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The works of Kornél Lánczos on the Theory of Relativity

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1. Introduction

Lánczos was a true acolyte of Einstein, and he kept returning to the study of relativity theory throughout his life. About thirty scientific publications, one-third of his full list, were written on this very subject. He had an open mind both to the geometrical and physical issues abundantly encountered in the theory.

He realized, for example {8}, that the redshift of light signals traveling in gravitational fields is, in fact, a Doppler-shift phenomenon as opposed to being a genuine gravitational effect. This contradicts {18} a statement, often made in general relativity. However, the Riemann tensor of the gravitational field does not appear anywhere in the redshift when expressed in terms of the four-velocities of the emitter and the observer.

He enlivened his radiant personal communication by characteristic gestures with his long fingers. The overall impression that he made arguing with the perpetual whirl of his hands was that he may have been born to live in an extraordinary neurotic state, a kind of like the *Angelman syndrome.*

Lánczos investigated a large number of research problems many of which are still of interest today. These include the formulation of junction conditions on the boundary surface of adjacent space-time regions. Known as the Lánczos–Israel boundary conditions {12, 7}, they are often used in constructing relativistic models of astrophysical bodies. He has written several papers on higher-derivative gravitational theories and their variational principles. His further contribution to the subject covers the asymptotic and exact symmetries in the formulation of conservation laws, gravitational radiation, tetrad methods and the Cauchy problem. Of special interest today is his work on the problem of motion, the potential he introduced for the Weyl tensor and his thoughts on a unified description of nature.

In a brief review of these latter works below, familiarity with the basic mathematical conventions of relativists will be an advantage for the reader. But for an added convenience, here is a summary. Space-time in general relativity is four-dimensional. Coordinates and geometrical quantities are labeled with indices assuming the values 0, 1, 2 or 3. The *Einstein convention* is used for the contraction of indices. For example, the contraction of the symmetric tensor density $\mathcal{T}_{ik}$ and the vector n^k is written, suppressing the symbol of summation over the repeated pair of indices,

$$\sum_{k=0}^{3} \mathcal{T}_{ik} n^k = \mathcal{T}_{ik} n^k. \tag{1.1}$$

Symmetrization of indexed quantities is denoted by enclosing the indices in parentheses. Skew symmetrization is similarly indicated by square brackets. Indices barred from the symmetrization are enclosed in bars.

Taking a covariant derivative (or a partial derivative) is indicated by a semicolon (or a comma, respectively) in the subscript. An example is the covariant derivative of the Lánczos potential L_{abc} when skewed in a pair of indices:

$$L_{ab[c;d]} = \frac{1}{2!}(L_{abc;d} - L_{abd;c}).$$

A convenient table to help comparison of the fluctuations in the notation is found on the red pages of {15}.

2. The problem of motion

In his 1941 paper {10}, Lánczos gave a detailed description of the motion of an isolated body in general relativity. The gist of this work is as follows.

A material body is said to be isolated in the sense that it is moving in vacuo. Other bodies may be present, but his treatment is not concerned with them. The gravitational equations inside the world tube of the particle are

$$G_{ik} = -8\pi T_{ik}.$$

Let Σ be the boundary of the world tube, with outward pointing unit normal n^i. Outside the boundary, the vacuum equations hold,

$$G_{ik} = 0$$

and G_{ik} is discontinuous across the hypersurface Σ. This discontinuity, however, is subject to the boundary condition on Σ, [*Cf.* (1.1)],

$$T_{ik} n^k = 0.$$

Inside the tube the conservation law holds for the matter

$$T^{ik}_{;k} = 0. \tag{2.1}$$

To write this in a detailed form, we introduce the tensor density

$$\mathcal{T}^{ik} = (-g)^{-1/2} T^{ik}.$$

We then have

$$\mathcal{T}^{ik}_{,k} = \Gamma^i \tag{2.2}$$

(where $\Gamma^i = -\Gamma^i_{rs} \mathcal{T}^{rs}$).

We now choose two space-like hypersurfaces $x^0 = a$ and $x^0 = b$, which enclose a section V_4 of the world tube. By Green's theorem,

$$\int_{x^0=b} \mathcal{T}^{i0} d^3x - \int_{x^0=a} \mathcal{T}^{i0}\, d^3x = \int_{V_4} \Gamma^i\, d^4x.$$

Passing to the limit $b \to a$ we get

$$\frac{d}{dx^0} \int \mathcal{T}^{i0}\, d^3x = \int \Gamma^i\, d^3x, \tag{2.3}$$

where the integrals are taken on any section $x^0 = a$.

By using the identity

$$(x^j \mathcal{T}^{ik})_{,k} = \mathcal{T}^{ij} + x^j \Gamma^i$$

we can derive other relations,

$$\frac{d}{dx^0} \int x^\alpha \mathcal{T}^{i0}\, d^3x = \int (\mathcal{T}^{i\alpha} + x^\alpha \Gamma^i)\, d^3x \tag{2.4}$$

$$\frac{d}{dx^0} \int (x^\alpha \mathcal{T}^{\beta 0} - x^\beta \mathcal{T}^{\alpha 0})\, d^3x = \int (x^\alpha \Gamma^\beta - x^\beta \Gamma^\alpha)\, d^3x \tag{2.5}$$

where the Greek indices refer to the three-space and range through the values 1, 2 and 3.

We now introduce a more compact new notation. The *four-momentum* of the body is

$$p^i = \int \mathcal{T}^{i0}\, d^3x.$$

The *angular momentum* of the body will be denoted

$$M^{\alpha\beta} = \int (x^\alpha \mathcal{T}^{\beta 0} - x^\beta \mathcal{T}^{\alpha 0})\, d^3x$$

and the *center of mass* of the body is

$$\bar{x}^\alpha = \frac{1}{p^0} \int x^\alpha \mathcal{T}^{00}\, d^3x.$$

We may then write Eqs. (2.3) and (2.5) in the form

$$\frac{dp^i}{dt} = \int \Gamma^i\, d^3x \tag{2.6}$$

$$\frac{dM^{\alpha\beta}}{dt} = \int (x^\alpha \Gamma^\beta - x^\beta \Gamma^\alpha)\, d^3x$$

where $t = x^0$. From (2.4),

$$\frac{d}{dt}(p^0 \bar{x}^\alpha) = p^\alpha + \int x^\alpha \Gamma^0\, d^3x.$$

Finally we find the physically very suggestive equation for the three-velocity of the center of mass

$$\frac{d\bar{x}^\alpha}{dt} = \frac{p^\alpha}{p^0} + \frac{1}{p^0} \int (x^\alpha - \bar{x}^\alpha)\Gamma^0\, d^3x. \tag{2.7}$$

In Eq. (2.6), the rate of change in the four-momentum and angular momentum is expressed in terms of quantities which may be regarded as the gravitational force and torque acting on the body. The last term in Eq. (2.7) expresses the amount by which the direction of the four-momentum diverges from the direction of the four-velocity. In case of a very small body, it is reasonable to neglect this term.

Synge {18} carried out an in-depth analysis of the above approach. He commented that the principal weakness of the treatment was due to its non-invariant character.

3. The Lánczos potential

In 1962, Lánczos {11} proposed that the Weyl tensor C_{abcd} (the traceless part of the Riemann tensor) in four dimensions can be given locally in terms of a potential L_{abc}. Later, however, Bampi and Caviglia {2} showed that his argument was flawed. These authors provided a new proof for the local existence of the Lánczos potential L_{abc}, which holds independently of the value of the metric signature. Yet another, spinorial derivation was presented by {6}.

The Lánczos potential has the symmetries

$$L_{abc} = -L_{bac}, \qquad L_{[abc]} = 0. \tag{3.1}$$

The Weyl tensor is given in terms of it in the form

$$\begin{aligned} C_{abcd} = {} & 2L_{ab[c;d]} + 2L_{cd[a;b]} - g_{a[c}\big(L_{|b|d]} + L_{|d|b]}\big) \\ & + g_{b[c}\big(L_{|a|d]} + L_{|d|a]}\big) + \frac{2}{3} g_{a[c}g_{d]b}L^{r}_{r} \end{aligned} \tag{3.2}$$

where

$$L_{ab} = 2L^{c}_{a[c;b]}.$$

This construction is analogous to that of the Maxwell tensor in terms of the vector potential. The Weyl tensor is invariant under the gauge transformations

$$L'_{abc} = L_{abc} + \chi_a g_{bc} - \chi_b g_{ac}$$

with χ_a an arbitrary four-vector. As with electromagnetism, various gauges can be introduced; one may set $L^{ab}{}_{b} = 0$ (this is known as the *algebraic gauge*) or $L_{abc}{}^{;c} = 0$ (the *differential gauge*).

The Lánczos potential can be utilized in the linearized theory of gravitation. Writing the metric in the form

$$g_{ab} = \eta_{ab} + h_{ab} \tag{3.3}$$

where η_{ab} is the flat metric and h_{ab} the perturbation and introducing the de Donder gauge

$$h^{ab}{}_{,b} = \frac{1}{2} h^{,a} \tag{3.4}$$

with $h = h^{ab}\eta_{ab}$, the *linearized* Lánczos potential reads

$$L_{abc} = \frac{1}{2}\big(h_{c[a,b]} - \frac{1}{6}\eta_{c[a}h_{,b]}\big). \tag{3.5}$$

Recently, there has been a revival of interest in the Lánczos potential. Novello and Neto {16} employed the linearized Lánczos potential as a model of a spin-2 field. {3} formulated the linearized gravitation theory in terms of the Lánczos potential. In a series of papers {1}, Edgar and his collaborators investigated the existence conditions of a Lánczos potential, using the Newmann–Penrose formalism.

4. The Lorentzian signature

Lánczos was deeply worried about the feature of general relativity that the signature of space-time was Lorentzian. He often argued with Einstein who was not as much concerned about this character of space-time geometry. According to Pythagoras' theorem, the distance of two neighboring points on the plane has the form

$$ds^2 = dx^2 + dy^2 \tag{4.1}$$

in Cartesian coordinates. In two more dimensions, one has

$$ds^2 = dx^2 + dy^2 + dz^2 + du^2. \tag{4.2}$$

However, in the physical world, one finds instead

$$ds^2 = dx^2 + dy^2 + dz^2 - du^2. \tag{4.3}$$

Although Einstein himself was not overtly worried about this state of affairs, he surmised that some deep mystery lurked behind the Lorentzian signature in (4.3). In Einstein's view, the essential feature of the theory was that the Riemann tensor describing the curvature of space-time is a covariant quantity and as such, it could be used in any coordinate system. From his point of view, the signature of the metric played a secondary role. However, in Lánczos's mind, the indefinite metric could not be a genuine ingredient of differential geometry since the latter is built upon the notion of small neighborhoods. With a Lorentzian metric, he argues, one cannot speak of two points being close together. In Riemannian geometry proper, zero separation means that the points coincide. However, in space-time, a pair of points can be at zero distance yet separated from each other in space by million light years. A photon that now hits one's eye from the Andromeda nebula left 3 million years before. And yet, the distance between the photon and our eyes has been zero during the whole travel. Why was there no interaction between the photon and us during the course of these 3 million years, save the moment of arrival – he asks {13}. Lánczos believed that if the notion of neighborhood lost its meaning, then differential geometry did not make sense.

In retrospect, the Lorentzian nature of the metric signature kept many researchers worried for a long time. Let us not forget that it had been a common practice right from the outset to use an imaginary time coordinate and in this way preserve, at least formally, the familiar definite form of the line element. But by now, a different viewpoint has, gradually, been adopted. This is best appreciated when recognizing that the Lorentzian geometry of space-time is a source of an abundantly rich structure and beauty. While today the topology of the manifold still occupies a central position in differential geometry, the edifice of an independent and elaborate theory of relativistic causality has grown to coexist with it. Causality theory yields deep insights into the global properties of space-time {5}. General relativity would be a good deal less appealing without the variegated world of causal phenomena.

In many other respects, Lánczos's scientific aspirations bore the influence of contemporary schools of thinking. He followed Einstein in spending much effort on attempting to geometrize all other properties of matter, including electromagnetism and quantum physics. He asked: *"With the enormous perspective allowing to interpret all material properties as special properties of space, how can it be that we can only derive* gravitation *from these very complicated relations, but both electromagnetism and quantum phenomena remain outside the scope?"*

He argued that there was a juncture in Einstein's argument leading to relativity theory where we must really take a different route to get the desired universal description of matter. This is *not* the description of the geometrical space by fundamental differential equations – a feature that he insisted on keeping. But in Lánczos's view, it is at the choice of the Lagrangian whence one must depart. He criticizes the Einstein-Hilbert Lagrangian $L = R$ on the grounds that it gives rise to a dimensioned action. In fact, this dimension is cm^2. Weyl pointed out in 1918 that it is nonsensical to seek the minimum of a dimensioned quantity since this can take any values with the appropriate choice of units. One can make the action dimensionless by choosing the Lagrangian to be quadratic in the curvature.

In 1938 Lánczos showed that the most general allowable Lagrangian can be brought to the form

$$L = R_{ik}R^{ik} - \sigma R^2 \tag{4.4}$$

where R_{ik} is the Ricci tensor, R the Einstein scalar and σ a free constant. He then considered the metric g_{ik} and the Ricci tensor as independent quantities. In this way he obtained 20 second-order differential equations for 20 unknowns. Originally, Lánczos hoped to unify gravitation and electromagnetism in this theory. To his disappointment, however, in the weak-field approximation the field equations reduce to the vacuum conditions $R_{ik} = 0$ of general relativity. Thus no room is left here for the electromagnetic field.

Later, in the sixties, Lánczos came to the idea that the class of solutions of his theory that is relevant to physics is *not* the one containing the weak-field limit. On the contrary, as he then held it, the required fields are strong periodic wave-like solutions. He conjectured that the period is the Planck length, $L_P = 10^{-32}$ cm. How come then that the world as we know is isotropic? To answer this, he resorted to the physics of the *isometric crystals* whose three principal axes are mutually orthogonal and they are equal in length. In an isotropic universe, the Ricci tensor and the metric are related by

$$R_{ik} = \lambda g_{ik}. \tag{4.5}$$

This is essentially Einstein's cosmological equation. In general relativity, the constant λ has the dimension (length)$^{-2}$. There λ is extremely small because the mean *'length'* of curvature of the universe is large. For Lánczos, on the other hand, this characteristic length is extremely small, whence λ must be large.

In this strong-field approximation, the computation of the Ricci tensor is quite unlike the procedure for weak fields. For the latter, the connection quantities are small, thus only the linear terms in the connection are kept. In the strong-field case, however, it is the *linear* terms in the curvature that can be dropped and the quadratic terms dominate. The metric does not determine the effect of the background geometry. Instead, it is the mean square of the first derivatives of the metric that does this.

Lánczos abandons the Lorentz signature of the metric and explores a genuine Riemannian geometry. He notes that the field equations derived from the Lagrangian (4.4) imply a constant scalar curvature R. He next observes that with the special choice $\sigma = 1/2$ in the Lagrangian, the Ricci-tensor can be substituted for by

$$P_{ik} = R_{ik} - \mu g_{ik} \tag{4.6}$$

where μ is a constant. If one chooses $\mu = \lambda$ then one would have $P_{ik} = 0$. The meaning of this is that the submetric does not give any contribution to the slow perturbations. In such circumstances, nothing would correspond to the Minkowski-like constants $(1, 1, 1, -1)$. Hence he concludes that macroscopically there must be a small deviation from perfect isotropy. He describes this deviation by adopting the diagonal elements of the macroscopic metric to have the mean values

$$(1 + \varepsilon, 1 + \varepsilon, 1 + \varepsilon, 1 - 3\varepsilon). \tag{4.7}$$

This form satisfies that the trace of the deviations is zero. He then asserts that the four 1's in the diagonal are unobservable and the effective metric becomes

$$[\eta_{ik}] = \operatorname{diag}(1, 1, 1, -3).$$

We see that this effective metric is indefinite.

The weak perturbations of the effective metric are to describe electromagnetism in this theory. Denoting these metric perturbations by h_{ik}, one can use the traceless and divergence-free property of the tensor P_{ik} to derive the following relations:

$$h_{ik}\eta^{ik} = 0 \tag{4.8}$$

$$h_{ik,m}\eta^{km} = 0 \tag{4.9}$$

where a comma in the subscript denotes partial derivative.

Lánczos proposes that the perturbations can be described by a vector φ_i as follows,

$$h_{ik} = \varphi_{i,k} + \varphi_{k,i}. \tag{4.10}$$

In his view, the vector φ_i should be interpreted as the four-potential of the Maxwell field,

$$F_{ik} = \varphi_{i,k} - \varphi_{k,i}.$$

According to this interpretation, the scalar constraint (4.8) is the Lorentz gauge condition and the vector constraint (4.9) becomes the wave equation.

Thirty years later, theoreticians picture the microstructure of space-time much the same way as Lánczos did. It is being acknowledged that the smooth nature of the manifold according the description of differential geometry is inadequate on a microscopic scale. Quantum fluctuations ripple the structure of the geometry more and more forcefully as we go down to the Planck scale.

Despite the general acceptance of this picture of quantum space-time, Lánczos's theory has submerged in oblivion, much the same way as Einstein's late unified theories did. In conclusion, let me try to outline the weaknesses of this bold attempt which are discernible from a perspective of the past three decades. At several points in this theory, apparently arbitrary assumptions have been made. The first of these is the assumption that the strong wave solutions of the field equations are *periodic.* One would not expect such periodicity in a stochastic description of the quantum fluctuations. Of course, this assumption of Lánczos enormously eases the task of describing the geometry in mathematical terms. A description, which is likely to provide a more realistic picture of physics, would require at least the full machinery of relativistic quantum field theory, and possibly more from beyond that theory. Another objection I would like to raise is to the way electromagnetism is geometrized in the model. Much of the standard model of fundamental interactions was unknown three decades ago. Today a minimal task would be seen to provide a coherent description of gravitational and electroweak phenomena. It is hard to assess if the past thirty years brought the dreams of theoreticians about a glorious completion any closer to coming true.

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