the zeros of g^* lie in the convex hull of S^*;

b) if S is a subset of the real axis, then S^* contains $n+1$ points which separate the n real zeros of $g^*(z)$;

c) if S is symmetric with respect to the real axis and the coefficients of g^* are real, then the complex zeros of g^* lie in the union of disks with diameters that have two conjugate points of S^* as endpoints.

A theorem of Edmond Laguerre states that if $f(z)$ is a polynomial of degree n with real coefficients and real zeros, and we divide the interval between two consecutive zeros into n equal parts, then $f'(z)$ has no zero in the two extreme subintervals. Paul Montel made this result more precise by proving that if the coefficients of $f(z)$ are real, z_k and z_{k+1} are real zeros of $f(z)$, and the real part of no other zero lies between z_k and z_{k+1}, then $f'(z)$ has no zero in the intervals $(z_k, z_k + \delta)$ and $(z_{k+1} - \delta, z_{k+1})$, where $\delta = (z_{k+1} - z_k)/n$. Gyula Szőkefalvi Nagy (Math. Természettud. Értesítő **53** (1935), 781–792, Acta Sci. Math. Szeged **8** (1936), 42–52) strengthened this result by showing that the same conclusion holds if $f(z)$ has no roots in the disk with diameter $[z_k, z_{k+1}]$, i.e. he replaced an infinite strip by a bounded disk.

In a similar vein Pál Turán (Acta Sci. Math. Szeged **11** (1946/48), 106–113; [184], No. 27) considers a polynomial $f(z)$ of even degree n with real coefficients such $f(-1) = f(1) = 0$. He assumes that $f(z)$ has no zeros in the open interval $]-1, +1[$ but makes no other assumptions on the zeros of $f(z)$. If ξ is the point where $\left|f(z)\right|$ achieves its maximum in $[-1, 1]$, then he proves that $|\xi| \leq \cos\frac{\pi}{n}$ and this bound is sharp. A more complicated estimate holds if n is odd.

Gyula Szőkefalvi Nagy (Tôhoku Math. J., **35** (1932), 126–135) finds still other regions which do not contain zeros of the derivative. Let $z_1 \in \mathbb{C}$ be a zero with multiplicity p of the polynomial $f(z)$. Assume that on either side of any straight line going through z_1 there lie at most s zeros of $f(z)$. Denote by D a closed disk with radius R, having z_1 on its boundary and containing no other zero of $f(z)$. Then $f'(z)$ has no zero in the disk D_1 touching D from the inside at z_1 and having radius $\frac{p}{p+s}R$. Gyula Szőkefalvi Nagy was fundamentally a geometer and the proofs of his theorems about polynomials have a strong geometric flavor: they belong to the "Geometry of Polynomials", the title of the book by Morris Marden [113] which discusses many of Gyula Szőkefalvi Nagy's results and lists 24 of his articles in the bibliography. Marden presents the results concerning polynomials also of other Hungarian authors: Elemér Bálint, Jenő Egerváry, Pál Erdős, Tibor Faragó, Lipót Fejér, Mihály Fekete, Péter Lax, István Lipka, Endre Makai, János Neumann, György Pólya, Tibor Radó, Ottó Szász, Pál Turán and István Vincze.

The theorems of Gauss–Lucas and Jensen are distant relatives of Michel Rolle's theorem which is also a statement concerning the position of a zero of the derivative with respect to the zeros of the function. Another basic fact of real analysis is the theorem of Bernhard Bolzano or the intermediate value theorem which does not hold in the complex case: $e^{i\pi} = -1$, $e^0 = 1$ but for no z in $\mathbb{C}$ is e^z equal to 0. For polynomials the situation is more favorable. Fekete (Jahresber. Deutsch. Math.-Verein., **34** (1926), 222–233) proved that if $f(z)$ is a polynomial of degree n and $f(z_1) = w_1 \neq f(z_2) = w_2$, then every value in the set of those points from which the segment $[w_1, w_2]$ can be seen under an angle $\geq \varphi$ is assumed by $f(z)$ in the set of those points from which $[z_1, z_2]$ can be seen under an angle $\geq \varphi/n$.

Elemér Bálint (Jahresber. Deutsch. Math.-Verein., **34** (1926), 233–237) proved that if $f(z)$ is a lacunary polynomial of the form (3) having "length" k, then every value in the disk with diameter having endpoints $w_1 = f(z_1)$ and $w_2 = f(z_2)$ is assumed by $f(z)$ in a disk with center $\frac{1}{2}(z_1 + z_2)$ and having radius $\leq Ck$, where C is an absolute constant.

Assume that the polynomial $f(z)$ of degree n assumes the values $w_1, \dots, w_n$ at the points $z_1, \dots, z_n$. Let P be the convex hull of the points z_j and Π the convex hull of the points w_j. Finally, let D be the smallest disk such that from every point of the boundary of D the polygon P can be seen under and angle $\leq \frac{\pi}{n}$. Gyula Szőkefalvi Nagy (Jahresber. Deutsch. Math.-Verein., **32** (1923), 307–309) proved that $f(z)$ assumes in D every value lying in Π.

Let me recall that the elementary symmetric functions $S_k = S_k(x_1, \dots, x_n)$, $0 \leq k \leq n$, are the polynomials in the n indeterminates $x_1, \dots, x_n$ defined by

$$(x - x_1) \dots (x - x_n) = S_0 x^n - S_1 x^{n-1} + \dots + (-1)^k S_k x^{n-k} + \dots + (-1)^n S_n,$$

i.e. $S_0 = 1$, $S_1 = x_1 + \dots + x_n$, $S_2 = x_1 x_2 + x_1 x_3 + \dots + x_{n-1} x_n$, ..., $S_n = x_1 x_2 \cdots x_n$.

Now let $x_1, \dots, x_n$ and $y_1, \dots, y_n$ be two n-tuples of indeterminates, set $s_k = S_k(x_1, \dots, x_n)$, $\sigma_k = S_k(y_1, \dots, y_n)$ and introduce the polynomial

$$G = \sum_{k=0}^{n} (-1)^k \frac{1}{\binom{n}{k}} \sigma_{n-k} s_k$$

in $2n$ indeterminates. The following result is due to J. H. Grace ([29], Th. VII, p. 11):

Let the complex numbers x_j and y_j $(1 \le j \le n)$ satisfy

$$G(x_1, \dots, x_n, y_1, \dots, y_n) = 0.$$

If all the y_j lie in the circular domain D, then at least one x_j lies in D.

This theorem and related results have been investigated in great detail by Gábor Szegő (Math. Z., **13** (1922), 28–55; [173], I, pp. 505–534). With Grace he considers polynomials of degree n having the form

$$A(x) = a_0 + \binom{n}{1} a_1 x + \cdots + \binom{n}{k} a_k x^k + \cdots + a_n x^n$$

and denotes the solutions of $A(x) = 0$ by $\alpha_1, \dots, \alpha_n$. Substituting the roots α_j for the indeterminates y_j and using that

$$(-1)^{n-k} a_n \sigma_{n-k} = \binom{n}{k} a_k,$$

the equation $G = 0$ becomes Szegő's "Faltungsgleichung"

$$A^\sharp(x_1, \dots, x_n) = a_0 s_0 + a_1 s_1 + \cdots + a_n s_n = 0, \tag{9}$$

where $s_k = S_k(x_1, \dots, x_n)$. Observe that $A^\sharp(x, \dots, x) = A(x)$. The above theorem then yields the form of Grace's theorem which Szegő calls the "Faltungssatz" and for which he gave a simple new proof ([173], I, p. 509):

Assume that the α_j all lie in the circular domain D and that $x_1, \dots, x_n$ satisfy (9). Then at least one of the x_j lies in D.

This result has several consequences, among then the usual formulation of the theorem of Grace: Consider a second polynomial

$$B(x) = b_0 + \binom{n}{1} b_1 x \cdots + \binom{n}{k} b_k x^k + \cdots + b_n x^n$$

of the same form, and denote its zeros by $\beta_1, \dots, \beta_n$. Replacing the variables x_j by the zeros β_j, the "Faltungsgleichung" becomes the condition

$$a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \cdots + (-1)^k \binom{n}{k} a_k b_{n-k} + \cdots + (-1)^n a_n b_0 = 0$$

expressing the fact that the two polynomials are "apolar". The assertion is that if $A(x)$ and $B(x)$ are apolar, and if all the zeros of $A(x)$ lie in the circular domain D, then at least one of the zeros $B(x)$ lies in D ([129], V. 145).

Another result is the following: Assume $|\alpha_j| \leq 1$ and $|\beta_j| < 1$ for $1 \leq j \leq n$, and let

$$C(x) = \sum_{k=0}^{n} \binom{n}{k} a_k b_k x^k$$

be the polynomial obtained by 'composition". Then every solution of $C(x) = 0$ lies in the interior of the unit disk. This result was also found by Jenő Egerváry (Acta Sci. Math. Szeged, **1** (1922), 39–45).

Let me quote one more theorem. Assume that all the roots α_j of $A(x)$ lie in the circular domain D. Then every root γ of $C(x)$ has the form $\gamma = -\delta\beta_j$, where $\delta \in D$ and β_j is one of the roots of $B(x)$.

For further discussion I refer to Chapter II, Nos. 2 and 3, pp. 11–14 of [29]. The composition polynomial was studied by N. G. de Bruijn and T. A. Springer, and Szegő's method was used by Lars Hörmander to extend Grace's theorem to polynomials of several variables over an algebraically closed field, see Chapters III and IV of [113].

4. Trigonometric polynomials, Toeplitz forms and a problem of Carathéodory

The names of Lipót Fejér and Frigyes Riesz are attached jointly to a theorem, to an inequality, and to a procedure.

The Fejér–Riesz theorem is about trigonometric polynomials, i. e., expressions of the form

$$\tau(t) = \sum_{k=0}^{n} (a_k \cos kt + b_k \sin kt)$$

($b_0 = 0$), or in the nowadays more popular complex writing

$$\tau(t) = \sum_{k=-n}^{n} c_k e^{ikt},$$

where n is the "order" of $\tau(t)$. We have $a_0 = c_0$, $a_k = c_k + c_{-k}$ and $b_k = i(c_k - c_{-k})$ for $k \geq 1$ so that $2c_{-k} = a_k + ib_k$, $2c_k = a_k - ib_k$. We shall assume that the coefficients a_k and b_k are real, i.e. that $c_{-k} = \bar{c}_k$ for $k \geq 1$ and in this case we can also write

$$\tau(t) = c_0 + 2\Re e \sum_{k=1}^{n} c_k e^{ikt}.$$

The theorem asserts that if $\tau(t) \geq 0$ for all $t \in \mathbb{R}$, then there exists a rational polynomial

$$g(z) = \gamma_0 + \gamma_1 z + \cdots + \gamma_n z^n$$

of degree n such that $\tau(t)$ equals $\left|g(z)\right|^2$ if we substitute $z = e^{it}$ ([129], VI.40). Fejér says that he conjectured the result for some time, presumably on the basis of his kernel

$$(10)\quad \frac{1}{2}(n+1) + n\cos t + (n-1)\cos 2t + \cdots + \cos nt = \frac{1}{2}\left(\frac{\sin(n+1)\frac{t}{2}}{\sin\frac{t}{2}}\right)^2$$

$$= \frac{1}{2}|1 + z + z^2 + \cdots z^n|^2,$$

but the proof was communicated to him orally by F. Riesz (J. Reine Angew. Math., **146** (1916), 53–82; [40], No. 51, vol. I). Riesz in an article printed immediately after Fejér's (pp. 83–87; [156], I4), and in which he applies the theorem, only says that the theorem was proved in Fejér's article. It seems that the two could not agree to write a joint paper on the subject. Curiously, the proof employs the geometric theorem on which the Carathédory–Fejér result of Section 2 is based.

The Fejér–Riesz theorem can be considered as a parametrization of the set of positive trigonometric polynomials with real coefficients a_k, b_k: to each $\tau(t)$ there correspond $n+1$ complex parameters (or $2n+2$ real parameters if we separate the real and imaginary parts of the γ_k). It has been extended to entire functions of exponential type which are in a sense generalizations of trigonometric polynomials (see [15], 7.5.1, p. 125, where a wrong reference is given).

As a first application, consider the set of all positive trigonometric polynomials with constant term $a_0 = c_0 = 1$. Since $c_k = \frac{1}{2\pi}\int_0^{2\pi} \tau(t)e^{-ikt}\,dt$, we have $|c_k| \leq \frac{1}{2\pi}\int_0^{2\pi} \tau(t)\,dt = c_0 = 1$ and so trivially $\tau(0) \leq 2n+1$. The parametrization $\tau(t) = \left|g(e^{it})\right|^2$ permits to find exactly the maximum of

$\tau(0)$. Indeed, $c_0 = \sum_{k=0}^{n} |\gamma_k|^2$ and $\tau(0) = |\gamma_0 + \gamma_1 + \cdots + \gamma_n|^2$. If we write $\gamma_k = \alpha_k + i\beta_k$, we have to determine the maximum of

$$\left(\sum_{k=0}^{n} \alpha_k\right)^2 + \left(\sum_{k=0}^{n} \beta_k\right)^2$$

under the constraint $\sum_{k=0}^{n} \alpha_k^2 + \sum_{k=0}^{n} \beta_k^2 = 1$. The usual methods of calculus yield that the maximum is achieved when

$$\alpha_0 = \alpha_1 = \cdots = \alpha_n, \quad \beta_0 = \beta_1 = \cdots = \beta_n.$$

In this case $\alpha_0^2 + \beta_0^2 = \gamma_0^2 = \frac{1}{n+1}$. Thus $\tau(0) \le n+1$ and $\tau(0) = n+1$ only for $\frac{2}{n+1}$ times the trigonometric polynomial (10). Considering $\tau(t + t_0)$ we obtain that $\tau(t) \le n+1$ for $t \in \mathbb{R}$ if τ is a positive trigonometric polynomial of order n having constant term 1.

The ideas of Fejér's paper were taken up by Gábor Szegő in an article (Math. Ann., **79** (1918), 323–339; [173], I, pp. 153–170) where orthonormal systems of polynomials on the unit circle were first considered. They were to play a central role in Szegő's research as József Szabados tells us elsewhere in this volume. For $z = e^{i\theta}$, $0 < r < 1$, let

$$p(\theta, r) = \frac{1 - r^2}{|z - r|^2} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Denote by $\tau(\theta)$ a positive trigonometric polynomial of order n which satisfies

$$\frac{1}{2\pi}\int_0^{2\pi} p(\theta, r)\tau(\theta)\, d\theta = 1.$$

Szegő proves that then $p(\theta, r)\tau(\theta) \le n + 1 + \frac{1+r}{1-r}$, and the upper bound is achieved for

$$\tau(\theta) = \frac{1+r}{n+1-r(n-1)}\left|1 + \frac{z-r}{1+r}\frac{z^n - 1}{z-1}\right|^2$$

when $\theta = 0$. If $r \to 0+$ we obtain the result of Fejér proved above. The proof uses a parametrization

$$\tau(t) = \left|\gamma_0\varphi_0(z) + \cdots + \gamma_0\varphi_n(z)\right|^2, \quad z = e^{it},$$

where $\big(\varphi_k(z)\big)$ is a sequence of polynomials orthonormal on $|z| = 1$ with respect to the weight $p(\theta, r)$.

Returning to Fejér's article, let $\tau(t)$ be a trigonometric polynomial of order n whose constant term is zero. Let $-m$ be the minimum and M the maximum of $\tau(t)$. Then

$$\frac{M-\tau(t)}{M} \quad \text{and} \quad \frac{\tau(t)+m}{m}$$

are positive trigonometric polynomials with constant term 1, and so by the result proved above

$$\frac{M+m}{M} \le n+1 \quad \text{and} \quad \frac{M+m}{m} \le n+1,$$

i.e. $m \le nM$ and $M \le nm$. If now $\tau(t)$ is an arbitrary trigonometric polynomial of order n, we call the maximum of $\tau(t) - a_0$ the "height" H and the maximum of $a_0 - \tau(t)$ the "depth" h of $\tau(t)$, i.e. $-h$ is the minimum of $\tau(t) - a_0$. Observing that the constant term of $\tau(t) - a_0$ is zero, we obtain a famous result of Fejér (which can also be proved in other ways):

The height of a trigonometric polynomial of order n is at most n times its depth, and its depth is at most n times its height.

Ottó Szász continued Fejér's investigations in four articles, the last one written in collaboration with Jenő Egerváry (Sitzungsber. Bayer. Akad.-Wiss. Math.-Phys. Kl. 1917, 317–320, ibid. 1927, 185–196, Math. Z., **1** (1918), 149–162, ibid., **27** (1928), 641–652; [172], pp. 656–683, 734–757). Some of his results are discussed in the report written by Fejér (Mat. Fiz. Lapok **37** (1930), 63–90; [40], No. 74, vol. II) when Szász was awarded the Kőnig Gyula prize in 1930.

I reproduce some very elegant and elementary proofs of certain inequalities concerning the coefficients of a positive trigonometric polynomial $\tau(t)$. The equality

$$\tau(t) = c_0 + \sum_{k=1}^{n} (c_k e^{ikt} + \bar{c}_k e^{-ikt}) = \left|\gamma_0 + \gamma_1 e^{it} + \cdots + \gamma_n e^{int}\right|^2$$

yields that $c_k = \sum_{r=0}^{n-k} \gamma_{k+r}\bar{\gamma}_r$ for $k \ge 0$ (note: Szász denotes by c_k our $2c_k$). Assume now that $c_0 = a_0 = \sum_{r=0}^{n} |\gamma_r|^2 = 1$. Using the arithmetic mean-geometric mean inequality we have

$$|c_k| \le \sum_{r=0}^{n-k} |\gamma_{k+r}\bar{\gamma}_r| \le \frac{1}{2}\sum_{r=0}^{n-k} \left(|\gamma_{k+r}|^2 + |\gamma_r|^2\right)$$

$$= \frac{1}{2}\left(|\gamma_0|^2 + \cdots + |\gamma_{n-k}|^2 + |\gamma_k|^2 + \cdots + |\gamma_n|^2\right).$$

Since $n-k<k$ if $k>\left[\frac{n}{2}\right]$ (where $[x]$ denotes the largest integer $\leq x$), we obtain that the coefficients of a positive trigonometric polynomial with $a_0=1$ satisfy the inequalities

$$|a_k-ib_k|\leq 1 \quad \text{if}\, k>\left[\frac{n}{2}\right],$$

$$|a_k-ib_k|\leq 2 \quad \text{for} \quad 1\leq k\leq n.$$

These inequalities were proved by Fejér for cosine polynomials, i.e., when $b_k=0$ for all k.

Still under the assumption that $a_0=c_0=1$ we have, using the inequality

$$\left(\frac{x_0+x_1+\cdots+x_n}{n+1}\right)^2\leq\frac{x_0^2+x_1^2+\cdots x_n^2}{n+1},$$

that

$$\sum_{k=0}^{n}|c_k|\leq\sum_{s=0}^{n}|\gamma_s|^2+\sum_{k=1}^{n}\sum_{s=0}^{n-k}|\gamma_{k+s}\bar{\gamma}_s|$$

$$=\frac{1}{2}+\frac{1}{2}\left(\sum_{s=0}^{n}|\gamma_s|^2+2\sum_{k=1}^{n}\sum_{s=0}^{n-k}|\gamma_{k+s}\bar{\gamma}_s|\right)$$

$$=\frac{1}{2}+\frac{1}{2}\left(\sum_{s=0}^{n}|\gamma_s|\right)^2\leq\frac{1}{2}+\frac{n+1}{2}\sum_{s=0}^{n}|\gamma_s|^2=\frac{n+2}{2}$$

and so

$$\sum_{k=0}^{n}|a_k-ib_k|\leq n+1.$$

In his report Fejér lists some of the more profound results of Szász and Egerváry. In particular he mentions the inequalities

$$|a_k-ib_k|\leq 2\cos\frac{\pi}{\left[\frac{n}{k}\right]+2}\min(h,H) \quad (1\leq k\leq n)$$

valid for any trigonometric polynomial with real coefficients. If $\tau(t)\geq 0$ and $a_0=1$, then $h\leq 1$; if furthermore $k>\left[\frac{n}{2}\right]$ then $\left[\frac{n}{k}\right]=1$ and since $\cos\frac{\pi}{3}=\frac{1}{2}$ we get again $|a_k-ib_k|\leq 1$.

Before I can tell you to what use Frigyes Riesz put the Fejér–Riesz theorem, I must go back to 1911 and speak about a problem of Carathéodory (Rend. Circ. Mat. Palermo **32** (1911), 193–217; [21], Bd. III, No. LIII, pp. 78–110): Find a condition on the $2n$ real numbers

$$a_1, b_1, a_2, b_2, \dots, a_n, b_n \tag{11}$$

so that there should exist a power series

$$\frac{1}{2} + \sum_{k=1}^{\infty} (a_k + ib_k) z^k$$

which has the numbers (11) as initial coefficients, converges for $r = |z| < 1$, and whose real part

$$u(r, \theta) = \frac{1}{2} + \sum_{k=1}^{\infty} r^k (a_k \cos k\theta - b_k \sin k\theta)$$

is a positive harmonic function ($z = re^{i\theta}$). Carathéodory proved that for this to happen the following condition is necessary and sufficient:

(C) The point in $\mathbb{R}^{2n}$, whose coordinates are the numbers (11), lies in the closed convex hull K_{2n} of the curve with parametric representation

$$x_1 = \cos\theta, \quad y_1 = -\sin\theta, \quad x_2 = \cos 2\theta, \quad y_2 = -\sin 2\theta, \quad \dots,$$
$$x_n = \cos n\theta, \quad y_n = -\sin n\theta.$$

Frigyes Riesz in an article on integral equations (Ann. Sci. École Norm. Sup. Paris **28** (1911), 33–62; [156], F3, vol. II, pp. 788–827) proved that the following condition is also necessary and sufficient:

(R) For all $(\xi_1, \eta_1, \dots, \xi_n, \eta_n) \in \mathbb{R}^{2n}$ the value of $a_1\xi_1 + b_1\eta_1 + \cdots + a_n\xi_n + b_n\eta_n$ is between the minimum and the maximum value of the trigonometric polynomial

$$\tau(t) = \sum_{j=1}^{n} (\xi_j \cos jt + \eta_j \sin jt).$$

To state the condition due to Otto Toeplitz let us write $\gamma_0 = 1, \gamma_k = a_k + ib_k$ and $\gamma_{-k} = \bar{\gamma}_k$ for $1 \leq k \leq n$. The following is necessary and sufficient for the existence in Carathéodory's problem:

(T) The "Toeplitz form"

$$H(\zeta_0, \zeta_1, \dots, \zeta_n) = \sum_{j=0}^{n} \sum_{k=0}^{n} \gamma_{j-k} \zeta_j \bar{\zeta}_k$$

is positive semi-definite.

In his 1916 paper quoted at the beginning of this section F. Riesz observes that the three conditions are obviously equivalent since they are all necessary and sufficient for the same fact, nevertheless he proves their equivalence in a direct way. This seems to be the first time that Toeplitz forms make an appearance in Hungarian mathematics. They were to have a central importance in the research of Gábor Szegő as I will indicate briefly below.

Riesz first remarks (as he already did in his 1911 article) that the equivalence of (C) and (R) is simply a restatement of the fact that the closed convex hull of a set is the intersection of the closed half-spaces containing it.

To prove (R) ⇔ (T) he starts by saying that (R) is clearly equivalent to:

(R′) Whenever the real numbers $\xi_0, \xi_1, \eta_1, \dots, \xi_n, \eta_n$ are such that the trigonometric polynomial

$$\xi_0 + \sum_{j=1}^{n} (\xi_j \cos jt + \eta_j \sin jt) \tag{12}$$

is positive, it follows that the expression

$$\xi_0 + a_1\xi_1 + b_1\eta_1 + \cdots + a_n\xi_n + b_n\eta_n \tag{13}$$

is positive.

Now assume that (12) is positive. Then by the Fejér–Riesz theorem there exists a polynomial

$$g(z) = \zeta_0 + \zeta_1 z + \cdots + \zeta_n z^n$$

such that

$$\xi_0 + \sum_{j=1}^{n} (\xi_j \cos jt + \eta_j \sin jt) = \left| g(z) \right|^2 = \sum_{j=0}^{n} \sum_{k=0}^{n} \zeta_j \bar{\zeta}_k e^{i(j-k)t}. \tag{14}$$

Substituting a_j for $\cos jt$ and b_j for $\sin jt$, i.e. γ_j for e^{ijt} and $\bar{\gamma}_j = \gamma_{-j}$ for e^{-ijt} $(j \geq 0)$, the left-hand side becomes (13) and the right hand yields $H(\zeta_0, \zeta_1, \ldots, \zeta_n)$. Let us e.g., assume that (R′) holds. For any $\zeta_0, \zeta_1, \ldots, \zeta_n$ (14) yields a positive trigonometric polynomial. By assumption (13) is positive. But after substitution (13) equals $H(\zeta_0, \zeta_1, \ldots, \zeta_n)$ which is therefore ≥ 0. The proof of (T) $\Rightarrow$ (R′) is similar.

In a joint article of Fejér and Carathéodory (Rend. Circ. Mat. Palermo **32** (1911), 218–239; [40], No. 39, vol. I, pp. 693–7l5; [21], Bd. III, No. LIV, pp. 111–138), printed immediately following the above quoted paper of Carathéodory, the authors use the results of that paper to answer questions of the following type:

(a) Let the real numbers

$$a_0, a_1, b_1, a_2, b_2, \ldots, a_n, b_n \tag{15}$$

and a number $R > 0$ be given. Consider the set of all harmonic functions $u(r, \theta)$ given by the series

$$u(r, \theta) = a_0 + \sum_{k=1}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta) \tag{16}$$

with the given initial coefficients which converge for $r < R$. Find the maximum of the least upper bounds of the functions belonging to the set.

(b) Let the numbers (15) be given, assume that among those with subscripts $1 \leq j \leq n$ at least one is different from zero, and let $m^* < a_0$. Find the largest value of $R > 0$ such that there exists a series (16) with the given initial coefficients which converges for $r < R$ and whose greatest lower bound is $\geq m^*$.

(c) Let $c_0, c_1, \ldots, c_n$ be given complex numbers, $c_0 \neq 0, 1$ and not all c_j with $1 \leq j \leq n$ equal to zero. Find the largest number $R > 0$ such that a power series $\sum_{j=0}^{\infty} c_j z^j$ with given initial coefficients converges for $|z| < R$ and represents a function which does not assume the values 0 and 1.

The last question is related to Landau's sharpening of the Picard theorem: Given $c_0, c_1 \neq 0$, there exists $R = R(c_0, c_1) > 0$ such that every power series $\sum_0^{\infty} c_k z^k$ which converges for $|z| < R$ assumes either the value 0 or the value 1 in $|z| < R$ ([74], III. 7, §6, p. 448).

To answer e.g., the question (a), Carathéodory and Fejér introduce the Toeplitz matrices ($\gamma_k = a_k + ib_k$, $\gamma_{-k} = \bar{\gamma}_k$)

$$\begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ \gamma_{-1} & \gamma_0 & \cdots & \gamma_{k-1} \\ \cdot & \cdot & \cdots & \cdot \\ \gamma_{-k} & \gamma_{-k+1} & \cdots & \gamma_0 \end{pmatrix}$$

and their determinants $D_k(\gamma_0, \gamma_1, \ldots, \gamma_k)$. It was Toeplitz who pointed out to them that the equivalent conditions (C) and (R) are also equivalent to (T), i.e. to the inequalities $D_k(1, \gamma_1, \ldots, \gamma_k) \geq 0$ for $1 \leq k \leq n$. More precisely, $(a_1, b_1, \ldots, a_n, b_n) \in \mathbb{R}^{2n}$ lies in the interior of K_{2n} if $D_k(1, \gamma_1, \ldots, \gamma_k) > 0$ for $1 \leq k \leq n$, and it lies on the boundary of K_{2n} if $D_k(1, \gamma_1, \ldots, \gamma_k) \geq 0$ for $1 \leq k \leq n-1$ and $D_n(1, \gamma_1, \ldots, \gamma_n) = 0$ With the help of this formulation a geometric argument gives the

Theorem. *Assume that the series* (16) *converges in* $r < 1$ *and has as initial coefficients the given numbers* (15). *The equation*

$$D_n\big(2(a_0 - \lambda), \gamma_1, \ldots, \gamma_n\big) = 0$$

in the unknown λ *has only real solutions. Let* λ_{n*} *be the smallest among the solutions and* λ_n^* *the largest one. Then* $\inf_{r<1} u(r, \theta) \leq \lambda_{n*}$ *and* $\sup_{r<1} u(r, \theta) \geq \lambda_n^*$. *There exists a function in the set of functions* (16) *considered whose greatest lower bound equals* λ_{n*} *and one whose least upper bound equals* λ_n^*.

Gábor Szegő wrote two articles in Hungarian on Toeplitz and Hankel forms in 1917–1918 (Math. Természettud. Értesítő **35** (1917), 185–122, **36** (1918), 497–538; [173], I, pp. 69–108, 113–148). The second of these articles appeared in English in the Translations of the American Mathematical Society ((2) **108** (1977), 1–36), and it is this translation which is reproduced in the Collected Papers. Most of the results of the two articles are treated again in his great 1920–21 paper published in two parts (Math. Z., **6** (1920), 167–202, **9** (1921), 167–190; [173], I, pp. 237–272, 279–302). He returned to the subject in 1952 (Comm. Sem. Math. Univ. Lund, Tome Suppl., Festskrift Marcel Riesz (1952), 228–238; [173], III, pp. 270–280), in a joint paper with Mark Kac and W. L. Murdock of 1953 (J. Rational Mech. Anal., **2** (1953), 767–800, **3**, 802–803; [173], III, pp. 333–367), and in his 1958 book with Ulf Grenander [59].

Let f be a positive integrable function in $[0, 2\pi]$ and assume that its arithmetic mean

$$A(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt$$

satisfies $A(f) > 0$. Denote by $\mu_n(f)$ the greatest lower bound of all expressions

$$\frac{1}{2\pi} \int_0^{2\pi} f(t) \big| P_n(e^{it}) \big|^2 \, dt, \tag{17}$$

where $P_n(z)$ varies in the set of all polynomials of the form

$$P_n(z) = 1 + \zeta_1 z + \zeta_2 z^2 + \cdots + \zeta_n z^n.$$

If we introduce the Fourier coefficients

$$c_j = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{ijt} \, dt,$$

the expression (17) becomes the Toeplitz form

$$\sum_{j=0}^{n} \sum_{k=0}^{n} c_{j-k} \zeta_j \bar{\zeta}_k, \tag{18}$$

where $\zeta_0 = 1$. As we have done above, we introduce the Toeplitz matrices

$$M_n(f) = \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_0 & \cdots & c_{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ c_{-n} & c_{-n+1} & \cdots & c_0 \end{pmatrix}$$

and their determinants $D_n(f) = \det M_n(f)$. By assumption $D_0(f) = c_0 = A(f) > 0$, and $D_n(f) > 0$ for $n \geq 1$ since (18) is positive definite by its definition. A well-known theorem on Hermitian forms implies that

$$\mu_n(f) = \inf_{\zeta_0 = 1} \sum_{j=0}^{n} \sum_{k=0}^{n} c_{j-k} \zeta_j \bar{\zeta}_k = \frac{D_n(f)}{D_{n-1}(f)}.$$

Since every polynomial $P_n(z)$ is a particular polynomial $P_{n-1}(z)$, the sequence $(\mu_n(f))$ of positive numbers is decreasing and therefore

$$\mu(f) = \lim_{n \to \infty} \mu_n(f)$$

exists.

Let the real numbers $\lambda_0^{(n)}, \lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ be the eigenvalues of the matrix $M_n(f)$. Then $D_n(f) = \lambda_0^{(n)}\lambda_1^{(n)} \dots \lambda_n^{(n)}$ and Szegő proves that

$$\mu(f) = \lim_{n\to\infty} \frac{D_n(f)}{D_{n-1}(f)} = \lim_{n\to\infty} \sqrt[n+1]{D_n(f)}$$

is equal to the geometric mean of f, i.e.

$$\lim_{n\to\infty} \sqrt[n+1]{\lambda_0^{(n)}\lambda_1^{(n)} \dots \lambda_n^{(n)}} = e^{\frac{1}{2\pi}\int_0^{2\pi} \log f(t)\, dt}$$

if $\log f$ is integrable. From this special case he deduces a general formula which shows that the eigenvalues behave like values of f at equidistant points, namely if F is, say, continuous or has only finitely many jump discontinuities, then

$$\lim_{n\to\infty} \frac{F\big(\lambda_0^{(n)}\big) + F\big(\lambda_1^{(n)}\big) + \dots + F\big(\lambda_n^{(n)}\big)}{n+1} = \frac{1}{2\pi}\int_0^{2\pi} F\big(f(t)\big)\, dt.$$

Later Szegő generalized his result to the case when in (17) the exponent 2 is replaced by $p > 0$. Andrei N. Kolmogorov and Mark G. Krein replaced $f(t)\, dt$ by a Stieltjes measure ([2], Appendix B).

The second part of the 1920–21 article is devoted to orthogonal polynomials, so I again refer to the corresponding Chapter.

5. The Fejér–Riesz inequality

The Fejér–Riesz inequality appeared in the only article the two authors published jointly (Math. Z., **11** (1921), 305–314; [40], No. 59, vol. II, pp. 111–120; [156], D5, vol. I, pp. 625–634).

The authors first observe that if f is a regular analytic function in the closed disk $|z| \le 1$, then it follows from Cauchy's integral theorem that the integral of $\big|f(z)\big|^2$ along $|z| = 1$ is at least twice as large as the integral along a diameter, i.e.,

(19) $$\int_{-1}^{1} \big|f(x)\big|^2\, dx \le \frac{1}{2}\int_0^{2\pi} \big|f(e^{i\theta})\big|^2\, d\theta.$$

Next they state that the inequality also holds for $|f|^p$ with any $p \geq 1$ instead of just $p = 2$, and prove it for $p = 1$. If f has no zeros in $|z| \leq 1$, the inequality follows from (19) applied to $\sqrt{f(z)}$. If $f(z)$ does have zeros, they asssume first that they lie all in the interior of the disk. Then there are only finitely many, say $a_1, \ldots, a_n$, and $f(z)$ can be divided by the "Blaschke product"

$$Q(z) = \frac{z - a_1}{1 - \bar{a}_1 z} \frac{z - a_2}{1 - \bar{a}_2 z} \cdots \frac{z - a_n}{1 - \bar{a}_n z}$$

mentioned in Section 3. Thus $f(z) = Q(z)g(z)$, where $\left|Q(z)\right| = 1$ for $|z| = 1$, hence $\left|Q(z)\right| \leq 1$ in the disk, and $g(z) \neq 0$ for $|z| \leq 1$. Finally they take care of the general case by considering circles $|z| = r < 1$ on which $f(z)$ has no zeros and letting $r \to 1$.

A first application is a simple proof of Hilbert's inequality

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_j a_k}{j+k} \leq \pi \sum_{j=1}^{\infty} |a_j|^2.$$

Applying the second inequality to $f'(z)$ one obtains

$$\int_{-1}^{1} \left| f'(z) \right| dz \leq \frac{1}{2} \int_0^{2\pi} \left| f'(e^{i\theta}) \right| d\theta$$

which has the following geometric interpretation: let $f(z)$ map $|z| < 1$ conformally onto a domain, and $|z| = 1$ onto its boundary which we assume to be a rectifiable Jordan curve Γ. Then the image of any diameter of $|z| \leq 1$ is at most half as long as Γ. Mapping the disk onto a very elongated ellipse shows that the factor $\frac{1}{2}$ is the best possible.

R. M. Gabriel proved a similar result: let L be a closed convex curve inside $|z| \leq 1$. Then

$$\int_L \left| f(z) \right| |dz| \leq 2 \int_0^{2\pi} \left| f(e^{i\theta}) \right| d\theta$$

and the constant 2 is the best possible. This does not contain the Fejér–Riesz inequality since a diameter counted twice in opposite directions can be considered a closed convex curve, and then the best factor is one. F. Carlson gave a common generalization to the two theorems: Let L be a rectifiable curve in $|z| \leq 1$ and denote by $V(z)$ the least upper bound of the sum of angles under which the line elements of L can be seen from the point z. Then

$$\int_L \left| f(z) \right| |dz| \leq \frac{1}{\pi} \int_0^{2\pi} \left| f(e^{i\theta}) \right| V(e^{i\theta})\, d\theta.$$

In particular, if $V(z) \leq M$ for all z with $|z| = 1$, then the right-hand side is $\leq \frac{M}{\pi} \int_0^{2\pi} \left| f(e^{i\theta}) \right| d\theta$. So if L is a diameter, then $M = \frac{\pi}{2}$, and if L is a closed convex curve in $|z| \leq 1$, then $M \leq 2\pi$.

Marcel Riesz in a note written in honor of his brother and Fejér on the occasion of their 70^{th} birthday (Acta Sci. Math. Szeged **12A** (1950), 53–56; [158], No. 50, pp. 794–797) proves Carlson's theorem using a double layer potential.

6. Boundary values, H^p spaces

The only article the Riesz brothers published jointly ($4^{\text{ème}}$ Congrès des Math. Scandinaves, Stockholm 1916 (1920), pp. 27–44; [156], D1, vol. I pp. 537–554; [158], No. 22, pp. 195–212) was inspired by the thesis of Pierre Fatou (Acta Math., **30** (1906), 335–400), the same memoir which gave Frigyes Riesz the "idea and the courage" to use the Lebesgue integral (Annales Inst. Fourier (Grenoble) **1** (1949), p. 29; [156], B16, p. 317). One of Fatou's main results was the following ([104], §5, pp. 35–42):

Let w be a function that is bounded and holomorphic in the unit disk $\mathbf{D} = \left\{ z \in \mathbb{C} : |z| < 1 \right\}$. Then for almost every point $e^{i\theta}$ of the circumference $\mathbf{T} = \left\{ z \in \mathbb{C} : |z| = 1 \right\}$ the limit $f(e^{i\theta}) = \lim_{r \to 1-} w(re^{i\theta})$ exists, i.e., those points $e^{i\theta}$ in which the limit does not exist form a set of Lebesgue measure zero on the circumference.

Fatou also proved that if w is not identically zero, then on any arc of $\mathbf{T}$ the set of those points $e^{i\theta}$ in which $f(e^{i\theta}) \neq 0$ has positive measure. He conjectured that the measure of the set on which the boundary value vanishes is zero but added that to prove this seems to be very difficult. Frigyes and Marcel Riesz say that Fatou underestimated the scope of his methods and present a proof of the conjecture "which is closely related to certain arguments of his work and is based directly on his results". Assume that $f(e^{i\theta}) = 0$ on a set $M \subset \mathbf{T}$ with measure $m > 0$. They construct a bounded holomorphic function $g(z)$ in $\mathbf{D}$ with $g(0) = 1$ such that $\left| g(z) \right|$ has boundary value $e^{A/m}$ on M and $e^{A/(m-2\pi)}$ on the complementary set

$M^c = \mathbf{T}\backslash M$, where A is a yet unspecified positive number. Then by Cauchy's theorem and by hypothesis

$$w(0) = \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{f(z)g(z)}{z}\,dz = \frac{1}{2\pi i}\int_{M^c} \frac{f(z)g(z)}{z}\,dz$$

and

$$\left|\int_{M^c} \frac{f(z)g(z)}{z}\,dz\right| \le e^{\frac{A}{m-2\pi}} \int_0^{2\pi} \left|f(e^{i\theta})\right| d\theta.$$

Let $A \to \infty$. Since $m - 2\pi < 0$ we get that $w(0) = 0$, i.e. $w(z)/z$ is holomorphic and bounded in $\mathbf{D}$. Repeating the argument, we see that $z^{-n}w(z) = 0$ at $z = 0$ for all $n \in \mathbb{N}$, hence w is identically zero.

Next, the Riesz brothers consider a holomorphic function F which maps the open unit disk $\mathbf{D} \subset \mathbb{C}$ onto a bounded domain Ω of a Riemann surface whose boundary Γ is a rectifiable curve. The mapping can be extended to yield a function $\mathbf{T} \to \Gamma$ which is of bounded variation because Γ is rectifiable. Let M be a closed subset of measure zero of $\mathbf{T}$. They construct a positive integrable function $\varphi(\theta)$ on $\mathbf{T}$ which has the value $+\infty$ on M and finite values on M^c. If u is a positive harmonic function in $\mathbf{D}$ with boundary values φ, and v is the harmonic function conjugate to u, set

$$g(z) = \frac{u + iv}{1 + u + iv}.$$

Then $\left|g(z)\right| < 1$ in $\mathbf{D}$, the boundary value of $\left|g(z)\right|$ is < 1 on M^c and $= 1$ on M, so

$$\lim_{n\to\infty} \int_{|z|=1} \big(g(z)\big)^n\, dF(z) = \int_M dF(z). \tag{20}$$

But

$$\int_{|z|=1} z^k\, dF(z) = -k\int_{|z|=1} z^{k-1}F(z)\,dz = 0$$

for $k > 0$ by Cauchy's theorem, and for $k = 0$ because of the factor k, so the integral on the left-hand side of (20) is zero. The integral on the right-hand side of (20) is the measure of $F(M)$, therefore the image under F of any subset of $\mathbf{T}$ of measure zero is a subset of Γ of measure zero. By a theorem credited to Stefan Banach ([71], (18.25), p. 288) a function of bounded variation is absolutely continuous if and only if it maps sets of measure zero into sets of measure zero, therefore F is absolutely continuous on $\mathbf{T}$.

The preceding proof shows the validity of the following result which is how the Theorem of F. and M. Riesz is most often quoted:

If F is a function of bounded variation on $\mathbf{T}$ and

$$\int_0^{2\pi} e^{in\theta}\, dF(e^{i\theta}) = 0$$

for $n = 1, 2, 3, \ldots$, then F is absolutely continuous.

Since functions of bounded variation correspond to signed measures, the theorem can also be stated in terms of measures. The work of the brothers Riesz was presented at the fourth Congress of Scandinavian Mathematicians in 1916 right in the middle of World War I. The Proceedings of the Congress were printed as late as 1920 and by error only in 50 copies. So the article of F. and M. Riesz, of which Paul Koosis says that every analyst should read it ([97], p. 40), was very difficult to find until a reprinted edition came out.

The Theorem of F. and M. Riesz spawned an enormous amount of research. It was generalized to several variables which was not straightforward because the obvious analogues are false. Abstract forms of the theorem became fundamental in the theory of function algebras and their generalizations. The output has not ceased and every year we see a number of articles about yet another generalization of the F. and M. Riesz Theorem.

Let me return to the article of Carathéodory and Fejér mentioned in Section 4. Let the $n + 1$ complex numbers

$$c_0, c_1, \ldots, c_n \tag{21}$$

be given. As above, consider the set $\mathcal{F}(\mathbf{c})$ of all functions $f(z)$ that are holomorphic in the closed unit disk $\mathbf{D} \cup \mathbf{T}$ and whose power series expansion $\sum_{k=0}^{\infty} c_k z^k$ starts with the coefficients (21). They prove that there exists a unique function $f_*(z)$ in $\mathcal{F}(\mathbf{c})$ for which $M[f] = \max_{|z| \le 1} \big| f(z) \big|$ is a minimum. This function $f_*(z)$ is determined by the following properties: it is meromorphic with at most n poles in the complex plane, has at most n zeros in $\mathbf{D}$, and on $\mathbf{T}$ its absolute value equals the constant $M[f_*]$.

H. T. Gronwall gave an elementary proof of this theorem which uses neither the geometry of convex bodies nor Toeplitz forms (Ann. of Math. (2) **16** (1914/15), 77–81).

Very soon after the lecture in Stockholm, Frigyes Riesz wrote a paper published in 1917 in Hungarian (Math. Természettud. Értesítő **35** (1917), 605–632; [156], D2, pp. 555–582) but only in 1919 in German (Acta Math.,

42 (1920), 145–171; [156], D3, pp. 583–609). In it he also considers the above class $\mathcal{F}(\mathbf{c})$ but asks for the function that minimizes

$$I[f] = \int_0^{2\pi} \left| f(e^{i\theta}) \right| d\theta.$$

If $F(z)$ is a primitive of $f(z)$, then the expression we want to minimize is

$$T[f] = \int_0^{2\pi} \left| F'(e^{i\theta}) \right| d\theta,$$

i.e. the arc length of the image of $\mathbf{T}$ under the map $F(z)$. He proves the following

Theorem. *There exists a unique function $f^*(z)$ in the set $\mathcal{F}(\mathbf{c})$ for which $I[f]$ is minimal. This minimizing function $f^*(z)$ is characterized by the following two properties:*

1) *it is a polynomial of degree at most $2n$,*

2) *its zeros can be organized in pairs such that either the two elements of the pair are equal and they lie outside $\mathbf{D}$, or the two are reflections of each other with respect to $\mathbf{T}$.*

To prove the existence of $f^*(z)$ the obvious idea is to take a sequence $f_k(z)$ in $\mathcal{F}(\mathbf{c})$ for which $I[f_k]$ converges to the greatest lower bound I^* of $I[f]$. However, in this way one cannot prove that the limit function is regular on the *closed* unit disk. Therefore Riesz considers a sequence $F_k(z)$ of primitives such that $T[F_k]$ converges to $T^* = \inf T[F]$. He proves directly that the limit $F^*(z)$ is absolutely continuous and its derivative is the required function $f^*(z)$. He remarks that the properties of $F^*(z)$ he just proved also follow from investigations he conducted with his brother Marcel Riesz (in the Hungarian version he adds: "a Privatdozent at the University of Stockholm") but those investigations reach much more deeply into the theory of Lebesgue integration then what is necessary for the special problem considered here.

Though Frigyes Riesz says that Gronwall's treatment of the Carathéodory–Fejér result is so elementary that it could hardly be simplified, at the end of the article he shows how their theorem follows from his.

The next step concerning the Fatou–F. Riesz–M. Riesz circle of ideas is quite spectacular and its consequences are felt to the present day. It was published in Hungarian as an exchange of letters between Gábor Szegő and

Frigyes Riesz (Math. Természettud. Értesítő **38** (1921), 113–127; [156], D4, pp. 610–624; [173], 21–7, I, pp. 421–435), followed by two papers in German, one by each author (Math. Z., **18** (1923), 117–124; [156], D7, pp. 645–653; Math. Ann., **84** (1921), 232–244; [173], 21–6, I, pp. 404–416). Szegő informed Riesz that using his result on Toeplitz forms and the geometric mean (see Section 4) he proved that if w is holomorphic in $\mathbf{D}$, not identically zero, and

$$\int_0^{2\pi} \left| w(re^{i\theta}) \right|^2 d\theta$$

is bounded as $r \to 1-$, then $f(e^{i\theta}) = \lim_{r\to 1-} w(e^{i\theta})$, which exists for almost every θ, is such that $\log \left| f(e^{i\theta}) \right|$ is integrable in $[0, 2\pi]$. Thus in particular $f(e^{i\theta})$ cannot be zero on a set of positive measure.

In his answer Riesz first gives a proof of Szegő's result using only the Jensen formula and no Toeplitz forms. Then he quotes a result of G. H. Hardy according to which if w is holomorphic in $\mathbf{D}$ and $p > 0$, then

$$\mathcal{M}_p[r; w] = \int_0^{2\pi} \left| w(re^{i\theta}) \right|^p d\theta$$

is an increasing function of r in $[0, 1)$, so it is either bounded or tends to $+\infty$ as $r \to 1-$. Hardy also proved that $\log \mathcal{M}_p[r; w]$ is a convex function of $\log r$, i.e., shares with $\mathcal{M}_\infty[r; w] = \max_\theta \left| w(re^{i\theta}) \right|$ the property expressed by the Hadamard three circles theorem ([104], Kap. 6, pp. 88–97). Riesz introduces the class $H^p(\mathbf{D})$ of holomorphic functions for which $\mathcal{M}_p[r; w]$ is bounded, and proves that every $w \in H^p(\mathbf{D})$ has a product decomposition $w(z) = g(z)h(z)$, where $g(z)$ also belongs to $H^p(\mathbf{D})$ and is nowhere zero, and $h(z)$ is the already mentioned Blaschke product which satisfies $\left| h(z) \right| = 1$ for almost every $|z| = 1$ (if w has infinitely many zeros in $\mathbf{D}$, then $h(z)$ is a convergent infinite product). Since $g(z)^{p/2}$ belongs to $H^2(\mathbf{D})$, it follows that if w belongs to $H^p(\mathbf{D})$ for some $p > 0$ and is not identically zero, then its boundary values $f(e^{i\theta})$ exist for almost every θ and $\log \left| f(e^{i\theta}) \right|$ is integrable. In particular, $f(e^{i\theta})$ can vanish at most on a set of measure zero.

The boundary values of functions $w \in H^p(\mathbf{D})$ form a function space $H^p(\mathbf{T})$ which for $p > 1$ is essentially $L^p(\mathbf{T})$ but for $p \leq 1$ has many important additional properties. Their generalizations to several variables have played a central role in harmonic analysis in the last fifty years (see [166], Chaps. III and IV).

Frederick Riesz was not lucky with the names of spaces. He discovered the L^p spaces and denoted them so in honor of Henri Lebesgue whose integral is essential for their definition; now everybody calls them Lebesgue spaces. In his great 1918 Acta Mathematica article he introduced complete normed spaces; the accepted terminology for them is now "Banach spaces." Then he introduced the spaces he called H^p in honor of G. H. Hardy who proved the theorem quoted above; now they are called Hardy spaces (though Ronald Coifman and Guido Weiss say: "...it could be argued fairly that the name 'Riesz' should be attached to these spaces", Bull. Amer. Math. Soc., **83** (1977), p. 570). On the other hand Bourbaki gave the name "espaces de Riesz" to lattice-ordered vector spaces but Riesz says: "...quelques auteurs français ... m'ont honoré en donnant mon nom à quelques notions (mais) je n'ai pas réussi à pénétrer suffisamment dans le langage et l'écriture créés par la société anonyme Bourbaki pour les comprendre entièrement" (Ann. Inst. Fourier (Grenoble), **1** (1949), p. 40; [156], B16, p. 338).

7. Kakeya's theorem, power series with monotone coefficients

The simplest result concerning monotone coefficients is the theorem of Eneström–Kakeya ([104], §2, p. 26; [129], II. 22):

If $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n > 0$, then no solution of

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0 \tag{22}$$

lies in the open disk $\mathbf{D} = \{ z \in \mathbb{C} : |z| < 1 \}$.

To prove it, one considers the expression

$$(1-z)P(z) = a_0 - (a_0 - a_1)z - (a_1 - a_2)z^2 - \cdots - (a_{n-1} - a_n)z^n - a_n z^{n+1}.$$

From here one concludes, as remarked by Simon Szidon (Sidon) (Acta Sci. Math. Szeged **9** (1938–1940), 244–246), as follows: introduce the positive numbers $p_k = a_k - a_{k+1}$ $(0 \leq k \leq n-1)$, $p_n = a_n$. Setting $z = 1$ we see that $a_0 = p_0 + p_1 + \cdots + p_n$, so (22) is equivalent to

$$\frac{p_0 z + p_1 z^2 + \cdots + p_n z^{n+1}}{p_0 + p_1 + \cdots + p_n} = 1.$$

The left hand side lies in the convex hull of the points z^k $(1 \leq k \leq n+1)$ and 1 does not lie in that convex hull if z, and therefore all z^k, lie in $\mathbf{D}$.

István Vincze (Mat. Lapok **10** (1959), 127–132) extended the Eneström–Kakeya theorem to complex coefficients as follows: Let the complex numbers $a_k = \alpha_k e^{i\theta} + \beta_k e^{i\varphi}$ satisfy $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n \geq 0$, $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n \geq 0$ and $|\theta - \varphi| < \pi$. Then no solution of (22) lies in the disk $|z| < |a_0|/(\alpha_0 + \beta_0)$.

If $\beta_0 = 0, \theta = 0$ we have of course $|a_0|/\alpha_0 = 1$. If $\theta = 0$, $\varphi = \frac{\pi}{2}$, so that α_k and β_k are the real and imaginary part of a_k, then

$$\frac{|a_0|}{\alpha_0 + \beta_0} \geq \frac{1}{\sqrt{2}},$$

so that (22) has no solution in the disk $|z| < 1/\sqrt{2}$.

Considering the expression $z^n P(\frac{1}{z})$ one can also state the Eneström–Kakeya theorem as follows: If $0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$, then all the solutions of (22) satisfy $|z| \leq 1$. If the inequalities between the a_k are strict, then the solutions lie in $|z| < 1$. Egerváry (Acta Sci. Math. Szeged **5** (1931), 78–82) proved the following generalization of the Eneström–Kakeya theorem: Let $a_k > 0$ for $0 < k < n$, let m be an integer satisfying $0 \leq m \leq n$, and assume that there exist numbers $R \geq r > 0$ such that

$$Rra_{k+1} - (R + r)a_k + a_{k-1} > 0 \tag{23}$$

for $k = 0, \ldots, m-1, m+1, \ldots, n$ $(a_{-1} = a_{n+1} = 0)$. Then m solutions of (22) lie in the open disk $|z| < r$ and $n - m$ solutions lie in the domain $|z| > R$. If for some values of k the inequality (23) holds with ≥ 0, then the solutions can also lie on the boundaries of the two regions.

If $a_0 > 0$ and $ra_{k+1} > a_k$ for $0 \leq k \leq n-1$ and some $r > 0$ then (23) is satisfied for a sufficiently large $R > r$ and for $0 \leq k \leq n-1$, so all solutions of (22) lie in the disk $|z| < r$; for $r = 1$ this is again the Eneström–Kakeya theorem.

Denote by $\Delta^1 a_k = a_k - a_{k+1}$ the first differences and by

$$\Delta^2 a_k = a_k - 2a_{k+1} + a_{k+2} = \Delta^1 a_k - \Delta^1 a_{k+1}$$

the second differences of the sequence $a_0, a_1, \ldots, a_n$. For $R = r = 1$ the inequality (23) with $\geq$ can be written $\Delta^2 a_k \geq 0$ and also

$$a_{k+1} \leq \frac{a_k + a_{k+2}}{2}.$$

The validity of the last inequality for $0 \leq k \leq n-1$ expresses the geometric fact that the graph in the rectangular coordinate system, given by the polygon whose sides are the segments joining a point (k, a_k) with $(k+1, a_{k+1})$, is convex. We then say simply thet the sequence $a_0, a_1, \dots, a_n, a_{n+1} = 0$ is convex.

Fejér (Jahresber. Deutsch. Math-Verein., **38** (1929), 231–238; [40], No. 70, vol. II, pp. 256–263) found a trigonometric analogue of the Eneström–Kakeya theorem. He considers a cosine polynomial

$$\tau(t) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kt$$

($a_n \neq 0$) and with the help of the formula

$$\cos t \cos kt = \frac{1}{2}\{ \cos (k-1)t + \cos (k+1)t\}$$

obtains the identity

$$(1 - \cos t)\tau(t) = \frac{1}{2}\Delta^1 a_0 - \frac{1}{2}\sum_{k=1}^{n} \Delta^2 a_{k-1} \cos kt - \frac{1}{2} a_n \cos (n+1)t.$$

If we assume that the sequence $a_0, a_1, \dots, a_n, a_{n+1} = 0, a_{n+2} = 0$ is convex, then it follows from this identity that $\tau(t) \geq 0$ for all t. It also follows that $a(0) > a_1 > \cdots > a_n > 0$ and that the cosine polynomal with a convex sequence of coefficients for which the first n coefficients are the smallest possible is a multiple of Fejér's signature polynomial (10), i.e.,

$$a_n \left\{ \frac{1}{2}(n+1) + n \cos t + (n-1) \cos 2t + \cdots + \cos nt \right\}.$$

Lipka (Acta Sci. Math. Szeged **9** (1938–1940), 69–77) observes first the following immediate consequence of Fejér's result: If the sequence $2a_0, a_1, \dots, a_n > 0, a_{n+1} = 0$ is convex, then equation (22) has no solution in $|z| < 1$. The convexity condition implies that $2a_0 \geq a_1$, while in the Eneström–Kakeya theorem $a_0 \geq a_1$ is required. Lipka proves that if $2a_0 < a_1$ but $a_1, a_2, \dots, a_n, a_{n+1}$ remains convex, then $P(z)$ has exactly one zero in $|z| < 1$. This follows from:

If each coefficient $a_1, a_3, \dots, a_n$ and at least one of a_0, a_2 is > 0, and the sequence $2a_1, (a_0 + a_2), a_3, \dots, a_n, a_{n+1} = 0$ is convex in such a way that no three vertices (k, a_k) are collinear, then $P(z) = 0$ has exactly one solution in $|z| < 1$.

Szegő (Trans. Amer. Math. Soc., **39** (1936), 1–17; [173], II, pp. 593–609) has some results of Eneström–Kakeya type concerning trigonometric polynomials. Consider for instance a cosine polinomial

$$\tau(t) = a_0 \cos nt + a_1 \cos (n+1)t + \cdots + a_{n-1} \cos t + a_n.$$

(i) If $a_0 > a_1 \geq \cdots \geq a_n \geq 0$, then $\tau(t)$ has only simple zeros in $-\pi < t < 0$. The positive zeros $t_1, t_2, \ldots, t_n$ satisfy the inequalities

$$\frac{2\nu - 1}{2n+1}\pi < t_\nu < \frac{2\nu + 1}{2n+1}\pi, \quad 1 \leq \nu \leq n.$$

(ii) If

$$2a_0 - a_1 > a_1 - a_2 \geq a_2 - a_3 \geq \cdots \geq a_{n-1} - a_n \geq a_n \geq 0$$

(this condition is satisfied whenever $a_0, a_1, \ldots, a_n, 0, 0$ is convex and not identically zero), then $\tau(t)$ has again n positive roots and this time they satisfy the stronger inequalities

$$\frac{2\nu - 1}{2n+1}\pi < t_\nu < \frac{\nu}{n}\pi, \quad 1 \leq \nu \leq n.$$

Szegő's results are the discrete analogues of theorems connected with a question investigated by Pólya, which we will discuss a little later.

Consider now an infinite sequence

$$c_0, c_1, \ldots, c_n, \ldots .$$

generalizing the notation introduced above, set $\Delta^0 c_n = c_n$, i.e, let the sequence of differences of order zero be the original sequence itself, and for $k \geq 1$ define inductively the sequence of differences of order k by $\Delta^k c_n = \Delta^{k-1} c_n - \Delta^{k-1} c_{n+1}$. Explicitly we have

$$\Delta^k c_n = \sum_{l=0}^{n} (-1)^l \binom{k}{l} c_{n+l}.$$

The sequence (c_n) is said to be monotone of order $k \in \mathbb{N}$ if $\Delta^l c_n \geq 0$ for $0 \leq l \leq k$ and all $n \in \mathbb{N}$.

Inspired probably by the Eneström–Kakeya theorem, Fejér considered in a number of publications during the 1930's (Z. Angew. Math. Mech., **13**

(1933), 80–88; [40], No. 81, vol. II, 479–492; Trans. Amer. Math. Soc., **39** (1936), 18–59; [40], No. 89, vol. II, 581–620; Proc. Cambridge Philos. Society **31** (1935), 307–316; [40], No. 90, vol. II, 621–631; Math. Természettud. Értesítő **54** (1936), 160–176; [40], No. 92, vol. II, 640–662; Acta Sci. Math. Szeged **8** (1936) 89–115; [40], No. 94, vol. II, 679–701; Math. Természettud. Értesítő **55** (1936), 1–27; [40], No. 95, vol. II, 702–725) also in collaboration with Szegő (Prace Matematyczno Fizyczne **44** (1935), 15–25; [40], No. 91, vol. II, 631–639; [173], No. 35–3, vol. II, 579–586) power series and trigonometric series whose coefficients form a sequence which are monotone of a certain order. Let me list some of their results concerning power series

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots$$

convergent in $|z| < 1$.

Fejér proved in the second paper quoted above that if the sequence $c_1, c_2, \ldots$ is monotone of order four then $f(z)$ is univalent ("schlicht") in $|z| < 1$. Szegő (Duke Math. J., **8** (1941), 559–564; [173], No. 41–2, vol. II, 797–802) improved this result by showing that the conclusion already holds if $c_1, c_2, \ldots$ is only assumed to be monotone of order three.

The example $1 + z + z^2 + \cdots + z^n + 0 + 0 + \cdots$ shows that the conclusion does not hold if the sequence of coefficients is monotone of order one. Both Szegő and Szidon (loc. cit.) gave examples of power series whose coefficients form a sequence monotone of order two and such that the map effectuated by the sum $f(z)$ is not univalent.

Theorem A. *If (c_n) is monotone of order two, then one has the chain of inequalities*

(24)
$$\left|f(z)\right| \geq \frac{\left|f(z) - s_0(z)\right|}{|z|} \geq \frac{\left|f(z) - s_1(z)\right|}{|z|^2} \geq \cdots \geq \frac{\left|f(z) - s_n(z)\right|}{|z|^{n+1}} \geq \cdots$$

for $|z| < 1$, where $s_n(z) = c_0 + c_1 z + \cdots + c_n z^n$ is the n^{th} partial sum of the power series.

Observe that we have then $|s_n| \leq 2\left|f(z)\right|$ for all n. This is remarkable because Fejér has earlier given an example of a power series for which $\left|f(z)\right| < 1$ in $|z| < 1$ but $\left|s_n(z)\right|$ is unbounded (Sitzungsber. München **40** (1940), 1–17; [40], No. 32, Vol. I, 573–583; [104], §3, p. 29). Szegő (Math. Z., **25** (1926), 172–187; [173], No. 26–3, I, 758–773) proved that if $c_n > 0$, c_{n+1}/c_n increases and $\sum c_n$ diverges, then also the inequalities $|s_n| \leq 2\left|f(z)\right|$ hold.

I want to reproduce Szegö's short and beautiful proof of Theorem A. In the first place, it is sufficient to prove the first inequality in (24) since the others then follow by iteration applied to the remainders $\sum_{\nu=n+1}^{\infty} c_\nu z^\nu$. One begins by proving $\left|f(z)\right| \geq \left|f(z) - c_0\right|$, which for geometric reasons is equivalent to $\Re e f(z) \geq \frac{1}{2}c_0$ since $c_0 > 0$ by hypothesis. An Abel transformation (which is Fejér's main tool in problems about coefficients monotone of higher order) gives

$$f(z) = \sum_{n=0}^{\infty} \Delta^2 c_n . \varphi_n(z),$$

where $\varphi_n(z) = (n+1) + nz + \cdots + z^n$. We know from (10) that $\Re e\varphi_n(z) \geq \frac{n+1}{2}$. On the other hand

$$f(0) = c_0 = \sum_{n=0}^{\infty} \Delta^2 c_n . \varphi_n(0) = \sum_{n=0}^{\infty} \Delta^2 c_n (n+1),$$

hence

$$\Re e f(z) = \sum_{n=0}^{\infty} \Delta^2 c_n . \Re e\varphi_n(z) \geq \sum_{n=0}^{\infty} \Delta^2 c_n \frac{n+1}{2} = \frac{c_0}{2},$$

which proves our claim. Now

$$\left|\frac{f(z) - c_0}{f(z)}\right| \leq 1,$$

$f(0) \neq 0$ and $f(z) - c_0$ vanishes at $z = 0$, so by the Schwarz lemma

$$\left|\frac{f(z) - c_0}{f(z)}\right| \leq |z|,$$

which is the inequality we wanted to prove.

Here are some more results:

If the sequence $\{nc_n : n = 1, 2, 3, \ldots\}$ is monotone of order two, then the image of $|z| < 1$ under $f(z)$ is starlike with respect to $f(0)$.

If the sequence $\{n^2 c_n : n = 1, 2, 3, \ldots\}$ is monotone of order two, then the image of $|z| < 1$ is convex.

If the sequence (c_n) is monotone of order $k+1$, then $f^{(l)}(x) \geq 0$ for $-1 < x < 1$ and $0 \leq l \leq k$.

Fejér also proved results concerning series of the form $\sum_{n=0}^{\infty} c_n z^{2n+1}$ whose coefficients are monotone of higher order.

Turán (Proc. Cambridge Philos. Soc., **34** (1938), 134–143; [184], No. 15) continued the investigations of Fejér and Szegő. Fejér proved that is $c_n \geq 0$ and $c_{n+1} - c_n \geq 0$, then the inequalities

$$\big|f(z) - s_n(z)\big| \geq \big|f(z) - s_{n+1}(z)\big| \tag{25}$$

hold in the disk $|z| < \frac{1}{2}$. Turán proves that if $c_{k!} = 1$, $c_{k!+1} = k!$ ($k \geq 1$ and $c_n = \frac{1}{n!}$ for $n \neq k!, k!+1$, then the inequalities (25) do not hold in any disk with center 0. On the other hand, there exists $\rho > 0$ depending only on max $\sqrt[n]{|c_n|}$ such that the means

$$S_n(z) = \frac{s_0(z) + s_1(z) + \cdots + s_n(z)}{n+1}$$

satisfy $\big|f(z) - S_n(z)\big| \geq \big|f(z) - S_{n+1}(z)\big|$ for $|z| < \rho$, $n = 1, 2, 3, \ldots$.

Consider the partial sums

$$g_n(z) = g_n^{(0)}(z) = 1 + z + z^2 + \cdots + z^n$$

of the infinite geometric progression, and the iterated partial sums defined inductively by

$$g_n^{(r)}(z) = g_0^{(r-1)}(z) + g_1^{(r-1)}(z) + \cdots + g_n^{(r-1)}(z)$$

($r = 1, 2, \ldots$; $n = 0, 1, 2, \ldots$). Fejér's proofs are based in part on the fact that the derivatives of order $\leq r$ of any $g_n^{(r)}(x)$ are strictly positive for $-1 < x < 1$, $n \geq r$. Fejér observed that to prove this it is enough to show that $D^r g_n^{(r)}(x) > 0$ for $-1 < x < 1, n \geq r$. Turán (Publ. Math. Debrecen **1** (1949), 95–97; [184], No. 42) strengthens this by showing that all the zeros of $D^r g_n^{(r)}(z)$ lie on the unit cercle $|z| = 1$; for $r = 1$ this was proved by Egerváry (Math. Z., **42** (1937), 221–230). Turán also proves that $D^r g_n^{(r)}(x) > 0$ for all $x \in \mathbb{R}$ if n is even, and that $D^r g_n^{(r)}(x)$ has a simple zero at $x = -1$ and is > 0 for all real $x \neq -1$ if n is odd.

Writing

$$\big|f(re^{i\theta})\big|^2 = f(re^{i\theta})f(re^{-i\theta}) = \sum_{n=0}^{\infty} P_n(\cos\theta) r^n$$

we have

$$P_n(\cos\theta) = \sum_{l=0}^{n} c_l c_{n-l} e^{i(2n-l)\theta}.$$

The expressions P_n are polynomials of degree n in the variable $x = \cos\theta$ which Fejér calls the generalized Legendre polynomials associated with $f(z)$ or with the sequence (c_n) (Math. Z., **24** (1925), 285–298; [40], No. 64, vol. II, 161–175). If $f(z) = (1-z)^{-\rho}$, then the $P_n(x)$ are the ultraspherical (or Gegenbauer) polynomials of index $\rho \in \mathbb{R}$. The special case $\rho = \frac{1}{2}$ yields the classical Legendre polynomials. For $\rho = 1$ we get the Čebishov polynomials

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},$$

and interpreting the case $\rho = 0$ carefully one gets the Čebishov polynomials $T_n(x) = \cos n\theta$.

Fejér proves that if the terms of the sequence (c_n) are positive and decrease with n, then the the generalized Legendre polynomials satisfy the inequalities

$$\left| P_n(x) \right| \leq \frac{4}{\sin\theta} c_{[n/2]},$$

where $[x]$ is the integral part of x. In the case of the classical Legendre polynomials this becomes

$$\left| P_n(\cos\theta) \right| \leq \frac{C}{\sqrt{n}\sin\theta}. \tag{26}$$

For the classical Legendre polynomials Stieltjes gave a proof of the inequality

$$\left| P_n(\cos\theta) - P_{n+2}(\cos\theta) \right| \leq \frac{C}{\sqrt{n}}.$$

Fejér shows that Stieltjes' proof also serves to obtain an analogous inequality for his generalized Legendre polynomials.

In the case of the classical Legendre polynomials the inequality

$$\left| P_n(\cos\theta) \right| \leq \frac{C}{\sqrt{n}\sqrt{\sin\theta}},$$

stronger than (26), also holds. Fejér gives a "short and elementary" proof of this inequality. It was, however, Szegő who, with the help of his result giving

$|s_n(z)| \le 2|f(z)|$ quoted above, proved that under the same hypotheses on (c_n) the generalized Legendre polynomials satisfy

$$|P_n(\cos\theta)| \le 8c_{[n/2]}|f(e^{2i\theta})|$$

for $0 < \theta < \pi, n \in \mathbb{N}$. He also showed that the asymptotic formula

$$P_n(\cos n\theta) = 2c_n \Re e e^{-in\theta} f(e^{2i\theta}) + \mathrm{o}(c_n)$$

holds uniformly in $\varepsilon < \theta < 2\pi - \varepsilon$ $(\varepsilon > 0)$ whenever $c_n > 0, (c_n)$ decreases, $\lim(c_{n+1}/c_n) = 1$ and $c_n = \mathrm{O}(c_{2n})$.

Fejér ([40], vol. II, p. 699 and p. 721) proved that if the sequence of coefficients (c_n) is monotone of order two, then the arithmetic means of the partial sums of the series $\sum_{n=0}^{\infty} P_n(\cos\theta)$, whose terms are the Legendre polynomials, are all posititive.

There is a treatment of Fejér's generalized Legendre polynomials in Szegő's book ([174], p. 134).

If in $U(z) = \int_0^\infty f(t)\cos zt\,dt$ we substitute

$$f(t) = \sum_{n=1}^{\infty} \left(4\pi^2 n^4 e^{\frac{9}{2}t} - 6\pi n^2 e^{\frac{5}{2}t}\right) e^{-n^2\pi e^{2t}},$$

then $U(z)$ becomes the Riemann function $\xi(z)$. It is known that the Riemann hypothesis is equivalent to the fact that $\xi(z)$ has only real zeros, and this was the motivation for György Pólya to investigate which entire functions defined by trigonometric integrals have only real zeros. He consecrated a number of articles to this problem, let me just refer to J. Reine Angew. Math., **158**(1927), 6–18 ([128], vol. II), where further references can be found. Of course such entire functions arise also in other situations, e.g., in the case of Bessel functions.

In an early article (Math. Z., **2** (1918), 352–383; [128], vol. II, pp. 166–197) Pólya considers a strictly positive increasing function $f(t)$ defined in $0 \le t < 1$ such that $\int_0^1 f(t)\,dt$ exists. By the Eneström–Kakeya theorem the polynomial

$$f(0) + f\left(\frac{1}{n}\right) z + \cdots + f\left(\frac{n-1}{n}\right) z^{n-1}$$

has all its zeros in $|z| \le 1$, and since $\frac{1}{n}\sum_{\nu=0}^{n-1} f\left(\frac{\nu}{n}\right) e^{\frac{\nu}{n}z}$ converges to $\int_0^1 f(t)e^{zt}\,dt$, it seems plausible that all the zeros of the function $W(z)$ defined by this integral lie in $\Re e z \le 0$. Pólya proves this not by passing to the limit but directly by imitating the proof of the Eneström–Kakeya theorem. Actually the zeros of $W(z)$ lie in the open half-plane $\Re e z < 0$ unless $f(t)$ is in the "exceptional case". i.e., it is a step-function having a finite number of jumps at points with rational abscissa.

Next assume that the coefficients of the polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ are real, $a_n > 0$, and that its zeros are all in $|z| < 1$ (i.e., we are in the situation of the conclusion of the Eneström–Kakeya theorem). Then the trigonometric polynomials

$$u(t) = \Re e P(e^{it}) = a_0 + a_1 \cos t + \cdots + a_n \cos nt$$

and

$$v(t) = \Im m P(e^{it}) = a_1 \sin t + \cdots + a_n \sin nt$$

have exactly $2n$ simple roots in the interval $0 \leq t < 2\pi$, consequently all their roots are real. For the proof (cf. also [129], III. 179) let $z_1, \ldots, z_n$ be the zeros of $P(z)$, each listed as often as its multiplicity indicates. Write $e^{it} - z_\nu = \rho_\nu(t) e^{i\psi_\nu(t)}$, where $\psi_\nu(t)$ increase by 2π as t increases from 0 to 2π. Then

$$P(e^{it}) = a_n \prod_{\nu=1}^{n} \rho_\nu(t) e^{i\Psi(t)} = R(t) e^{i\Psi(t)},$$

where $\Psi(t) = \sum_{\nu=1}^{n} \psi_\nu(t)$ increases by $2n\pi$ as t goes from 0 to 2π. Since $u(t) = R(t) \cos \Psi(t)$ and $v(t) = R(t) \sin \Psi(t)$, the claim is proved because both $\cos\theta$ and $\sin\theta$ have $2n$ zeros in any half-open interval of length $2n\pi$.

Keeping the above hypotheses concerning $f(t)$, introduce with Pólya the entire functions

$$U(z) = \int_0^1 f(t) \cos zt \, dt, \quad V(z) = \int_0^1 f(t) \sin zt \, dt.$$

It follows from the Eneström–Kakeya theorem, combined with the proposition just proved, that the polynomials

$$U_n(z) = \frac{1}{n} \sum_{\nu=0}^{n-1} f\left(\frac{\nu}{n}\right) e^{\frac{\nu}{n^2}} \cos \frac{\nu z}{n} \quad \text{and} \quad V_n(z) = \frac{1}{n} \sum_{\nu=0}^{n-1} f\left(\frac{\nu}{n}\right) e^{\frac{\nu}{n^2}} \sin \frac{\nu z}{n}$$

have only real zeros. Passing to the limit, Pólya obtains that the functions $U(z), V(z)$, and more generally the functions $\alpha U(z) + \beta V(z)$, where α, β are real constants not both zero, have only real roots. If $f(t)$ is not in the "exceptional case", then the zeros are simple, furthermore $U(z)$ and

$V(z)$ have no common zero because otherwise $W(z)$ would have a purely imaginary zero, which is excluded by the first reult.

In Szegő's Transaction article discussed above, in which he presents inequalities for zeros of trigonometric polynomials, he says: "the elementary inequalities ... lead in a direct way to a theorem of Pólya, giving even a slightly more precise result". He proves the following:

Let α, β be two real constants, not both zero, and set $\alpha + i\beta = \rho e^{i\delta}$ ($\rho > 0$, $0 < \delta \leq 2\pi$). Then the entire function $\alpha U(z) + \beta V(z)$ has only real simple zeros; every interval $\left(\left(k-\frac{1}{2}\right)\pi\right)+\delta$, $\left(\left(k+\frac{1}{2}\pi\right)+\delta\right)$, $k = 0, \pm 1, \pm 2, \ldots$, except the one containing the origin, contains exactly one zero. (If $\alpha = 0$, i.e. $\delta = \frac{\pi}{2}$, then $V(z) = 0$ at $z = 0$ and there are no zeros in $(0, \pi)$.) The only exception is when $f(t)$ is a step function with jumps at the points

$$t = 1 - \frac{2\pi h}{(k-\frac{1}{2})\pi + \delta},$$

h, k integers; then the zeros of $\alpha U(z) + \beta V(z)$ are still real and lie in the closed intervals $\left[\left(k-\frac{1}{2}\right)\pi + \delta, \left(k+\frac{1}{2}\right)\pi + \delta\right]$.

Both Pólya and Szegő have further results concerning the regularity with which the zeros of $U(z)$ and $V(z)$ are distributed. For instance if $f(t)$ satisfies the additional condition to be convex, then $V(0) = 0$ and $V(z)$ has exactly one simple zero in each interval $\left(k\pi, \left(k+\frac{1}{2}\right)\pi\right)$, $k = 1, 2, 3, \ldots$.

Pólya points out that if $f(t) > 0$ and is decreasing, then $\int_0^\infty f(t) \sin zt\, dt$ does not vanish for any $z > 0$, and $\int_0^\infty f(t) \cos zt\, dt$ has no real zeros. A decreasing $f(t)$ figures also in the following theorem of Alfréd Rényi (C.R. Acad. Bulgare Sci., **3** (1950), 9–11; [151], No. 38, I, pp. 199–201) which generalizes results of L. Ilieff:

Let n and m be two positive integers such that $n + m$ is odd. Let $f(t)$ be n times differentiable in $0 < t \leq 1$ and satisfy the following conditions: $f(1) = 0, f^{(k)}(1) = 0$ for $1 \leq k \leq n-1, f^{(2k+1)}(0) = 0$ for $1 \leq 2k+1 \leq n-1$, the function $t^{-m} f^{(n)}(t)$ is positive, increasing and integrable in $0 \leq t \leq 1$. Then $U(z)$ and $V(z)$ have only real zeros.

8. Power series: singularities and analytic continuation

György Pólya begins his influential article "Untersuchungen über Lücken und Singularitäten von Potenzreihen" (Math. Z., **29** (1929), 549–640; [128], I, pp. 363–454) by saying that it is directed towards the "Hadamardsche Aufgabe der Funktionentheorie", i.e., its purpose is to obtain from properties of the sequence $c_0, c_1, \dots, c_n, \dots$ conclusions concerning the behavior of the function $f(z)$ represented by the power series

$$\sum_{n=0}^{\infty} c_n z^n \tag{27}$$

in the open disk where it converges.

The most classical example is of course the theorem found by Augustin Louis Cauchy, and made precise by Jacques Hadamard, according to which if

$$\limsup \sqrt[n]{|c_n|} = l,$$

then $f(z)$ has a singularity at some point z_0 with $|z_0| = 1/l$. This theorem is usually stated saying the (27) has a radius of convergence $r = 1/l$.

Equally famous is the gap theorem of Hadamard: if (λ_k) is a sequence of positive integers such that

$$\frac{\lambda_{k+1}}{\lambda_k} \geq q \tag{28}$$

for some $q > 1$ and all $k \geq 1$ then the "lacunary series"

$$\sum_{k=1}^{\infty} c_k z^{\lambda_k} \tag{29}$$

cannot be continued analytically beyond its circle of convergence, i.e., the function it represents has a singularity at every point of the said circle.

Pál Turán (*On the gap theorem of Fabry*, Hungarica Acta Math., **1** (1947), 21–29; [184], No. 30) says: "...one of the most exciting parts of the Weierstrassian theory of functions is the group of those theorems which draw conclusions from the lacunary distribution of the exponents λ_k

to the impossibility of analytical continuation over the convergence-circle, i.e.... the gap theorems". Hadamard's condition (28) was weakened to

$$\lim_{k\to\infty}(\lambda_{k+1}-\lambda_k)=\infty \tag{30}$$

and Edouard Fabry introduced the even weaker condition

$$\lim_{k\to\infty}\frac{\lambda_k}{k}=\infty \tag{31}$$

([104], §19, Satz 2, p. 83).

It seems to have been known for a long time that the deep reason why Hadamard's gap theorem holds is the inequality

$$\max_{0\le\theta\le 2\pi}\left|\sum_{\nu=1}^{n}c_\nu e^{i\lambda_\nu\theta}\right|\le C(q,\delta)\max_{\alpha\le\theta\le\alpha+\delta}\left|\sum_{\nu=1}^{n}c_\nu e^{i\lambda_\nu\theta}\right|, \tag{32}$$

and Norbert Wiener proved in 1933 that the gap theorem under condition (30) follows from an inequality which is the L^2-norm analogue of (32). He and R. E. A. C. Paley asked: from which similar inequality does the Fabry theorem follow? Turán (loc. cit.) showed that the answer is (32) provided the factor $C(q,\delta)$ is replaced by

$$\left(\frac{48\pi}{\delta}\right)^n.$$

This more precise inequality is one of the first applications of Turán's "new method" of which more will be said below. Actually, in his book ([187], Section 20, pp. 219–220) Turán proves the generalization of Fabry's theorem to Dirichlet series, first proved by Ottó Szász (Math. Ann., **85** (1922), 99–110; [172], pp. 503–514).

Let me list some early results "of classical beauty" (Landau) due to Hungarian mathematicians about power series.

Assume that the radius of convergence of (27) is $0<r<\infty$. Giulio Vivanti stated and Alfred Pringsheim proved that if $c_n\ge 0$ for all (large) n, then the function represented by (27) in $|z|<r$ has a singularity at the point $z=r$. Indeed, the Cauchy-Hadamard theorem and a simple calculation show that the radius of convergence of

$$\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}\left(\frac{r}{2}\right)\left(z-\frac{r}{2}\right)^n$$

equal to $\frac{1}{2}r$ ([104], §17).

In a long paper (J. Math. Pures Appl. (6) **5** (1909), 327–413) Pál Dienes, who later wrote two books on power series ([27], [28]), showed that it is sufficient to assume that the c_n lie in an angular domain with vertex 0 and opening $< \pi$.

Ottó Szász (loc. cit.) considered

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} \Re e c_n . z^n,$$

where he assumed that the two power series have the same radius of convergence $0 < r < \infty$. If $g(z)$ has a singularity at $z = r$, so does $f(z)$. From here the theorem of Dienes follows immediately: Assume, as we may, that $-\alpha \leq \arg c_n \leq \alpha$, where $\alpha < \frac{\pi}{2}$. Then $0 \leq \Re e c_n \leq |c_n| \leq \Re e c_n / \cos\alpha$, so the two power series have the same radius of convergence r, and $g(z)$ has a singularity at $z = r$ by the Vivanti–Pringsheim Theorem. Again Szász proved the result even for Dirichlet series.

Let $f(z)$ be the function represented by the power series (27) in the disk $|z| < r$ and assume that $\left|f(z)\right| \leq M$ there. In analogy with his theorem on trigonometric series, Fejér proved that if

$$s_n(z) = c_0 + c_1 z + \cdots + c_n z^n$$

is the partial sum of order n of (27), then

$$\left|s_0(z) + s_1(z) + \cdots + s_n(z)\right| \leq (n+1)M \tag{33}$$

for $n \geq 0$ and $|z| \leq r$ (special case of Theorem VII in Rend. Circ. Mat. Palermo **38** (1914), 79–97; [40], No. 46, I, pp. 783–802; see also [104], §1). Ottó Szász (Math. Z., **1** (1918), 163–183; [172], 446–466) and Issai Schur proved that the conditions

$$\left|s_0(z)\right| + \left|s_1(z)\right| + \cdots + |s_n| \leq (n+1)M$$

and

$$\left|s_0(z)\right|^2 + \left|s_1(z)\right|^2 + \cdots + \left|s_n(z)\right|^2 \leq (n+1)M^2$$

for $|z| \leq r$ are equivalent to (33) and to $\left|f(z)\right| \leq M$ ([104], §1).

On the other hand, modifying his example of a continuous function with a divergent Fourier series, Fejér gave an example of a function analytic in

$|z| < 1$, continuous (hence bounded) in $|z| \leq 1$, for which the partial sums of its power series expansion are unbounded (Sitzungsber. Bayrische Akad. Wiss. Math.-Phys. Kl., **40** (1910), 1–17; [40], No. 32, I, pp. 573–583; [104], §3).

Assume that the series

$$\sum_{n=0}^{\infty} c_n \tag{34}$$

whose terms are complex numbers, is Cesàro-summable of order 1, i.e., that the sequence of arithmetic means

$$\frac{s_0 + s_1 + \cdots + s_n}{n+1}$$

of the partial sums $s_k = c_0 + c_1 + \cdots + c_k$ converges as $n \to \infty$. The Tauberian theorem of Hardy and Landau states that if furthermore

$$c_n/n > -K \tag{35}$$

for some $K > 0$ and all n, then (34) is convergent. Therefore the Dirichlet–Jordan criterion for the convergence of the Fourier series of a function of bounded variation follows from Fejér's summability theorem.

Fejér replaced the Tauberian condition (35) by

$$\sum_{n=0}^{\infty} n|c_n|^2 < \infty.$$

Simple examples show that neither condition implies the other. Landau proves ([104], §13) that instead of Cesàro summability of (34) it is sufficient to assume Abel-summability.

This theorem has the following consequence: Let the function $f(z)$ represented by (27) be regular and univalent in the disk $|z| < 1$. If the area

$$\int\int_{|z|<1} \left| f'(z) \right|^2 dx\, dy = \pi \sum_{n=0}^{\infty} n|c_n|^2$$

of the image of the unit disk is finite, then $\sum_{n=0}^{\infty} c_n e^{in\varphi}$ converges for almost every value of φ (Schwarz–Festschrift, Mathematische Abhandlungen (1914), 42–53; [40], No. 49, I, pp. 813–822).

In general there is no connection between the behavior of $f(z)$ and the convergence of its power series expansion on the circle of convergence: $\sum z^n$ diverges for all $|z| = 1$, its sum $(1-z)^{-1}$ is regular at $z = -1$ and has a pole at $z = 1$; $\sum n^{-1}z^n$ converges at $z = -1$ and $-\log(1-z)$ is regular there; $\sum n^{-2}z^n$ converges for all $|z| = 1$ but by Vivanti–Pringsheim has a singularity at $z = 1$. Clearly if (27) converges at a single point of its circle of convergence $|z| = r$, then necessarily $\lim_{n\to\infty} c_n r^n = 0$. It was a remarkable result of Pierre Fatou that if $f(z)$ is regular at a point z_0 of the circle of convergence, then the necessary condition $c_n|z_0|^n \to 0$ is also sufficient, i.e., $\sum c_n z_0^n$ converges.

Marcel Riesz (J. Reine Angew. Math., **140**(1911), 89–99; Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen, math.-phys. Klasse 1916, 62–65; [158], No. 9, pp. 76–86, No. 17, pp. 145–148) gave a simple proof of Fatou's theorem ([104], §13) and extended it in several ways (Ark. Mat. Astr. Fys. 11, no. 12 (1916), 1–16; [158], No. 18, 149–164). For the sake of brevity, assume without loss of generality that the radius of convergence is 1. If $\lim_{n\to\infty} c_n = 0$ and $f(z)$ is regular on the closed arc $\{e^{i\varphi} : \alpha \le \varphi \le \beta\}$, then the convergence of $\sum c_n e^{in\varphi}$ is uniform for $\alpha \le \varphi \le \beta$. If $\lim_{n\to\infty} c_n/n^k = 0$ for some $k \ge 0$, then $\sum c_n e^{in\varphi}$ is Cesàro-summable of order k, uniformly on every closed arc on which $f(z)$ is regular. If c_n/n^k is only bounded, then the Cesàro-sums of order k of $\sum c_n e^{in\varphi}$ are bounded. He also generalized these results to Dirichlet series (C.R. Acad. Sci. Paris **149** (1909, 909–912; [158], No. 7, pp. 62–64).

Answering a question raised by Gösta Mittag-Leffler at the 1908 International Mathematical Congress in Rome, Marcel Riesz proved the following theorem (Rend. Circ. Mat. Palermo **30** (1910), 339–345; [158], No. 8, 65-71; [104], §12): Let $R > 1$, $0 < \alpha < \frac{\pi}{2}$ and S the closed set defined by the inequalities $|z| \le R$, $\alpha \le \arg(z-1) \le 2\pi - \alpha$. Assume that $f(z)$ is continuous on S and that it is regular on S with the exception of $z = 1$. If for $|z| < 1$ the series (27) converges to $f(z)$, then it converges uniformly to $f(z)$ on $|z| = 1$, and in particular $f(1) = \sum c_n$.

Fatou conjectured that given a power series (27) with radius of convergence 1, there exists a sequence $\varepsilon_n = \pm 1$ such that $\sum \varepsilon_n c_n z^n$ cannot be continued analytically beyond the unit circle. Pólya sent a proof of the conjecture in a letter to Adolf Hurwitz, who answered with another proof. The two letters were published in a joint article it is one of Pólya's first publications on power series (Acta Math., **40** (1916), 173–183; [128], I, pp. 17–21). Landau ([104], §20) calls the result Pólya's theorem and reproduces the proof of Hurwitz. Later Pólya (Acta Sci. Math. Szeged **12B** (1950), 199–203; [128], I, pp. 720–724) writing in honor of the 70th birthday of Fejér and F. Riesz showed that if (27) converges but is not a polynomial, then $\varepsilon_n = \pm 1$ can be chosen in such a way that $\sum \varepsilon_n c_n z^n$ satisfies no algebraic differential equation.

As is apparent from the above references, Landau's delightful little book [104] treats the work of several Hungarian mathematicians. A similar source of some material is a small book by Hadamard, whose second edition, written in collaboration with Szolem Mandelbrojt, appeared in 1926 [64]. The second edition of Landau's book was published in 1929, the same year as the fundamental paper of Pólya quoted at the beginning of this section appeared. Ludwig Bieberbach in his 1955 Ergebnisse volume [14] reports on the "very lively" progress made in the course of the preceding twenty-five years "in great part under the influence of Pólya's first gap-paper". Bieberbach discusses the work of many Hungarian mathematicians: Manó Beke, Dienes, Erdős, Fekete, Pólya, Marcel Riesz, Ottó Szász, Szegő, Turán. In particular Pólya's name appears on 36 out of the 155 pages of the book.

Pólya approaches the study of lacunary power series through the theory of entire functions of exponential type. The book [15] of Ralph P. Boas, which appeared at the same time as [14], contains a succinct presentation of the theory in Chapter 5, however, Boas himself says on p. 789 of [128], vol. I that Chapter 2 of Pólya's 1929 paper "is still a nearly complete and very readable exposition".

Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

be an entire function and set

$$M(r) = \max_{|z|=r} \left| f(z) \right|.$$

The order $0 \le \rho \le \infty$ of $f(z)$ is defined by

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

If $0 < \rho < \infty$, then the type of $f(z)$ is the number

$$\tau = \limsup_{r\to\infty} r^{-\rho} \log M(r).$$

According to the terminology introduced by Pólya, $f(z)$ is of exponential type if either $\rho = 1$ and τ is finite, or if $\rho < 1$, i.e., if

$$\left| f(z) \right| \le e^{a|z|}$$

for some $a > 0$ and large $|z|$. If

$$h(\varphi) = \limsup_{r\to\infty} r^{-1} \log \left| f(re^{i\varphi}) \right|$$

is the Phragmén–Lindelöf indicator function of $f(z)$, then the intersection of the half-planes

$$x\cos\varphi + y\sin\varphi \le h(\varphi) \quad (z = x + iy)$$

as φ varies, is the indicator diagram $\mathcal{D}$ of $f(z)$, and its reflection $\overline{\mathcal{D}}$ in the real axis is the conjugate indicator diagram. The function

$$F(z) = \frac{c_0}{z} + \frac{1!c_1}{z^2} + \frac{2!c_2}{z^3} + \cdots + \frac{n!c_n}{z^{n+1}} + \cdots \tag{36}$$

is regular outside $\overline{\mathcal{D}}$, and the series (36) converges for $|z| > \tau$. It is called the Borel–Laplace transform of $f(z)$ and is given by

$$F(z) = \int_0^\infty f(t)e^{-zt}\,dt$$

for $x > \tau$. Inversion gives gives the Pólya representation

$$f(z) = \frac{1}{2\pi i}\oint_C F(\zeta)e^{\zeta z}d\zeta,$$

where C is a contour containing $\overline{\mathcal{D}}$ in its interior.

Let (λ_k) be a sequence of real numbers such that $\lambda_0 > 0$ and $\lambda_{k+1} - \lambda_k \ge c > 0$. For $t \ge 0$ let $N(t)$ be the number of the λ_k with $\lambda_k \le t$. The *density* of the sequence is defined by

$$D = \lim_{k\to\infty} \frac{k}{\lambda_k} = \lim_{t\to\infty} \frac{N(t)}{t}$$

if it exists. Thus the condition in Fabry's gap theorem means that $D = 0$. The *upper density* of (λ_k) given by

$$\overline{D} = \limsup_{k\to\infty} \frac{k}{\lambda_k}$$

always exists, and so does the *lower density* $\underline{D}$ obtained when lim sup is replaced by lim inf. Pólya introduces further the *maximum density*

$$\overline{\Delta} = \lim_{s \to 1-} \limsup_{t \to \infty} \frac{N(t) - N(st)}{t - st}$$

and the *minimum density* $\underline{\Delta}$ in whose definition liminf replaces limsup. One always has

$$\underline{\Delta} \leq \underline{D} \leq \overline{D} \leq \overline{\Delta}$$

and all four are equal if D exists.

André Bloch pointed out that there is a strong analogy between the study of singularities of a function on the circle of convergence of its power series expansion, and the study of the Julia directions of an entire function. Let me recall that α is a Julia direction (sometimes called a Picard direction) of $f(z)$ if $f(z)$ assumes every complex value with at most one exception in the angular region $\alpha - \delta \leq \varphi \leq \alpha + \delta, r \geq 0$, where $z = re^{i\varphi}$ and $\delta > 0$ is arbitrarily small. E.g., $\pm\frac{\pi}{2}$ are Julia directions for e^z. Pólya presents a parallel treatment of the two questions. One of the main results in Chapter 3 of his 1929 paper is the following:

Theorem IV. *Let*

$$G(z) = \sum c_{\lambda_k} z^{\lambda_k} \quad (c_{\lambda_k} \neq 0)$$

be an entire function of order ρ and type τ, and let the maximum density of the exponents λ_k be $\overline{\Delta}$ (Pólya calls it the maximum density of the non-vanishing coefficients, or sometimes simply of the coefficients: "maximale Koeffizientendichte"). Writing

$$M(r; \alpha, \beta) = \max_{\alpha \leq \varphi \leq \beta} \left| G(re^{i\varphi}) \right| \tag{37}$$

for $\alpha < \beta$, define

$$\rho(\alpha, \beta) = \limsup_{r \to \infty} \frac{\log \log M(r; \alpha, \beta)}{\log r}$$

and

$$\tau(\alpha, \beta) = \limsup_{r \to \infty} r^{-\rho(\alpha,\beta)} \log M(r; \alpha, \beta).$$

If $\beta - \alpha > 2\pi\overline{\Delta}$, then $\rho(\alpha, \beta) = \rho$ and $\tau(\alpha, \beta) = \tau$.

In order to obtain a theorem concerning the singularities of a lacunary power series he uses the following result which he ascribes to Émile Borel:

Lemma a. *Let*

$$f(z) = c_0 + c_1 z + \cdots + c_n z^n + \cdots$$

be an entire function of order 1 *and type* $0 < \tau < \infty$. *We have*

$$h(\alpha) = \limsup_{r\to\infty} r^{-1} \log \left| f(re^{i\alpha}) \right| = \tau$$

if and only if the half-line $z = re^{i\alpha}$ $(0 \le r < \infty)$ *meets the circle of convergence of*

$$\frac{1}{z} F\left(\frac{1}{z}\right) = c_0 + 1! c_1 z + 2! c_2 z^2 + \cdots + n! c_n z^n + \cdots$$

in a singular point.

This leads to the famous result

Theorem IVa. *Let the maximum density of the nonvanishing coefficients of a power series with finite radius of convergence be* $\overline{\Delta}$. *Then every closed arc of the circle of convergence, whose central angle equals* $2\pi\overline{\Delta}$, *contains a singular point of the function represented by the power series.*

$\overline{\Delta} = 0$ is equivalent to $D = 0$, so Fabry's gap theorem is a special case. The function $(1 - z^k)^{-1} = \sum z^{kn}$ with $D = 1/k$ illustrates Theorem IVa nicely.

In order to obtain a theorem concerning Julia lines, Pólya uses the following result of Bieberbach:

Lemma b. *Let* $G(z)$ *be an entire function of infinite order, and* $M(r; \alpha, \beta)$ *as in* (37). *If* α *is such that*

$$\limsup_{r\to\infty} \frac{\log\log M(r; \alpha - \delta, \alpha + \delta)}{\log r} = \infty$$

for any $\delta > 0$, *then* α *is a Julia direction of* $G(z)$.

He obtains:

Theorem IVb. *If $f(z)$ is an entire function of order ∞, and the maximum density of the nonvanishing coefficients of its power series expansion is $\overline{\Delta}$, then every closed angle with opening $2\pi\overline{\Delta}$ contains a Julia direction.*

Pólya proves a theorem (Theorem II) which is of the same nature as Theorem IV, and deduces from it with the help of Lemma a the Vivanti–Pringsheim–Dienes theorem. The parallel result is:

If all the coefficients of the power series expansion of an entire function of infinite order lie in an angular domain with vertex 0 and opening $< \pi$, then the direction of the positive real axis is a Julia direction.

Related to Lemma b is the comparision of $\log M(r)$ and of $\log M(r; \alpha, \beta)$ for small $\beta - \alpha$ when the power series is lacunary. The study of this problem was pursued by Turán and by Tamás Kővári, see Chapter 21 of [187] and the references quoted there. Pál Erdős and Kővári (Acta Math. Acad. Sci. Hungar., **7** (1957), 305–317) proved that for any maximum modulus $M(r) = \max_{|z|=r} \left| f(z) \right|$ of an entire function there exists a series $N(r) = \sum \gamma_n r^n$ with $\gamma_n \geq 0$ such that $e^{-\varepsilon} < M(r)/N(r) < e^{\varepsilon}$ with $\varepsilon = 0.005$.

Let $f(z)$ be represented by the power series (27) and set $\mu_n = \inf \frac{M(r)}{r^n}$. By the Cauchy inequality $|c_n| \leq \mu_n$. Vincze (Acta. Sci. Math. Szeged **19** (1958), 129–140) proved that $\sum \frac{|c_n|}{\mu_n} = \infty$, i.e. $|c_n|$ cannot always be much smaller than μ_n.

Let $f(z) = \sum c_k z^{\lambda_k}$ have radius of convergence $R > 0$ and assume that the λ_k satisfy the Fabry gap condition. For any $z_0 \neq 0$ with $|z_0| < R$ the expansion

$$\sum a_n (z - z_0)^n \tag{38}$$

has only one singularity of $f(z)$ on its circle of convergence C, namely the point where C touches $|z| = R$. Therefore (38) cannot be a lacunary series satisfying the Fabry condition. More precisely, Kővári (J. London Math. Soc., **34** (1959), 185–194) proved with a geometric argument using Theorem IVa of Pólya that if the radius of convergence of $f(z) = \sum c_n z^n$ is 1 and the maximum density of its nonvanishing coefficients is $\overline{\Delta}$, while for $z_0 = re^{i\alpha} \neq 0$ $(r < 1)$ the maximum density of the nonvanishing coefficients of $\sum a_n (z - z_0)^n$ is $\overline{\Delta}_0$, then $\overline{\Delta} + \overline{\Delta}_0 \geq 1 - \frac{1}{\pi} \arcsin r$.

Pólya conjectured that the power series expansions of an entire function $f(z)$ at two distinct points cannot both have Fabry gaps. This was proved by Kató (Catherine) Rényi (Acta Math. Acad. Sci. Hungar., **7** (1956), 145–150). For $a \in \mathbb{C}$ denote by $Z_a(n)$ the number of terms of the sequence $f(a), f'(a), \ldots, f^{(n)}(a)$ which are equal to zero. The Fabry gap condition at a means that

$$\lim_{n\to\infty} \frac{Z_a(n)}{n} = 1.$$

Kató Rényi proves that if $a \neq b$, then

$$\liminf_{n\to\infty} \frac{Z_a(n) + Z_b(n)}{n} \le 1.$$

In particular the power series of a periodic entire function (e.g. e^z, $\cos z$, $\sin z$) cannot have Fabry gaps at any point. In a later article (ibid., **8** (1957), 227–233) she proved that if $f(z)$ has finite order $\rho \ge 1$ then

$$\liminf_{n\to\infty} \frac{Z_a(n) + Z_b(n) - n}{n^{1-\frac{1}{\rho+\varepsilon}}} \le 0$$

for any $\varepsilon > 0$, and if $f(z)$ has furthermore finite type $\tau \ge 0$, then

$$\liminf_{n\to\infty} \frac{Z_a(n) + Z_b(n) - n}{n^{1-\frac{1}{\rho}}} \le \frac{|b-a|}{2} e^2 \left(\frac{\tau}{e}\right)^{\frac{1}{\rho}}.$$

Kató Rényi returned to the topic several times, and studied also lacunary power series of two variables (Colloq. Math., **11** (1964), 165–171). This is one of the first articles of a Hungarian mathematician on analytic functions of several complex variables. Turán encouraged the research in this direction, which then produced contributions by younger mathematicians, e.g., László Lempert and László Sztacho.

The proofs of Kató Rényi's theorems are related to another area of Pólya's interests: "The zeros of derivatives of a function and its analytic character". In 1942 he gave an address with this title to the American Mathematical Society (Bull. Amer. Math. Soc., **49** (1943), 178–191; [128], II, pp. 394–407). In this lecture he summarized the known results, and also stated some new results and conjectures. One of the theorems stated without proof, and used by Kató Rényi is the following: Let $f(z)$ be an entire function which is real for real z, and denote by $\mathcal{N}(n)$ the number of zeros of $f^{(n)}(z)$ in the closed interval $[0,1]$. Then

$$\liminf_{n\to\infty} \frac{\mathcal{N}(n)}{n} = 0.$$

It was proved by Erdős and Alfréd Rényi (Acta Math. Acad. Sci. Hungar., **7** (1956), 125–141) even when $\mathcal{N}(n)$ denotes the number of zeros is $|z| \le 1$. Later (ibid., **8**(1957), 223–225) they proved that if $f(z)$ is an arbitrary entire function, and $r = H(s)$ is the inverse function of $s = \log M(r)$, then

$$\liminf_{n\to\infty} \frac{\mathcal{N}(n).H(n)}{n} \le e^2,$$

and if $f(z)$ is of finite order $\rho \ge 1$, then the right hand side can be replaced by $e^{2-1/\rho}$.

In his lecture Pólya introduced the set of all points z such that in any neighborhood of z infinitely many derivatives of $f(z)$ vanish. Kővári (Mat. Lapok **7** (1956), 106–108) gave an example of an entire function such that the zeros of all the successive derivatives are dense in $\mathbb{C}$. Erdős (ibid., **7** (1956), 214–217) proved that given an arbitrary sequence (z_k) of complex numbers, and a sequence $n_1 < n_2 < \cdots$ of integers such that the complementary sequence is infinite, there exists an entire function $f(z)$ such that $f^{(n_k)}(z_k) = 0$ for $k = 1, 2, \dots$.

In a joint paper Alfréd and Kató Rényi (J. Analyse Math., **14** (1965), 303–310) proved that if $f(z)$ is a non-constant entire function and $P(z)$ is a polynomial of degree ≥ 3, then $f\big(P(z)\big)$ cannot be periodic. However, $f(z) = e^{\sqrt{z}} + e^{-\sqrt{z}}$ is entire and $f(z^2)$ is periodic.

The fourth chapter of Pólya's great article appeared only in 1933 and in a different journal (Ann. of Math. (2) **34** (1933), 731–777; [128], I, pp. 543–589). Let

(a) $$f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$$

(b) $$g(z) = b_0 + b_1 z + \cdots + b_n z^n + \cdots$$

(c) $$h(z) = a_0 b_0 + a_1 b_1 z + \cdots + a_n b_n z^n + \cdots .$$

Hadamard's composition theorem states that if γ is a singularity of $h(z)$, then $\gamma = \alpha\beta$, where α is a singularity of $f(z)$, and β is a singularity of $g(z)$. Émile Borel proved that if the pole α is the only singularity on the circle of convergence of (a), and the pole β is the only singularity on the circle of convergence of (b), then $\alpha\beta$ is the unique singularity on the circle of convergence of (c) and it is a pole of $h(z)$. Georg Faber proved a somewhat more general statement, and Pólya announced in 1927 without proof that the product of two isolated singular points is a singular point.

To go beyond this result Pólya introduces the following definitions:

Let the power series (38) represent $f(z)$ in $|z - z_0| < R$, and let α be a singular point of $f(z)$ on the circle of convergence. The singularity α is said to be *almost isolated* for the series (38) if there exists a neighborhood of α in which there is no other singular point of $f(z)$ except possibly on the straight half-line joining z_0 with α.

A singular point α of $f(z)$ on the circle of convergence of (38) is said to be *isolable* if in any neighborhood of α there exists a simple closed curve surrounding α along which the function $f(z)$ defined by (38) can be continued analytically.

His main result is:

Theorem C. *If on the circle of convergence of the series* (a) *there is a unique singularity α of $f(z)$ which is almost isolated for* (a), *and on the circle of convergence of the series* (b) *there is a unique singularity β of $g(z)$ which is isolable, then the point $\gamma = \alpha\beta$ is singular for $h(z)$ and it is the only singularity on the circle of convergence of the power series* (c).

The proof is based on two auxiliary results which have a great interest on their own.

Theorem A. *If on the circle of convergence of a power series there is a unique singular point, and this singular point is almost isolated for the power series, then the upper density $\overline{D}$ of the nonvanishing coefficients is* 1.

Theorem B. *If the lower density $\underline{D}$ of the nonvanishing coefficients of a power series is* 0, *then the domain of existence of the function represented by the power series is a simply connected domain in* $\mathbb{C}$.

Later Pólya published proofs of the converses of Fabry's gap theorem and of Theorem B (Trans. Amer. Math. Soc., **52** (1942), 65–71; [128], I, pp. 713–719): Let (λ_k) be an increasing sequence of positive integers. If $\liminf_{k\to\infty}(\lambda_k/k) < \infty$, i.e. $\overline{D} > 0$, then there exists a power series $\sum a_k z^{\lambda_k}$ whose radius of convergence is 1 but for which the circle $|z| = 1$ is not a natural boundary; if $\limsup_{k\to\infty}(\lambda_k/k) < \infty$, i.e. $\underline{D} > 0$, then there exists a power series $\sum a_k z^{\lambda_k}$ whose radius of convergence is 1 and which defines a multivalent analytic function (hence its domain of definition is not a simply connected part of $\mathbb{C}$). Erdős gave an elementary proof of the first result (Trans. Amer. Math. Soc., **57** (1945), 102–104).

Pólya (Comment. Math. Helv., **7** (1934/35), 201–221; [128], I, pp. 593–613) also studied the following question of Hadamard type: how must the coefficients c_n be constituted in order that the function defined by (27) have the following properties: it is single-valued on the Riemann surface of $\sqrt[p]{z-1}$, it is regular at all points excluding the points of ramification $z = 1$ and $z = \infty$, and it vanishes at $z = \infty$.

Many of Pólya's results, in part with new proofs, can be found in the book of Vladimir Bernstein [13] generalized to Dirichlet series, which are the series (29) after substituting $z = e^{-s}$, where the λ_k do not have to be integers. Pólya (Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen Math.-Phys. Kl. 1927, 187–194; [128], I, pp. 309–317) considers Dirichlet series with complex exponents λ_k and proves that if they satisfy the Fabry condition $k/\lambda_k \to 0$, then the domain of existence of the function defined by the series is convex. He explains that in the case when the λ_k are positive integers this yields the Fabry gap theorem.

9. Turán's "new method"

"An idea, which is used once, is a trick. If it is used a second time, it becomes a method" – say Pólya and Szegő in the Preface of [129]. Turán's idea, that too many consecutive power-sums of n complex numbers cannot simultaneously be small, occurs first as a hypothesis in "Über die Verteilung von Primzahlen (I)" (Acta Sci. Math. Szeged **10** (1941), 81–104; [184], No. 23).

In 1912 Landau stated as one of the main problems of the theory of prime numbers to prove that between x^2 and $(x+1)^2$ there is always a prime. Denoting, as usual, by $\pi(x)$ the number of primes $p \le x$, one asks more generally for an estimate of $\pi(x + x^\theta) - \pi(x)$ as $x \to \infty$.

As every reader of these lines knows, the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it \in \mathbb{C},$$

converges for $\sigma > 1$ and defines the Riemann $\zeta(s)$-function, which is analytic in the whole complex plane with the exception of $s = 1$, where it has a simple pole. It follows from the functional equation discovered by Riemann that $\zeta(s) = 0$ for $s = -2k$ $(k = 1, 2, 3, \ldots)$; these are the trivial zeros. It is known that all the other zeros lie in the strip $0 < \sigma < 1$, and that there are infinitely many zeros ρ with $\Re e\rho = \frac{1}{2}$. The million dollar question is the Riemann hypothesis: all the nontrivial zeros of $\zeta(s)$ lie on $\sigma = \frac{1}{2}$.

To approach this problem F. Carlson introduced in 1920 the function $N(\alpha, T)$ which equals the number of zeros of $\zeta(s)$ in the rectangle $\alpha \le \sigma < 1$, $0 < t \le T$. A. E. Ingham proved in 1937 that if

$$N(\alpha, T) = \mathrm{O}\big(T^{b(1-\alpha)} \log^B T\big) \tag{39}$$

holds uniformly for $\frac{1}{2} \le \alpha \le 1$, then

$$\pi(x + x^\theta) - \pi(x) \sim \frac{x^\theta}{\log x} \tag{40}$$

for $\theta > \frac{1}{b}$. Observe that according to Riemann and H. von Mangold we have $N(\frac{1}{2}, T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi}$, so that b cannot be less than 2.

The Riemann hypothesis implies the Lindelöf hypothesis ([181], Chap. XIII):

$$\zeta(\frac{1}{2} + it) = \mathrm{O}\big(|T|^\varepsilon\big) \tag{41}$$

for any $\varepsilon > 0$. The converse implication does not hold (op. cit. p. 279). Ingham proved that in (39) one can take $b = 2 + 4c, B = 5$, where c is the greatest lower bound of all numbers ε for which (41) holds. Thus if the Lindelöf hypothesis is true, then $c = 0$, so in (39) one has the optimal $b = 2$: this is called the *density hypothesis* ([187], p. 359). In this case (40) is true for $\theta > \frac{1}{2}$.

Now Turán says that the behavior of $\zeta(s)$, and in particular the Lindelöf hypothesis, is inextricably connected with the distribution of primes. Therefore he proves

$$N(\alpha, T) \le T^{2(1-\alpha)} \exp(13 \log^{0.18} T)$$

under a hypothesis that has nothing to do with prime numbers:

Let $|z_j| \le 1$ for $1 \le j \le n$. Then

$$\max_{l(n) \le \nu \le u(n)} \big| z_1^\nu + \cdots + z_n^\nu \big| > \exp(-n^{0.09}),$$

where $l(n) = n^{3/2}(1 - n^{-0.42})$, $u(n) = n^{3/2}$. It is clearly visible on page 98 of Turán's article how this inequality is used in formula (35b).

László Kalmár called the following statement the *quasi-Riemann hypothesis:* One can find a number $\frac{1}{2} \le \alpha < 1$ such that $\zeta(s)$ has only finitely many zeros in the half-plane $\sigma > \alpha$. Turán (Izv. Akad. Nauk SSSR, Ser. Mat., **11** (1947), 197–262; [184], No. 31) gave a necessary and sufficient condition for the quasi-Riemann hypothesis to hold. The manuscript was received on December 2, 1945, he lectured on the subject in Budapest on February 7, 1944, so he worked on the paper during the darkest days of World War II. The condition in question is the existence of numerical constants $c > 0$ and $C > 0$ such that for $t > 0$, $N \in \mathbb{N}$ the condition

$$0 < c < t^{10} \le \frac{1}{2}N \le N_1 < N_2 \le N$$

implies

$$\left| \sum_{N_1 \le p \le N_2} e^{it \log p} \right| \le C t^{-\frac{1}{2}} N e^{23(\log \log N)},$$

where p is prime (cf. [187], Section 33). The statement about consecutive power-sums has been promoted from hypothesis to Lemma XII, it is a somewhat weaker form of Turán's "Second Main Theorem", see below.

So the "trick" has become a "method"! Other applications soon followed: to lacunary power series, as we saw in the preceding section, to the quasi-analyticity of functions having an expansion into trigonometric series with "small" coefficients (C.R. Acad. Sci. Paris **224** (1947), 1750–1752; [184], No. 29), to the distribution of real roots of almost periodic polynomials (Publ. Math. Debrecen **1** (1949/1950), 38–41; [184], No. 40), etc. Already in 1949 Turán lectured in Prague with the title "On a new method in the analysis with applications", and in 1953 his book with a similar title appeared simultaneously in Hungarian and in German. It lists twelve previous papers of the author in which the power-sum method is used. An expanded version in Chinese appeared in 1956. Several Hungarian mathematicians, mostly students and later collaborators of Turán, joined him in solving the fascinating problems which arose: István Dancs, Gábor Halász, János Komlós, Endre Makai, János Pintz, András Sárközy, Vera Sós, Mihály Szalay, Endre Szemerédi. But the theory also had an influence outside Hungary: F. V. Atkinson, A. A. Balkema, N. G. de Bruijn, J. D. Buchholz, J. W. S. Cassels, D. Gaier, J. M. Geysel, H. Leenman, D. J. Newman, S. Uchiyama and H. Wittich contributed to it. Furthermore Alfred J. van der Poorten and R. Tijdeman wrote their doctoral dissertations on the subject, the first at the University of New South Wales (Sydney, Australia), the second at the University of Amsterdam.

A considerably augmented English edition of *"On a New Method of Analysis and its Applications"* [187] appeared in 1984, eight years after the premature death of the author. Nine sections had their final versions written by Halász, and thirteen by Pintz. The book has two parts; Part I (16 sections) deals with minimax problems concerning power-sums, concentrating on those results which then have applications in Part II (42 sections). Each part ends with a long section on open problems.

Let me state two of the three Main Theorems. Set $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, for $\nu \in \mathbb{N}$, write $s_\nu(z) = z_1^\nu + \cdots + z_n^\nu$, and introduce the generalized power-sums

$$g_\nu(z) = b_1 z_1^\nu + \cdots + b_n z_n^\nu,$$

where the b_j $(1 \le j \le n)$ are complex constants.

First Main Theorem. *Assume that* $\min_{1\le j\le n} |z_j| = 1$. *Then for* $m \in \mathbb{N}$ *we have*

$$\max_{m+1\le\nu\le m+n} \left|g_\nu(z)\right| \ge C(m,n)\left|\sum_{j=l}^{n} b_j\right|,$$

where $C(m,n) = \left(\frac{m}{2e(m+n)}\right)^n$.

de Bruijn and Makai proved that the best posible value of $C(m,n)$ is $P(m,n)^{-1}$ with

$$P(m,n) = \sum_{j=0}^{n-1} \binom{m+j}{j} 2^j.$$

Second Main Theorem. *Assume that* $1 = |z_1| \ge |z_2| \ge \cdots \ge |z_n|$. *Then*

$$\max_{m+1\le\nu\le m+n} \left|g_\nu(z)\right| \ge 2\left(\frac{n}{8e(m+n)}\right)^n \min_{1\le j\le n} |b_1 + \cdots + b_j|.$$

In special cases stronger lower bounds can be found. Thus if $b_1 = \cdots = b_n = 1$, then we have:

Theorem. *Assume that* $\min_{1\le j\le n} |z_j| = 1$. *Then*

$$\min_z \max_{1\le\nu\le n} \left|s_\nu(z)\right| = 1.$$

The minimum is achieved when the z_j *are* n *vertices lying on the unit circle of a regular* $(n+1)$*-gon.*

I cannot resist the temptation to prove at least the first part of the Theorem. Set $\prod_1^n(\zeta - z_j) = \zeta^n + a_1\zeta^{n-1} + \cdots + a_n$. The assumption yields $|a_n| = |z_1 \ldots z_n| \geq 1$, so $|a_l| = \max|a_j| \geq 1$. The Newton–Girard formulas (also called Newton–Waring formulas, or Newton formulas, see Heinrich Weber, Lehrbuch der Algebra, vol. I, §46)

$$s_\nu(z) + a_1 s_{\nu-1}(z) + \cdots + a_{\nu-1}s_1(z) + \nu a_\nu = 0, \quad 1 \leq \nu \leq n,$$

yield

$$l|a_l| = \bigl| s_l(z) + a_1 s_{l-1}(z) + \cdots + a_{l-1}s_1(z)\bigr|$$
$$\leq \bigl(1 + |a_1| + \cdots + |a_{l-1}|\bigr) \max_{1\leq\nu\leq n} \bigl|s_\nu(z)\bigr| \leq l|a_l| \max_{1\leq\nu\leq n}\bigl|s_\nu(z)\bigr|,$$

so indeed $\max_{1\leq\nu\leq n}\bigl|s_\nu(z)\bigr| \geq 1$. ■

After an introduction (Sections 17–19), the first applications of Part II are to Complex Function Theory (Sections 20–26). The results already mentioned on lacunary series and on quasi-analyticity can be found here. Other applications are to Borel summability, to the value distribution of entire functions satisfying a linear differential equation, to linear combinations of entire functions, etc. Following this, the topics covered are: Differential Equations (Sections 27–28), Numerical Algebra (Sections 29–31), Markov Chains (Section 32), and the largest portion of Part II (Sections 33–57) is devoted to Analytic Number Theory, Here we find the topics discussed earlier: the density hypothesis, the quasi-Riemann hypothesis, but also the remainder term in the prime number formula, the least prime in an arithmetic progression, and mainly the joint creation of Turán and Stefan Knapowski: comparative prime number theory. The primary object of the study is the analytic function $\zeta(s)$ and its cousins, the Dirichlet $L(s;\chi)$-series, however, I will not transcribe the results but refer to the accessible and eminently readable book [187].

10. Power series: behavior on the circle of convergence

Pál Turán has given a number of results and examples concerning power series which are independent of his “new method”.

If the power series $\sum c_n z^n$ converges in $|z| < 1$ and represents there a function which is continuous in $|z| \leq 1$, then $\sum |c_n|^2 < \infty$. There exist, however, functions for which the series $\sum |c_n|^{2-\varepsilon}$ diverges for any $\varepsilon > 0$. This phenomenon is named after Torsten Carleman whose article appeared in 1918. But he and authors before him (E. Fabry, G. H. Hardy, S. Bernstein, J. E. Littlewood) listed in Turán's note (Bull. Amer. Math. Soc., **54** (1948), 932–936; [184], No. 37) only consider the analogous phenomenon for trigonometric series. It was Ottó Szász (Math. Z., **8** (1920), 222–236; [172], pp. 481–496) who first stated it explicitly for power series. Turán quotes two articles of Simon Szidon in which the divergence of $\sum \left| c_{n_k} \right|^{2-\varepsilon}$ is examined for an increasing sequence (n_k) of integers.

Turán gives an example for the Carleman phenomenon which requires only elementary (though not simple) calculations, and does not need van der Corput's Lemma as the example given in [203] (Chapter 5, (4.11), p. 200). Ottó Szász pointed out to Turán that completely elementary examples are also furnished by the reasoning he and S. Minakshisundaram used in their paper (Trans. Amer. Math. Soc., **61** (1947), 36–53; [172], pp. 1054–1071).

A Möbius transformation

$$z \mapsto w = \mu(z) = c\frac{z - z_0}{1 - \bar{z}_0 z},$$

where $|c| = 1$ and $|z_0| < 1$, maps the unit disk $|z| < 1$ bijectively and conformally onto $|w| < 1$, and the circle $|z| = 1$ onto $|w| = 1$. For simplicity take $c = 1$. The inverse transformation is

$$w \mapsto z = \mu^{-1}(w) = \frac{w - w_0}{1 - \bar{w}_0 w},$$

where $w_0 = -z_0 = \mu(0)$. Let

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a function which is holomorphic in the unit disk, and set

$$f_2(w) = f_1\big(\mu^{-1}(w)\big) = \sum_{n=0}^{\infty} b_n w^n.$$

Turán (Publ. Inst. Math. (Beograd) **12** (1958), 19–26; [184], No. 103) gave an example of a series $f_1(z)$ which converges at $z = 1$ but the series $f_2(w)$

does not converge at the corresponding point $w = \mu(1)$. Turán also proved that if $f_1(z)$ is Abel-summable at $z = 1$, then $f_2(w)$ is Abel-summable at $w = \mu(1)$.

László Alpár, Turán's friend and disciple, devoted several articles to this situation. In a paper written in Hungarian (Mat. Lapok **11** (1960), 312–322) he shows that there exists an $f_1(z)$ such that $\sum |a_n| < \infty$ but $\sum |b_n|$ diverges. What makes this result interesting is the fact that $\sum |a_n| < \infty$ implies $\sum |a_n|^2 < \infty$, i.e. that $f_1(z) \in L^2$ on $|z| = 1$ and thus also $\sum |b_n|^2 < \infty$. Alpár asks whether there exists a number $0 < \varepsilon < 1$ such that $\sum |b_n|^{2-\varepsilon}$ converges. Gábor Halász (Publ. Math. Debrecen **14** (1967), 63–68) gave a negative answer by proving the following theorem:

Given z_0 and $0 < \omega(n)$ which tends to $+\infty$ as $n \to \infty$, there exists a function $f_1(z)$ such that $\sum |a_n| < \infty$ but

$$\sum_{n=0}^{\infty} |b_n|^{2-\frac{\omega(n)}{\log n}} = \infty.$$

However, as he shows in a footnote, if $k \geq 0$, then

$$\sum_{n=0}^{\infty} |b_n|^{2-\frac{k}{\log n}} < \infty.$$

In a second paper (ibid., **15** (1968), 23–31) Halász proves that if there exists a decreasing sequence (A_n) such that $|a_n| \leq A_n$ and $\sum A_n < \infty$, then also $\sum |b_n| < \infty$. If, however, $\sum A_n = \infty$, then there exists (a_n) with $|a_n| \leq A_n, \sum |a_n| < \infty$ but $\sum |b_n| = \infty$.

Alpár wrote a sequence of eight articles in French on the subject. The last one appeared in Studia Sci. Math. Hungar., **1** (1966), 379–388 and contains references to the preceding ones. Motivated by his result on Abel-summability, Turán asked whether the same holds for Cesàro-summability. Alpár gave a negative answer. More precisely, he proved that if $k \geq 0$ and $\sum a_n$ is Cesàro-summable of order k, then $\sum b_n \mu(1)^n$ is Cesàro-summable of order $k + \frac{1}{2}$ but not necessarily of smaller order. If $\alpha_n^{(k)}$ is the n^{th} Cesàro-sum of order k of $\sum a_n$ and $\beta_n^{(k+\delta)}$ is the n^{th} Cesàro-sum of order $k + \delta$ of $\sum b_n \mu(1)^n$, then one has a linear relation

$$\beta_n^{(k+\delta)} = \sum_{\nu=0}^{\infty} \gamma_{n\nu}^{(k,\delta)} \alpha_\nu^{(k)}.$$

The idea of Alpár was to show that the matrix $\left(\gamma_{n\nu}^{(k,\delta)}\right)$ satisfies the Toeplitz–Schur regularity conditions if $\delta \geq \frac{1}{2}$ but not if $\delta < \frac{1}{2}$.

Alpár also considered the analogous problems when instead of power series one studies expansions into a series of Faber polynomials.

As we have seen in Section 8, Fejér has given an example of a function $f(z)$ holomorphic in $|z| < 1$, continuous in $|z| \leq 1$ whose Taylor series $\sum c_n z^n$ diverges at $z = 1$. If $|c_n| \leq \frac{1}{n}$ for all n, then $\sum c_n z^n$ converges uniformly in $|z| \leq 1$. Turán (Mat. Lapok, **10** (1959), 278–282; [184], No. 113) uses the method of Fejér to show that if $\omega(n)$ is a positive sequence which tends increasingly to $+\infty$ as $n \to \infty$, then there exists a function $f(z) = \sum c_n z^n$ in $|z| < 1$, continuous in $|z| \leq 1$ such that $|c_n| \leq \frac{\omega(n)}{n}$ but $\sum c_n$ diverges.

Let $f(z)$ be an entire function of order ρ, let $b \in \mathbb{C}$, and denote by z_ν the points (counted with multiplicity), where $f(z_\nu) = b$. Denote by $\rho(b)$ the exponent of convergence of (z_ν), i.e., the number such that

$$\sum_\nu \frac{1}{|z_\nu|^\alpha}$$

converges for $\alpha > \rho(b)$ and diverges for $\alpha < \rho(b)$; a number b_0 for which $\rho(b_0) < \rho$ is called a *Borel exceptional value* of $f(z)$.

This concept can be generalized. Let φ be a positive function defined for $0 \leq x < \infty$, strictly decreasing to zero. A number $b \in \mathbb{C}$ is a φ-exceptional value of $f(z)$ if $\sum \varphi(|z_\nu|) < \infty$. In a joint article Alpár and Turán (Publ. Math. Inst. Hungar. Acad. Sci., **A6** (1960), 157–164; [184], No. 120) show that for any function φ of the above kind there exists an entire function of infinite order that has no φ-exceptional values.

Let (λ_k) be a sequence of positive integers which satisfies the gap condition

$$\lambda_{k+1} - \lambda_k \geq \gamma$$

for some fixed $\gamma \geq 1$. Assume that the power series

$$f(z) = \sum_{k=0}^{\infty} c_k z^{\lambda_k}$$

has radius of convergence 1. If writing $z = re^{i\theta}$ the limit

$$f(\theta) = \lim_{r \to 1-0} f(re^{i\theta})$$

exists almost everywhere and belongs to L^2 on an arc of length greater than $2\pi/\gamma$, then $f(\theta)$ exists everywhere and belongs to L^2 on $[0, 2\pi]$. The result can be found in [203] (Chapter V, (9.1), p. 222), where the notes contain the following remark: "Nothing seems to be known about possible extensions to classes L^p, $p \neq 2$" (p. 380).

In the book [135] published in honor of the 75th birthday of Pólya, the champion of gap theorems, there is a contribution by Erdős and Rényi (pp. 110–116) and one by Turán (pp. 404–409) addressing this problem for $q > 2$.

The first two use probability theory. They consider the exponents λ_k as random variables and prove that with probability one there exists a function $f(\theta)$ whose Fourier series $\sum c_k \cos \lambda_k \theta$ satisfies $\lambda_{k+1} - \lambda_k \to \infty$, belongs to L^2 in $|\theta| \leq \pi$, is bounded in $\delta \leq |\theta| \leq \pi$ for every $\delta > 0$, but does not belong to any L^q with $q > 2$ on $|\theta| \leq \pi$.

Turán constructs for any $q > 6$ an explicit lacunary power series such that

$$\lambda_{k+1} - \lambda_k > \frac{1}{2}\lambda_k^{1/(q+6)},$$

and for which $f(\theta) \in L^q\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ but $f(\theta)$ is not in $L^q(0, 2\pi)$.

11. Pólya–Schur functions

One of the earliest publications of György Pólya has the title "Über ein Problem von Laguerre" (Rend. Circ. Mat. Palermo **34** (1912) 89–120; [128], II, pp. 1–32); it is in fact an exchange of letters between him and Mihály Fekete. Much later Pólya wrote a paper with almost the same title: "Über einen Satz von Laguerre" (Jahresber. Deutsch. Math.-Verein., **38** (1929), 161–168; [128], II, pp. 314–321). It is the preoccupation with problems left open by Edmond Laguerre which led to the class we now call Pólya–Schur functions. There is a nice account of their theory in the little book of Nikola Obrechkoff [125].

Let $f(z)$ be an entire function of finite order ρ, denote by (z_k) the sequence of its zeros different from 0 counted according to their multiplicities, and set $|z_k| = r_k$. The genus p *of the sequence* (r_k) is the smallest integer such that

$$\sum_{k=1}^{\infty} \frac{1}{r_k^{\nu}}$$

converges for $\nu \geq p+1$. The function has the product representation

$$f(z) = z^m e^{Q(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{\frac{z}{z_k} + \frac{1}{2}\left(\frac{z}{z_k}\right)^2 + \cdots + \frac{1}{p}\left(\frac{z}{z_k}\right)^p},$$

where $Q(z)$ is a polynomial of degree $q \leq \rho$. Laguerre called the genus *of the function* the larger of the two integers p and q. Completing the proofs and weakening the hypotheses of theorems stated by Laguerre, Pólya proved the following results (Rend. Circ. Mat. Palermo **36** (1913), 279–295; Nachr. Ges. Wiss. Göttingen 1913, 325–330; [128], II, pp. 54–70, 71–75):

Let $P_l(z)$ ($l \in \mathbb{N}$) be a sequence of polynomials which converges uniformly in a disk $|z| \leq R$ to a function $F(z)$.

I. If the zeros of all the polynomials $P_l(z)$ are > 0, then $F(z)$ is an entire function of the form $e^{-\beta z}G(z)$, where $G(z)$ is of genus zero and $\beta \geq 0$. If $F(z)$ is not identically zero, then $P_l(z)$ converges to $F(z)$ in the whole plane and the convergence is uniform in every bounded domain.

More generally ([125], pp. 13–14): If the zeros of the $P_l(z)$ lie in an angular domain W with opening $< \pi$, then $F(z) = e^{-\beta z}G(z)$, where $G(z)$ has genus 0, and β lies in the domain $\overline{W}$ which is symmetric to W with respect to the real axis.

II. If the zeros of the $P_l(z)$ are all real, then $F(z)$ is a Pólya–Schur function, i.e., an entire function of genus 1 multiplied by a Gauß density function $e^{-\gamma z^2}$ ($\gamma \geq 0$).

Actually I. follows easily from a theorem of Hurwitz ([104], p. 17) and the Hadamard factorization. The proof of II. is "weniger einfach" (less simple).

Pólya wrote the immediately following article (Rend. Circ. Mat. Palermo **37** (1914), 297–302; [128], II, pp. 76–83) in collaboration with Egon Lindwart. They prove that if $z_{l1}, z_{l2}, \ldots, z_{ll}$ are the zeros of $P_l(z)$ and if there exists $M > 0$ such that

$$\sum_{j=1}^{l} \frac{1}{|z_{lj}|^k} \leq M$$

for some $k > 0$, then $F(z)$ is an entire function of genus $\leq [k]$; if k is an integer, $F(z) = e^{\gamma z^k}G(z)$, where the genus of $G(z)$ is $\leq k-1$.

The authors list several consequences, e.g., the following suggested by Fekete: Write $z_{ls} = r_{ls}e^{i\theta_{ls}}$ and assume that the θ_{ls} belong to the union of the r closed intervals $\left[(4t-1)\frac{\pi}{2r}, (4t+1)\frac{\pi}{2r}\right]$, $t = 0, 1, \ldots, r-1$. Then $F(z)$

has genus $\leq 2r$. Furthermore if z_k denotes again the zeros $\neq 0$ of $F(z)$ and $|z_k| = r_k$, then $\sum r_k^{-2k} < \infty$.

Another corollary of the theorem of Lindwart and Pólya was given the following stronger form by Ottó Szász (Bull. Amer. Math. Soc., **49** (1943), 377–383; [172], pp. 1390–1396): Assume that the zeros of each $P_l(z)$ lie in a half-plane containing 0 on its boundary, which can vary with l. If P_l converges to $F(z)$ on a set which has a finite limit point and the coefficients of the $P_l(z)$ are bounded, then $P_l(z)$ converges to $F(z)$ uniformly on every bounded domain and

$$F(z) = e^{\alpha+\beta z+\gamma z^2} \prod \left(1 - \frac{z}{z_k}\right) e^{-\frac{z}{z_k}},$$

where $\sum 1/|z_k|^2 < \infty$.

The problem of Laguerre, investigated by Pólya, received a very general treatment in the 1949 Leiden thesis of Jacob Korevaar. He assumes that the z_{lj} belong to an arbitrary subset of $\mathbb{C}$ and characterizes $F(z) = \lim_{l\to\infty} P_l(z)$. An account of his results can be found in [31], pp. 261–272.

Then Issai Schur got into the picture. At the origin is the following theorem by E. Malo which appeared of all places in the Journal de Mathématiques Spéciales (4) **4** (1895), 7: Assume that the zeros of the polynomial

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m \quad (a_m \neq 0) \tag{42}$$

are all real, and that the zeros of the polynomial

$$b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n \quad (b_n \neq 0) \tag{43}$$

are all real and of the same sign. Set $k = \min(m, n)$. Then the zeros of

$$a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots + a_k b_k z^k \tag{44}$$

are all real. If $m \leq n$ and $a_0 b_0 \neq 0$, then the zeros of (44) are distinct.

Schur proved (J. Reine Angew. Math., **144** (1914), 75–88) a result he calls "composition theorem" and which asserts that under the same hypotheses as before the zeros of

$$0! a_0 b_0 + 1! a_1 b_1 z + 2! a_2 b_2 z^2 + \cdots + k! a_k b_k z^k$$

are all real. If $m \leq n$, $a_0 b_0 \neq 0$ the same conclusion holds as above.

From the composition theorem Malo's result follows with a neat little trick (§3 of loc. cit.).

In their joint article (J. Reine Angew. Math., **144** (1914), 89–113; [128], II, pp. 100–124; [163], II, No. 24, pp. 56–69) Pólya and Schur say that a sequence

$$\text{(A)} \qquad \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$$

of real numbers is a *factor sequence of the first kind* if given any polynomial (42) whose zeros are all real, the polynomial

$$\alpha_0 a_0 + \alpha_1 a_1 z + \alpha_2 a_2 z^2 + \cdots + \alpha_m a_m z^m$$

has only real zeros. Similarly a sequence

$$\text{(B)} \qquad \beta_0, \beta_1, \beta_2, \ldots, \beta_n, \ldots$$

is a *factor sequence of the second kind* if for any polynomial (43) whose zeros are all real and have the same sign (i.e., are all positive or all negative), the polynomial

$$\beta_0 b_0 + \beta_1 b_1 z + \beta_2 b_2 z^2 + \cdots + \beta_n b_n z^n$$

has only real zeros.

Clearly a factor sequence of the first kind is also one of the second kind but not conversely. It was Laguerre who gave the first examples of factor sequences.

Pólya and Schur start with giving algebraic criteria for factor sequences. For instance (A) is a factor sequence of the first kind if and only if the polynomials

$$\alpha_0 + \binom{n}{1}\alpha_1 z + \binom{n}{2}\alpha_2 z^2 + \cdots + \alpha_n z^n$$

have only real zeros of the same sign. In one direction this follows from the fact that $(1+z)^n$ has -1 as its only zero and from Descartes' rule of signs.

Let $\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots$ be a sequence of real numbers. The authors prove that

$$\frac{\gamma_0}{0!}, \frac{\gamma_1}{1!}, \frac{\gamma_2}{2!}, \ldots, \frac{\gamma_n}{n!}, \ldots$$

is a factor sequence of the first kind if and only if the following condition is satisfied: whenever (42) has only real zeros and (43) only real zeros with the same sign, the polynomial

$$\gamma_0 a_0 b_0 + \gamma_1 a_1 b_1 z + \gamma_2 a_2 b_2 z^2 + \cdots + \gamma_k a_k b_k z^k$$

has only real zeros. Since the sequence where $\alpha_n = 1$ for all n is obviously a factor sequence of the first kind, this yields Schur's composition theorem.

In order to give transcendental criteria for factor sequences, Pólya and Schur introduce two classes of entire functions with real Taylor coefficients.

A function

$$\Phi(z) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} z^k \tag{45}$$

with real zeros having the same sign is of type (I.) if $\Phi(z)$ or $\Phi(-z)$ has the representation

$$\Phi(z) = \frac{\alpha_r}{r!} z^r e^{\beta z} \prod_{\nu=1}^{\infty} (1 + \gamma_\nu z)$$

with $\alpha_r \neq 0$, $\beta, \gamma_\nu \geq 0$ (i.e. if on some disk $|z| \leq R$ it is the uniform limit of a sequence of polynomials having only real zeros of the same sign).

A function

$$\Psi(z) = \sum_{k=0}^{\infty} \frac{\beta_k}{k!} z^k \tag{46}$$

whose zeros are all real is of type (II.) if it has the representation

$$\Psi(z) = \frac{\beta_r}{r!} z^r e^{\beta z - \gamma z^2} \prod_{\nu=1}^{\infty} (1 + \delta_\nu z) e^{-\delta_\nu z},$$

where $\beta_r \neq 0, \gamma \geq 0, \beta$ and δ_ν real (i.e. if on some disk it is the uniform limit of a sequence of polynomials having only real zeros).

Theorem. (A) *is a factor sequence of the first kind if and only if* (45) *is of type* (I.). (B) *is a factor sequence of the second kind if and only if* (46) *is of type* (II.).

Now a new motif enters in the form of the Hermite–Poulain theorem: Let the polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (a_n \neq 0)$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_n z_n + b_{n+1} z^{n+1} + \cdots + b_{n+m} z^{n+m},$$

where $b_0, b_1, \ldots, b_n > 0$, have only real zeros. Then the polynomial

$$b_0 P(z) + b_1 P' z + b_2 P''(z) + \cdots + b_n P^{(n)}(z)$$

has only real zeros.

In a short note (Vierteljahrschr. Naturforsch. Ges. Zürich **61** (1916), 546–548; [128], II, p. 163–165) Pólya gives a geometric proof of the fact that under the same hypotheses the curve

$$F(x, y) = b_0 P(y) + b_1 x P'(y) + b_2 x^2 P''(y) + \cdots + b_n x^n P^{(n)}(y) = 0$$

has n real points of intersection with any straight line $sx - ty + u = 0$, provided that $s \geq 0$, $t \geq 0$, $s + t > 0$ and u is real.

The special case $s = 1$, $t = 0$, $u = -1$, i.e. $x = 1$, yields the Hermite–Poulain result. The case $s = 0$, $t = 1$, $u = 0$, i.e. $y = 0$, gives the Schur composition theorem. Finally, $s = t = 1$, $u = 0$, i.e. $x = y$, gives an example of Pólya–Schur according to which

$$b_0 P(z) + b_1 z P'(z) + b_2 z^2 P''(z) + \cdots + b_n z^n P^{(n)}(z)$$

has only real zeros. Conversely, the general theorem can be deduced from the three special cases by changes of variables.

In an earlier article (J. Reine Angew. Math., **145** (1915), 224–249; [128], II, pp. 128–153) Pólya made some elementary remarks related to the Hermite–Poulain theorem, and generalized it to certain pairs of entire functions. Changing slightly the hypotheses and the notation, let

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

be a polynomial with only real roots, $a_0 \neq 0$ real, $a_n \neq 0$, $n \geq 1$, and let $G(z)$ be a polynomial with real coefficients having exactly r real roots. Then the following hold concerning the polynomial

$$H(z) = F(\partial) G(z) = a_0 G(z) + a_1 G'(z) + a_2 G''(z) + \cdots + a_n G^{(n)}(z) :$$

(i) $H(z)$ has $r+2k$ real zeros, $k \in \mathbb{N}$;

(ii) If $r \geq 1$, then $H(z)$ has at least one real zero with odd multiplicity, hence assumes for real z both positive and negative values;

(iii) If $r \geq 2$, then $H(z)$ has at least two distict real zeros;

(iv) If $G(z)$ has only real zeros, then the multiple zeros of $H(z)$ are also multiple zeros of $G(z)$

(v) If $r \geq 1$ and $F(z)$ has only positive zeros, then $H(z)$ has a real zero with odd multiplicity which is larger than the largest real zero of $G(z)$;

Let $\Phi(z)$ be an entire function of type (I.), where in (45) the coefficients α_k are positive, and let $\Psi(z)$ be of type (II.). Then the series

$$\sum_{\nu=0}^{\infty} \frac{\alpha_\nu}{\nu!} z^\nu \Psi^{(\nu)}(z) \quad \text{and} \quad \sum_{\nu=0}^{\infty} \frac{\beta_\nu}{\nu!} z^\nu \Phi^{(\nu)}(z)$$

converge and represent entire functions of type (II.).

The second paper referred to at the beginning of this section appeared immediately following an article of Obrechkoff who gave a simplified proof of a theorem of Hurwitz according to which the function $\sqrt{z}^\nu J_{-\nu}(2\sqrt{z})$ has exactly $[\nu]$ negative zeros if $\nu \geq 0$. In the first part of his proof Obrechkoff uses an algebraic theorem of Laguerre, in the second part also the differential equation of the Bessel function $J_{-\nu}(z)$. Pólya shows that the whole proof can be based on ideas of Laguerre if one transports them from polynomials to entire functions. He proves namely the following theorem: Let

$$g(z) = a_0 + a_1 z + a_z z^2 + a_3 z^3 + \cdots$$

be either a polynomial of degree m with only positive zeros, or an entire function of type (I.) with positive zeros. In the algebraic case set $J = [0, m]$, in the transzendental case $J = [0, \infty)$. Let $G(z)$ be an entire function of type (II.) which in J has exactly s simple zeros such that the distance between two consecutive zeros is ≥ 1. Then

$$a_0 G(0) + a_1 G(1) z + a_2 G(2) z^2 + a_3 G(3) z^3 + \cdots$$

has exactly $m - s$ strictly positive zeros in the algebraic case, and it has s negative zeros in the transzendental case.

Since

$$\sqrt{z}^\nu J_{-\nu}\big(2\sqrt{z}\,\big) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \frac{1}{\Gamma(n+1-\nu)},$$

setting $g(z) = e^{-z}$ and $G(z) = \big(\Gamma(z+1-\nu)\big)^{-1}$ the theorem of Hurwitz follows.

Pólya says that the content of the paper is a portion of an investigation on which he published two notes (C.R. Acad. Sci. Paris **183** (1926), 413–414, 467–468; [128], II, pp. 261–264). The full proofs appeared in the dissertation of E. Benz (Comment. Math. Helv., **7** (1934), 243–289). Let L be an operation determined by a sequence $l_0, l_1, \ldots, l_n, \ldots$, which associates with each polynomial $P(z)$ the polynomial

$$LP(z) = l_0 P(z) + l_1 P'(z) + l_2 P''(z) + \cdots .$$

Setting $L(z) = l_0 + l_1 z + l_2 z^2 + \cdots$ one can write

$$LP(z) = L(\partial)P(z).$$

Pólya proposes to determine the class of operations L such that whenever $P(z)$ has all its roots in the convex domain K, then so does $LP(z)$. Pólya mentions that there are three equivalent necessary and sufficients conditions, the third of which involves a product decomposition of $L(z)$. When K is the lower half-plane, then $L(z)$ has to be a Pólya–Schur function with zeros in the upper half-plane. In the second note Pólya considers the case when $P(z)$ is a Dirichlet polynomial

$$P(z) = a_0 e^{\lambda_0 z} + a_1 e^{\lambda_1 z} + \cdots + a_n e^{\lambda_n z}$$

and

$$LP(z) = L(\lambda_0) a_0 e^{\lambda_0 z} + L(\lambda_1) a_1 e^{\lambda_1 z} + \cdots + L(\lambda_n) a_n e^{\lambda n z}.$$

In a joint paper of Pólya with André Bloch (Proc. London Math. Soc. (2) **33** (1932), 102–114; [128], II, pp. 336–348) the authors consider polynomials of the form

$$P(z) = 1 + \varepsilon_1 z + \varepsilon_2 z^2 + \cdots + \varepsilon_n z^n,$$

where each coefficient has one of three values -1, 0 or 1. These are now called *partition polynomials*, and for recent literature concerning them see the article by Morley Davidson (J. Math. Anal. Appl., **269** (2002), 431–443). There are 3^n such polynomials, so there are some which have a maximum number of zeros in the open interval (0,1). Denote this maximum number by π_n; clearly $0 \le \pi_n \le n$, and π_n increases with n. It is easy to see that $\pi_n = \mathrm{o}(n)$ and the authors say that the crucial question is whether π_n is of order as low as $\log n$. They find that the answer is negative. They prove that there exists a constant $A.0$ such that

$$\frac{1}{A}\frac{n^{1/4}}{(\log n)^{1/2}} < \pi_n < A\frac{n\log\log n}{\log n}$$

for $n \geq 3$.

Bloch and Pólya feel that "this question seems very particular and rather out of the way", therefore they present a number of examples and observations which motivate it. The first example is

$$\left(\frac{1}{p}\right) + \left(\frac{2}{p}\right) z + \left(\frac{3}{p}\right) z^2 + \cdots + \left(\frac{p-1}{p}\right) z^{p-2}, \tag{47}$$

where p is an odd prime number and the coefficients are the Legendre symbols. This is precisely the polynomial from which the Fekete–Pólya correspondence mentioned at the beginning of this section sets out. If p is such that (47) has no zeros in the interval $(0,1)$, then the Dirichlet series $\sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\frac{1}{n^s}$ has no positive real zeros, and the problem is to decide for which primes p is this the case. Fekete conjectured that (47) has no zeros in $(0,1)$. Pólya (Jahresber. Deutsch. Math.-Verein., **28** (1919), 31–40; [128], III, pp. 76–85) disproved the conjecture by a simple calculation which showed that for $p = 67$ and for $p = 167$ the polynomial has two zeros between 0 and 1.

Another example is the partial sums S_n of the power series

$$z - z^2 - z^3 + z^4 + z^5 + z^6 + z^7 - z^8 - \cdots - z^{15} + z^{16} + \cdots, \tag{48}$$

where on plus sign is followed by 2 minus signs, then 4 plus signs, 8 minus signs, etc. The number of zeros in (0,1) of S_n in $\log n/\log 2 + \mathrm{O}(1)$. This is related to two earlier articles of Pólya. The first (Nachr. Ges. Wiss. Göttingen 1930, 19–27; [128], I, pp. 459–467) is about the sign of the remainder term in the prime number theorem. Completing a result of Landau, he finds an upper bound for the smallest $\gamma > 0$, where $\zeta = \beta + i\gamma$ is a pole with maximal β of the function

$$\Phi(s) = \int_0^{\infty} \omega(u)u^{-s}\,du.$$

The bound is given in terms of the asymptotic behavior of the number of changes of sign of $\omega(u)$ in $0 < u \leq x$ as $x \to \infty$. The investigation is continued in a paper (Proc. London Math. Soc. (2) **33** (1932), 85–101; [128], I, pp. 521–537) which is printed preceding immediately the Bloch–Pólya article. Let (a_n) be a bounded sequence of real numbers, and set

$$D(s) = a_1 1^{-s} + a_2 2^{-s} + \cdots + a_n n^{-s} + \cdots ,$$

$$P(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots .$$

The Dirichlet series converges for $\Re e s > 1$ and the two functions are related by the formula

$$\Gamma(s) D(s) = \int_0^\infty P(e^{-x}) x^{s-1}\, dx.$$

Assume that $D(s)$ is meromorphic for $\Re e s > b$, where $b < 1$ and that, if $p(n)$ is the number of zeros of

$$P_n(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n$$

in the interval $(0, 1)$, then there exists an increasing sequence (n_m) of integers such that

$$\lim_{m\to\infty} \frac{\log n_{m+1}}{\log n_m} = 1, \quad p(n_m) = \mathrm{O}(\log n_m) \quad \text{as} \quad m \to \infty.$$

Then either $\Gamma(s)D(s)$ is holomorphic for $\Re e s > b$ or it has a pole $\beta + i\gamma$ with maximal β such that

$$0 \le \gamma \le \pi \limsup_{m\to\infty} \frac{p(n_m)}{\log n_m}. \tag{49}$$

The example (48) is used to show that equality on the right hand side of (49) can be attained.

12. Conformal mapping, complex interpolation

The most important concept to which the names of Lipót Fejér and Frigyes Riesz are attached was not published by the two either separately or jointly. It appeared in a note of Tibor Radó (Acta Sci. Math. Szeged **1** (1922/23), 240–251), and it is the *"Fejér–Riesz procedure"* for the proof of the Riemann mapping theorem. Radó's note is reproduced in part in Fejér's collected works ([40], II, pp. 841–842) and in its entirety in the works of Riesz ([156], pp. 1483–1494).

Carathéodory, who simplified slightly the procedure, writes the following (Bull. Calcutta Math. Soc., **20** (1928), 125–134; [21], III. pp. 300–301): "About ... the main theorem of conformal mapping I must say a few words. After the insufficiency of Riemann's original proof was recognized, the miraculously beautiful but very complicated methods of proof developed by H. A. Schwarz were the only paths to this theorem. Since about twenty years in rapid succession a large series of shorter and better proofs was proposed; but it was reserved for the Hungarian mathematicians L. Fejér and F. Riesz to return to the basic idea of Riemann and to relate again the solution of the problem of conformal mapping with a solution of a variational problem. But they did not choose a variational problem which, like Dirichlet's principle, is extraordinarily difficult to treat but one for which the existence of a solution is clear. In this way a proof came about which is only a few lines long and which was immediately adopted by all newer textbooks." In Fejér's collected works Turán quotes this passage in the original German ([40], II, pp. 842–843).

Riemann's mapping theorem states that if G is a simply connected domain in $\mathbb{C}$ having at least two boundary points and a is a point in G, then there exists a univalent holomorphic function f mapping G onto the disk $\{\zeta : |\zeta| < \rho\}$ and satisfying $f(a) = 0$, $f'(a) = 1$; the radius ρ and the function f are uniquely determined. An elementary transformation shows that we may assume G to be bounded and take $a = 0$. Radó explains the Fejér–Riesz procedure as follows: Consider all bounded, holomorphic functions f which map G univalently into the ζ-plane and satisfy $f(0) = 0$, $f'(0) = 1$. Such functions exist, e.g., $f(z) = z$. Set $M(f) = \sup_{z\in G} |f(z)|$ and let ρ be the greatest lower bound of all numbers $M(f)$. There exists a sequence (f_n) such that $M(f_n) \to \rho$. By Montel's theory of normal families (or – as we say now – compactness) a subsequence of (f_n) converges uniformly on every compact subset of G to a univalent, holomorphic function f satisfying $f(0) = 0, f'(0) = 1$ and $|f(z)| < \rho$ for $z \in G$. If the image of G under f does not fill out the whole disk $|\zeta| < \rho$, then a square root transformation due to Carathéodory and Koebe yields a function $F(z)$ which has the required properties and is such that $M(F) < M(f)$, which is impossible.

Radó realized that simple connectedness is not made use of in the proof of the Fejér–Riesz procedure, and he proves with it the so-called

"Grenzkreissatz". Also this proof is simplified in the article of Carathéodory quoted above.

An often quoted result of Tibor Radó (Acta Sci. Math. Szeged **2** (1925), 101–121) states that every Riemann surface satisfies the second axiom of countability. This is interesting because Heinz Prüfer has given an example of a two-dimensional differentiable manifold which does not satisfy the axiom: the countability is a consequence of the conformal structure, Radó also pointed out that, as a consequence of countability, every Riemann surface can be triangulated.

A short note of Henri Cartan has the title "Sur une extension d'un théorème de Radó" (Math. Ann., **125** (1952), 49–50; [22], II, pp. 667–668). The theorem referred to can be found in a paper (Math. Z., **20** (1924), 1–6) whose main result asserts that there exist open Riemann surfaces F which cannot be continued, i.e., there exists no Riemann surface G such that F is conformally equivalent to a proper subdomain of G.

Radó's "theorem" in which Cartan is interested is, however, the following Lemma in Radó's article:

Let G be a simply connected domain in the unit disk D which is distinct from D. Let $f(z)$ be holomorphic in D and assume that at every boundary point of G which lies in the interior of D the function $f(z)$ has boundary value zero. Then $f(z) \equiv 0$.

Peter Thullen (Math. Ann., **111** (1935), 137–157) gave a new proof and a generalization of the theorem. Then Heinrich Behnke and Karl Stein (Math. Ann., **124** (1951), 1–16) extended it to n variables (Satz 1). They use Radó's result and even reproduce its proof. Cartan found a very simple proof of the general theorem. He uses potential theory and does not need Radó's result. His assertion, slightly different from that of Behnke-Stein, is as follows:

Let M be an n-dimensional complex analytic manifold. Let g be a continuous, complex-valued function defined on M, and assume that g is holomorphic at each point z where $g(z) \neq 0$. Then g is holomorphic on M.

If $n = 1$ we obtain Radó's result setting $g(z) = f(z)$ for $z \in G$ and $g(z) = 0$ if $z \in D \setminus G$.

Conformal mapping and interpolation is the subject of a note of Fejér (Göttinger Nachrichten 1918, 319–333; [40], II, 100–111). Let C be a continuous, simple, closed curve in $\mathbb{C}$ and $z_1^{(k)}, \dots, z_k^{(k)}$ points on it for $k \in \mathbb{N}$. Fejér gives a sufficient condition for the following to happen:

If $f(z)$ is a function holomorphic inside C and on C itself, and if $L_k(z;f)$ is the Lagrange interpolation polynomial of F at the points $z_l^{(k)}$ $(1 \le l \le k)$, then $L_k(z;f)$ converges uniformly to $f(z)$ inside C as $k \to \infty$.

Let $\Phi(z)$ map conformally the exterior of C onto $|\zeta| > 1$ and satisfy $\Phi(\infty) = \infty$. Then Fejér's condition requires that the points $\zeta_l^{(k)} = \Phi\bigl(z_l^{(k)}\bigr)$ be the vertices of a regular k-gon inscribed in $|\zeta| = 1$. At the end of his note Fejér remarks that it would be sufficient to require that the $\zeta_l^{(k)}$ be uniformly distributed in the sense of Hermann Weyl.

The subject was taken up by László Kalmár in a prize essay he wrote as a student and which became his doctoral dissertation (Mat. Fiz. Lapok **33** (1926), 120–149). To describe his results, we change slightly our notation. Let $\Psi(z)$ be the unique holomorphic function which maps the exterior of C onto the exterior of a circle $|\zeta| = R$ and which satisfies

$$\lim_{z\to\infty} \frac{\Psi(z)}{z} = 1.$$

The uniquely determined radius $R = R_e(E)$ is the exterior mapping radius of the closure E of the inside of C. Let $\bigl(z_1^{(k)}, \dots, z_{n_k}^{(k)}\bigr)$ $(k \in \mathbb{N})$ be a sequence of n_k-tuples of points on C (the $z_j^{(k)}$, $1 \le j \le n_k$ do not have to be distinct, in that case $L_k(z;f;)$ denotes the Lagrange–Hermite interpolation polynomials). Denote by $\psi_k(z)$ that branch of the function

$$\Bigl\{\bigl(z - z_1^{(k)}\bigr)\bigl(z - z_2^{(k)}\bigr)\cdots\bigl(z - z_{n_k}^{(k)}\bigr)\Bigr\}^{1/n_k}$$

outside the curve C which satisfies

$$\lim_{z\to\infty} \frac{\psi_k(z)}{z} = 1.$$

For $0 \le a < b \le 2\pi$ denote by $\nu_k(a,b)$ the number of those points

$$\Psi\bigl(z_j^{(k)}\bigr) = Re^{i\theta_j^{(k)}}, \quad 1 \le j \le n_k,$$

whose arguments $\theta_j^{(k)}$ lie in $a \le \theta < b$. The following are equivalent:

a) $\lim_{k\to\infty} L_k(z;f) = f(z)$ uniformly for every function $f(z)$ holomorphic inside and on C, i.e., the points $z_j^{(k)}$ are "well-interpolating";

b) $\lim_{k\to\infty} \psi_k(z) = \Psi(z)$ outside C;

c)

$$\lim_{k\to\infty} \frac{\nu_k(a,b)}{n_k} = \frac{b-a}{2\pi}$$

for all $0 \le a < b \le 2\pi$.

For the equivalence of a) and b) the points $z_j^{(k)}$ can be chosen anywhere in E.

A different approach to finding well-interpolating points was found by Fekete using the concept of transfinite diameter introduced by him (Math. Z., **17** (1923), 246–249). Let E be a closed bounded set in $\mathbb{C}$, and $n \ge 2$ an integer. Denote by $\delta_n(E)$ the root of order

$$\binom{n}{2}$$

of the maximum of all expressions

$$\big|(z_1 - z_2)(z_1 - z_3)\cdots(z_2 - z_3)\cdots(z_{n-1} - z_n)\big| \tag{50}$$

as the $z_1, z_2, \ldots, z_n$ vary in E. The sequence of positive numbers $\delta_n(E)$ tends decreasingly to the transfinite diameter $\delta(E)$ of E.

It is a remarkable fact that $\delta(E)$ coincides with $R_e(E)$, and with the logarithmic capacity $c(E)$ (Szegő, Math. Z., **21** (1924), 203–208; [173], I, pp. 637–642). Consider furthermore the set $\mathcal{P}_n$ of all polynomials with leading coefficient 1, and denote by $M_n(E)^n$ the greatest lower bound of $\max_{z\in E} \big|P_n(z)\big|$ as $P_n(z)$ varies in $\mathcal{P}_n$. The "Čebishov constant" $M(E) = \lim_{n\to\infty} M_n(E)$ is also equal to $\delta(E)$.

For $n \ge 2$ the points (z_j^n) in E $(1 \le j \le n)$ for which (50) achieves its maximum are called *Fekete points.* Fekete proved that if E is the inside a continuous, simple, closed curve C together with C itself, then the Fekete points are well-interpolating (Z. Angew. Math. Mech., **6** (1926), 410–413).

This implies that the points $\Psi(z_j^n)$ are uniformly distributed. Kővári and Pommerenke obtained precise results about the distribution of Fekete points (Mathematika **15** (1968), 70–75, **18** (1971), 40–49).

13. Epilogue

And here, my friends, I cease. There is, however, much, much more. I hope I gave you a taste of some beautiful classical mathematics and the desire to read more about it. Fortunately this is easy, the works of Fejér, F. and M. Riesz, Pólya, Rényi, Szegő, Szász, Turán have appeared collected together (see Bibliography). They were not only titans of mathematics but also masters of exposition.

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