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Orthogonal Polynomials

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The theory of orthogonal polynomials plays an important role in many branches of mathematics, such as approximation theory (best approximation, interpolation, quadrature), special functions, continued fractions, differential and integral equations. The notion of orthogonality originated from the theory of continued fractions, but later became an independent (and possibly more important) discipline. Among the contributors to the theory of orthogonal polynomials, we can find such outstanding mathematicians as Abel, Chebyshev, Fourier, Hermite, Laguerre, Laplace, Legendre, Markov, and Stieltjes, just to name a few. Beginning with Gábor Szegő, Hungarian mathematicians like Pál Erdős, Pál Turán, Géza Freud, Ervin Feldheim and others have made essential contributions to the flourishing theory of orthogonal polynomials in the last century. At this point I would like to mention two names who have made considerable efforts to propagate the work of the above mentioned Hungarian mathematicians: Richard Askey and Doron Lubinsky.

A considerable part of the material to be presented below is based on the classical book of Szegő [174]. Since its first publication in 1939, this monograph has reached three more editions, and until now is the most comprehensive, most quoted source on the subject. Another source of information concerning more recent developments in the theory of orthogonal polynomials is the monograph [47] of Géza Freud and a survey paper by Paul Nevai (J. Approx. Theory 48 (1986), pp. 3–167).

Problems connected with interpolation and mechanical quadrature on the roots of orthogonal polynomials are considered in another chapter written by Péter Vértesi.

Let α be a real valued increasing function on $\mathbf{R}$. We call such an α a *distribution function* if it assumes infinitely many values and the improper

Stieltjes integrals

$$c_n = \int_{\mathbf{R}} x^n \, d\alpha(x), \qquad n = 0, 1, \ldots \tag{1}$$

(called *moments*) exist and are finite. If α is absolutely continuous we write $d\alpha(x) = w(x)\,dx$, and call w a *weight function.* Let $\mathcal{P}_n$ and $\mathcal{P}_n^c$, $n = 0, 1, \ldots$ be the set of real and complex algebraic polynomials of degree at most n, respectively. Associated with each distribution function α, there is a unique sequence of *orthonormal polynomials* $p_n \in \mathcal{P}_n$, $\deg p_n = n$, $n = 0, 1, \ldots$ with positive leading coefficients γ_n and having the property

$$\int_{\mathbf{R}} p_n(x) p_m(x) \, d\alpha(x) = \delta_{mn}, \qquad m, n = 0, 1, \ldots .$$

Of course, the region of the above integration can be restricted to the *support* of the distribution α which, in general, may be a finite interval, the half line, or $\mathbf{R}$.

We list some important properties of orthogonal polynomials.

1. p_n, $n = 0, 1, \ldots$ are linearly independent.

2. All roots of p_n are real, simple, and lie in the interior of the support of α (which is a finite or infinite interval $[a, b]$).

3. With the help of the moments (1), the representation

$$p_n(x) = (D_{n-1} D_n)^{-1/2} \begin{vmatrix} c_0 & c_1 & c_2 & \ldots & c_n \\ c_1 & c_2 & c_3 & \ldots & c_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & c_{n+1} & \ldots & c_{2n-1} \\ 1 & x & x^2 & \ldots & x^n \end{vmatrix}$$

holds, where $D_n = \det\,[c_{i+j}]_{i,j=0,\ldots,n}$, $n = 1, 2, \ldots$.

4. There is a three term recurrence relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \qquad n = 2, 3, \ldots, \tag{2}$$

where

$$a_n = \frac{\gamma_{n-1}}{\gamma_n} \quad \text{and} \quad b_n = \int_{\mathbf{R}} t p_n(t)^2 \, d\alpha(t).$$

5. The Christoffel–Darboux formula

$$K_n(x,y) := \sum_{k=0}^{n-1} p_k(x)p_k(y) = \frac{\gamma_{n-1}}{\gamma_n}\,\frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y} \tag{3}$$

holds.

Note also the following important property of K_n:

$$\lambda_n(x) := \frac{1}{K_n(x,x)} = \inf_{p(x)=1,\ p\in\mathcal{P}_{n-1}} \int_a^b p^2(t)\,d\alpha(t). \tag{4}$$

The latter is called the nth *Christoffel function* associated with $d\alpha$. Perhaps this is the most important notion in connection with orthogonal polynomials, since results concerning this function have a deep effect on various fields of approximation theory, harmonic and numerical analysis.

6. The Gauss–Jacobi quadrature formula

$$\int_{\mathbf{R}} p\,d\alpha = \sum_{k=0}^{n} \lambda_k p(x_k)$$

valid for every polynomial $p \in \mathcal{P}_{2n-1}$, where $x_k = x_{kn}$ are the roots of the orthogonal polynomial p_n, and

$$\lambda_k = \int_a^b \frac{p_n(x)}{p_n'(x_k)(x-x_k)}\,dx = \frac{1}{K_n(x_k,x_k)} > 0, \qquad k = 0,1,\dots,$$

are the Cotes numbers.

With each function f such that $\int_{\mathbf{R}} f^2\,d\alpha < \infty$, we can associate the nth partial sum

$$S_n(d\alpha, f, x) := \sum_{k=0}^{n} f_k p_k(x), \tag{5}$$

of the orthogonal expansion, where

$$f_k := \int_{\mathbf{R}} f p_k\,d\alpha, \qquad k = 0,1,\dots .$$

Of particular interest are the classical orthogonal polynomials, when

$$w(x) = \begin{cases} (1-x)^\alpha(1+x)^\beta \ (\alpha,\beta > -1) & \text{(Jacobi polynomials on } [-1,1],\\ x^\alpha e^{-x} \quad (\alpha > -1) & \text{(Laguerre polynomials on } [0,\infty),\\ e^{-x^2} & \text{(Hermite polynomials on } \mathbf{R}). \end{cases}$$

What makes these cases interesting is the existence of second order linear differential equations

$$(1-x^2)p_n''(x)+\big[\beta-\alpha-(\alpha+\beta+2)x\big]\,p_n'(x)+n(n+\alpha+\beta+1)p_n(x)=0,$$
$$xp_n''(x)+(\alpha+1-x)p_n'(x)+np_n(x)=0,$$
$$p_n''(x)-2xp_n'(x)+2np_n(x)=0,$$

respectively. These differential equations (which do not exist in the general case) make all considerations much easier. Hence our knowledge concerning the classical orthogonal polynomials is much more exhaustive.

The special case $\alpha=\beta=0$ of the above mentioned Jacobi polynomials, i.e. the Legendre polynomials, were generalized by Leopold Fejér in the following way. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a generator function analytic in a neighborhood of the origin with all a_n real. Consider

$$\big|\,f(re^{i\theta})\big|^2=\sum_{n=0}^{\infty}p_n(\cos\theta)r^n,$$

where

$$p_n(\cos\theta)=\sum_{k=0}^{n}a_ka_{n-k}\cos(n-2k)\theta$$

is a polynomial of degree n of the variable $\cos\theta$. These are generalizations of the Legendre polynomials which are obtained in the special case $f(z)=(1-z)^{-1/2}$. More generally, for $f(z)=(1-z)^{-\lambda}$, $\lambda>-1/2$, we obtain the ultraspherical Jacobi polynomials (with the above notation, for $\alpha=\beta=\lambda-1/2$). Imposing proper monotony and asymptotic conditions on the sequence $\{a_n\}$, one can obtain polynomials p_n with different interesting properties (see Ch. 6.5 of Szegő's monograph [174]). The most intriguing question in this context is whether the polynomials p_n are orthogonal with respect to some weight function. Ervin Feldheim (Izv. Akad. Nauk SSSR, Ser. Math. **5** (1941), 241–248) and independently I. L. Lanzewizky showed that in case of orthogonality, these polynomials p_n can be rescaled to C_n to satisfy a three term recurrence relation of the form (2):

$$2x(1-\beta q^n)C_n(x)=(1-q^{n+1})C_{n+1}(x)+(1-\beta^2q^{n-1})C_{n-1}(x)$$

for $n\ge 1$, where

$$C_0(x)=1,\quad C_1(x)=2x\frac{1-\beta}{1-q},\qquad |\beta|,|q|<1.$$

(When $\beta = q^\lambda$ we obtain the ultraspherical polynomials mentioned above.) Although Feldheim was unable to obtain explicit representation for these polynomials, as well as for the weight function, this was a crucial step in characterizing these polynomials. Later the polynomials were found explicitly by Richard Askey who did pioneering work in the theory of so-called q-polynomials (cf. e.g. R. Askey, M. Ismail, Studies in Pure Math., P. Erdős editor, Birkhäuser, Basel, 1983, pp. 55–78.).

The notion of orthogonal polynomials was extended to complex domains mainly by the pioneering work of Szegő (cf. [174], Chs. XI and XVI). The most interesting case is the unit circle $|z| = 1$. Let μ be an increasing 2π periodic function which takes infinitely many values in any interval of length 2π. Then there is a uniquely determined sequence of complex polynomials $\phi_n(z) = \kappa_n z^n + \dots$, $\phi_n \in \mathcal{P}_n$, $\kappa > 0$ real numbers, such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_m(z)\overline{\phi_n(z)}\, d\mu(\theta) = \delta_{mn}, \qquad z = e^{i\theta}; \quad m, n = 0, 1, \dots .$$

Szegő has shown that properties of these polynomials are somewhat simpler than those of the real orthogonal polynomials, to which they are related (in case of a finite interval). Such properties enable us to derive statements for real orthogonal polynomials from those of complex ones.

All roots of the complex orthonormal polynomials lie in the open unit circle. The problem of distribution of roots in the unit circle is a delicate question. Turán (J. Approx. Theory **29** (1980), 23–85.) raised the question whether the accumulation points of the roots of the orthogonal polynomials can fill up the whole unit disk. I (Acta Math. Acad. Sci. Hungar. **33** (1979), 197–210.) gave a partial answer to this question by constructing a weight function for any given $\varepsilon > 0$ such that the two dimensional Lebesgue measure of the above mentioned accumulation points greater than $\pi - \varepsilon$. The complete positive answer to Turán's problem was given later by M. P. Alfaro and L. Vigil (J. Approx. Theory **53** (1988), 195–197.).

Besides the three term recurrence relation (2), complex orthogonal polynomials on the unit circle obey the simpler two term forward-backward recurrence relations

$$\kappa_n z\phi_n(z) = \kappa_{n+1}\phi_{n+1}(z) - \phi_{n+1}(0)\phi^\star_{n+1}(z)$$

and

$$\kappa_n \phi_{n+1}(z) = \kappa_{n+1}\phi_n(z) + \phi_{n+1}(0)\phi^\star_n(z),$$

where $\phi_n^\star(z) := z^n\overline{\phi}_n(z^{-1})$ are the so-called *reciprocal polynomials,* where the bar indicates that the coefficients of the corresponding polynomial are conjugated (cf. Szegő [174]). These formulas play an important role in constructing a simpler algorithm for the prediction of a stationary time series (cf. [173], pp. 43–44).

In order to discover further properties of orthogonal polynomials, we have to consider the analog

$$K_n(z,u) := \sum_{k=0}^{n-1}\overline{\phi_k(z)}\phi_k(u) = \frac{\overline{\phi_n^\star(z)}\phi_n^\star(u) - \overline{\phi_n(z)}\phi_n(u)}{1-\overline{z}u}$$

of the Christoffel–Darboux formula (3). The analogue of the Christoffel function (4) is

$$\omega_n(d\mu,z) := K_n(z,z)^{-1} = \left[\sum_{k=0}^{n-1}\left|\phi_k(z)\right|^2\right]^{-1} \tag{6}$$

$$= \inf_{\substack{p\in\mathcal{P}_{n-1}^c\\ p(z)=1}} \frac{1}{2\pi}\int_0^{2\pi}\left|p(u)\right|^2 d\mu(t), \qquad u=e^{it}.$$

Let $0 \le g \in L_1$ be a positive measurable function in $[0,2\pi]$. The *Szegő function* associated with g is defined as

$$D(g,z) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi}\frac{u+z}{u-z}\log g(t)\,dt\right\}, \qquad u=e^{it}, \quad |z|<1. \tag{7}$$

$D(g,0)\neq 0$ in $|z|<1$, $D(g)$ is square integrable in the unit disk, $D(g,0)>0$, the radial limit $D(g,e^{it})$ exists, and $\left|D(g,e^{it})\right|^2 = g(t)$ almost everywhere. Szegő [174] proved that if μ is differentiable and $\log\mu' \in L_1$, then the asymptotic relation

$$\lim_{n\to\infty}\phi_n^\star(z) = D(\mu',z^{-1})$$

holds uniformly in every compact subset of $|z|<1$. Moreover,

$$\lim_{n\to\infty} z^{-n}\phi_n(z) = \overline{D(\mu',z^{-1})}, \qquad |z|>1,$$

and

$$\lim_{n\to\infty}\kappa_n = D(\mu',0)^{-1},$$

or equivalently

$$\lim_{n\to\infty}\int_0^{2\pi}\Big[\big|\phi_n(z)\big|\mu'(t)^{1/2}-1\Big]^2\,dt=0,\qquad z=e^{it}.$$

The latter result was later extended by Attila Máté, Paul Nevai, and Vilmos Totik to the case when $\log\mu'\in L_1$ is replaced by the weaker condition that $\mu'>0$ almost everywhere.

As for the behavior of the orthogonal polynomials on the unit circle, Szegő and U. Grenander [59] showed that

$$\lim_{n\to\infty}\Big[\phi_n(e^{it})-e^{int}\overline{D^{-1}(\mu',e^{it})}\Big]=0 \tag{8}$$

uniformly in t, provided $0<m\le\mu'(t)\le M<\infty$ and the modulus of continuity $\omega(\mu',h)$ of μ' is of order $\big(\log(1/h)\big)^{-\beta}$, $\beta>1$. The latter condition was later weakened to $\int_0^\pi\frac{\omega(\mu',h)}{h}\,dh<\infty$ by Freud, who also gave similar conditions for the pointwise convergence in (8) (cf. [47]). The idea of localization of the asymptotic behavior of orthogonal polynomials is due to Tamás Frey (Mat. Sbornik **49** (1959), 133–180.).

The Szegő function (7) plays role in solving *Szegő's generalized extremal problem:* find

$$\omega(d\mu,z):=\lim_{n\to\infty}\omega_n(d\mu,z),\qquad |z|<1$$

(note that

$$\omega(d\mu,z)=\begin{cases}\dfrac{\mu(z)}{2\pi}, & \text{if } |z|=1,\\[2mm] 0, & \text{if } |z|>1,\end{cases}$$

where $\mu(z)$ is the $d\mu$-measure of the point z).

Szegő found this limit function when μ is absolutely continuous. The case when μ is not necessarily absolutely continuous was proved by A. N. Kolmogorov and generalized by M. G. Krein for $L^p_{d\mu}$ measures ($p>0$) in (6) instead of $L^2_{d\mu}$-measure.

In the above quoted work of Szegő and Grenander it is proved that

$$\omega(d\mu,z)=\big(1-|z|^2\big)\big|D(\mu',z)\big|^2,\qquad |z|<1$$

for every measure $d\mu$ on the unit circle.

The considerations resulting in the asymptotic formula (8) in the case of the unit circle led Szegő to the investigation of the asymptotic behavior of the Christoffel function (see (4)) in case of the interval $[-1,1]$. He proved that if the support of $d\alpha$ is in $[-1,1]$ and the derivative of the function

$$\frac{1}{|\sin t|\alpha'(\cos t)} \geq 1$$

is in some Lipschitz class, then (cf. Freud [47], p. 269)

$$\lambda_n^{-1}(x) = n - \frac{1}{2} + \operatorname{Re}\left[e^{i\xi}\frac{D'(\mu', e^{i\xi})}{D(\mu', e^{i\xi})}\right]$$
$$+ \frac{1}{2\sin\xi}\operatorname{Im}\left[e^{(2n-1)i\xi}\frac{D(\mu', e^{i\xi})}{\overline{D(\mu', e^{i\xi})}}\right] + o(1), \qquad x = \cos\xi,$$

where

$$\mu(t) := (\operatorname{sgn} t)\big[\alpha(1) - \alpha(\cos t)\big], \qquad |t| \leq \pi.$$

Purely in terms of orthogonal polynomials on the interval $[-1,1]$, a complete and general asymptotic of the Christoffel function can be found in a paper by Paul Nevai {1}. He proved that if $\log\alpha'(\cos t) \in L_1$ then

$$\lim_{n\to\infty} n\lambda_n(x) = \pi\alpha'(x)(1-x^2)^{1/2} \tag{9}$$

for almost all x in an interval where $1/\alpha'(x) \in L_1$. In addition, if $x \in (-1,1)$, α is absolutely continuous in a neighborhood of x and α' is continuous at x then (9) holds. The condition on α later was relaxed to $\log\alpha' \in L_1$ by Máté, Nevai and Totik (Annals of Math. **134** (1991), 433–453.).

Convergence problems of orthogonal expansions can be simplified by considering only Fourier series. This idea of *equiconvergence theorems* was developed by Alfréd Haar (Math. Annalen **78** (1917), 121–136.), and later generalized by Szegő (Math. Zeitschrift **12** (1921), 61–94.) and Freud ([47], p. 260.). The latter proved the following: Let $f \in L^2$ with respect to an absolutely continuous distribution function α whose support is in $[-1,1]$ such that its derivative has a positive polynomial minorant and is in a Lipschitz class. Further let $f^\star$ be that function which coincides with f in a neighborhood of f and zero otherwise. With the notation (5) and $d\alpha_0(x) = (1-x^2)^{-1/2}\,dx$ we have

$$\lim_{n\to\infty}\big[S_n(d\alpha, f, x) - S_n(d\alpha_0, f^\star, x)\big] = 0.$$

Note that the convergence properties of $S_n(d\alpha, f)$ depend only values of f taken in an arbitrarily small interval around the fixed point x. Under some additional conditions the above results can be made uniform in x.

The theory of orthogonal polynomials on infinite intervals is significantly different from that on a finite interval. Until the 1960's not much was known about this topic, except possibly the Hermite, Laguerre and a few other polynomials. Amazingly, while Szegő has made pioneering work in the theory on finite intervals, he was not interested in carrying over his ideas to infinite intervals. It was Freud who founded the now flourishing theory of orthogonal polynomials on the real line, and the corresponding representative polynomials are named after him. In the rest of this chapter we will describe this theory, except problems connected with interpolation and quadrature which are considered in the above mentioned work of Péter Vértesi.

Freud's aim was to extend the theory of best approximation, Jackson–Bernstein type estimates to the real axis. The natural way to do this was to explore properties of orthogonal polynomials, since the expectation was that orthogonal expansions, Cesàro and de la Vallée-Poussin means, may serve as near-best approximation.

What Freud did was to start from Hermite polynomials (orthogonal with respect to the weight e^{-x^2}), and generalize this weight. He considered weights of the form

$$w(x) = e^{-Q(x)} \tag{10}$$

where the even, twice differentiable function $Q > 0$ behaves like a polynomial at infinity, and $xQ'(x)$ is increasing. More exactly, we assume that

$$0 < \alpha = \liminf_{x\to\infty} \frac{\big(xQ'(x)\big)'}{Q'(x)} \le \limsup_{x\to\infty} \frac{\big(xQ'(x)\big)'}{Q'(x)} < \infty. \tag{11}$$

An important role is played by the number q_n which is defined (by Freud) as the unique solution of the equation

$$q_n Q'(q_n) = n, \qquad n = 1, 2 \dots .$$

We mention that a closely related number a_n (which is of the same order of magnitude as q_n) was later determined independently by Mhaskar, Rahmanov and Saff as the unique positive solution of the integral equation

$$\frac{1}{\pi}\int_{-a_n}^{a_n} \frac{tQ'(t)\,dt}{\sqrt{a_n^2 - t^2}} = n, \qquad n = 1, 2, \dots .$$

The number a_n describes the behavior of polynomials at infinity, namely the so-called infinite-finite range relation (this terminology was coined by D. Lubinsky)

$$\max_{|x|\le a_n} w(x)\big|p_n(x)\big| = \max_{x\in\mathbf{R}} w(x)\big|p_n(x)\big|$$

holds for every polynomial $p_n \in \mathcal{P}_n$. While this result tells us that the behavior of a weighted polynomial can be learned by studying it on a finite interval (whose length is independent of the polynomial and depends only on the weight and the degree), the number a_n plays the same role in considering L_p norms: we have

$$\left(\int_{\mathbf{R}} w(x)\big|p_n(x)\big|^p\,dx\right)^{1/p} \le c\left(\int_{-a_n}^{a_n} w(x)\big|p_n(x)\big|^p\,dx\right)^{1/p}, \tag{12}$$

for $p_n \in \mathcal{P}_n$, $p \ge 1$ where $c > 0$ are constants depending only on the weight. This infinite-finite range inequality was discovered by Freud for $p = 2$ and Q =polynomial, and later generalized for weights (10) with the property (11) by D. Lubinsky and others. As a further generalization, I (Advanced Problems in Constructive Approximation, Birkhäuser 2002, pp. 223–236.) considered weights with infinitely many zeros on $\mathbf{R}$. Let

$$0 < t_1 < t_2 < \dots, \qquad \lim_{k\to\infty} t_k = \infty$$

and

$$m_k > 0, \quad k = 1, \dots, \qquad \liminf_{k\to\infty} m_k > 0$$

be two sequences of real numbers satisfying the condition

$$\sum_{k=1}^{\infty} \frac{m_k}{t_k^{\varrho+\varepsilon}} < \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{m_k}{t_k^{\varrho-\varepsilon}} = \infty$$

for some $\varrho \ge 0$ and all $\varepsilon > 0$. Now consider the weight

$$w(x) = e^{-Q(x)} \prod_{k=1}^{\infty} \left|1 - \left(\frac{x}{t_k}\right)^{2q}\right|^{m_k}, \qquad q = [\varrho/2] + 1,$$

where Q again satisfies (11). Here the infinite product converges uniformly in each compact subset of $\mathbf{R}$. Then (12) still holds for $p \ge 1$ provided $0 \le \varrho < \alpha$.

(12) is the key tool for investigating practically all problems arising in connection with orthogonal polynomials on infinite intervals.

The most important step in developing the theory of orthogonal polynomials on infinite intervals, just like in case of finite intervals, is to determine the exact order of magnitude of the Christoffel functions (4). Freud did this in the 1960's for the Hermite weight e^{-x^2}, using ad hoc methods. What he found was an interesting phenomenon. Namely, while

$$\max_{|x|\le(1-\varepsilon)\sqrt{2n}} e^{x^2}\lambda_n(x)\sim n^{-1/2} \tag{13}$$

for any fixed $0<\varepsilon<1$, at the same time

$$\max_{x\in\mathbf{R}} e^{x^2}\lambda_n(x)\sim n^{-1/6}. \tag{14}$$

In other words, the behavior of the Christoffel function (and, in fact, that of the orthogonal polynomials) is different near the critical point $a_n=\sqrt{2n}$. This "irregularity" caused problems in finding the optimal value of the weighted Lebesgue constant of Lagrange interpolation (for details see the chapter written by Péter Vértesi).

The key issue to overcome the difficulties in generalizing the above result e.g. for weights (10) satisfying (11) is the infinite-finite range inequality (12). Freud's original idea to generalize (13)–(14) for weights (10) when Q is restricted to polynomials of the form x^{2m} ($m\ge 1$ is an integer), was to use one-sided approximation for these weights, where it was essential that Q be a polynomial. Later he was able to remove this restriction and generalize (13)–(14) to

$$\min_{x\in\mathbf{R}}\frac{\lambda_n(x)}{w(x)}\ge\frac{ca_n}{n}\quad\text{and}\quad\max_{|x|\le c_1a_n}\frac{\lambda_n(x)}{w(x)}\le\frac{c_2a_n}{n}$$

for weights (10) satisfying (11). Freud's idea was to approximate w by polynomials in the sense

$$p_n^2(x)\sim w,\qquad |x|\le c_1a_n,\quad p_n\in\mathcal{P}_n.$$

The next important task is to find a good approximation for functions with the property that $\lim_{|x|\to\infty}f(x)w(x)=0$. Freud proved the weighted L_p-boundedness of the $(C,1)$-means of Fourier series, i.e.

$$\left\{\int_{\mathbf{R}}\left|\frac{1}{n}\sum_{k=0}^{n}S_k(w,f,t)w(t)\right|^p dt\right\}^{1/p}\le K\left\{\int_{\mathbf{R}}\left|f(t)w(t)\right|^p dt\right\}^{1/p},$$

$1 \le p \le \infty$, for weights (10) under some conditions which basically ensure that Q behaves like a polynomial at infinity. This boundedness easily implies that the de la Vallée-Poussin means

$$\frac{1}{n} \sum_{k=n+1}^{2n} S_k(w, f, x)$$

converge (in the weighted norm) in the order of best weighted approximation.

In order to obtain so-called *converse* theorems, that is statements about structural properties of functions deduced from the order of best approximation, one needs Bernstein–Markov type inequalities. These inequalities restrict the order of magnitude of the derivative of polynomials in terms of the weighted norm and the degree of the polynomial. The classical Markov–Bernstein inequality on the finite interval $[-1, 1]$ states that

$$\max_{|x|\le 1} \bigl|p_n'(x)\bigr| \le \frac{cn^2}{1 + n\sqrt{1-x^2}} \max_{|x|\le 1} \bigl|p_n(x)\bigr|$$

for all $p_n \in \mathcal{P}_n$. Freud proved the L_p-version of this for Hermite weights:

$$\biggl(\int_{\mathbf{R}} e^{-x^2} \bigl|p_n'(x)\bigr|^p \, dx \biggr)^{1/p} \le c\sqrt{n} \biggl(\int_{\mathbf{R}} e^{-x^2} \bigl|p_n(x)\bigr|^p \, dx \biggr)^{1/p}, \quad 1 \le p \le \infty.$$

Later this was generalized to

(15)

$$\frac{\bigl(\int_{\mathbf{R}} e^{-|x|^\alpha} \bigl|p_n'(x)\bigr|^p \, dx \bigr)^{1/p}}{\bigl(\int_{\mathbf{R}} e^{-x^2} \bigl|p_n(x)\bigr|^p \, dx \bigr)^{1/p}} \le \begin{cases} cn^{1-1/\alpha} & \text{if } \alpha > 1, \\ c \log n & \text{if } \alpha = 1, \\ c & \text{if } 0 < \alpha < 1, \end{cases} \qquad 1 \le p \le \infty$$

(for $\alpha \ge 2$ see Freud (J. Approx. Theory **19** (1977), 22–37.), for $1 < \alpha < 2$ see Levin and Lubinsky (J. Approx. Theory **49** (1987), 149–169.), and for $0 < \alpha \le 1$ see Nevai and Totik (Constr. Approx. **2** (1986), 113–127.)). All these inequalities are sharp in the order of magnitude.

A further generalization of (15) is given by Levin and Lubinsky (SIAM J. Math. Anal. **21** (1990), 1065–182.) for weights (10)–(11) and $p = \infty$: then the right hand side of (15) becomes

$$c \int_1^{Q^{[-1]}(n)} \frac{Q(t)}{t^2} \, dt.$$

The sharpness of this "Markov factor" was proved by Kroó and Szabados (J. Approx. Theory **83** (1995), 41–64.).

Another class of polynomial inequalities is the so-called Nikolski-type inequalities. These compare L_p and L_q norms of polynomials. Concerning Freud weights $e^{-|x|^\alpha}$ with $\alpha > 0$, Nevai and Totik proved the following:

$$\frac{\left(\int_{\mathbf{R}} e^{-|x|^\alpha}\left|p_n(x)\right|^p dx\right)^{1/p}}{\left(\int_{\mathbf{R}} e^{-|x|^\alpha}\left|p_n(x)\right|^q dx\right)^{1/q}} \le \begin{cases} cn^{1/\alpha} & \text{if } p \le q, \\ cn^{1-1/\alpha} & \text{if } p > q,\ \alpha > 1, \\ c\log n & \text{if } p > q,\ \alpha = 1, \\ c & \text{if } p > q,\ 0 < \alpha < 1, \end{cases}$$

where $c > 0$ depends only on p and q. The proof uses the infinite-finite range inequality and estimates for the Christoffel function.

In the 1970's Freud made two remarkable conjectures concerning the recursion coefficients (2) and the greatest zero x_{1n} of the orthogonal polynomials with respect to the weight $w(x) = e^{-|x|^\alpha}$, $\alpha > 1$. These are the following:

$$\lim_{n\to\infty} n^{-1/\alpha} a_n = \left[\frac{\Gamma(\alpha/2)\Gamma(\alpha/2+1)}{\Gamma(\alpha+1)}\right]^{1/\alpha} \tag{16}$$

and

$$\lim_{n\to\infty} n^{-1/\alpha} x_{1n} = 2\left[\frac{\Gamma(\alpha/2)\Gamma(\alpha/2+1)}{\Gamma(\alpha+1)}\right]^{1/\alpha}. \tag{17}$$

Besides their beauty, these limit relations have a practical use in the theory of polynomial approximation. Using ad hoc methods, Freud was able to prove (16) for $\alpha = 2$, 4 and 6. His method breaks down for $\alpha \ge 8$. Eventually, A. Máté, P. Nevai and T. Zaslavsky (Trans. Amer. Math. Soc. **287** (1985), 495–505.) proved for any even integer α the asymptotic expansion

$$n^{-1/\alpha} a_n = \sum_{j=0}^{\infty} c_j n^{-2j},$$

where c_0 is the right hand side of (16).

As for (17), it is easy to see that (16) implies it (but not conversely). (17) in full generality ($\alpha > 1$ is a real number) was proved by E. A. Rakhmanov (Math. USSR Sbornik **47** (1984), 155–193.). A similar asymptotic relation for the kth root of the orthogonal polynomials was proved by Máté, Nevai, and Totik when α is an even integer.

Saff and Totik [160] established a very general result concerning the limit distribution of zeros of orthogonal polynomials with respect to certain measures. Let α be a finite Borel measure with support consisting of infinitely many points in $[0,1]$, and let the zeros of the corresponding nth orthogonal polynomial $p_{n,\alpha}$ be $x_1(\alpha,n),\dots,x_n(\alpha,n)$. In general, the limit distribution of these roots does not exist, so one should analyze the weak* limit of the "zero measures"

$$\nu(p_{n,\alpha}) = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i(\alpha,n)} \tag{18}$$

in the space of all unit Borel measures supported on $[0,1]$, where δ_t is the unit mass at t. The *carrier* of α is any Borel set whose complement has zero α-measure, and the *minimal carrier capacity* of α is $c_\alpha = \inf\bigl\{\operatorname{cap}(C) \mid C$ is a carrier of $\alpha\bigr\}$, where "cap" means logarithmic capacity. Now if $c_\alpha > 0$ and C is a minimal carrier of α, then any weak* limit ν of the zero distributions (18) satisfies

$$C\,\text{“}\subseteq\text{”}\,\mathrm{MAX}\,U^\nu \quad \text{and} \quad \operatorname{supp}(\nu) \subseteq \bar{C} \tag{19}$$

where "$\subseteq$" means inclusion except for a set of zero capacity, and $\mathrm{MAX}\,U^\nu$ is the set of maximum points of the logarithmic potential function

$$U^\nu(z) = \int_0^1 \log\frac{1}{|z-t|}\,d\nu(t)$$

associated with the limit distribution ν. Conversely, if $C \subseteq [0,1]$ is of positive capacity and $\mathcal{M}_C$ is the set of probability measures ν satisfying (19), then there is a measure α such that C is a minimal carrier of α and $\mathcal{M}_C = \bigl\{\nu \mid \nu$ is a weak* limit point of the measures $\nu(p_{n,\alpha})\bigr\}$.

Finally, we mention that recently the notion of orthogonality e.g. on the interval $[-1,1]$ was extended to *varying weights,* mostly due to the work of Vilmos Totik. Let u be a measurable function satisfying the so-called Szegő condition

$$\int_{-1}^{1} \frac{\log u(t)}{\sqrt{1-t^2}}\,dt > -\infty,$$

and let w_n be a sequence of weights (in the sense discussed above). Then the kth orthonormal polynomial $p_{n,k}$ with positive leading coefficient is defined by

$$\int_{-1}^{1} p_{n,k}(x)p_{n,m}(x)w_n^{2n}(x)u^2(x)\,dx = \delta_{k,m}, \qquad k,n,m = 0,1,\dots .$$

Totik [182, Ch. 14] gave asymptotics for these polynomials, and discussed their fundamental properties.

In this survey on orthogonal polynomials we tried to recite the most outstanding results achieved by Hungarian mathematicians. There are many other contributors (among them, of course, foreigners who were inspired mostly by the work of Szegő and Freud), but perhaps the mentioned results convince the reader that Hungarians were always in the forefront of research in this significant area of mathematics.

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