BOLYAI SOCIETY
MATHEMATICAL STUDIES, 14

A Panorama of Hungarian Mathematics in the Twentieth Century, pp. 29–54.

Constructive Function Theory: I. Orthogonal Series

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1. The Riesz–Fischer Theorem

Let us consider the family of measurable functions defined on a Lebesgue measurable subset E of finite or infinite measure of the real line $\mathbb{R} := (-\infty, \infty)$. The functions may take real or complex values. The function space $L^2(E)$ consists of all measurable functions f whose squares $|f|^2$ are integrable in the Lebesgue sense. By the Schwarz inequality, f will then be integrable on the subsets of finite measure. Let us endow $L^2(E)$ with the inner product and norm

$$(f \mid g) := \int_E f(x)\overline{g(x)}\, dx \quad \text{and} \quad \|f\| := \sqrt{(f \mid f)},$$

respectively. Then $L^2(E)$ becomes a normed linear space whose norm is derived from the inner product. We say that a sequence $(f_n : n = 1, 2, \ldots)$ of functions in $L^2(E)$ converges in the mean to a function f in $L^2(E)$ if

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

Observe that the limit function f is uniquely determined, up to a set of measure zero. In fact, if

$$\lim_{n\to\infty} \|f_n - f\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|f_n - g\| = 0,$$

*This survey paper was partially supported by the Hungarian National Foundation for Scientific Research under Grants TS 044 782 and T 046 192.

then by the triangle inequality we have

$$\|f - g\| \le \|f_n - f\| + \|f_n - g\| \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, $\|f - g\| = 0$, which means that $f(x) - g(x) = 0$ almost everywhere (in abbreviation: a.e.).

The Riesz–Fischer theorem states that the classical Cauchy convergence criterion is valid in the case of this mean convergence notion. As a consequence, the space $L^2(E)$ is complete, and this fact turned out to be of basic importance in the theory of Hilbert spaces.

Riesz–Fischer theorem. *Given a sequence (f_n) of functions in $L^2(E)$, then in order that there exist a function in $L^2(E)$ to which it converges in the mean it is necessary and sufficient that*

$$\|f_m - f_n\| \to 0 \quad \textit{as} \quad m, n \to \infty.$$

Frigyes Riesz and Ernst Fischer proved this theorem in 1907, independently of one another. Both of them published it in the Comptes Rendus Acad. Sci. Paris. However, Riesz submitted his paper two months earlier than Fischer. See [156, Vol. 1, C2, C5, pp. 378–381, 389–395], and also [203, Vol. 1, pp. 127–128, 377].

A system $(\phi_n : n = 1, 2, \ldots)$ of functions in $L^2(E)$ is called orthogonal if none of the ϕ_n vanishes a.e. and

$$(\phi_k \mid \phi_\ell) = 0 \quad \text{whenever} \quad k \ne \ell.$$

The system (ϕ_n) is called orthonormal (in abbreviation: ONS) if, besides the above condition of orthogonality, we also have

$$\|\phi_n\| = 1, \quad n = 1, 2, \ldots .$$

Clearly, each orthogonal system (ϕ_n) can be made normal by substituting $\big(\phi_n / \|\phi_n\|\big)$ for (ϕ_n).

Given a function f in $L^2(E)$, we form its expansion (one may say: generalized Fourier) coefficients with respect to the ONS (ϕ_n) as follows:

$$c_n := (f \mid \phi_n), \quad n = 1, 2, \ldots .$$

The series

$$\sum_{n=1}^{\infty} c_n \phi_n(x)$$

is called the orthogonal expansion (or we may say: generalized Fourier series) of f with respect to (ϕ_n). By the Bessel inequality, we have

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \|f\|^2.$$

Since

$$\left\| \sum_{k=m}^{n} c_k \phi_k \right\|^2 = \sum_{k=m}^{n} |c_k|^2 \to 0 \quad \text{as} \quad m, n \to \infty,$$

by the Riesz–Fischer theorem, the series $\sum c_k \phi_k$ converges in the mean to some function g in $L^2(E)$. Since mean convergence implies weak convergence, we have

$$(g \mid \phi_\ell) = \lim_{n\to\infty} \sum_{k=1}^{n} c_k(\phi_k, \phi_\ell) = c_\ell := (f \mid \phi_\ell),$$

whence

$$(f - g \mid \phi_\ell) = 0, \quad \ell = 1, 2, \ldots;$$

that is, $f - g$ is orthogonal to every ϕ_ℓ of the given ONS.

The ONS $(\phi_n,\ n = 1, 2, \ldots)$ is called maximal (sometimes called complete) if it cannot be enlarged in the sense that whenever a new function not vanishing a.e., say ψ, is added to it, then the new system $(\psi, \phi_n\ :\ n = 1, 2, \ldots)$ is no longer orthogonal. To sum up the above reasoning, we obtain the following theorem: Given a complete ONS in $L^2(E)$, every function f in $L^2(E)$ admits an expansion convergent in the mean:

$$f = \sum_{n=1}^{\infty} (f \mid \phi_n)\phi_n.$$

We say that a set of functions in $L^2(E)$ is total (sometimes called closed) if it determines the entire space in the sense that every function f in $L^2(E)$ can be approximated arbitrarily closely, in the mean, by the linear combinations of the elements of this set. We may say briefly that a total set of functions in $L^2(E)$ spans the whole space $L^2(E)$. By the Bessel identity

$$\left\| f - \sum_{k=1}^{n} (f \mid \phi_k)\phi_k \right\|^2 = \|f\|^2 - \sum_{k=1}^{n} \left| (f \mid \phi_k) \right|^2,$$

it is easy to see that the notions of "maximal" and "total" systems (ϕ_n) are equivalent; furthermore, these are equivalent to the validity of the Parseval formula

$$\|f\|^2 = \sum_{n=1}^{\infty} \big|(f \mid \phi_n)\big|^2, \quad f \in L^2(E).$$

Combining these reasonings gives the following.

Riesz–Fischer theorem for orthogonal systems. *Given an orthonormal system $(\phi_n : n = 1, 2, \ldots)$ in $L^2(E)$ (complete or not) and an arbitrary sequence $(a_n : n = 1, 2, \ldots)$ of complex numbers such that*

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty,$$

the orthogonal series $\sum a_n \phi_n$ converges in the mean to some function g in $L^2(E)$ such that the Fourier coefficients of g with respect to ϕ_n are these a_n:

$$(g \mid \phi_n) = a_n, \quad n = 1, 2, \ldots .$$

It is this form as the Riesz–Fischer theorem was originally stated and proved by Frigyes Riesz. This theorem was one of those impressive theorems which first demonstrated the effectiveness of the integral introduced by Lebesgue in 1902.

We note that if the system (ϕ_n) is maximal, then the function g in the above theorem is uniquely determined. If the system (ϕ_n) is not maximal, then the expansion $\sum (g \mid \phi_n)\phi_n$ is the orthogonal projection of the function g onto the closed subspace spanned by the functions of the system (ϕ_n).

The Riesz–Fischer theorem explains why we usually start with a sequence (a_n) of numbers in the space ℓ^2, instead of a function f in the space $L^2(E)$ in Sections 5–7 below.

Frigyes Riesz played an outstanding role in the development of the powerful new theory of functional analysis, which emerged from the theory of real and complex functions, linear algebra and topology, etc. As a matter of fact, the emerging functional analysis preceded the present form of the modern linear algebra. For example, the notion of duality first became clear for Banach spaces. Up to the 1940's, the finite dimensional vector spaces were identified with their duals.

2. Riesz Typical Means

Almost everywhere convergence of an orthogonal series is a much more complicated question than its convergence in the mean. We shall discuss the problem of the pointwise convergence of the partial sums in Section 5 in connection with the work by Károly Tandori.

Instead of convergence of the partial sums of an orthogonal series, one may consider various summability methods to assign a reasonable sum (if any) to the given series. The so-called Riesz typical means were introduced by Marcel Riesz [158, Parts 5, 6, pp. 55–58, 59–61] in 1909 as follows.

Let $\lambda = (\lambda_n : n = 0, 1, \ldots)$ be a sequence of real numbers such that

$$0 \le \lambda_0 < \lambda_1 < \lambda_2 < \ldots \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \infty, \tag{2.1}$$

and let α be a positive number. Given a series $\sum_{n=0}^{\infty} u_n$ of real or complex numbers, the so-called Riesz typical means of type λ and of order α are defined by

$$R(\omega; \lambda, \alpha) := \sum_{\lambda_k \le \omega} \left(1 - \frac{\lambda_k}{\omega}\right)^{\alpha} u_k, \quad \omega > 0.$$

The series $\sum u_n$ is said to be summable to some L by the Riesz method of type λ and of order α if the finite limit

$$\lim_{\omega\to\infty} R(\omega; \lambda, \alpha) = L$$

exists. M. Riesz introduced these means for the study of the behaviour of Dirichlet series $\sum a_n e^{-\lambda_n z}$, where (a_n) is a given sequence of complex numbers and z is a complex variable.

In the particular case where $\omega := n + 1$ and $\lambda_n := n$, we get

$$R_n(\alpha) := R\big(n + 1; (n), \alpha\big) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^{\alpha} u_k,$$

which is fairly similar to the expression of the nth Cesàro mean of order α of the series $\sum u_n$ provided α is a positive integer:

$$C_n(\alpha) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)\left(1 - \frac{k}{n+2}\right) \ldots \left(1 - \frac{k}{n+\alpha}\right) u_k, \quad n = 0, 1, \ldots .$$

However, the parameter ω of the Riesz method approaches ∞ in a continuous way.

Let us consider another special case where $\omega := \lambda_n$ and $\alpha := 1$. Then

$$R(\lambda_n; \lambda, 1) := \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_n - \lambda_k) u_k = \frac{1}{\lambda_n} \sum_{k=0}^{n-1} (\lambda_{k+1} - \lambda_k) s_k,$$

where $s_k := \sum_{j=0}^{k} u_j$ is the kth partial sum. If, in addition, $\lambda_n := n$, then $R_n(1)$ is the first arithmetic mean of the partial sums.

Now, we return to the general case and introduce the notation

$$T(\omega; \lambda, \alpha) := \sum_{\lambda_k \le \omega} (\omega - \lambda_k)^{\alpha} u_k,$$

while in case $\alpha = 0$ set

$$T_\lambda(\omega) := T(\omega; \lambda, 0) = \sum_{\lambda_k \le \omega} u_k,$$

the latter may be called the partial sum function. Clearly, we have

$$R(\omega; \lambda, \alpha) = \frac{T(\omega; \lambda, \alpha)}{\omega^{\alpha}}.$$

We can express $T(\omega; \lambda, \alpha)$ in terms of a Riemann–Stieltjes integral as follows:

$$T(\omega; \lambda, \alpha) = \int_{\lambda_0}^{\omega} (\omega - \tau)^{\alpha} dT_\lambda(\tau) \quad \text{if} \quad \lambda_0 > 0,$$

while in case $\lambda_0 = 0$ we have to add the term $\omega^{\alpha} u_0$ to the right-hand side. By integration by parts, in either case we obtain

$$T(\omega; \lambda, \alpha) = \alpha \int_0^{\omega} (\omega - \tau)^{\alpha - 1} T_\lambda(\tau)\, d\tau.$$

Let us observe that the integral on the right-hand side coincides, up to a constant, with the Riemann–Liouville integral

$$(I^{\alpha} f)(\omega) := \frac{1}{\Gamma(\alpha)} \int_0^{\omega} (\omega - \tau)^{\alpha - 1} f(\tau)\, d\tau,$$

which is used for defining the fractional integral of f of order α, when it is applied to $f = T_\lambda$. Here $\Gamma(\alpha)$ is the Euler gamma function. From the identity

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

it follows that

$$R(\omega;\lambda,\alpha) = \frac{\Gamma(\alpha+1)}{\omega^\alpha}(I^\alpha T_\lambda)(\omega).$$

On the other hand, for the binomial coefficient

$$A_n^{(\alpha)} := \binom{n+\alpha}{n},$$

it follows from the Stirling formula that

$$\lim_{n\to\infty}\frac{A_n^{(\alpha)}}{n^\alpha} = \frac{1}{\Gamma\alpha+1)}, \quad \alpha > 0.$$

Hence, for the nth Cesàro mean $C_n(\alpha)$ we obtain the following asymptotic equality:

$$C_n(\alpha) := \frac{S_n^{(\alpha)}}{A_n^{(\alpha)}} \approx \frac{\Gamma(\alpha+1)}{n^\alpha}S_n^{(\alpha)},$$

where

$$S_n^{(\alpha)} := \sum_{k=0}^{n} A_{n-k}^{(\alpha)} u_k = \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)} s_k.$$

In the special case where α runs over the positive integers, the following recursive definition is equivalent to the above one:

$$S_n^{(j)} := \sum_{k=0}^{n} S_k^{(j-1)} \quad \text{for} \quad j = 1, 2, \ldots, \quad \text{and} \quad S_n^{(0)} := s_n.$$

To sum up, the "integral of the partial sum function T_λ of fractional order α" occurs in the case of the Riesz typical means, while the "α times iteration of the partial sum s_n" comes in for the Cesàro means. This makes plausible the conjecture that the Cesàro method of summability of order α is equivalent to the Riesz method of summability of type $\lambda = (\lambda_n := n)$ and of order α whenever ω runs over the positive real numbers. This conjecture turned out to be true [158, Part 9, pp. 72–75], but its proof is rather complicated. (See the detailed proof for integer α in [68, §5.16, p. 113].)

On the other hand, the equivalence of the Cesàro and Riesz methods of summability is no longer true if ω runs over only the positive integers.

Marcel Riesz discovered the above summability method, named after him, in 1909, but he but he gave a detailed treatment of it only in 1915 in hiscitep [66] with G. H. Hardy.

The Riesz method of summability became of great significance in the theory of multiple Fourier series, as Jean-Pierre Kahane tells us in the Section "Commutative Harmonic Analysis". Here we confine ourselves to mention another result in the theory of orthogonal series. Given any orthogonal series

$$\sum_{n=1}^{\infty} a_n \phi_n(x) \quad \text{with} \quad \sum |a_n|^2 < \infty,$$

there exists a sequence $\lambda = (\lambda_n)$ satisfying the conditions in (2.1) such that the orthogonal series in question is summable a.e. by the Riesz method $R_n(\lambda_n; \lambda, 1)$.

Marcel Riesz achieved several significant results and each of them opened the way to new branches of analysis. Among others, the famous Riesz–Thorin convexity theorem started the development of the interpolation theory of linear operators, indispensable in modern harmonic analysis. M. Riesz also revealed the behavior of the harmonic conjugate to functions in the Lebesgue spaces L^p, where $1 < p < \infty$. Last but not least, the celebrated Riesz brothers' theorem on the unit circle surprisingly connected the theory of Fourier series with measure theory, thereby initiated a large number of deep results in both classical and abstract harmonic analysis.

3. The Haar Orthogonal System

An interesting and very useful orthonormal system was constructed by Alfréd Haar [62, B1, pp. 45–87] in his doctoral dissertation in 1909. The functions are defined on the interval $[0, 1]$ and are conveniently labelled by two indices:

$$\chi_0^{(0)}(x); \chi_0^{(1)}(x); \chi_1^{(1)}(x), \chi_1^{(2)}(x); \ldots; \chi_n^{(1)}(x), \ldots, \chi_n^{(2^n)}(x); \ldots .$$

To go into details, the first of them, $\chi_0^{(0)}(x)$ is identically equal to 1; while for $n \geq 0$ and $1 \leq k \leq 2^n$, we set

$$\chi_n^{(k)}(x) := \begin{cases} 2^{n/2} & \text{for } x \in \left(\dfrac{k-1}{2^n},\ \dfrac{k-1/2}{2^n}\right), \\ -2^{n/2} & \text{for } x \in \left(\dfrac{k-1/2}{2^n},\ \dfrac{k}{2^n}\right), \\ 0 & \text{for } x \in \left(\dfrac{\ell-1}{2^n},\ \dfrac{\ell}{2^n}\right) \text{ with } \ell \neq k,\ 1 \leq \ell \leq 2^n; \end{cases}$$

furthermore, at the points of discontinuity let $\chi_n^{(k)}(x)$ be equal to the arithmetic mean of the values assumed on the two adjacent intervals; and at the points $x = 0$ and $x = 1$ let $\chi_n^{(k)}(x)$ take the same value as on the interval $(0, 2^{-n-1})$ and $(1 - 2^{-n-1}, 1)$, respectively.

It is easy to see that the Haar system is an ONS on the interval $[0, 1]$. Let $f(x)$ be a function integrable on $[0, 1]$ in the Lebesgue sense. Its expansion with respect to the Haar system

$$f(x) \sim a_0^{(0)}\chi_0^{(0)}(x) + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} a_n^{(k)}\chi_n^{(k)}(x), \tag{3.1}$$

where

$$a_0^{(0)} := \int_0^1 f(t)\,dt, \quad a_n^{(k)} := \int_0^1 f(t)\chi_n^{(k)}(t)\,dt,$$

has remarkable properties of representing the function $f(x)$. The following theorem was theorem was proved citep Haar [62, B1, B2, pp. 45–87, 88–103].

Convergence theorem of Haar. (i) *If $f(x)$ is integrable in the Lebesgue sense on the interval $(0, 1)$, then its Haar expansion (3.1) converges to $f(x)$ a.e.*

(ii) *The Haar expansion (3.1) converges to $f(x)$ at each point of continuity of $f(x)$, and converges uniformly on each interval on which $f(x)$ is uniformly continuous.*

Statement (ii) expresses one of the most interesting features of the Haar system that every continuous function on the interval $[0, 1]$ is uniformly approximated by the partial sums of its expansion with respect to the discontinuous functions of the Haar system.

Denote by $s_n^{(k)}(x)$ the partial sum of the series in (3.1) ending with the term $a_n^{(k)}\chi_n^{(k)}(x)$. Then by introducing the kernel

$$K_n^{(k)}(x,t) := \chi_0^{(0)}(x)\chi_0^{(0)}(t) + \chi_0^{(1)}(x)\chi_0^{(1)}(t) + \ldots + \chi_n^{(k)}(x)\chi_n^{(k)}(t),$$

we get the following representation:

$$s_n^{(k)}(x) = \int_0^1 f(t)K_n^{(k)}(x,t)\,dt.$$

The convergence theorem of Haar follows in a natural way from the nice properties of the kernel $K_n^{(k)}(x,t)$. It turns out that if x is not a dyadic rational number in $(0,1)$, then $K_n^{(k)}(x,t)$ differs from zero only on an interval $I_n^{(k)}$ (containing x) of length $\left|I_n^{(k)}\right| = 2^{-n}$ or 2^{-n-1}, on which it takes the value 2^n or 2^{n+1}, respectively. Therefore, we have

$$s_n^{(k)}(x) = \frac{1}{\left|I_n^{(k)}\right|}\int_{I_n^{(k)}} f(t)\,dt. \tag{3.2}$$

If x is a dyadic rational number in $(0,1)$, then

$$K_n^{(k)}(x,t) = \begin{cases} 2^n & \text{on } I_n^{(k)} := (x-2^{-n-1}, x), \\ 2^{n-1} & \text{on } J_n^{(k)} := (x, x+2^{-n}), \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, in this case we have

$$s_n^{(k)}(x) = \frac{1}{2\left|I_n^{(k)}\right|}\int_{I_n^{(k)}} f(t)\,dt + \frac{1}{2\left|J_n^{(k)}\right|}\int_{J_n^{(k)}} f(t)\,dt. \tag{3.3}$$

If $x=0$ or $x=1$, then the first or the second term on the right-hand side in (3.3) should be deleted, respectively.

If we let n tend to infinity, the intervals $I_n^{(k)}$ and $J_n^{(k)}$ contract into the point x and

$$\lim_{n\to\infty} s_n^{(k)}(x) = F'(x),$$

provided that the derivative of the integral

$$F(u) := \int_0^u f(t)\,dt$$

exists at the point $u = x$. As it is well known, the derivative $F'(x)$ exists at almost every x. So, statement (i) of the convergence theorem of Haar follows at once. The proof of statement (ii) also follows almost immediately from (3.2) and (3.3).

As a by-product of the above reasoning, we can conclude that the Haar system is maximal. Indeed, if each expansion coefficient of a function $f(x)$ equals zero in (3.1), then each partial sum $s_n^{(k)}(x) \equiv 0$. Thus, by statement (i) above, we necessarily have $f(x) = 0$ a.e. even in the case when $f(x)$ is only integrable. This proves the maximality of the Haar system not only in $L^2(0,1)$, but also in the larger space $L(0,1)$.

The Haar wavelet basis evolved from the classical Haar system, was the first prototype of the wavelet theory, one of the major events in Harmonic Analysis started in the 1980's. Signal processing, image compression, and many other areas of applied mathematics have been revolutionized because of wavelet theory. We note that the Nobel Prize winning physicist Dénes Gábor essentially discovered the notion of wavelets and applied them successfully in his physical computations already in 1946, decades before the rigorous mathematical theory.

In 1933, Alfréd Haar also solved the problem of the existence of an invariant measure on topological groups, thus giving one of the most powerful tools to all subsequent investigations of such groups. In the Section on "Noncommutative Harmonic Analysis", Jonathan Rosenberg will tell the details of this story.

4. The Saturation Problem for the Fejér Means

Let f be a 2π-periodic (real or complex-valued) function, integrable in the Lebesgue sense on the torus $[-\pi, \pi)$, in symbol: $f \in L_{2\pi}$. The Fourier series of f is given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \tag{4.1}$$

where

$$a_k := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt, \quad b_k := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt$$

are the Fourier coefficients. Denote by

$$s_n(f,x) := \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

the nth partial sum, and by

$$\begin{aligned}\sigma_n(f,x) &:= \frac{1}{n+1}\sum_{k=0}^{n} s_k(f,x)\\ &= \frac{1}{2}a_0 + \sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)(a_k \cos kx + b_k \sin kx)\end{aligned}$$

the first arithmetic, so-called Fejér mean, of series (4.1).

In 1900 Lipót Fejér proved that if a 2π-periodic function is continuous, then the means $\sigma_n(f,x)$ converge uniformly to $f(x)$. In 1905, Lebesgue proved that if $f \in L_{2\pi}$, then the means $\sigma_n(f,x)$ converge to $f(x)$ a.e. For more details see the Section "Commutative Harmonic Analysis" written by Jean-Pierre Kahane.

We recall that the conjugate series to (3.1) is defined by

$$\sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx). \tag{4.2}$$

In 1911, I. I. Privalov proved that the first arithmetic means of series (4.2) (the so-called conjugate Fejér means) also converge a.e. for each $f \in L_{2\pi}$. This limit is called the conjugate function to f and denoted by $\tilde{f}$. This $\tilde{f}$ is not necessarily integrable in the Lebesgue sense. However, if $\tilde{f} \in L_{2\pi}$, then (4.2) is the Fourier series of $\tilde{f}$.

The conjugate function $\tilde{f}$ can be represented in the form of a Cauchy principal value integral as follows:

$$\begin{aligned}\tilde{f}(x) &= (\text{P.V.})\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x-t)}{2\tan\frac{t}{2}}\,dt\\ &=: \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\varepsilon}^{\pi}\frac{f(x-t)-f(x+t)}{2\tan\frac{t}{2}}\,dt,\end{aligned}$$

where the limit exists a.e. whenever $f \in L_{2\pi}$. This explains why $\tilde{f}$ is sometimes called the periodic Hilbert transform of f.

A 2π-periodic function f is said to satisfy the uniform Lipschitz condition of order $\alpha > 0$, in symbol: $f \in \operatorname{Lip}_{2\pi} \alpha$, if

$$\omega(f,\delta) := \sup\Big\{\big|\, f(x_1) - f(x_2)\big| \ : \ |x_1 - x_2| \le \delta\Big\} = \mathcal{O}(\delta^{\alpha}), \quad 0 \le \delta < 2\pi.$$

Only the case $0 < \alpha \le 1$ is interesting: if $\alpha > 1$, then $\omega(f,\delta)/\delta$ tends to zero with δ. Consequently, in this case $f'(x)$ exists and is zero everywhere, and f is constant. The function $\omega(f,\delta)$, $0 \le \delta < 2\pi$, is called the modulus of continuity of f. It is clear that a function f is uniformly continuous if and only if $\omega(f,\delta)$ tends to zero with δ. On the other hand, f belongs to $\operatorname{Lip}_{2\pi} 1$ if and only if f is the antiderivative of a bounded function.

It is easy to check that if $f \in \operatorname{Lip}_{2\pi} \alpha$, then

$$\big\|\, \sigma_n(f) - f \big\|_C := \max_x \big|\, \sigma_n(f,x) - f(x)\big|$$

$$= \begin{cases} \mathcal{O}(n^{-\alpha}) & \text{if } \ 0 < \alpha < 1, \\ \mathcal{O}(n^{-1}\ln n) & \text{if } \ \alpha = 1. \end{cases}$$

In 1941, György Alexits {1} proved the following remarkable characterization.

Theorem of Alexits. *A necessary and sufficient condition for the relation*

$$\big\|\, \sigma_n(f) - f \big\|_C = \mathcal{O}(n^{-1}) \tag{4.3}$$

is that the conjugate function $\tilde{f} \in \operatorname{Lip}_{2\pi} 1$.

This theorem becomes even more significant in the light of the following simple observation: If

$$\big\|\, \sigma_n(f) - f \big\|_C = o(n^{-1}), \tag{4.4}$$

then we have necessarily $f \equiv$ constant. In fact, since

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \big[\, f(x) - \sigma_n(f,x)\big] \cos kx = ka_k/(n+1), \quad 1 \le k \le n,$$

relation (4.4) implies that the left-hand side here is $o(n^{-1})$, which means that $a_k = 0$ for $k \ge 1$. Analogously, we have $b_k = 0$ for $k \ge 1$. Consequently, $f \equiv \frac{1}{2}a_0$ as we have claimed.

Using the terminology introduced by Jean Favard, we may say that the Fejér means are saturated with saturation order $\mathcal{O}(n^{-1})$. The collection of functions f for which relation (4.3) is satisfied, is called the saturation class for the Fejér means. Now, the above theorem of Alexits says that the saturation class for the Fejér means consists of those functions f for which $\tilde{f} \in \mathrm{Lip}_{2\pi} 1$.

At this point, we have to correct a historical misunderstanding in the literature. Unfortunately, it had been a widespread belief that G. Alexits proved only the sufficiency part in the above theorem; that is, if $\tilde{f} \in \mathrm{Lip}_{2\pi} 1$, then (4.3) is satisfied. For example, A. Zygmund in his monograph [203, Vol. 1, p. 377] writes the following: "The sufficiency part... was proved by Alexits, the necessity by Zamansky." However, reading the text of {1} makes it clear that G. Alexits did prove both the necessity and the sufficiency part even in the case of the more general $(C, \alpha \geq 1)$ Cesàro means. We emphasize that Alexits' paper appeared 8 years earlier than that of Marc Zamansky {21}. What could have been the reason for this misunderstanding? We guess that the answer lies in the fact that Alexits' paper appeared in a Hungarian periodical in 1941, the third year of World War II.

The first international recognition of Alexits' paper was given by R. A. De Vore {3, pp. 59–60}, who wrote the following: "In 1941, G. Alexits in a now classical theorem gave a characterization of the saturation class for Fejér operators (the 'o' saturation theorem for Fejér operators goes back to Zygmund). This paper of Alexits was the beginning of the study of saturation of convolution operators. In 1949, J. Favard gave a general formulation of the phenomenon of saturation for convolution operators. Later, M. Zamansky initiated a systematic study of saturation for convolution with trigonometric polynomials. In particular, Zamansky was able to recover the Alexits theorem."

Last but not least, Alexits was one of the creators of the Hungarian orthogonal school in the middle of the twentieth century, who envisaged several far-reaching problems in the theory of orthogonal series and approximation theory. He had numerous disciples in Hungary as well as all over the world. His monograph [8] on orthogonal series appeared in German in 1960, translated into English the following year, and into Russian in 1963. It has became the standard reference book for researchers in the theory of orthogonal series.

Motivated by the notion of strong $(C, 1)$-summability introduced by G. H. Hardy and J. E. Littlewood in 1913, Alexits {2} introduced the notion of strong approximation by Fourier series in 1963, and by raising several problems in subsequent papers he laid the foundation of the so-called strong approximation, a new branch of approximation theory. The strong approximation by Fourier series as well as by general orthogonal series have been studied, while considered not only Fejér but also Cesàro and other means. We refer the reader to the monograph of László Leindler [108].

5. Almost Everywhere Convergence of Orthogonal Series

One of the main problems in the theory of orthogonal series is to characterize the pointwise behavior of the orthogonal series

$$\sum_{k=1}^{\infty} a_k \phi_k(x) \tag{5.1}$$

in terms of its coefficients a_k, where (a_k) is a sequence of real numbers and $\big(\phi_k(x)\big)$ is an arbitrary ONS on a finite interval, say $(0,1)$ for the sake of simplicity.

By the Schwarz inequality and the Beppo Levi theorem, the condition

$$\sum_{k=1}^{\infty} |a_k| < \infty \tag{5.2}$$

implies even the absolute convergence of series (5.1) a.e. But this requirement is too strong for a more delicate study of convergence problems. On the other hand, the requirement

$$\sum_{k=1}^{\infty} a_k^2 < \infty \tag{5.3}$$

is indispensable, because in the special case where $\big(\phi_k(x)\big)$ is the familiar Rademacher system, the orthogonal series (5.1) converges a.e. if and only if (5.3) is satisfied. Thus, the useful convergence tests lie between conditions (5.2) and (5.3).

(i) *Sufficient condition: Rademacher* (1922), *Menshov* (1923). The most important convergence test was discovered nearly simultaneously by Hans Rademacher {13} and D. E. Menshov {8}.

Rademacher–Menshov theorem. *If*

$$\sum_{k=1}^{\infty} a_k^2 (\log k)^2 < \infty, \tag{5.4}$$

then the orthogonal series (5.1) *converges a.e.*

Here and in the sequel the logarithm is to the base 2, but any base greater than 1 would be appropriate.

It was D. E. Menshov {8} who first observed that condition (5.4) is the best possible in general.

Theorem of Menshov. *If* $\big(w(k) \,:\, k = 1, 2, \dots\big)$ *is an increasing sequence of positive numbers with*

$$w(k) = o(\log k) \quad as \quad k \to \infty,$$

then there exist a sequence (a_k) *of numbers and an* ONS $\big(\phi_k(x)\big)$ *such that*

$$\sum_{k=1}^{\infty} a_k^2 w^2(k) < \infty \tag{5.5}$$

and the orthogonal series (5.1) *diverges everywhere.*

(ii) *Necessary condition: Tandori* (1957). The breakthrough in the convergence (or one may say, in the divergence) theory of orthogonal series was achieved by Károly Tandori {14}, who proved the following deep complement of the Rademacher–Menshov theorem.

Divergence theorem of Tandori. *If*

$$|a_1| \geq |a_2| \geq \dots \tag{5.6}$$

and

$$\sum_{k=1}^{\infty} |a_k|^2 (\log k)^2 = \infty, \tag{5.7}$$

then there exists an ONS $\big(\phi_k(x)\big)$ *depending on* (a_k) *such that the orthogonal series* (5.1) *diverges everywhere.*

It is easy to see that the theorem of Menshov above is a consequence of the divergence theorem of Tandori.

Combining the Rademacher–Menshov theorem with the divergence theorem of Tandori gives that if (a_k) satisfies (5.6), then condition (5.4) is not only sufficient, but also necessary that the orthogonal series (5.1) converge a.e. for all ONS $\big(\phi_k(x)\big)$.

Starting with {14}, which contains the divergence theorem, K. Tandori wrote a series of ten papers entitled "Über die orthogonalen Funktionen I–X", which is the longest of its kind ever published in the "Acta Scientiarum Mathematicarum". By constructing delicate counterexamples, he gave a great impetus to the further development of the theory of orthogonal series. His contribution is commensurable with the one D. E. Menshov exerted 30 years earlier. The majority of these results are collected in the monograph by G. Alexits [8], which serves as a by G. Alexits [8], which serves as a reference book even nowadays.

The divergence theorem of Tandori has several consequences. We cite here only one of them. It follows from the Rademacher–Menshov theorem and the familiar Kronecker lemma that if $\big(\lambda(k) \;:\; k = 1, 2, \dots\big)$ is an increasing sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \frac{(\log k)^2}{\lambda^2(k)} < \infty, \tag{5.8}$$

then for every ONS $\big(\phi_k(x)\big)$ we have

$$\lim_{n\to\infty} \frac{1}{\lambda(n)} \sum_{k=1}^{n} \phi_k(x) = 0 \quad \text{a.e.} \tag{5.9}$$

Now, from his divergence theorem K. Tandori {14} deduced that condition (5.8) is the best possible for the validity of (5.9).

Corollary. *If* $\big(\lambda(k)\big)$ *is an increasing sequence of positive numbers such that*

$$\sum_{k=1}^{\infty} \frac{(\log k)^2}{\lambda^2(k)} = \infty,$$

then there exists an ONS $\big(\phi_k(x)\big)$ *such that*

$$\limsup_{n\to\infty} \frac{1}{\lambda(n)} \left| \sum_{k=1}^{n} \phi_k(x) \right| = \infty \quad \textit{a.e.}$$

In particular, set

$$\lambda(n) := \left\{ n(\log n)^3 (\log\log n)^{1+\varepsilon} \right\}^{1/2}.$$

Then (5.9) holds true for $\varepsilon > 0$, but it is no longer true for $\varepsilon = 0$. In a certain sense, (5.9) can be related to the "strong law of large numbers" well known in probability.

(iii) *Synthesis: Tandori* (1964). Denote by $\mathfrak{M}$ the class of those sequences $\mathfrak{a} = (a_k)$ of real numbers for which the orthogonal series (5.1) converges a.e. for all ONS $\big(\phi_k(x)\big)$, where the set of measure zero of divergence points may depend on the particular ONS $\big(\phi_k(x)\big)$. For each $N = 1, 2, \ldots$ introduce the quantity

$$I(a_1, a_2, \ldots, a_N) := \sup \int_0^1 \left(\max_{1 \le n \le N} \left| \sum_{k=1}^{n} a_k \phi_k(x) \right| \right)^2 dx,$$

where the supremum is taken over all ONS $\big(\phi_k(x)\big)$. The following characterization of $\mathfrak{M}$ is due to K. Tandori {15}.

Characterization theorem of Tandori. *A sequence $\mathfrak{a} = (a_k)$ of numbers belongs to $\mathfrak{M}$ if and only if*

$$\|\mathfrak{a}\| := \lim_{N\to\infty} I^{1/2}(a_1, a_2, \ldots, a_N) < \infty. \tag{5.10}$$

Furthermore, the set $\mathfrak{M}$ is a reflexive Banach space with respect to the usual vector operations and the norm defined in (5.10).

The significance of this theorem is that it reduces the problem of a.e. convergence of general orthogonal series to the problem of evaluating the norm defined in (5.10). The Rademacher–Menshov theorem provides an upper estimate, while the divergence theorem of Tandori provides a lower estimate for this norm, the latter being valid only in the monotonicity case (5.6).

Corollary. *There exist positive constants C_1 and C_2 such that for every sequence $\mathfrak{a} = (a_k)$ of numbers we have*

$$\|\mathfrak{a}\|^2 \le C_1 \sum_{k=1}^{\infty} a_k^2 (\log 2k)^2;$$

and for every sequence $\mathfrak{a} = (a_k)$ *of numbers satisfying condition* (5.6), *we have*

$$\|\mathfrak{a}\|^2 \geq C_2 \sum_{k=1}^{\infty} a_k^2 (\log 2k)^2.$$

The space $\mathfrak{M}$ enjoys the following remarkable property. Let $\mathfrak{a} = (a_k)$ and $\mathfrak{b} = (b_k)$ be sequences of numbers for which

$$|a_k| \leq |b_k|, \quad k = 1, 2, \dots .$$

If $\mathfrak{b} \in \mathfrak{M}$, then $\mathfrak{a} \in \mathfrak{M}$. If $\mathfrak{a} \notin \mathfrak{M}$, then $\mathfrak{b} \notin \mathfrak{M}$.

(iv) *Sign type* ONS: *Kashin* (1976). In the examples of divergent orthogonal series given by D. E. Menshov {9} and K. Tandori {14}, the orthonormal functions ϕ_k can be chosen to be uniformly bounded:

$$\big|\phi_k(x)\big| \leq C, \qquad 0 \leq x \leq 1; \quad k = 1, 2, \dots, \tag{5.11}$$

with some constant C.

The most important case is when $C = 1$. In this case, we necessarily have

$$\big|\phi_k(x)\big| = 1, \qquad 0 \leq x \leq 1; \quad k = 1, 2, \dots;$$

and the ONS $\big(\phi_k(x)\big)$ is said to be sign type. Now, B. S. Kashin {6} improved the divergence theorem of Tandori as follows.

Divergence theorem of Kashin. *If conditions* (5.6) *and* (5.7) *are satisfied, then there exists a sign type* ONS $\big(\phi_k(x)\big)$ *depending on* (a_k) *such that the orthogonal series* (5.1) *diverges everywhere.*

Later K. Tandori {16} gave an extremely short proof of this surprising result.

Let $1 \leq C \leq \infty$ and denote by $\mathfrak{M}(C)$ the class of those sequences $\mathfrak{a} = (a_k)$ of real numbers for which the orthogonal series (5.1) converges a.e. for all ONS $\big(\phi_k(x)\big)$ satisfying condition (5.11). It is clear that if $1 < C_1 < C_2 < \infty$, then

$$\mathfrak{M}(\infty) \subseteq \mathfrak{M}(C_2) \subseteq \mathfrak{M}(C_1) \subseteq \mathfrak{M}(1).$$

Now, K. Tandori {17} proved that

$$\mathfrak{M}(C) = \mathfrak{M}(1), \quad 1 \leq C < \infty.$$

In other words, there is no difference between the class of all sign type ONS and the class of all uniformly bounded ONS as to the convergence a.e. of the orthogonal series (5.1). But the problem of whether $\mathfrak{M}(\infty) = \mathfrak{M}(1)$ is still open.

(v) *Rademacher–Menshov inequality revisited: Móricz and Tandori* (1996). The following sharpened form has been proved by Ferenc Móricz and Károly Tandori {11}.

Convergence theorem of Móricz and Tandori. *If for some* $0 < \varepsilon \le 2$ *we have*

$$\sum_{m=0}^{\infty} \sum_{k\in I_m} a_k^2 (\log k)^{\varepsilon} \left(\log \frac{2A_m^2}{a_k^2}\right)^{2-\varepsilon} < \infty, \tag{5.12}$$

where

$$I_m := \{2^m+1, 2^m+2, \dots, 2^{m+1}\} \tag{5.13}$$

and

$$A_m^2 := \sum_{k\in I_m} a_k^2, \quad m = 0, 1, \dots,$$

then the orthogonal series (5.1) *converges a.e.*

Denote by $\sum_\varepsilon$ the sum of the series occurring in (5.12). It is not difficult to show that if $0 \le \delta < \varepsilon \le 2$ and $\sum_\varepsilon < \infty$, then $\sum_\delta < \infty$ as well. Thus, the smaller ε is, the weaker is condition (5.12). Furthermore, if the sequence (a_k) satisfies condition (5.6) and $\sum_\varepsilon < \infty$ for some $0 \le \varepsilon \le 2$, then $\sum_\varepsilon < \infty$ for all $0 \le \varepsilon \le 2$.

It is interesting to point out that condition (5.12) for $\varepsilon = 0$ no longer guarantees the a.e. convergence of the orthogonal series (5.1). However, condition (5.12) for $\varepsilon = 0$ is necessary in order that the orthogonal series (5.1) converge a.e. for all ONS $\big(\phi_k(x)\big)$. Even somewhat more was shown by Károly Tandori {18}: If the orthogonal series $\sum a_k\phi_k(x)$ converges a.e. for all ONS $\big(\phi_k(x)\big)$, then

$$\sum_{k=1}^{\infty} a_k^2 \left(\log_+ \frac{1}{a_k^2}\right)^2 < \infty, \tag{5.14}$$

where

$$\log_+ u := \begin{cases} \log u & \text{if } u \ge 2 \\ 1 & \text{if } u < 2. \end{cases}$$

Obviously, (5.14) implies (5.12) for $\varepsilon = 0$. To sum up, condition (5.12) for some $\varepsilon > 0$ is sufficient, while for $\varepsilon = 0$ is necessary in order that the orthogonal series (5.1) converge a.e. for all ONS $\big(\phi_k(x)\big)$. Furthermore, these conditions are equivalent in the special case when (a_k) satisfies condition (5.6).

6. Cesàro Summability of Orthogonal Series

Similarly to trigonometric Fourier series, one can expect better convergence behavior of orthogonal series if ordinary convergence is replaced by $(C,1)$-summability. In fact, this is the case, but the improvement is less than that in the case of trigonometric series.

We recall that the first arithmetic or $(C,1)$-mean of the orthogonal series (5.1) with partial sum

$$s_n(x) := \sum_{k=1}^{n} a_k \phi_k(x)$$

is defined by

$$\sigma_n(x) := \frac{1}{n}\sum_{k=1}^{n} s_k(x) = \sum_{k=1}^{n}\left(1 - \frac{k-1}{n}\right) a_k\phi_k(x), \quad n = 1, 2, \ldots .$$

The whole theory of $(C,1)$-summability of orthogonal series is based on the following observations of A. N. Kolmogorov {7} and Stefan Kaczmarz {4}, respectively.

Theorem of Kolmogorov and Kaczmarz. *If condition* (5.3) *is satisfied, then*

$$\lim_{m\to\infty} \big\{ s_{2^m}(x) - \sigma_{2^m}(x)\big\} = 0 \quad \text{a.e.}$$

and

$$\lim_{m\to\infty} \max_{2^m < k < 2^{m+1}} \big|\sigma_k(x) - \sigma_{2^m}(x)\big| = 0 \quad \text{a.e.}$$

As a corollary we obtain that, under condition (5.3), the orthogonal series (5.1) is $(C,1)$-summable a.e. if and only if the subsequence $\big(s_{2^m}(x) : m = 0, 1, \ldots\big)$ of the partial sums converges a.e. Now, it is a simple consequence of the Rademacher–Menshov theorem that if

$$\sum_{m=1}^{\infty}\Bigg(\sum_{k\in I_m} a_k^2\Bigg)(\log m)^2 < \infty, \tag{6.1}$$

where I_m is defined in (5.13), then the partial sums $s_{2^m}(x)$ of the orthogonal series (5.1) converge a.e. It is clear that (6.1) is equivalent to the condition

$$\sum_{k=4}^{\infty} a_k^2 (\log\log k)^2 < \infty. \tag{6.2}$$

Combining this observation with the theorem of Kolmogorov and Kaczmarz yields the following.

Theorem of Menshov and Kaczmarz. *If condition* (6.2) *is satisfied, then the orthogonal series* (5.1) *is* $(C, 1)$*-summable a.e.*

This theorem was first proved by D. E. Menshov {10} and S. Kaczmarz {5}. D. E. Menshov {10} also proved that condition (6.2) is the best possible.

Theorem of Menshov. *If* $\big(w(k) : k = 1, 2, \dots\big)$ *is an increasing sequence of positive numbers such that*

$$w(k) = o(\log\log k) \quad \text{as} \quad k \to \infty,$$

then there exists a sequence (a_k) *of numbers and an* ONS $\big(\phi_k(x)\big)$ *such that condition* (5.5) *is satisfied and the orthogonal series* (5.1) *is nowhere* $(C, 1)$*-summable.*

K. Tandori {19} also sharpened this result of Menshov into a necessary and sufficient condition for certain sequences (a_k) of numbers as follows.

Theorem of Tandori. *If* (a_k) *is a sequence of numbers for which*

$$\sqrt{k}|a_k| \ge \sqrt{k+1}|a_{k+1}|, \quad k = 1, 2, \dots$$

and

$$\sum_{k=4}^{\infty} a_k^2 (\log\log k)^2 = \infty,$$

then there exists an ONS $\big(\phi_k(x)\big)$ *depending on* (a_k) *such that the orthogonal series* (5.1) *is nowhere* $(C, 1)$*-summable.*

7. Unconditional Convergence of Orthogonal Series

One of the deepest results of K. Tandori {20} completely solved the problem of unconditional convergence of general orthogonal series. Since a general ONS $\big(\phi_k(x)\big)$ (unlike the classical trigonometric system, for example) has no a priori given arrangement of its terms, it is quite natural to study the question of convergence of the orthogonal series (5.1) in every rearrangement of its terms; that is, the convergence of the rearranged series

$$\sum_{\ell=1}^{\infty} a_{k(\ell)}\phi_{k(\ell)}(x), \tag{7.1}$$

where $\big\{k\,:\,\ell\to k(\ell)\,:\,\ell=1,2,\dots\big\}$ is a bijective mapping of the set of the positive integers (so-called permutation). By a.e. convergence of series (7.1), we mean the term "almost everywhere" in the sense that the set of measure zero of divergence points may vary with every rearrangement. Otherwise, the a.e. convergence in every rearrangement would reduce to the a.e. absolute convergence. Now, the aforementioned theorem of Tandori can be formulated as follows. Let

$$\nu_p := 2^{2^p} \quad \text{and} \quad J_p := \{\nu_p+1, \nu_p+2, \dots, \nu_{p+1}\}, \quad p = 0, 1, \dots .$$

We agree to say that the orthogonal series (5.1) converges unconditionally a.e. if the series in (7.1) converges a.e. for each permutation $\big\{k\,:\,\ell\to k(\ell)\,:\,\ell=1,2,\dots\big\}$.

Unconditional convergence theorem of Tandori. *Let*

$$|a_1^*| \ge |a_2^*| \ge \dots \ge |a_k^*| \ge \dots$$

be a decreasing rearrangement of the sequence (a_k) *of numbers. Then the orthogonal series* (5.1) *converges unconditionally a.e. if and only if*

$$\sum_{p=0}^{\infty}\Big\{\sum_{k\in J_p}|a_k^*|^2(\log k)^2\Big\}^{1/2} < \infty.$$

The idea of studying unconditional convergence is due to Wladislaw Orlicz {12}, who proved the first basic result in this direction. His result, which is actually a consequence of the above theorem of Tandori, is formulated in terms of the so-called Weyl multiplier.

Corollary. *Let* $\big(\lambda(k) \,:\, k = 1, 2, \dots\big)$ *be an increasing sequence of positive numbers.*

(i) (due to Orlicz). *If*

$$\sum_{p=0}^{\infty} \frac{1}{\lambda(\nu_p)} < \infty, \quad \textit{or equivalently} \quad \sum_{m=1}^{\infty} \frac{1}{m\lambda(2^m)} < \infty, \tag{7.2}$$

and

$$\sum_{k=1}^{\infty} a_k^2 (\log k)^2 \lambda(k) < \infty, \tag{7.3}$$

then the orthogonal series (5.1) *converges unconditionally a.e.*

(ii) (due to Tandori). *On the other hand, if*

$$\sum_{p=0}^{\infty} \frac{1}{\lambda(\nu_p)} = \infty, \quad \textit{or equivalently} \quad \sum_{m=1}^{\infty} \frac{1}{m\lambda(2^m)} = \infty,$$

then there exist a sequence (a_k) *of numbers and an* ONS $\big(\phi_k(x)\big)$ *such that condition* (7.3) *is satisfied and the orthogonal series* (5.1) *diverges a.e. in some rearrangement of its terms.*

For instance, if

$$\lambda(k) := (\log \log k)^{1+\varepsilon}, \quad k = 4, 5, \dots,$$

then condition (7.2) holds whenever $\varepsilon > 0$, but fails to hold when $\varepsilon = 0$. The above corollary can be reformulated as follows: The sequence $\big((\log k)^2 \lambda(k)\big)$ is a Weyl multiplier for the unconditional convergence of orthogonal series for $\varepsilon > 0$, while it is not for $\varepsilon = 0$.

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