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Probability Theory

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In the early sixties György Pólya gave a talk in Budapest where he told the following story.

He studied in his teens at the ETH (Federal Polytechnical School) in Zürich, where he had a roommate. It happened once that the roommate was visited by his fiancée. From politeness Pólya left the room and went for a walk on a nearby mountain. After some time he met the couple. Both the couple and Pólya continued their walks in different directions. However, they met again. When it happened the third time, Pólya had a bad feeling. The couple might think that he is spying on them. Hence he asked himself what is the probability of such meetings if both parties are walking randomly and independently. If this probability is big then Pólya might claim that he is innocent.

To give an answer to the above question one has to build a mathematical model. Pólya's model is the following.

Consider a random walk on the lattice $\mathbb{Z}^d$. This means that if a moving particle is in $x \in \mathbb{Z}^d$ in the moment n then at the moment $n+1$ the particle can move with equal probability to any of the $2d$ neighbours of x independently of how the particle achieved x. (The neighbours of an $x \in \mathbb{Z}^d$ are those elements of $\mathbb{Z}^d$ which have $d-1$ coordinates equal to those of x and the remaining coordinate differs by $+1$ or -1.) Let S_n be the location of the particle after n steps (i.e., at the moment n) and assume that $S_0 = 0$.

Then Pólya {55} proved

Theorem 1.

$$\mathbf{P}\{S_n = 0 \quad i.o.\} = \begin{cases} 1 & \text{if } d \le 2, \\ 0 & \text{if } d \ge 3. \end{cases}$$

(i.o. = infinitely often).

This theorem clearly means that a random walker returns to his or her starting point infinitely often with probability 1 in the plane. It easily implies that two independent random walkers will meet infinitely often with probability 1 in the plane. Hence Pólya was innocent.

Another nice problem studied by Pólya is the following:

Let an urn contain M red and $N-M$ white balls. Draw a ball at random, replace the drawn ball and at the same time place into the urn R extra balls with the same colour as the one drawn. ($R=\pm 1, \pm 2, \dots$ in case of negative R we remove from the urn R balls of the same colour.) Then we draw again a ball and so on. What is the probability of the event that in n drawings we obtain a red ball exactly k times? Let this event be denoted by A_k. Of course we assume that at every drawing each ball of the urn is selected with the same probability. Pólya evaluated the distribution $\{\mathbf{P}(A_k)\}$. It is called Pólya distribution (see F. Eggenberger–G. Pólya, {17}).

Felix Hausdorff in 1913 asked how far can a particle go from its starting point in n steps. In the case $d=1$ A. I. Khinchine (1923) proved that the distance can be $(2n\log\log n)^{1/2}$ but not more. In fact

Theorem 2.

$$\limsup_{n\to\infty}\frac{S_n}{(2n\log\log n)^{1/2}} = -\liminf_{n\to\infty}\frac{S_n}{(2n\log\log n)^{1/2}} = 1 \quad \textit{a.s.}$$

(a.s. = almost surely).

Consequently the distance of the particle from its starting point will be infinitely often more than $(1-\varepsilon)(2n\log\log n)^{1/2}$ but it will be only finitely many times more than $(1+\varepsilon)(2n\log\log n)^{1/2}$ for any $\varepsilon>0$. Clearly this theorem implies Pólya's theorem in case $d=1$.

Paul Lévy asked how can we obtain an even sharper version of Khinchine's theorem. Pál Erdős' answer in {18} is the following.

Theorem 3. *Let $a(n)$ be an increasing function and $d=1$. Then*

$$\mathbf{P}\{S_n \ge n^{1/2}a(n) \ \ \textit{i.o.}\} = \begin{cases} 1 & \textit{if } \ A=\infty, \\ 0 & \textit{if } \ A<\infty, \end{cases}$$

where

$$A=\sum_{n=1}^{\infty}\frac{a(n)}{n}\exp\left(-\frac{a^2(n)}{2}\right).$$

For example Theorem 3 implies that $S_n \geq (2n \log\log n)^{1/2}$ a.s. i.o.

The theory of random walks became one of the most popular topics of probability theory and undoubtedly Erdős was one of the most important contributors to this topic, especially in the multidimensional case. Now we formulate some of the results of Erdős on random walks.

Consider the last return $R(n)$ of a random walk to its starting point before its n-th step, i.e., let

$$R(n) = \max\{k \,:\, 0 \leq k \leq n,\ S_k = 0\}.$$

Let $d = 1$. Then Theorem 1 clearly implies that

$$\lim_{n\to\infty} R(n) = \infty \quad \text{a.s.}$$

and

$$\mathbf{P}\big\{\, R(n) = n \ \text{ i.o.}\big\} = 1.$$

Kai-Lai Chung and Erdős in {6} asked how small $R(n)$ can be. They proved

Theorem 4. *Let $d = 1$ and let $f(x)$ be an increasing function for which $\lim_{x\to\infty} f(x) = \infty$, $x/f(x)$ is increasing and $\lim_{x\to\infty} x/f(x) = \infty$. Then*

$$\mathbf{P}\left\{R(t) \leq \frac{t}{f(t)} \quad \textit{i.o.}\right\} = \begin{cases} 1 & \textit{if } \ I = \infty, \\ 0 & \textit{if } \ I < \infty, \end{cases}$$

where

$$I = \int_1^\infty \frac{dx}{x\big(f(x)\big)^{1/2}}.$$

This theorem clearly implies that $R(n)$ can be smaller than $n(\log n)^{-2}$ infinitely often but $R(n)$ can be smaller than $n(\log n)^{-2-\varepsilon}$ $(\varepsilon > 0)$ only finitely many times.

Arieh Dvoretzky and Erdős in {15} asked: how many points will be visited by a random walk in $\mathbb{Z}^d$ $(d \geq 2)$ during its first n steps. Let $V(n)$ be the number of different vectors among $S_1, S_2, \ldots, S_n$, i.e., $V(n)$ is the number of visited points. They proved

Theorem 5.

$$\lim_{n\to\infty} \frac{V(n)}{\mathbf{E}V(n)} = 1 \quad \text{a.s.},$$

where

$$\mathbf{E}V(n) \sim \begin{cases} \dfrac{\pi n}{\log n} & \text{if } d = 2, \\ n\gamma_d & \text{if } d \geq 3, \end{cases}$$

γ_d is a sequence of strictly positive constants, and $\mathbf{E}$ *denotes the expected value.*

The following question can be considered as the converse of the above question of Dvoretzky and Erdős: How many times is a "typical" point of $\mathbb{Z}^d$ visited up to time n? By Theorem 1 in case $d \geq 3$ the answer is a finite random variable (r.v.). In case $d = 2$ the answer is a random sequence converging to ∞ as $n \to \infty$. In fact Erdős and S. J. Taylor {34} proved:

Theorem 6. *Let $\xi(n)$ be the number of visits of $0 \in \mathbb{Z}^2$ before n, i.e.,*

$$\xi(n) = \#\{k : 0 \leq k \leq n,\ S_k = 0\}.$$

Then

$$\lim_{n\to\infty} \mathbf{P}\big\{\xi(n) < x \log n\big\} = 1 - e^{-\pi x}.$$

As we have said, in case $d \geq 3$, by Theorem 1 any fixed point is visited only finitely many times. However, some randomly chosen point will be visited many times. Let

$$\xi(x, n) = \#\{k : 0 \leq k \leq n,\ S_k = x\}$$

and

$$\zeta(n) = \max_{x \in \mathbb{Z}^d} \xi(x, n).$$

Then Erdős and Taylor, in their above mentioned paper, proved:

Theorem 7. *Let $d \geq 3$. Then*

$$\lim_{n\to\infty} \frac{\zeta(n)}{\log n} = \gamma_d \quad \text{a.s.}$$

where γ_d is the same constant as in Theorem 5.

It is easy to see that the path of a random walk crosses itself infinitely many times with probability 1 for any $d \geq 1$. We mean that there exists an infinite sequence $\{U_n, V_n\}$ of pairs of positive integer valued r.v.'s such that $S(U_n) = S(U_n + V_n)$, and $0 \leq U_1 < U_2 \ldots$, $(n = 1, 2, \ldots)$, where $S(U_n) = S_{U_n}$. However, we ask the following question: will selfcrossings occur after a long time? For example, we ask whether the crossing $S(U_n) = S(U_n + V_n)$ will occur for every $n = 1, 2, \ldots$ if we assume that V_n converges to infinity with great speed and U_n converges to infinity much slower. In fact Erdős and Taylor {35} proposed the following two problems:

Problem A. Let $f(n) \uparrow \infty$ be a positive integer-valued function. What are the conditions on the rate of increase of $f(n)$ which are necessary and sufficient to ensure that the paths $\{S_0, S_1, \ldots, S_n\}$ and $\left\{S_{n+f(n)}, S_{n+f(n)+1} \ldots\right\}$ have points in common for infinitely many values of n with probability 1?

Problem B. A point S_n of a path is said to be "good" if there are no points common to $\{S_0, S_1, \ldots, S_n\}$ and $\{S_{n+1}, S_{n+2}, \ldots\}$. For $d = 1$ or 2 there are no good points with probability 1. For $d \geq 3$ there might be some good points: how many are there?

As far as Problem A is concerned, they (Erdős–Taylor) proved

Theorem 8. *Let $f(n) \uparrow \infty$ be a positive integer-valued function and let E_n be the event that the paths*

$$\{S_0, S_1, \ldots, S_n\} \quad \textit{and} \quad \left\{S_{n+f(n)+1}, S_{n+f(n)+2}, \ldots\right\}$$

have points in common. Then

(i) *for $d = 3$, if $f(n) = n\big(\varphi(n)\big)^2$ and $\varphi(n)$ is strictly increasing, one has*

$$\mathbf{P}\{E_n \ \ \textit{i.o.}\} = 0 \quad \textit{or} \quad 1 \tag{1}$$

depending on whether $\sum_{k=1}^{\infty} \left(\varphi(2^k)\right)^{-1}$ converges or diverges,

(ii) *for $d = 4$, if $f(n) = n\chi(n)$ and $\chi(n)$ is strictly increasing, we have* (1) *depending on whether $\sum_{k=1}^{\infty} \left(k\chi(2^k)\right)^{-1}$ converges or diverges,*

(iii) *for $d \geq 5$, if*

$$\sup_{m \geq n} \frac{f(m)}{m} \geq C \frac{f(n)}{n}$$

(for some $C > 0$), we have (1) depending on whether

$$\sum_{n=1}^{\infty} \bigl(f(n)\bigr)^{(2-d)/2}$$

converges or diverges.

For Problem B they (Erdős–Taylor) have as an answer:

Theorem 9. *For $d \geq 3$ let $G^{(d)}(n)$ be the number of integers r $(1 \leq r \leq n)$ for which $(S_0, S_1, \ldots, S_r)$ and $(S_{r+1}, S_{r+2}, \ldots)$ have no points in common. Then*

(i) $d = 3$. *For any $\varepsilon > 0$*

$$\mathbf{P}\bigl\{G^{(3)}(n) > n^{1/2+\varepsilon} \quad i.o.\bigr\} = 0.$$

(ii) $d = 4$.

$$\mathbf{P}\left\{0 = \liminf_{n\to\infty} \frac{G^{(4)}(n)\log n}{n} \leq \limsup_{n\to\infty} \frac{G^{(4)}(n)\log n}{n} \leq C\right\} = 1$$

with some positive constant C.

(iii) $d \geq 5$.

$$\lim_{n\to\infty} \frac{G^{(d)}(n)}{n} = \tau_d \quad a.s.$$

where τ_d is an increasing sequence of positive numbers with $\tau_d \uparrow 1$ as $d \to \infty$.

Random walk is the simplest mathematical model for Brownian motion that is well known in physics. However, it is not a very realistic model. In fact the assumption, that the particle goes at least one unit in a direction before turning, is hardly satisfied by the real Brownian motion observed in nature.

In a more realistic model the particle makes instantaneous steps, that is a continuous time scale is used instead of a discrete one. Such a model is the Wiener process or frequently called Brownian motion.

Now we mention a few results, on the Wiener process.

The first one, which characterizes the modulus of continuity, was proved by P. Lévy {48}.

Theorem 10. *Let $W(t) \in \mathbb{R}^1$ $(t \geq 0)$ be a Wiener process. Then*

$$\lim_{h\to 0} \frac{\sup_{0\leq t\leq 1-h} \left|W(t+h)-W(t)\right|}{(2h\log 1/h)^{1/2}} = 1 \quad a.s.$$

A stronger version of this Theorem was given by Chung, Erdős and T. Sirao {7}.

Theorem 11. *Let $f(x)$ be a continuous increasing function and let*

$$\mathcal{J}(f) = \int_1^\infty f^2(t) \exp\left(-\frac{1}{2} f^2(t)\right) dt.$$

Then

$$\sup_{0\leq t\leq 1-h} \left|W(t+h)-W(t)\right| \begin{cases} \leq h^{1/2} f(h^{-1}) & \text{if } \mathcal{J}(f) < \infty, \\ \geq h^{1/2} f(h^{-1}) & \text{if } \mathcal{J}(f) = \infty \end{cases}$$

a.s. if h is small enough and $W(t) \in \mathbb{R}^1$.

For example

$$\sup_{0\leq t\leq 1-h} \left|W(t+h)-W(t)\right| \leq h^{1/2}\big(2\log h^{-1} + (5+\varepsilon)\log\log h^{-1}\big)^{1/2}$$

a.s. if $\varepsilon > 0$ and h is small enough and the above inequality a.s. does not hold if $\varepsilon \leq 0$.

Beside numerous physical applications of the Wiener process it has important applications in mathematical analysis. The most important ones are in the theory of partial differential equations.

A very classical result claims that the sample paths of a Wiener process are nowhere differentiable with probability one. It has been well known for a long time that there exist continuous, nowhere differentiable functions. However, one thinks that it is a very rare phenomenon. The fact that a Wiener process is nowhere differentiable can be interpreted by saying that almost all continuous functions are nowhere differentiable. A similar fact is that almost all continuous functions are nowhere monotone.

A function $f(x)$ $(0 < x < 1)$ is said to be monotone at x_0 if there exists an $0 < \varepsilon = \varepsilon(x_0) < \min(x_0, 1-x_0)$ such that

$$f(u) < f(x_0) \quad \text{for any} \quad x_0 - \varepsilon < u < x_0$$

and

$$f(u) > f(x_0) \quad \text{for any} \quad x_0 < u < x_0 + \varepsilon$$

or

$$f(u) > f(x_0) \quad \text{for any} \quad x_0 - \varepsilon < u < x_0$$

and

$$f(u) < f(x_0) \quad \text{for any} \quad x_0 < u < x_0 + \varepsilon.$$

It is not easy at all to construct a continuous nowhere monotone function. However, Dvoretzky, Erdős and Shizuo Kakutani in {16} proved

Theorem 12. *With probability one the sample paths of a Wiener process are nowhere monotone.*

Let $X_1, X_2, \dots$ be independent, identically distributed r.v.'s (i.i.d.r.v.'s) with $\mathbf{E}X_1 = 0$, $\mathbf{E}X_1^2 = 1$ and let F be their distribution function. Let $Y_1, Y_2, \dots$ be i.i.d. normal $(0,1)$ r.v.'s and put $S_n = \sum_{i=1}^n X_i$, $T_n = \sum_{i=1}^n Y_i$. Then one of the most classical result of probability theory, the central limit theorem, states that

$$\lim_{n\to\infty} \mathbf{P}\{n^{-1/2}S_n < y\} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{y} e^{-u^2/2}\, du$$

or equivalently

$$\lim_{n\to\infty} \left(\mathbf{P}\{n^{-1/2}S_n < y\} - \mathbf{P}\{n^{-1/2}T_n < y\}\right) = 0$$

i.e., the limiting behaviours of S_n and T_n are the same. In other words, as times goes on, S_n forgets about the distribution function F where it comes from.

A similar phenomenon was observed by Erdős and Mark Kac in {20} and {21}. They investigated the limits $(n \to \infty)$ of the distribution functions

$$\mathbf{P}\Big\{n^{-1/2} \max_{1\le k\le n} S_k < y\Big\},$$

$$\mathbf{P}\Big\{n^{-1/2} \max_{1\le k\le n} |S_k| < y\Big\},$$

$$\mathbf{P}\Big\{n^{-2} \sum_{k=1}^{n} S_k^2 < y\Big\},$$

$$\mathbf{P}\Big\{n^{-3/2} \sum_{k=1}^{n} |S_k| < y\Big\}$$

and they observed that these limit distributions also do not depend on F. Hence a program for finding the above limits may be carried out in two steps. First, they should be evaluated for a specific distribution F, and then one should show that the above considered functionals of $\{S_k\}$ do not remember the initially taken distribution. They called this method of proof the invariance principle and their papers initiated a new methodology for proving limit laws in probability theory. A very essential new step in this methodology is the invention of the strong invariance principle due to V. Strassen {75} which is capped by János Komlós–Péter Major–Gábor Tusnády {45}, see also Major {49}.

Their most important result is

Theorem 13. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d.r.v.'s defined on a rich enough probability space $\{\Omega, \mathcal{S}, \mathbf{P}\}$. Assume that $R(t) = \mathbf{E} \exp(tX_1)$ exists in a neighbourhood of $t = 0$. Then there exists a Wiener process $\big\{W(t), t \geq 0\big\}$ defined on Ω such that*

$$\big|S_n - W(n)\big| = O(\log n) \quad a.s.$$

The analogues of Problems A and B can also be formulated for a Wiener process. Very important results in this area were obtained by Dvoretzky, Erdős, Kakutani and Taylor in the fifties.

Up to now we mentioned the names of two Hungarian probabilists, Erdős and Pólya. They were certainly Hungarian but Pólya lived mostly abroad and Erdős between '38 and '56 also was abroad. The first Hungarian probability school in Hungary was founded by Károly Jordán. His work is closely related to statistics. Hence his work is treated in the Mathematical Statistics Section.

The *first revolution* in probability in Hungary took place after the second world war when Alfréd Rényi founded his probability school.

Rényi originally was interested in number theory rather than in probability. He wanted to study the method of Yu. V. Linnik (Linnik's large sieve) and he went to Leningrad in '46 to study with him. There he gained insight into connections between number theory and probability, and when he returned to Hungary in '47 he also was already a real probabilist.

His first papers which belong to pure probability deal with the Poisson process. They start with an article written jointly with János Aczél and the physicist Lajos Jánossy {44} and an article with the same title written by Rényi alone (*ibid.*, **2** (1951), 83–98). Let $X(t)$ be a process defined for

$t \geq 0$ with $X(0) = 0$ and such that $X(t) - X(s)$ takes positive integer values for $t > s$. Write $W_k(t) = \mathbf{P}\{X(t) = k\}$, and make the following assumptions:

A) $X(t)$ is homogeneous, i.e., $\mathbf{P}\{X(t_2) - X(t_1) = k\} = W_k(t)$ whenever $t_2 - t_1 = t$.

B) $X(t)$ has independent increments, i.e., $X(t_n) - X(t_{n-1}), X(t_{n-2}) - X(t_{n-3}), \ldots, X(t_2) - X(t_1)$ $(n = 3, 4, \ldots)$ are independent random variables $(t_1 < t_2 \leq t_3 < t_4 \leq \ldots \leq t_{n-1} < t_n)$.

C) The "events" are rare, i.e.

$$\lim_{t \to 0} \frac{W_1(t)}{1 - W_0(t)} = 1.$$

From these hypotheses the authors deduce that

$$W_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

for some $\lambda > 0$, i.e., $X(t)$ is a Poisson process. The three assumptions are natural and minimal, in particular the differentiability of $W_k(t)$ is not required. The proof uses functional equations – a speciality of Aczél – instead of differential equations.

The three authors prove that if condition C is omitted, then $X(t)$ is a "composed Poisson process", i.e., of the form

$$X(t) = X_1(t) + 2X_2(t) + \ldots + nX_n(t) + \ldots,$$

where the $X_n(t)$ are independent Poisson processes. Again using functional equations, they determine $W_k(t)$.

The fact that a Poisson process is Markovian is proved and applied in a number of papers of Rényi. One of these is: Rényi, Lajos Takács {73}.

A further result in this direction investigates the discontinuities of a process of independent increments. It turns out that the number of discontinuities of "any size" are independent. More precisely András Prékopa and Rényi {72} proved:

Theorem 14. *If the process ξ_t of independent increments is weakly continuous, i.e., for every $\varepsilon > 0$*

$$\lim_{\Delta \to 0} \mathbf{P}\{|\xi_{t+\Delta} - \xi_t| \geq \varepsilon\} = 0$$

uniformly in $t \in (0,1)$ and $I_1, I_2, \dots, I_r$ are pairwise disjoint subintervals of $[0,1]$ with positive distances from the origin, then the random variables

$$\nu(I_1), \nu(I_2), \dots, \nu(I_r)$$

are independent, where $\nu(I)$ denotes the random variable giving the number of discontinuities of magnitudes $h \in I$.

In his paper {71} Rényi investigated the converse of these results. Consider a point process $\xi(E)$ on the real line which is Poisson in the sense that the distribution of $\xi(E)$ is Poisson for certain subsets E of the real line. Then Rényi proved that $\xi(E)$ is a Poisson process indeed, i.e., $\xi(E_1)$ and $\xi(E_2)$ are independent if E_1 and E_2 are disjoint.

The Poisson process was also investigated by Pólya {56}. He considered a stone at the bottom of a river which lies at rest for such long periods that its successive displacements are practically instantaneous. Then he observed that the total displacement within a time interval $(0,t)$ might be treated as a compound Poisson process.

Another favorite topic of Rényi was the study of the properties of empirical processes (see Rényi {59} and György Hajós, Rényi {43}). The main results of these papers show that an ordered sample can be described via exponential random variables. Let $0 < E_1 < E_2 < \dots < E_n$ be an ordered sample obtained from a sequence of independent, exponential random variables with mean 1. Further let

$$\delta_{k+1} = (n-k)(E_{k+1} - E_k).$$

It is easy to see that $\delta_1, \delta_2 \dots, \delta_n$ are independent, exponentially distributed with mean 1, and

$$E_k = \frac{\delta_1}{n} + \frac{\delta_2}{n-1} + \dots + \frac{\delta_k}{n-k+1}.$$

Hence $\{E_k\}$ is a Markov chain. This simple observation was enough for Rényi to get some important results on order statistics and empirical distributions. For example he proved

Theorem 15.

$$\lim_{n\to\infty} \mathbf{P}\{nE_k < x\} = \int_0^x \frac{t^{k-1}e^{-t}}{(k-1)!}\,dt.$$

Replacing the sequence $E_1 < E_2 < \dots < E_n$ by an ordered sample of uniform $(0,1)$ random variables, similar results can be obtained. For further details on Rényi's work in this regard we refer to the Section on Mathematical Statistics.

Rényi as a probabilist never forgot his number theorist origin. He worked occasionally in pure number theory but was mostly interested in applications of probability to number theory. He realized that a great difficulty in the application of probability to number theory comes from the fact that it is meaningless to say: choose an integer from the set of all positive integers with equal probability. As an example he says in {60}: let $U(n)$ denote the number of different prime divisors of n; let $\pi_k(N)$ denote the number of those natural numbers $n \leq N$ for which $U(n) = k$. By a theorem of Erdős {19}

$$\frac{\pi_k(N)}{N} \sim \frac{(\log\log N)^k}{k!\log N} := p_k.$$

Hence it looks natural to say that the probability that a randomly chosen integer has k divisors is p_k. However, this sentence is meaningless in Kolmogorov's theory of probability. Rényi decided to build a new theory where this sentence was going to have a meaning i.e., where unbounded measures may occur. Rényi says: "Unbounded measures occur in statistical mechanics, quantum mechanics, in some problems of mathematical statistics, in integral geometry, in number theory etc. At first glance it seems that unbounded measures can play no role in probability theory. But if we observe more attentively how unbounded measures are really used in all the cases mentioned above, we see that unbounded measures are used only to calculate conditional probabilities as the quotient of the values of the unbounded measure of two sets. Clearly in a theory in which unbounded measures are allowed, conditional probability must be taken as the fundamental concept."

Hence Rényi built up such an axiomatic theory. Further number theoretical application of this theory can be found in Rényi {63, 68}.

The *second revolution* in probability in Hungary took place in '56 when Erdős started to work intensively together with Rényi and his students.

Once in 1958 Rényi arrived in his Department in the University and asked his assistants: "Do you know why trees do not grow up to the heavens?" With this question he announced that he and Erdős started to deal with random graphs. In fact their problem is the following (Erdős–Rényi {22}):

Let us suppose that n labelled vertices $P_1, P_2, \ldots, P_n$ are given. Let us choose at random an edge among the $\binom{n}{2}$ possible edges, so that all these edges are equiprobable. After this let us choose an other edge among the remaining $\binom{n}{2} - 1$ edges, and continue this process so that if already k edges are fixed, any of the remaining $\binom{n}{2} - k$ edges have equal probabilities to be chosen as the next one. We shall study the "evolution" of such a random graph $\Gamma_{n,N}$ if the number of chosen edges, N, is increased. In this investigation we endeavour to find what is the "typical" structure at a given stage of evolution (i.e., if N is equal, or asymptotically equal, to a given function $N(n)$ of n). By a "typical" structure we mean such a structure the probability of which tends to 1 if $n \to \infty$ when $N = N(n)$.

If N is very small compared with n, namely if $N = o\big(\sqrt{n}\big)$ then it is very probable that $\Gamma_{n,N}$ is a collection of isolated points and isolated edges, i.e., that no two edges of $\Gamma_{n,N}$ have a point in common. As a matter of fact the probability that at least two edges of $\Gamma_{n,N}$ shall have a point in common is

$$1 - \frac{\binom{n}{2N}(2N)!}{2^N N! \binom{\binom{n}{2}}{N}} = O\left(\frac{N^2}{n}\right).$$

If however $N \sim c\sqrt{n}$ where $c > 0$ is a constant not depending on n, then the appearance of trees of order 3 will have a probability which tends to a positive limit as $n \to +\infty$, but the appearance of a connected component consisting of more than 3 points will be still very improbable. If N is increased while n is fixed, the situation will change only if N reaches the order of magnitude of $n^{2/3}$. Then trees of order 4 (but not of higher order) will appear with a probability not tending to 0.

Clearly if N becomes larger and larger then we obtain more and more connected graphs. For example if N is about n then the graph contains a cycle of order k for any $k \geq 3$.

In this paper on their aim the authors write:

In the present paper we consider the evolution of a random graph in a more systematic manner and try to describe the gradual development and step-by-step unravelling of the complex structure of the graph $\Gamma_{N,N}$ when N increases while n is a given large number.

We succeeded in revealing the emergence of certain structural properties of $\Gamma_{n,N}$. However a *great deal* remains to be done in this field.

A typical result of this paper characterizes the size of the greatest tree of $\Gamma_{n,N}$. In fact they prove

Theorem 16. *Let $\Delta_{n,N}$ denote the number of points of the greatest tree which is a component of $\Gamma_{n,N}$. Suppose $N = N(n) \sim cn$ with $c \neq 1/2$. Let ω_n be a sequence tending arbitrarily slowly to $+\infty$. Then we have*

$$\lim_{n\to+\infty} \mathbf{P}\left(\Delta_{n,N(n)} \geq \frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) + \omega_n\right) = 0$$

and

$$\lim_{n\to+\infty} \mathbf{P}\left(\Delta_{n,N(n)} \geq \frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) - \omega_n\right) = 1$$

where

$$e^{-\alpha} = 2ce^{1-2c}$$

i.e., $\alpha = 2c - 1 - \log 2c$ and thus $\alpha > 0$.

Hence the answer of the question of Rényi is: The trees cannot grow up to the heavens because when the size of a tree will be larger than $\log n$ then a triangle will appear in it.

Erdős and Rényi returned to this problem occasionally {25} and many of their students attacked the "great deal" remained to be done in this field (e.g. Komlós, Endre Szemerédy {46}, Béla Bollobás, T. I. Fenner, A. M. Frieze {5}, Lajos Pósa {57}, Bollobás {4}).

Another problem of Rényi which initiated intensive research is the so-called stochastic geyser problem.

Let $X_1, X_2, \ldots$ be i.i.d. positive and bounded r.v.'s let $\{S_n\}$ be their partial sum sequence; can one then determine the distribution function of X_i with probability one, observing only the sequence $\{[S_n]\}$? This problem was motivated by the following story: Robinson Crusoe had a geyser on his island, which kept on erupting at random time points. After he had observed the number of eruptions per day for a long time, it occurred to him that he should now be able to predict the geyser's behaviour, i.e., he should be able to estimate the distribution function of the time length between two eruptions.

It is trivial that the sequence $\{[S_n]\}$ determines with probability one the expectation $\mathbf{E}X_1$ and the variance $\operatorname{Var} X_1$. However, the determination of higher moments even, is not trivial at all.

We now formulate a more general form of the geyser problem. Let $X_1, X_2, \ldots$ be i.i.d.r.v. and let $F(\cdot)$ be their distribution function. Put

$$V_n = S_n + R_n,$$

where $\{R_n\}$ is also a random variable sequence, not necessarily independent of S_n. Then we can ask whether it is possible to determine the distribution function $F(\cdot)$ with probability one via some Borel function of $\{V_n;\ n = 1, 2, \ldots\}$. In statistical terminology $\{R_n;\ n = 1, 2, \ldots\}$ can be viewed as a random error sequence when trying to observe S_n in order to estimate $F(\cdot)$. Answering this question Pál Bártfai {2} proved

Theorem 17. *Assume that the moment generating function* $R(t) = \int e^{tx} dF(x)$ *of* X_1 *exists in a neighbourhood of* $t = 0$ *and* $R_n \stackrel{\text{a.s.}}{=} o(\log n)$. *Then, given the values of* $\{V_n;\ n = 1, 2, \ldots\}$, *the distribution function* $F(\cdot)$ *is determined with probability one, i.e., there exists a r.v.* $L(x) = L(V_1, V_2, \ldots; x)$ *measurable with respect to the* σ*-algebra generated by* $\{V_n\}$ *such that for any given real* $x, L(x) \stackrel{\text{a.s.}}{=} F(x)$.

Bártfai also conjectured that his result is the best possible in the sense that if $R_n = O(\log n)$ and $R(t)$ exists in a neighbourhood of $t = 0$ then the distribution function $F(\cdot)$ is not determined by $\{V_n\}$.

This conjecture was proved later on by Komlós–Major–Tusnády (1975). In fact Theorem 13 easily implies Bártfai's conjecture.

Theorem 17 claims the existence of a functional L which determines F but it does not give any hint how to find it. Later on Erdős and Rényi gave a surprising solution of this problem. In fact they posed a problem in {27} which has nothing to do with the above mentioned question.

In connection with a teaching experiment in mathematics, Tamás Varga posed a problem which has also nothing to do with the Stochastic Geyser Problem. A solution of a more general form of it, however, turned out to be also an answer to the latter. The experiment goes like this: his class of secondary school children is divided into two sections. In one of the sections each child is given a coin which they then throw two hundred times, recording the resulting head and tail sequence on a piece of paper. In the other section the children do not receive coins but are told instead that they should try to write down a "random" head and tail sequence of length two hundred. Collecting these slips of paper, he then tries to subdivide them into their original groups. Most of the time he succeeds quite well. His secret is that he had observed that in a randomly produced sequence of length two hundred, there are, say, head-runs of length seven. On the other hand, he had also observed that most of those children who were to write down an imaginary random sequence are usually afraid of putting down runs of longer than four. Hence, in order to find the slips coming from the coin tossing group, he simply selects the ones which contain runs longer than five.

This experiment led Varga to ask: What is the length of the longest run of pure heads in n Bernoulli trials?

An answer to this question was given by Erdős and Rényi in {27}; they proved

Theorem 18. *Let $X_1, X_2, \ldots$ be i.i.d.r.v., each taking the values ± 1 with probability $1/2$. Put $S_0 = 0$, $S_n = X_1 + \ldots + X_n$. Then for any $c \in (0,1)$ and for almost all $\omega \in \Omega$ there exists an $n_0 = n_0(c, \omega)$ such that*

$$\max_{0 \le k \le n - [c \lg n]} \left(S_{k+[c \lg n]} - S_k \right) = [c \lg n],$$

if $n > n_0$, where $\lg$ is the logarithm of base 2.

That is, this theorem guarantees the existence of a run of length $[c \lg n]$ for every $c \in (0,1)$ with probability one if n is large enough.

On the other hand, in the same paper, they also showed for $c > 1$ that the above equality can only hold for a finite number of values of n with probability one. They proved

Theorem 19. *With the above notation one has*

$$\max_{0 \le k \le n - [c \lg n]} \frac{S_{k+[c \lg n]} - S_k}{[c \lg n]} \stackrel{\text{a.s.}}{\to} \alpha(c),$$

where $\alpha(c) = 1$ for $c \le 1$, and, if $c > 1$, then $\alpha(c)$ is the only solution of

$$\frac{1}{c} = 1 - h\left(\frac{1+\alpha}{2}\right),$$

with $h(x) = -x \lg x - (1-x) \lg (1-x)$, $0 < x < 1$; the herewith defined $\alpha(\cdot)$ is a strictly decreasing continuous function for $c > 0$ with $\lim_{c \searrow 1} \alpha(c) = 1$ and $\lim_{c \to \infty} \alpha(c) = 0$.

They also gave the following generalization of Theorem 19:

Theorem 20. *Let $X_1, X_2, \ldots$ be i.i.d.r.v.'s with mean zero and a moment generating function $R(t) = \mathbf{E} e^{tX_1}$, finite in a neighbourhood of $t = 0$. Let*

$$\rho(x) = \inf_t e^{-tx} R(t),$$

the so-called Chernoff function of X_1. *Then for any* $c > 0$ *we have*

$$\max_{0 \le k \le n-[c\log n]} \frac{S_{k+[c\log n]} - S_k}{[c\log n]} \xrightarrow{\text{a.s.}} \alpha(c),$$

where

$$\alpha(c) = \sup\{x : \rho(x) \ge e^{-1/c}\}.$$

Clearly this theorem gives a concrete functional L to determine F (c.f. Theorem 17).

Now, we turn to some number theoretical applications of probability, an area investigated frequently by Erdős and Rényi.

Let $q \ge 2$ be an integer. Then, as it is well-known, every real number x $(0 \le x \le 1)$ can be represented in the form

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{q^n},$$

where the n-th "digit" $\varepsilon_n(x)$ may take values $0, 1, \ldots, q-1$. The classical Borel strong law of large numbers claims that for almost all real numbers $0 \le x \le 1$ the relative frequency of the numbers $0, 1, 2, \ldots, q-1$ among the first n digits of the q-adic expansion of x tends to $1/q$ as $n \to \infty$.

A possible generalization of the q-adic expansion is the so-called Cantor's series. Let $q_1, q_2, \ldots$ be an arbitrary sequence of positive integers, restricted only by the condition $q_n \ge 2$. Then the Cantor's series of x is

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{q_1 q_2 \ldots q_n}$$

where the n-th digit $\varepsilon_n(x)$ may take the values of $1, 2, \ldots, q_n - 1$.

Let $f_n(k, x)$ denote the number of those digits among $\varepsilon_1(x), \ldots, \varepsilon_n(x)$ which are equal to k $(k = 0, 1, 2, \ldots)$, i.e., put

$$f_n(k, x) = \sum_{\varepsilon_j(x)=k,\ 1 \le j \le n} 1.$$

Let us put further

$$Q_n = \sum_{j=1}^{n} \frac{1}{q_j}$$

and

$$Q_{nk} = \sum_{j=1,\ q_j>k}^{n} \frac{1}{q_j}.$$

Then a possible generalization of Borel's theorem is proved by Rényi {61} claims that for almost all $0 < x < 1$ we have

$$\lim_{n\to\infty} \frac{f_n(k,x)}{Q_{nk}} = 1$$

for those values of k for which

$$\lim_{n\to\infty} Q_{nk} = \infty.$$

For those values of k for which Q_{nk} is bounded, $f_n(k,x)$ is bounded for almost all x.

Erdős and Rényi {23} studied the behaviour of

$$\max_k f_n(k,x)$$

i.e., that of the frequency of the most frequent number among the first n digits in the case when $\lim_{n\to\infty} Q_n = \infty$. It turns out that the behaviour of $\max_k f_n(k,x)$ is very sensitive to the properties of the sequence $\{q_n\}$.

In the case when

$$\sum_{n=1}^{\infty} \frac{1}{q_n} < \infty,$$

the Cantor series has a strikingly different behaviour. It is studied by Erdős, Rényi {24}.

Other investigated expansions of the real numbers are the so-called Engel's and Sylvester's series (Erdős, Rényi, Péter Szüsz {28}, Rényi {70}) and the Cantor's products (Rényi {64}).

A very general expansion is treated by Rényi {62}.

As we have said earlier, Rényi first met probability through Linnik and he was especially interested in Linnik's large sieve. He returned later to this problem occasionally.

In the original formulation of Linnik the large sieve asserts that if we take any sequence S_N consisting of $Z \le N$ positive integers and if Y denotes the number of those primes $p \le \sqrt{N}$ for which all the elements of the sequence

S_N are contained in $p(1-\varepsilon)$ residue class mod p, where $0 < \varepsilon < 1$, then one has

$$Y \leq \frac{20\pi N}{\varepsilon^2 Z}.$$

Rényi {58} also proved that if Z is not too small compared with N then the elements of the sequence S_N not only occupy "almost all" residue classes mod p with respect to most primes $p \leq \sqrt{N}$ but are almost uniformly distributed in the p residue classes mod p for most primes $p \leq \sqrt{N}$.

Even more precise results are given by Rényi ({65} and {66}). The last paper in this subject (Erdős, Rényi {26}) gives a number of applications of the large sieve method.

Among the numerous very different subjects which Rényi was interested in, finally we mention two further ones. One of these is the measure of dependence (Rényi {67}). In this paper he discusses and compares certain quantities which are used to measure the strength of dependence between two random variables. He formulates seven rather natural postulates which should be fulfilled by a suitable measure of dependence. It is shown that most of the known measures of dependence fulfill these conditions. Among them the so-called maximal correlation is studied in detail.

Finally we mention a very elementary problem. In classical probability theory many identities and inequalities are known for probabilities of arbitrary events. The best known theorem of this type is the Poincaré theorem: Let $A_1, A_2, \ldots, A_n$ be arbitrary events and

$$S_0 = 1, \quad S_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \mathbf{P}\{A_{i_1} A_{i_2} \ldots A_{i_k}\} \quad (k = 1, 2, \ldots, n).$$

Then

$$\mathbf{P}\{\bar{A}_1 \bar{A}_2 \ldots \bar{A}_n\} = \sum_{k=0}^{n} (-1)^k S_k.$$

Rényi {69} was interested to find a general method of proof of such formulas.

Let $F_1 = f_1(A_1, \ldots, A_n), F_2 = f_2(A_1, \ldots, A_n), \ldots, F_N = f_N(A_1, \ldots, A_n)$ where f_i's are Boole functions. Consider the linear inequality

$$\sum_{i=1}^{N} \alpha_i \mathbf{P}\{F_i\} \geq 0$$

where $\alpha_1, \alpha_2, \ldots, \alpha_N$ are given real numbers. The result of Rényi tells us that if we want to prove the above inequality for any sequence of events

$A_1, A_2, \ldots, A_n$ then it is enough to verify it in the 2^n special cases when some of them are the empty set and the others are the complete probability space. Later on János Galambos and Rényi {37} proved a similar theorem for quadratic inequalities.

Till his death (1970) Rényi was the spiritual and in some sense the administrative head of the Hungarian probability school. After his death Erdős took over as the spiritual leader. He suggested the most important problems to the Hungarian probabilists. One of those was a continuation of the Erdős–Rényi paper: On a new law of large numbers.

In fact Theorems 18, 19, 20 do not give an exact answer of the question of Varga. In a joint paper Erdős and Pál Révész {29} proved in a very precise sense that the length of the longest head run in n Bernoulli trials is $\lg n$.

Theorem 21. *Let Z_N be the length of the longest head-run till N. Then*

$$Z_N \geq [\lg N - \lg\lg\lg N + \lg\lg e - 2 - \varepsilon]$$

for all but finitely many N if $\varepsilon > 0$ but

$$Z_N \leq [\lg N - \lg\lg\lg N + \lg\lg e - 1 + \varepsilon] \quad \text{i.o. a.s.}$$

However, for some N the r.v. Z_N can be larger than the above given bounds.

Theorem 22. *Let $\{a_n\}$ be a sequence of positive numbers and let*

$$A\big(\{a_n\}\big) = \sum_{n=1}^{\infty} 2^{-a_n}.$$

Then

$$Z_N \leq a_N \quad \text{a.s.}$$

for all but finitely many N if $A\big(\{a_n\}\big) < \infty$ but

$$Z_N > a_N \quad \text{i.o. a.s.}$$

if $A\big(\{a_n\}\big) = \infty$.

It is also interesting to ask what the length is of the longest run containing at most T ($T = 1, 2, \ldots$) tails. Denote by $Z_N(T)$ this length. Then the four results of Theorems 21 and 22 can be generalized for this case. Here we mention only one of them:

$$Z_N(T) \geq [\lg N + T \lg\lg N - \lg\lg\lg N - \lg T! + \lg\lg e - 1 + \varepsilon] \quad \text{i.o. a.s.}$$

for any $\varepsilon > 0$.

Among the further Hungarian results going in this direction we mention only a few.

Komlós, Tusnády {47}.

Sándor Csörgő {12}.

Miklós Csörgő, J. Steinebach {11}.

Tamás F. Móri {53}.

Paul Deheuvels, Erdős, Karl Grill, Révész {13}.

Móri {54}.

Endre Csáki, Antónia Földes, Komlós {9}.

Erdős and Taylor {36} beside their many interesting new results proposed a number of unsolved problems. In the eighties new efforts were taken to solve these problems. One of them is the so-called covering problem.

We say that the disc

$$Q(r) = \{x \in \mathbb{Z}^2,\ \|x\| \leq r\}$$

is covered by the random walk $\{S_k\}$ in time n if for each $x \in Q(r)$ there exists an integer $k \leq n$ such that $S_k = x$. Let $R(n)$ be the largest integer for which $Q\big(R(n)\big)$ is covered in time n. Erdős and Taylor presented the conjecture that $R(n)$ is about $\exp\big((\log n)^{1/2}\big)$. This fact was proved by Erdős, Révész {32} and by Peter Auer, Révész {1}. The fundamental result is the following:

$$\exp\big((\log n)^{1/2}(\log\log n)^{-1/2-\varepsilon}\big) \leq R(n) \leq \exp\big(2(\log n)^{1/2}\log\log\log n\big) \quad \text{a.s.}$$

for all but finitely many n.

Having the above inequality we can say that the Erdős–Taylor conjecture is correct, i.e., the radius of the largest circle around the origin, covered in time n is about $\exp\big((\log n)^{1/2}\big)$. It is natural to ask: how big is the radius $r(n)$ of the largest circle in $\mathbb{Z}^2$ not surely around the origin, which is covered in time n. One expects that $r(n)$ cannot be much larger than $R(n)$. However, by Erdős–Révész {33} we have

Theorem 23. *Let*

$$\psi_0 = \frac{1}{50} \quad \textit{and} \quad \chi_0 = 0,42.$$

Then for any $0 < \psi < \psi_0 < \chi_0 < \chi$ *we have*

$$n^\psi \le r(n) \le n^\chi \quad \textit{a.s.}$$

for all but finitely many n.

A survey on covering problems is: Révész {74}.

Let $\{S_n\}$ be a random walk on $\mathbb{Z}^d$ and let

$$\xi(x,n) = \#\{k \,:\, 0 \le k \le n,\ S_k = x\}, \quad (x \in \mathbb{Z}^d)$$

$$\zeta(n) = \max_{x\in\mathbb{Z}^d} \xi(x,n).$$

A point $z_n \in \mathbb{Z}^d$ is called a favourite value at moment n if the particle visits z_n most often during the first n steps, i.e.,

$$\xi(z_n, n) = \zeta(n).$$

Erdős liked to write papers on different subjects of mathematics with the title: "Problems and results on …". He has only one such paper in probability (Erdős, Révész {31}).

In this paper, among others, there are a few problems mentioned on the favourite values. Here we recall one of those.

One can easily observe that for infinitely many n there are two favourite values and also for infinitely many n there is only one favourite value with probability one. More formally speaking let F_n be the set of favourite values, i.e.,

$$F_n = \big\{ z \,:\, \xi(z,n) = \zeta(n) \big\}$$

and let $|F_n|$ be the cardinality of F_n. Then the question is: whether 3 or more favourite values can occur i.o., i.e.,

$$\mathbf{P}\big\{ |F_n| = r \ \text{ i.o.} \big\} = 1?$$

It turned out that this innocent looking question is very hard. In fact it is still open. The strongest result is due to Bálint Tóth {76} who proved

$$\mathbf{P}\{\,|F_n| \geq 4 \ \text{ i.o.}\} = 0.$$

Let

$$f_n = \max\{\,|x| \,:\, x \in F_n\}$$

be the largest favourite value. The question how big f_n can be was studied by Erdős and Révész {30} who proved that

$$\limsup_{n\to\infty} \frac{f_n}{(2n \log\log n)^{1/2}} = 1 \quad \text{a.s.}$$

Bass and Griffin {3} studied the much harder question: how small f_n can be.

The mentioned Erdős–Taylor paper also contains very nice results and problems on the properties of

$$\zeta(n) = \max_{x\in\mathbb{Z}^d} \xi(x,n).$$

The one dimensional results are well known. In fact we have

$$\limsup_{n\to\infty} \frac{\xi(0,n)}{(2n \log\log n)^{1/2}} = \limsup_{n\to\infty} \frac{\zeta(n)}{(2n \log\log n)^{1/2}} = 1 \text{ a.s.}$$

if $d = 1$. In case $d = 2$ Erdős and Taylor proved

Theorem 24. *Let $f(x)$ resp. $g(x)$ be a decreasing resp. increasing function for which*

$$f(x)\log x \nearrow \infty, \quad g(x)(\log x)^{-1} \searrow 0.$$

Then

$$\xi(0,n) \leq \pi^{-1} g(n) \log n \quad \text{a.s.}$$

for all n large enough if and only if

$$\int_2^\infty \frac{g(x)}{x \log x} e^{-g(x)} dx < \infty$$

and

$$\xi(0,n) \geq f(n) \log n \quad \text{a.s.}$$

for all n large enough if and only if

$$\int_2^\infty \frac{f(x)}{x\log x}dx < \infty.$$

Also in case $d=2$ they presented the following conjecture

$$\lim_{n\to\infty}\frac{\zeta(n)}{(\log n)^2} = \frac{1}{\pi} \quad \text{a.s.}$$

This conjecture was proved by Amir Dembo, Yuval Peres, Jay Rosen, Ofer Zeitouni {14}.

Theorem 4, in case $d=1$, gives a lower estimate of the last return $R(n)$ of a random walk to its starting point before its n-th step. Let

$$R^*(n) = \max\big\{ k : 1 < k < n \text{ for which there exists a } 0 < j < n-k \text{ such that } \xi(j+k)-\xi(j)=0\big\}$$

be the length of the longest zero-free interval. Remember that

$$\xi(n) = \#\{k : 0 \le k \le n,\ S_k = 0\}.$$

It is easy to see that replacing $R(n)$ by $R^*(n)$, Theorem 4 remains true in its original form. For example we have

$$R^*(n) \ge n - \frac{n}{(\log n)^2} \quad \text{i.o. a.s.}$$

However for some n, $R^*(n)$ can be much smaller than the above lower estimate. Endre Csáki, Erdős and Révész in {8} asked how small $R^*(n)$ can be. As an answer of this question we proved:

Theorem 25. *Let $f(n)$ be an increasing function for which*

$$f(n) \nearrow \infty, \quad \frac{n}{f(n)} \nearrow \infty \quad (n\to\infty).$$

Then

$$\mathbf{P}\left\{R^*(n) \le \beta\frac{n}{f(n)} \quad \text{i.o.}\right\} = \begin{cases} 1 & \text{if } J=\infty, \\ 0 & \text{if } J<\infty \end{cases}$$

where

$$J = \sum_{n=1}^\infty \frac{f(n)}{n}\exp\big(-f(n)\big)$$

and $\beta = 0.85403\ldots$ *is the root of the equation*

$$\sum_{k=1}^{\infty} \frac{\beta^k}{k!(2k-1)} = 1.$$

As a consequence of this theorem we mention that

$$\liminf_{n\to\infty} \frac{\log\log n}{n} R^*(n) = \beta \quad \text{a.s.}$$

The path of a random walk between two zeros is called an excursion. Then Theorems 4 and 23 tell us that for any $\varepsilon > 0$ the length of the longest excursion not surely completed before n is

$$R^* \begin{cases} \le n - \dfrac{n}{(\log n)^{2+\varepsilon}} & \text{a.s. if } n \text{ is big enough,} \\ \ge n - \dfrac{n}{(\log n)^2} & \text{i.o. a.s.,} \\ \ge (1-\varepsilon)\beta \dfrac{n}{\log\log n} & \text{i.o. a.s.,} \\ \le (1+\varepsilon)\beta \dfrac{n}{\log\log n} & \text{a.s. if } n \text{ is big enough.} \end{cases}$$

Besides studying the length of the longest excursion $R^*(n)$, it looks interesting to say something about the second, third ... etc. longest excursions. Let $R_1^*(n) \ge R_2^*(n) \ge \ldots \ge R^*_{\xi(n)+1}(n)$ be the length of the second, third etc. longest excursions. Then we have

Theorem 26. *For any fixed* $k = 1, 2, \ldots$ *we have*

$$\liminf_{n\to\infty} \frac{\log\log n}{n} \sum_{j=1}^{k} R_j^*(n) = k\beta \quad \text{a.s.}$$

Theorem 4 tells us that for some n nearly the whole random walk $\{S_k\}_{k=0}^n$ is one excursion. Theorem 24 tells us that for some n the random walk consists of at least $\beta^{-1}\log\log n$ excursions. These results suggest the question: For which values of $k = k(n)$ will the sum $\sum_{j=1}^{k} R_j^*(n)$ be nearly equal to n? In fact we formulated two questions:

Question 1. For any $0 < \varepsilon < 1$ let $\mathcal{F}(\varepsilon)$ be the set of those functions $f(n)$ $(n = 1, 2, \ldots)$ for which

$$\sum_{j=1}^{f(n)} R_j^*(n) \geq n(1-\varepsilon)$$

with probability 1 except finitely many n. How can we characterize $\mathcal{F}(\varepsilon)$?

Question 2. Let $\mathcal{F}(o)$ be the set of those functions $f(n)$ $(n = 1, 2, \ldots)$ for which

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^{f(n)} R_j^*(n) = 1 \quad \text{a.s.}$$

How can we characterize $\mathcal{F}(o)$?

Studying the first question we have

Theorem 27. *For any $0 < \varepsilon < 1$ there exists a $C = C(\varepsilon) > 0$ such that*

$$C \log\log n \in \mathcal{F}(\varepsilon).$$

Concerning Question 2, we have the following result:

Theorem 28. *For any $C > 0$*

$$f(n) = C \log\log n \notin \mathcal{F}(o)$$

and for any $h(n) \nearrow \infty$ $(n \to \infty)$

$$h(n) \log\log n \in \mathcal{F}(o).$$

Up to now we mostly concentrated on the results of Erdős and Rényi and their students. For a recent review of Erdős's work in probability and statistics we refer to Miklós Csörgő {10}.

We now consider the works of a few Hungarian probabilists whose results are not so strongly connected to the Erdős–Rényi school. In fact we give a short survey of the works of Béla Gyires, Pál Medgyessy and József Mogyoródi.

Gyires' most important contribution to probability theory is the foundation of the theory of stationary matrix valued processes and the solution of some extrapolation problems {38, 40, 41}. In this theory block Toeplitz matrices generated by matrix valued functions play an important role. He generalized results due to Szegő and to Helson and Lowdenslager.

In {39} he proves an interesting generalization of a central limit theorem for a sequence $\zeta_n = \xi_1 + \cdots + \xi_n$, where $\xi_k = \theta^{(k)}_{\eta_{k-1},\eta_k}$ with mutually independent random variables $\theta^{(k)}_{i,j}$ $(i,j = 1,\ldots,p;\ k = 1,2,\ldots)$ which are independent of the Markov chain $\{\eta_n\}$ with states $1,\ldots,p$. He shows that if the chain is ergodic and the conditional distribution functions $\mathbf{P}\{\xi_k < x \mid \eta_{k-1} = i\}$ have zero mean and finite second moments and satisfy a condition of the Lindeberg type then it satisfies the central limit theorem.

He also developed a systematic investigation of the decomposability problems of distribution functions. The main results can be found in his book {42}. The problem is to give conditions for a distribution function F to be a mixture of a given stochastic kernel G with a weight function H from a certain given set of distribution functions.

Medgyessy was mostly interested in the decomposition of superpositions of distribution functions. The superposition of distributions is a frequently used operation in probability. Let $F_1, F_2, \ldots, F_N$ resp. $p_1, p_2, \ldots, p_N$ be sequences of distributions resp. real numbers. The function

$$G(x) = \sum_{k=1}^{N} p_k F_k(x)$$

will be called a superposition of F_k. Assume also that F_i's $(i = 1,2,\ldots,N)$ are elements of a class of distributions containing a finite number of parameters. Then the problem is the following: Given the superposition $G(x)$ determine the parameters of the components F_k when their analytic form is known (only their parameters should be determined by the aid of $G(x)$).

Working through 8 years on this topic Medgyessy wrote a book {50}.

Mogyoródi was also a student of Rényi. However, his research area moved away from Rényi's school. He was mostly interested in martingales and Orlicz and Hardy spaces.

First we recall the definition of the Orlicz space. A random variable X defined on a probability space $\{\Omega, \mathcal{S}, \mathbf{P}\}$ belongs to the Orlicz space $L^\Phi(\Omega, \mathcal{S}, \mathbf{P})$ if there exists a constant $a > 0$ such that $\mathbf{E}\Phi\big(a^{-1}|X|\big) \le 1$ where Φ is a Young-function. The L^Φ-norm of X is:

$$\|X\|_\Phi = \inf\Big\{a \,:\, a > 0,\ \mathbf{E}\Phi\big(a^{-1}|X|\big) \le 1\Big\}.$$

Now we give the definition of the Hardy space $\mathcal{H}_\Phi$. Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$ be a sequence of σ-fields with $\lim_{n\to\infty} \mathcal{F}_n = \mathcal{S}$. Consider the martingale $X_n = \mathbf{E}(X \mid \mathcal{F}_n)$ and the martingale-differences $d_i = X_{i+1} - X_i$. We say that $X \in L^1$ belongs to $\mathcal{H}_\Phi$ if

$$S = \left(\sum_{i=1}^{\infty} d_i^2 \right)^{1/2} \in L^\Phi.$$

In general a sequence $\Theta = (\Theta_1, \Theta_2, \dots)$ belongs to the Banach space $\delta\mathcal{H}_\Phi$ if

$$\left(\sum_{i=1}^{\infty} \Theta_i^2 \right)^{1/2} \in L^\Phi.$$

Mogyoródi in {51} gave a characterization of the linear functionals on Hardy spaces, similar to Riesz' characterization of the linear functionals on Hilbert spaces. It is known that if (Φ, Ψ) is a pair of conjugate Young-functions and A is a bounded linear functional on L^Ψ then there exists a random variable $X \in L^\Phi$ such that $AY = \mathbf{E}XY$ for any $Y \in L^\Psi$. The main result of this paper concludes that under mild conditions we have:

$$AY = \lim_{n\to\infty} \mathbf{E}X_n Y_n,$$

where $X_n = \mathbf{E}(X \mid \mathcal{F}_n)$ and $Y_n = \mathbf{E}(Y \mid \mathcal{F}_n)$.

In an other paper {52} Mogyoródi investigates the properties of $X_n^* = \max_{0\le k\le n} X_k$ and those of $X^* = \sup_{k\ge 0} X_k$.

Let $\Phi(\cdot)$ be a Young-function with derivative $\varphi(\cdot)$ and let $\xi(x) = x\varphi(x) - \varphi(x)$.

One of the main results of the paper concludes that for any $\rho > 1$ we have

$$\mathbf{E}\xi \left(\frac{X_n^*}{\rho \|M_n\|_\Phi} \right) \le \frac{1}{\rho - 1},$$

where $\{M_n\}$ is the martingale in the Doob decomposition of X_n, i.e., $M_n = X_n + A_n$, where $\{A_n\}$ is a unique and predictable sequence of random variables.

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