BOLYAI SOCIETY
MATHEMATICAL STUDIES, 14

A Panorama of Hungarian Mathematics in the Twentieth Century, pp. 385–413.

Differential Geometry

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In the thirties of the 19^{th} century János Bolyai and Nikolai Ivanovič Lobacevskii created the hyperbolic geometry. Thus they proved that not only the Euclidean but also other geometries may exist. Concerning its geometrical importance, this discovery can be compared to the change which replaced the Ptolemaic geocentric concept of astronomy by the heliocentric point of view of Copernicus. Hyperbolic geometry opened new horizons. Indeed, only 30 years had to pass, and in Göttingen, in the presence of the elder Gauss, Bernhard Riemann (1826–1866) announced in his habilitation lecture (Über die Hypothesen die der Geometrie zu Grunde liegen) the basic concepts of the new geometry later named after him. His main idea joins Gauss' work.

Let us consider the hypersurface

$$\phi : U^2 \to E^3, \quad (u^1, u^2) \mapsto x^i(u^1, u^2) \qquad i = 1, 2, 3 \tag{1}$$

of the Euclidean space $E^3(x)$. According to Gauss the arc length s_E of the curve $C = \phi \circ C^*$, $C^* : I \to U^2$, $t \in (a,b) = I \mapsto u^\alpha = u^\alpha(t)$, $\alpha = 1, 2$ has (in modern notation), the form

$$s = s_E = \int_a^b \sqrt{\sum g_{\alpha\beta}\big(u(t)\big)\,\dot u^\alpha \dot u^\beta}\,dt, \quad \alpha, \beta = 1, 2. \tag{2}$$

If $\phi \circ U^2 \subset E^3$ is the plane $E^2(x^1, x^2)$ (i.e. $x^3(u^1, u^2) = 0$), then s_E gives the Euclidean arc length of C expressed in the curvilinear coordinate system (u) of E^2, where the

$$g_{\alpha\beta}(u) = \sum_\sigma \frac{\partial x^\sigma}{\partial u^\alpha} \frac{\partial x^\sigma}{\partial u^\beta} \tag{3}$$

are derived from the functions (1) describing the transition to the curvilinear system (u). Riemann's idea was to give $g_{\alpha\beta}$ $\big(\operatorname{Det} |g_{\alpha\beta}| \neq 0\big)$ arbitrarily, and to define the arc length by the integral (2). Today this is called the Riemannian arc length s_V of the curve C. s_V produces the Euclidean arc length in the plane E^2 related to the curvilinear coordinate system (u^1, u^2), i.e. the geometry defined by s_V is Euclidean iff (3), considered as a system of partial differential equations for the given $g_{\alpha\beta}(u)$ and the unknown functions $x^\alpha(u^1, u^2)$, is solvable. However, this occurs rarely. Hence, Riemann's geometry gives the Euclidean geometry as a special case only. If we start with an n-dimensional manifold M in place of the E^2, and give on M a tensor g of type $(0, 2)$ (in local coordinates by $g_{ik}(x)$), then we obtain the Riemannian manifold $V^n = (M, g)$.

This lecture of Riemann was first published only after his death, in 1868, in the volume of his collected works. However, in this lecture one can find certain signs of the Finsler geometry too. The integrand of (2) is a special positive valued function $\mathcal{L}(u, \dot{u})$ positively homogeneous of degree 1 in $\dot{u}$. If we are given such a function on M, and define the arc length in the form

$$s_F := \int_a^b \mathcal{L}\big(u(t), \dot{u}(t)\big)\, dt,$$

then we arrive at a still more general geometry. In 1918 Paul Finsler obtained such a geometry (see his Göttingen thesis "Über Kurven und Flächen in allgeneinen Räumen" written under the supervision of Constantin Carathéodory). He called this a geometry with general metric, and later it was designated by others by the shorter name of Finsler geometry. This geometry is the most general, under certain natural requirements, among those geometries for which the arc length is the integral of the infinitesimal distance. According to Shiing-shen Chern Finsler geometry is nothing other than Riemannian geometry without the quadratic restriction on the function $\mathcal{L}^2$. He sees in this the geometry of the new century. The architect of the early part of Finsler geometry was Ludwig Berwald, the excellent professor of the Charles University in Prague, who later came to a tragic end during his deportation in the Lodz (Litzmannstadt) Ghetto. He laid the foundation of this geometry between 1920 and 1940. His pupil and later private-docent of Prague University was Ottó Varga, who after the German occupation of Prague came to Kolozsvár, and later to Debrecen.

It is well known that every differential geometry, and so the Finsler and the Riemannian geometry too, has two key concepts: the notion of metric and the parallelism of vectors. The fundamental function $\mathcal{L}(x, y)$, $x \in M$, $y \in T_xM$ determines the metric of the Finsler manifold $F^n = (M, \mathcal{L})$, $\mathcal{L}(x, y) = \|y\|_F$ gives the Finsler norm of the vector $y \in T_xM$, and $\mathcal{L}(x, dx) = \|dx\|_F$ the Finsler distance between the points x and $x + dx$. Also $\mathcal{L}$ makes each tangent space T_xM into a Minkowski space (i.e. a normed vector space). The endpoints of the unit vectors of T_xM form a convex and centrally symmetric hypersurface $\mathcal{I}(x)$ called an indicatrix. In a V^n they are ellipsoids, and unit spheres in an E^n. The parallelism of the vectors y of the tangent bundle $TM = \{(x, y)\}$ is defined by a linear connection. This is a mapping $\varphi : T_xM \to T_{x+dx}M$ between the vectors of the neighbouring tangent spaces, the differential dy being linear in $y : \varphi(y) = y + dy$, $dy^i = \sum_{j,k} \Gamma^i_{jk}(x) y^j \, dx^k$. Among these mappings those which preserve the length of the vectors are most applicable (they are called metrical), and take indicatrices into indicatrices. Nevertheless while two ellipsoids (indicatrices of a Riemannian space) can always be taken into each other by a linear transformation, this is not true for indicatrices of a Finsler space. Thus in a Finsler space among the vectors of the tangent spaces there does not exist, in general, a linear and metrical connection, and this makes impossible the development of an absolute differential calculus and curvature theory similar to the Riemannian geometry. This difficulty was surmounted by Élie Cartan in 1931. He osculated the indicatrix $\mathcal{I}(x_0)$ in each direction y_0 by an ellipsoid of equation $\sum g_{ik}(x_0, y_0) y^i y^k = 1$. He considered the vectors at line-elements (x, y). These are the Finsler vectors: $\xi^i(x, y)$. Then he defined their norm by $\big\| \xi(x, y) \big\|^2 = \sum g_{ik}(x, y) \xi^i(x, y) \xi^k(x, y)$, and thus he was able to introduce a metrical and linear connection. After several additional requirements this connection is unique, and nowadays it is called Cartan connection.

At this time began Varga's scientific carrier which, with a few exceptions, focused on Finsler geometry or questions connected with it. A deep geometric thinking was a characteristic feature of his scientific activity. In one of his early works {43} he gives a completely geometric construction which leads without any further requirement to the just mentioned metrical and linear Cartan connection. Let us consider a line-element field $x^i = x^i(\tau)$, $\dot{x}^i = \dot{x}^i(\tau)$, $\tau \in [\tau_1, \tau_2] = T$ and along this a vector field $\xi^i\big(x(\tau), \dot{x}(\tau)\big)$. From each point $x(\tau_0)$, $\tau_0 \in T$ there goes out in the direction of $\dot{x}(\tau_0)$ a unique geodesic of the Finsler space F^n. These yield a 1-parameter family, γ_1, of geodesics. γ_1 can be completed by further geodesics of the F^n to a family, γ_2, such that γ_2 covers one-fold a small tube $\mathfrak{B}$ around the curve $x(\tau)$. This completion can be done in several different ways. The tangents, $r(x)$, of the geodesics at the points $x \in \mathfrak{B}$ determine a line-element field, $\big(x, r(x)\big)$, and by

$$\gamma_{ij}(x) := g_{ij}\big(x, r(x)\big) \qquad g_{ij}(x,y) := \frac{1}{2}\,\frac{\partial^2 \mathcal{L}^2}{\partial y^i \partial y^j}$$

a Riemannian space V^n on $\mathfrak{B}$. This V^n osculates the F^n in the sense that the geodesics of V^n resp. F^n starting out from $x \in \mathfrak{B}$ in the direction $r(x)$ osculate each other in the second order, and the $\dot{x}(\tau)$ turn out to be parallel along $x(\tau)$ in V^n. Now $\xi^i\big(x(\tau), \dot{x}(\tau)\big)$ will be called parallel in the F^n if $\xi^i(\tau) := \xi^i\big(x(\tau), \dot{x}(\tau)\big)$ is parallel along $x(\tau)$ in the V^n. Moreover, Varga shows that with an appropriate choice of the extension of γ_1 to γ_2 we can show that the Finsler connection obtained on $\mathfrak{B}$ is just the Cartan connection.

With the problem of the linear connection of Finsler vector fields, Varga has already dealt earlier in his Ph.D. thesis. Slightly before publishing his thesis, in 1933 there appeared in the C.R. Paris a short announcement by Cartan, whose contents were explained in detail in his famous booklet "Les espaces de Finsler" (Actualités Scientifiques et Industrielles No. 79, Paris, Hermann, 1934) which is considered still the foundation of the Cartan theory of Finsler spaces. Varga's thesis showed a considerable overlap with this, so only a summary was published in the Prague journal Lotos (vol. **84** (1936), 1–4). The above discussed osculating Riemannian space represents another geometric solution of the problem leading to the same result.

He was able to apply with success the osculation of an F^n by a Riemannian space to other problems also. As well known, the sectional curvature, $R(x,p)$, of a Riemannian space V^n at a point x and plane position p is the curvature of the 2-dimensional V^2 induced by V^n on the subspace ϕ^2 consisting of the geodesics of V^n tangent to p at x, and this curvature equals the Gaussian curvature of the surface representing V^2 in Euclidean three-space. This $R(x,p)$ was generalized and transferred into the Finsler space F^n partly on the basis of its formal expression, and partly on the basis of its role in certain variational problems. This generalization, the Riemann–Berwald curvature $R(x,v;X)$ (today called flag curvature) of the F^n, is defined at a line-element (x,v) and a vector X defined at this (x,v). Varga has shown in {46} that $R(x,v;X)$ too is the curvature of a 2-dimensional subspace F^2 induced by F^n on the subspace ϕ^2 consisting of geodesics of F^n tangent to the plane-position $(v,X) = p$, similarly to the case of the Riemannian geometry. He considered the geodesic C starting from x in the direction of X (belonging to both F^2 and F^n), and constructed a Riemannian space V^n osculating F^n along this C (this is a little different from the previous osculating Riemannian space). Then he proved that the curvature $\bar{R}(x)$ of the $\bar{V}^2 := V^n \upharpoonright \phi^2$ (i.e. the restriction of V^n to ϕ^2) equals $R(x,v;X)$. On the other hand he proved that this $\bar{R}(x)$ equals the Finsler curvature $S(x,v)$ (introduced by Finsler) at x in the direction of v of the above F^2. Thus $R(x,v;X) = S(x,v)$. This shows a complete analogy to the Riemannian case.

Finsler geometry is a most natural generalization of Riemannian geometry, using notions and apparatus which may be more sophisticated than that of Riemannian geometry, but is essentially similar and closely related to it. Therefore it is of basic interest to see how and to what extent the notions and theorems of Riemannian geometry can be transferred and extended to Finsler geometry. Ottó Varga has important achievements in this direction, especially concerning Finsler spaces of scalar or constant curvature. Riemannian spaces of constant curvature which are near Euclidean spaces have an exceptional importance. Their first well known characterization was given by Beltrami according to whom V^n is of constant curvature iff it admits a geodesic mapping φ onto an affine space A^n such that φ takes every geodesic of the V^n into a straight line of A^n. This is equivalent to the vanishing of the projective curvature tensor of Weyl or the property that the difference vector $\mathcal{P}\xi - \xi$ of any vector ξ and its parallel translated $\mathcal{P}\xi$ along an infinitesimal parallelogram Π lies in the same parallelogram Π (up to quantities of the third order in the measure of the area of the parallelogram). It is clear that this last property also characterizes the projectively flat affinely connected spaces (spaces with a linear connection, but without Riemannian metric). Among Finsler spaces or affinely connected line-element spaces in the sense of O. Varga {47} there are spaces of constant- and also of scalar-curvature. Berwald called an F^n of scalar curvature $R(x,v)$ if $R(x,v;X)$ is independent of X, and of constant curvature if $R(x,v)$ is independent of v. The independence of $R(x,v)$ of v implies its independence of x too. Varga showed in {49} that Finsler spaces of scalar or constant curvature can also be characterized in a quite similar way. According to his results, F^n is of scalar curvature iff $\mathcal{P}\xi - \xi$ belongs to the vector space spanned by Π and ξ, and F^n is of constant curvature iff $\mathcal{P}\xi - \xi$ lies in Π. From his calculation it also follows that one can build the whole curvature theory on the main curvature tensor, T, introduced by him and defined by $T^i_{jk\ell} = R^i_{jk\ell} - \sum_{s,m} A^i_{s\ell} R^s_{jkm} \ell^m$, where the $R^i_{jk\ell}$ are the components of the first curvature tensor of Cartan, $A^i_{s\ell}$ those of the torsion tensor, and $\ell^m = v^m$ are the components of the unit line-element.

Varga also gave other interesting characterizations of Finsler spaces of constant curvature. An $F^n = (M, \mathcal{L})$ makes any of its k-dimensional $(k < n)$ submanifolds $N \subset M$ into a Finsler space $F^k = (N, \tilde{\mathcal{L}})$. A curve $C \subset N \subset M$ has the same arc length in F^n and in F^k, but a geodesic of F^k between two points $p, q \in N \subset M$ is not, in general, a geodesic of the F^n, for in M there may exist curves between p and q shorter than C. A submanifold in which every geodesic is at the same time a geodesic of the embedding space is called a totally geodesic submanifold. This is a generalization of the Euclidean k-dimensional planes. In a Euclidean space E^n there exists through each point and every plane position a totally geodesic submanifold (a k-plane). In a V^n or F^n this is not so. Varga showed that among the Finsler spaces this holds exactly for the spaces of constant curvature. In the case of constant negative curvature the metric induced on these totally geodesic submanifolds is Euclidean.

Of fundamental importance are his results concerning the angular-metric. For two unit vectors ξ and η at the same line-element (x_0, v_0) the angle $\varphi = \measuredangle(\xi, \eta)$ is defined by the Euclidean metric at the given line-element

$$\cos^2 \varphi = \sum_{i,k} g_{ik}(x_0, v_0)\xi^i \eta^k, \qquad g_{ik}(x, y) = \frac{1}{2}\,\frac{\partial^2 \mathcal{L}^2}{\partial y^i \partial y^k}.$$

These g_{ik} induce on the indicatrix $\mathcal{I}(x_0) \subset T_{x_0}M$ a Riemannian space V^{n-1}. If the direction of the unit vectors $\xi(x_0, v)$ and $\eta(x_0, \hat{v})$ coincides with the direction of their line-elements, i.e. we have $\xi(x_0, v) = av$ and $\eta(x_0, \hat{v}) = b\hat{v}$, $a, b \in R$; and moreover $\hat{v}$ is sufficiently near to v: $\hat{v} = v + dv$, then for $\measuredangle(\xi, \eta) \equiv \measuredangle(v, v + dv) = d\varphi$ we can put

$$\cos^2 d\varphi = \sum_{i,k} g_{ik}(x_0, v)\, dv^i\, dv^k = \|dv\|_V.$$

Thus the measure of the infinitesimal angle equals the Riemannian measure of the corresponding arc. So this V^{n-1} and its metric play a distinguished role in the angular metric of the F^n. Varga has shown in {51} that the curvature tensor of this V^{n-1} is the sum of the curvature tensor S (the third curvature tensor of Cartan) and the metric tensor of the bivectors of the V^{n-1}. The geometric meaning of the first and second curvature tensor R and P was already known, however, the geometric role of S was revealed by this result. He also gave a simple criterion for this V^{n-1} to be of constant curvature.

We cannot discuss in detail his numerous other results, yet we mention a few here. He had valuable results on Minkowski geometry. He gave a very clever and direct geometric derivation of the Euclidean connection in the Minkowski geometry making no use of the roundabout way of Finsler geometry, see {44}. He showed that if a hypersurface of a Minkowski space has constant normal curvature at every line-element, then the geometry induced by the Minkowski space on the hypersurface is a Finsler geometry of constant curvature with respect to the induced connection, see {54}. He achieved a number of results concerning hypersurfaces. Each of them is a little masterpiece of Finsler geometry. He studied Finsler spaces which generalize the non-Euclidean spaces, see {45}; the metrizability of affinely connected line-element spaces, i.e. the possibility of endowing a Finsler connection with a Finsler metric such that parallel vector fields have constant Finsler norms, and obtained nice results for them, see {50}. He studied Hilbert geometry (the generalization of the Cayley–Klein model of hyperbolic geometry). This not-Riemannian geometry has the important property that geodesics are straight lines. He gave an analytic characterization of all functions which represent the arclength of this geometry in {53}. He has important theorems concerning the decomposition of a Finsler space into a product of other spaces, see {52}; the coincidence of the induced and the intrinsic connection on a hypersurface of an F^n, etc.

Although the main field of Varga's activity was Finsler geometry, he also obtained notable results concerning Riemann spaces V^n of constant curvature. It was known for a long time that if through any hyperplane position of a V^n totally geodesic hypersurfaces can be laid, i.e. if the plane axiom is fulfilled, then the space is of constant curvature. But this criterion-like quality does not separate a) the spaces of constant negative and b) the spaces of constant positive curvature. Varga showed in {55} that in case of a) two hyperplanes can be laid through any plane position so that the geometry induced on them by the V^n is Euclidean (they are paraspheres), and in the case of b) a totally geodesic hyperplane can be laid through any plane position which is turned by the V^n into a V^{n-1} of constant curvature. These qualities are characteristic. He could extend the above mentioned criterion-like quality of the V^n of constant curvature to Finsler spaces, i.e. he also proved that the F^n of constant curvature are characterized by the property that through any hyperplane-position a totally geodesic hyperplane can be laid, see {56}.

He also had several works on integral geometry (Math. Z., **40** (1935), 384–405; **41** (1936), 768–784; **42** (1937), 710–736; Acta Sci. Math. Szeged,

9 (1939), 88–102; etc.). They are products of his collaboration with W. Blaschke in Hamburg in 1934–35. In these he found parameter transformation and motion invariant measures, i.e. geometric densities on different sets of geometrical configurations, and disclosed the relations existing between them, i.e. with the aid of the Crofton formulae, he established relations between integral invariants.

Varga was always intent on seizing and putting into relief the geometric meaning behind the often rather complicated formalism of Finsler geometry. This is a characteristic feature of his scientific work. His mathematical thinking was guided by simple geometric ideas. If we divide the history of Finsler geometry into three periods: I) the beginning, 1918–1940; II) development of the local theory, 1940–1970; III) use of intrinsic tools and global theory, 1970–, then we can state that Varga played a decisive role in the second period. He also created a school of differential geometry in Debrecen which continues even now, and he was one of the founders of the journal Publicationes Mathematicae Debrecen in 1949.

András Rapcsák was a colleague and collaborator of O. Varga at Debrecen University. The field of his research was line-element (or support-element) spaces and connected areas. His first investigations were related to normal coordinate systems. A fortunate choice of a coordinate system can considerably facilitate the treatment of a geometrical problem. Such good coordinate systems are in Euclidean or affine spaces the Cartesian or the polar coordinate system. In affinely connected or Riemannian spaces there exist in general no Cartesian coordinate systems, but we have an analogue of the polar coordinate system called normal coordinate system in which the equations of the geodesics (in affinely connected spaces geodesic means auto-parallel curve) starting from the origin are linear. A coordinate system is a device of investigation only. Geometrical statements must be independent of it, they must express relations between invariant geometrical objects. The importance of normal coordinates stems from the fact that by them normal affinors and with their aid a complete system of invariants can be obtained, i.e. such invariants may be gained by which any other invariant of the space is expressible. This parallels the Erlanger Program of Felix Klein. However, Jesse Douglas proved that the above normal coordinates do not exist in general in line-element spaces. Thus it seemed that insurmountable obstacles stood in the way of employing this successful method for the determination of a complete system of invariants of a line-element space, a task of fundamental importance from a theoretical point of view. It turned out that the negative result was due to the inappropriate manner of carrying over the notion of geodesics. The appropriate notion, the quasigeodesic introduced by Varga is a curve $x(s)$ whose tangents are parallel translated with respect to a field of line-elements $\big(x(s), \ell(s)\big)$, where the $\ell(s)$ are parallel along $x(s)$:

$$\frac{d^2x^i}{ds^2} = \sum_{k,j} -\overset{*}{\Gamma}{}_k{}^i{}_j(x,\ell)\frac{dx^k}{ds}\frac{dx^j}{ds}, \qquad \frac{d\ell^i}{ds} = \sum_{k,j} -\overset{*}{\Gamma}{}_k{}^i{}_j\,\ell^k\frac{dx^j}{ds}.$$

Quasigeodesics reduce in point spaces to geodesics. Quasigeodesics cover one-fold a region of the $2n$-dimensional tangent bundle TM^n, and so are suited for introducing normal coordinates in line-element spaces. With their aid Varga showed that the coefficients determining the affine connection of a line-element space, L^n, the main curvature tensor and the partial- and covariant-derivatives of these, form a complete system of invariants of the L^n, see {48}. Rapcsák proved that these normal coordinates can be characterized by the same properties by which H. S. Ruse characterized the normal coordinates in Riemannian spaces. He also succeeded in establishing an invariant Taylor series (i.e. a Taylor series, where the coefficients are invariants) in Finsler spaces, see {35}. In order to obtain a complete system of differential invariants in affinely connected point- and Riemannian-spaces normal coordinates were already applied also by Ruse, T. Y. Taylor and Oscar Veblen. Rapcsák successfully used the normal coordinate systems introduced through quasigeodesics for obtaining a complete system of invariants in Cartan spaces, see {36}. These spaces originate with and are named after Élie Cartan, and concerning their structure they are very similar to Finsler spaces. While in an F^n a vector ξ is defined at a line-element (x, v), in a Cartan space the support element of a vector ξ is a point x and a hyperplane u through $x : \xi(x, u)$. The fundamental metric function of the Cartan space has the form $\mathcal{L}(x, u)$, and it has properties similar to that of an F^n.

The geodesics (or autoparallel curves) of an n-dimensional manifold (endowed with an appropriate geometric structure) form a $2n - 2$ parameter family $x^i = x^i(t; a^1, \dots, a^{2n-2})$. The members of the family are called paths. The investigation of paths in point spaces was initiated by J. Douglas in 1928. In line-element spaces the more involved theory was developed by Rapcsák {34}. Here paths form a $3n - 3$ parameter family of curves. Quasigeodesics of an F^n yield an example for such a family. He developed the affine connection theory of these paths. This is a little different from that of the F^n. He established curvature and torsion tensors and gave a method to resolve the equivalence problem of these spaces. The equivalence problem poses the question of whether two differential-geometric spaces are the same, but in different coordinate systems or whether their geometries are basically different. He also posed and answered the inverse question: given in a line-element space a $3n-3$ parameter family of curves $\big(x(t), v(t)\big)$, does there exist an F^n in which these curves become quasigeodesics?

A Euclidean space E^n makes any $n-1$-dimensional submanifold H also a metric space. However a geodesic of H (with respect to the induced metric) is in general not a geodesic (straight line) of the E^n. If, however, this happens for every geodesic of H, then H is called totally geodesic, and it is a hyperplane. H is also a hyperplane if its unit normals are parallel in E^n. Both of these properties can be carried over into Finsler spaces, moreover in the first case geodesics can be replaced by quasigeodesics. In an F^n these three properties yield three different notions: hyperplanes of I, II and III kind. We must mention that here unit normal vectors of H are considered at line-elements tangent to H, i.e. H is considered as an $n-1$-dimensional line-element space. Another difference is that in F^n these hyperplanes do not exist unrestrictedly (through every point and plane-position). Rapcsák showed in {37} that A): hyperplanes of the I kind exist unrestrictedly in an F^n iff F^n is of scalar curvature and projectively flat. This last property means that in an appropriate coordinate system the geodesics coincide with the straight lines or equivalently: there exists a smooth mapping $\varphi : F^n \to E^n$ such that the image of each geodesic is a straight line. Since in a V^n the condition of the unrestricted existence of totally geodesic hyperplanes is that V^n is of constant curvature (Friedrich Schur), and this is equivalent to the property that V^n is projectively flat (Enrico Bompiani), Rapcsák's result is the Finsler geometric counterpart of this famous theorem; B): hyperplanes of the II kind exist unrestrictedly iff F^n is projectively flat and the torsion tensor A satisfies the relation $A_{\alpha\beta\gamma|0} = 0$; C): hyperplanes of the III kind exist unrestrictedly iff F^n is a V^n of constant curvature.

J. M. Wegener and E. T. Davies investigated hyperplanes H, where normals are considered at line-elements transversal to H. In this case H is a point space. If these normals are parallel along H, then H is called a hyperplane of the IV kind. Rapcsák showed in {38} that in an F^n with vanishing projective curvature hyperplanes of the IV kind exist unrestrictedly iff F^n is of constant curvature. This means a considerable sharpening of a similar result of Wegener, who considered an F^n of scalar curvature and found a sufficient condition only. Rapcsák also found conditions for the metric induced by an F^n on hyperplanes of the IV kind to be the metric of a Riemannian space of constant curvature or the metric of an E^n.

Two affinely connected (point) spaces $L_1(x)$ and $L_2(u)$ allow, in general, no path preserving mapping $\varphi : L_1 \to L_2$, where paths (i.e. geodesics) of L_1 are taken into paths of L_2. An important question is, when do they allow a path-preserving mapping? If we perform a coordinate transformation $\bar{x}^i = \bar{x}^i(u)$ on L_2 such that the $\bar{x}^i$ equal the coordinates x^i of the original point, then the question takes the form: how can we change (deform) the connection on L_1, so that the paths remain unchanged. Rapcsák's studies in {39} led in a natural way to this question on line-element spaces. He solved the problem by giving necessary and sufficient relations (not complicated, but not quite short ones) between the Weyl and Douglas tensors of the original and deformed spaces. One of his nicest results {40} gives an elegant and amazingly simple answer to the question: what relation must exist between the metric functions $\mathcal{L}(x, y)$ and $\bar{\mathcal{L}}(x, y)$ of two Finsler spaces F^n and $\bar{F}^n$ with common base manifold in order that their geodesics be the same? According to his result this is the case iff

$$\bar{\mathcal{L}}_{|i} - \sum_s \frac{\partial \bar{\mathcal{L}}_{|s}}{\partial v^i} v^s = 0. \tag{4}$$

This result seems to be independent of $\mathcal{L}$. But this is not true, for the operator $|i$ denotes the Berwald covariant derivation in F^n determined by $\mathcal{L}$. Berwald too has dealt with this problem. His answer is also very simple, but it does not contain the functions $\mathcal{L}$ and $\bar{\mathcal{L}}$, at least not in an explicit form. Continuing his considerations Rapcsák was able to answer in {41} the following important and interesting questions closely related to the above ones. A): given a $2n - 2$ parameter family of curves, when does there exist an F^n (i.e. a fundamental function $\mathcal{L}$) such that the geodesics of F^n are just the curves of the given family? This means the metrizability of the path-space. B): When do the geodesics of an F^n coincide with the geodesics of a Riemannian space V^n? C): When are the geodesics of an F^n straight lines in an appropriate coordinate system? This last question is the Finslerian version of Hilbert's 4th problem. The answers are similar to (4).

Arthur Moór also belonged to the Debrecen school of Finsler geometry. He dealt not only with Finsler geometry, but also with many related fields. He was a very productive mathematician who used and applied the apparatus of differential geometry with utmost ease and efficiency. He wrote more than hundred papers all of them, with a few exceptions, in German, similarly to Ottó Varga, András Rapcsák and L. Berwald.

He started with the investigation of several special Finsler spaces, special concerning the dimension or the metric function. He gave a necessary and sufficient condition in order that the curvature scalar K of an F^2 be constant, and gave an explicit form of the metric function $\mathcal{L}(x,y;\dot{x},\dot{y})$ along a geodesic if $K=\text{const.}$, see $\{15\}$. $\mathcal{L}=f/g$, where $f=\sum_{k=0}^{N} a_k(x,y)\dot{x}^{N-k}\dot{y}^k$ and $g=\sum_{k=0}^{N-1} b_k(x,y)\dot{x}^{N-k-1}\dot{y}^k$ (i.e. f and g are homogeneous polynomials in $\dot{x}$ and $\dot{y}$ with coefficients dependent on x and y) is a special metric (fundamental) function of an F^2 investigated first by Moór. He determined for an F^2 with such an $\mathcal{L}$ the main scalar $\mathcal{J}$ and the curvature scalar $\mathfrak{K}$ in case of $N=2$ or 3, see $\{16\}$. If $g\equiv 1$ then $\mathcal{L}$ is not first order homogeneous in $\dot{x}$ and $\dot{y}$, but $\sqrt[N]{f}$ is. Such F^2 with $N=3$ were investigated by J. M. Wegener. Moór investigated such F^2 in case of $N=4$, see $\{17\}$. Of course this can be generalized to dimension n. A Finsler space with $\mathcal{L}=\sqrt[N]{f_N(y)}$, where f_N is a homogeneous polynomial in $y^1,\dots,y^n$ of order N with coefficients dependent on $x^1,\dots,x^n$ is an interesting type of Finsler space, and many special cases of it are investigated. If $N=n$ and

$$f_n(y)=y^1y^2\dots y^n\Big(=\prod_{A=1}^{n} y^A\Big), \tag{5}$$

then F^n is a Minkowski space, for $\mathcal{L}=\sqrt[n]{f_n(y)}$ is independent of x and in the case of $n=2$ it is a pseudo-Riemannian space with an indefinite Lorentz metric and non-convex indicatrix. Let us replace each y^A, $A=1,2,\dots,n$ by a linear form $\sum\limits_{m=1}^{n} S_m^A(x)y^m$. An F^n with the metric function

$$\mathcal{L}(x,y)=\Big(\prod_{A=1}^{n}\sum_{m=1}^{n} S_m^A(x)y^m\Big)^{\frac{1}{n}} \tag{6}$$

is called a Finsler space with Berwald–Moór metric (G. S. Asanov: [11], p. 53). It has an interesting and important geometric interpretation. Suppose $\det\left|S_m^A(x)\right|\neq 0$, and denote the inverse matrix by $S_A^m(x)$: $\sum_m S_m^A(x)S_B^m(x)$ $=\delta_B^A$. Then to each vector $y=(y^1,\dots,y^m)\in T_xM$ there exist scalars λ^A such that $y^m=\sum\limits_{A=1}^{n} S_A^m(x)\lambda^A$, and hence λ^A equals $\sum_m S_m^A(x)y^m$, which can be considered as components of y in the base $S_1^m,\dots,S_n^m$. Then $\mathcal{L}^n(x,y)=\prod_{A=1}^{n}\lambda^A=\frac{V(\mathcal{P})}{V(S_A)}$, where $V(\mathcal{P})$ means the volume of the parallelotope $\mathcal{P}$ whose edges are parallel to the base vectors and whose diagonal is the vector

y. $V(S_A)$ is the volume of the parallelotope, S_A, spanned by S^m_A. Hence the Finsler measure of the vector $y(x)$ is

$$\left\| y(x) \right\|_F = \mathcal{L}(x,y) = \left(\frac{V(\mathcal{P})}{V(S_A)} \right)^{\frac{1}{n}} .$$

Thus, in this F^n the Finsler length of a vector is measured by volumes. The length can be deduced from the area. If $f_n(y)$ of (5) is replaced by $\big(\mathcal{F}(y)\big)^n$, where $\mathcal{F}$ is an arbitrary first order homogeneous function, then (6) gets the form $\mathcal{L}(x,y) = \mathcal{F}\Big(\sum_m S^1_m(x) y^m, \dots, \sum_m S^n_m(x) y^m \Big)$. This is the renowned 1-form metric investigated by M. Matsumoto, H. Shimada and others. It has a number of beautiful properties, e.g. it allows a linear metrical connection for the vectors of the tangent bundle (connection in a Finsler space without line-elements, i.e. point Finsler spaces). The Berwald–Moór metric is clearly a simple special case of this.

The well known theorem of A. Deicke states that an F^n with vanishing torsion vector A_i and positive fundamental function $\mathcal{L}$ is Riemannian. The Berwald–Moór metric (6) was up to now the only known example, where $A_i = 0$, yet the space is not Riemannian (for $\mathcal{L}$ is not everywhere positive). The Berwald–Moór metric has still a number of interesting properties: the signature of its metric tensor g_{ij} is $(+ - - \dots)$, $\det |g_{ij}|$ is independent of y, etc.

Finsler and Cartan spaces have very similar structures, as it was mentioned a few pages earlier. In an F^n all vectors and geometric objects are defined at line-elements $(x, \dot{x})$ (i.e. at a point x and a direction or oriented line $\dot{x}$), while in a Cartan space vectors and geometric objects are given at (oriented) hyperplane-elements (x, u), where u is an $n-1$ dimensional linear subspace, i.e. a hyperplane in the tangent space T_xM through the origin having an equation $\sum \mu_i y^i = 0$, where the y^i are coordinates in T_xM and μ_i is the normal of u. Thus the coordinates of (x,u) are (x^i, μ_i). A duality between Finsler and Cartan spaces was investigated by L. Berwald. Somewhat differently from him, Moór in {18} and {21} called a Finsler and a Cartan space dual if

$$\mu_i = \frac{1}{\sqrt{g^*}} \sum_k g^*_{ik} \dot{x}^k \quad \text{and} \quad \dot{x}^i = \sqrt{g} \sum_k g^{ik} \mu_k \tag{7}$$

establish a $1-1$ correspondence between the line-elements $(x, \dot{x})$ of the Finsler space and the hyperplane-elements (x, u) of the Cartan space having the same underlying manifold. Here $g^{ik}(x,\mu)$ is the metric tensor of the Cartan space and $g(x,\mu)$ is its determinant. The corresponding objects of the F^n are denoted by an asterisk $*$. If between a Finsler and a Cartan space (7) establishes a duality, then $g^*_{ij}(x,\dot{x}) = g_{ij}(x,\mu)$ and $\mathcal{L}^*(x,\dot{x}) = \mathcal{L}(x,\mu)$ at the corresponding elements, and also the torsion vectors $A^*_i(x,\dot{x})$ and $A_i(x,\mu)$ vanish. With Berwald $A^*_i = A_i = 0$ was a requirement. For Moór this is a consequence of the existence of the given dual mapping. Because of $A^*_i = A_i = 0$ volume elements independent of $\dot{x}$, resp. μ, exist and the dual mapping is volume preserving. It should be mentioned that according to Deicke's theorem $A^*_i = 0$ yields that F^n is a Riemannian space, however this holds only if $\mathcal{L}^*(x,\dot{x})$ is everywhere positive, which is not the case in general.

Moór investigated Varga's osculating Riemannian space V^n (reviewed 11 pages earlier) in dual Finsler and Cartan spaces, see {19} and {20}. He proved that the Riemannian spaces osculating the dual Finsler and Cartan spaces along a 1-parameter family of line-elements

$$\big(x(t),\dot{x}(t)\big), \quad \text{resp. } \big(x(t),\mu(t)\big) \tag{8}$$

are the same. From this it follows that if $\xi^i\big(x(t),\dot{x}(t)\big)$ and $\xi^i\big(x(t),\mu(t)\big)$ are corresponding vector fields along (8) in the two dual spaces, then their invariant derivatives with respect to the dual Finsler and Cartan spaces are also the same. Moreover, under a mild further condition, the curvature tensor R^V of the osculating V^n coincides along (8) with Varga's main curvature tensor $T(x,\dot{x})$ of the F^n and also with the curvature tensor $\bar{R}(x,\mu)$ of the Cartan space.

$\dot{x}^i$ and μ_i in $(x,\dot{x})$ and (x,μ) are vector densities (of weight 0, resp. -1). Thus, by replacing of $\dot{x}$ or μ by a vector density u of certain weight p one obtains a common generalization of Finsler and Cartan spaces which can be given by a metric function $\mathcal{L}(x,u)$. Investigations concerning such so called general metric spaces $\mathfrak{R}_n$ were initiated by J. A. Schouten, J. Haantjes, E. T. Davies and R. S. Clark.

The detailed development of the geometry of the generalized spaces $\mathfrak{R}_n$ was completed by Moór in several papers. First he succeeded in expanding his above sketched duality to general vector density spaces in {21}. Two spaces $\mathfrak{R}_n$ and $\tilde{\mathfrak{R}}_n$ are called by him dual, if between the elements (x,u), resp. $(x,\tilde{u})$, of the two spaces there exists a mapping $\tilde{u}^i = \varphi^i(x,u)$ such that $g_{ik}(x,u) = \tilde{g}_{ik}(x,\tilde{u})$. Then by Varga's osculating Riemannian space method he obtained results similar to his duality theory between Finsler and Cartan spaces.

In {22} he developed the curvature theory of the $\mathfrak{R}_n$, and determined four curvature tensors. One of them, $\bar{R}_j{}^i{}_{k\ell}$, reduces in an F^n to Varga's main curvature tensor T, and another: $R_j{}^i{}_{k\ell}$ generalizes Riemann's curvature tensor of a V^n. He also found two curvature invariants: $B(x,u,\eta)$ and $\bar{B}(x,u,\eta)$ depending on u and another vector η and generalizing the Riemann–Berwald curvature tensor of the F^n. In the case of an $\mathfrak{R}_n$ of scalar curvature, i.e. when B and $\bar{B}$ are independent of η, he obtained results which are counterparts of several theorems of Berwald in Finsler spaces.

Conformal transformations

$$\tilde{g}_{rs}(x,u) = e^{2\sigma(x)} g_{rs}(x,u), \quad g_{rs} = \frac{1}{2}\frac{\partial^2 \mathcal{L}^2}{\partial u^r \partial u^s}$$

of the metric of an $\mathfrak{R}_n$ were investigated first by R. S. Clark. Moór in {26} obtained a generalization partly by starting directly from the g_{rs} which in Clark's work is deduced from the fundamental function $\mathcal{L}(x,u)$ similarly to Finsler and Riemannian geometry; partly by replacing $\sigma(x)$ by $\sigma(x,u)$. Then he developed the conformal geometry of the $\mathfrak{R}_n$ in this general setting.

A Finsler space is determined by its fundamental function $\mathcal{L}(x,y)$, and the basic geometric objects, e.g. the metric tensor

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 \mathcal{L}^2}{\partial y^i \partial y^i} \tag{9}$$

are derived from $\mathcal{L}$. However one can start directly with the $g_{ij}(x,y) = g_{ji}(x,y)$. This is obviously a more general case, because for a given g_{ij} (9) is, in general, not solvable for $\mathcal{L}$. Moór was always intent on new generalizations. Between 1955 and 1960 his attention turned to these so called general metric line-element spaces which he denoted by $\mathfrak{L}_n$. Finsler spaces built directly on $g_{ij}(x,y)$ have been investigated since then by a number of authors; in particular by R. Miron, and many Roumanian and Japanese geometers. Moór investigated these spaces from several points of view, see {23}. First he constructed the connection theory of the $\mathfrak{L}_n$ developing a metrical connection with the new essential generalization that the connection coefficients are allowed to be not symmetric. Dropping the symmetry of the connection coefficients is essential. In Finsler, and thus also in Riemannian spaces, where connection coefficients are symmetric, autoparallel curves coincide with geodesics. However in case of their asymmetry these curves may be different. Moór gave several concrete examples for the $\mathfrak{L}_n$, and characterized the case when these $\mathfrak{L}_n$ reduce to an F^n. He investigated the infinitesimal transformations

$$\bar{x}^i = x^i + \xi^i(x)\delta t \quad \text{which yield} \quad \bar{y}^i = y^i + \sum_r y^r \frac{\partial \xi^i}{\partial x^r}\delta t \tag{10}$$

of the $\mathfrak{L}_n$, and determined the corresponding Lie derivation denoted by Δ. Transformations generated by the velocity vectors $\xi^i(x)$ of (10) are motions if they preserve length. This is assured by the Killing equations

$$\Delta g_{ik} = 0. \tag{11}$$

He determined the integrability condition of the Killing equations (the unknowns are the ξ^i), and determined the paths of the motions. He called a motion a translation if its paths are also autoparallels. He found a simple explicit condition for this case. In an F^n the paths of a translation intersect a geodesic under the same angle. In an $\mathfrak{L}_n$ this is not so. According to his result the condition for this is the vanishing of two tensors A_{oor} and σ_{ioo}.

The number of the Killing equations (11) is $\frac{1}{2}n(n+1)$. This is the maximum of the number r of the independent parameters a solution of (11) may have. H. C. Wang showed that if in an F^n one has $r \geq \frac{1}{2}n(n-1)+1$, then it is a V^n. In an $\mathfrak{L}_n$ this is not true. Here the relation

$$\sum_{i,j} \frac{\partial^3}{\partial y^v \partial y^s \partial y^r} g_{ij}(x,y) y^i y^j = 0$$

is a necessary, but not sufficient condition for this. Moór investigated the case $r = \frac{1}{2}n(n+1)$ and found simple additional conditions in order that a) the curvature $\bar{R} := \sum_i \bar{R}_i{}^i$, $\bar{R}_i{}^s := \sum_{k,\ell} g^{ks}\bar{R}_i{}^\ell{}_{k\ell}$ of the $\mathfrak{L}_n$ vanish; b) $\mathfrak{L}_n$ is of scalar curvature; or c) $\bar{R}_{koij}$ has an interesting special form. He published his results in a number of sometimes comprehensive papers.

F. Schur's famous result states: if the sectional curvature $R(x,p)$ of a Riemannian space is independent of the plane position p, then it is independent of the point x too, i.e. R is a constant. This is also true in an F^n of scalar curvature. However this is not so in an $\mathfrak{L}_n$. Moór found a nice characterization of the $\mathfrak{L}_n$ of scalar curvature in which Schur's theorem holds, see {24}.

Geodesic deviation plays an important role in the theory of Riemannian and also in that of Finsler spaces. Moór derived the equation of autoparallel deviation in an $\mathfrak{L}_n$ and obtained conditions for the case that autoparallel curves of an $\mathfrak{L}_2$ have an envelope, see {25}. Also he developed the anholomic geometry (where the bases of T_xM are not the tangents of the coordinate lines) of the $\mathfrak{L}_n$-s, see {28}.

Albert Einstein was able to derive gravitation from the structure of a 4-dimensional Riemannian space. To this end he used up the Riemann curvature tensor and there remained no more geometric objects which would reflect the impact of the electromagnetic field. More general spaces may offer more possibility for the incorporation of electromagnetic field. Moór and his coauthor János Horváth, the physicist from Szeged, discussed in a long paper {11} the possibility of creating a unified field theory within the frame of a Finsler space F^n.

In 1949–1953 for the purpose of studying the quantum theory of wave spaces with the aid of operator calculus H. Yukawa developed a bilocal space theory in which the objects of the space are defined at pairs of points. Moór and Horváth converted this space into a general metrical line-element space $\mathfrak{L}_n$, and gave an interpretation of Yukawa's theory in this frame (Entwicklung einer Feldtheorie begründet auf einen allgemeinen metrischen Linienelementraum I. and II. Indag. Math., **17** (1955), 421–429, 581–587).

In the theory of affine or metrical point- or line-element-spaces the absolute derivation $\frac{D}{dt} = \sum_k \frac{dx^k}{dt}\nabla_k$ of covariant and contravariant vectors is performed with the same connection coefficients. In 1958–61 T. Otsuki introduced a general connection in which this does not hold. Moór called this the Otsuki connection, and devoted a number of papers to its investigation. He coupled the Otsuki connection with a recurrent metric tensor g_{ij} which satisfies $\nabla_k g_{ij} = \gamma_k(x) g_{ij}$ and thus obtained a Weyl–Otsuki space {33}. He investigated in the Otsuki and Weyl–Otsuki spaces transformation groups, geodesic deviation, and considered anholonomic coordinate systems. He also considered Finsler–Otsuki spaces, their duality, etc.

An object at a point with coordinates x^i $(i = 1, \dots, n)$ is defined by certain numbers $\Omega_1, \dots, \Omega_N$ called components of the object. If its components $\Omega_\alpha,\ \alpha = 1, \dots, N$ in an other coordinate system $(\bar{x})$ can be calculated from the original $\Omega_\alpha(x)$ and the coordinate transformation $\bar{x}^i = \bar{x}^i(x)$ according to a given rule, then Ω is a geometric object; e.g. vectors, tensors, connection coefficients, etc. (cf. [3]). Moór investigated many problems of the theory of geometric objects. For example: The covariant derivative $\nabla_k v^i$ of the vector v^i is a tensor $T^i{}_k$ depending on v^i, $\frac{\partial v^i}{\partial x^k}$ and the connection coefficients $\Lambda_j{}^i{}_k$. One can ask, in what form can a tensor $T^i{}_k$ be composed from

v^i, $\frac{\partial v^i}{\partial x^k}$ and $\Lambda_j{}^i{}_k$; i.e. which are the possible forms of a "covariant derivative" in the above sense. He solved a number of problems of the following type: to construct a tensor from given geometric objects of different kinds in {29}; to construct covariant derivatives from the metric tensor; connection coefficients from covariant vectors; etc. He also investigated geometric objects and covariant derivatives in line-element spaces.

A tensor T is recurrent if its covariant derivative equals the tensorial product of a covariant vector λ and the tensor itself: $\nabla T = \lambda \otimes T$. Among Riemannian manifolds V^n those which are near Euclidean space enjoy a special interest. The nearest class consists of V^n of constant curvature (i.e. non-Euclidean spaces). Next are the locally symmetric V^n characterized by $\nabla R = 0$ (where R is the curvature tensor of the V^n), and after that the V^n of recurrent curvature. They were investigated in detail by H. S. Ruse and A. G. Walker. Moór extended the investigations to affinely connected spaces L_n (spaces with parallelism, but without metric), to metrical spaces with not symmetric coefficients, generalized metric spaces $\mathfrak{R}_n$ and general line-element spaces $\mathfrak{L}_n$. He wrote 15 papers on this topic, investigated spaces with recurrent torsion, and F^n with recurrent metric tensor: $\nabla_k g_{ij} = \lambda_k g_{ij}$. Metric point spaces with this property are the Weyl spaces. They found important applications in the field theories of physics. Moór also investigated spaces with double recurrence given by the property $\nabla\nabla R = T \otimes R$, where T is a tensor of type $(0,2)$. It is easy to see that simple recurrence always yields double recurrence, but not conversely. To convey the flavour of his results in this field we mention two of them chosen arbitrarily. 1) He proved that if an affinely connected space L_n of recurrent curvature (with possibly not symmetric connection coefficients) splits into two factors: $L_n = L_r \times L_{n-r}$, then one of them is flat, i.e. its curvature tensor vanishes, see {27}. 2) An F^3 is always recurrent with respect to the total curvature tensor, provided there exists in it an absolute parallelism of the line-elements, see {30}.

Also interesting are his investigations concerning equivalent variational problems. He called two variational problems

$$(12) \qquad \text{a)} \quad \delta \int_a^b F(x, \dot{x}, \ddot{x})\, dt = 0 \qquad \text{and} \qquad \text{b)} \quad \delta \int_a^b F^*(x, \dot{x}, \ddot{x})\, dt = 0$$

equivalent if their solutions (their geodesics) are the same, i.e. if

$$(13) \qquad \mathcal{E}_i(F) = 0 \Longleftrightarrow \mathcal{E}_i(F^*) = 0, \quad \mathcal{E}_i \equiv \frac{\partial}{\partial x^i} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}^i} + \frac{d}{dt}\frac{\partial}{\partial \ddot{x}^i}.$$

A. Kawaguchi has considered and investigated spaces in which the arc length of a curve $x^i(t)$ is defined by $\int_a^b F\big(x, \dot{x}, \dots, x^{(r)}\big)\, dt$. So (12,a,b) are variational problems of Kawaguchi spaces with $r = 2$, and (13) yields their equivalence. This equivalence means that the identity mapping between the two spaces given by F and F^* is a geodesic mapping. For $r = 1$ one obtains Finsler spaces. In this case $\frac{\partial F}{\partial \ddot{x}^i} = 0$. Moór investigated the clearly equivalent variational problems, where

$$\text{(14)} \qquad \text{a)} \quad \mathcal{E}_i(F^*) = \lambda(x)\mathcal{E}_i(F),$$

$$\text{resp.} \quad \text{b)} \quad \mathcal{E}_i(F^*) = \sum_k \lambda_i^k \mathcal{E}_k(F), \ \det|\lambda_i^k| \neq 0$$

and obtained results concerning the form of F and F^*, see {31}, e.g. in the case of (14,a) with $\lambda \neq$ const., if F and F^* do not depend on $\ddot{x}$ (i.e. in the case of Finsler spaces) he obtained that

$$F^*(x, \dot{x}) = \lambda(x)F(x, \dot{x}) + \sum_k S_k(x)\dot{x}^k, \quad F(x, \dot{x}) = \sum_k a_k(x)\dot{x}^k,$$

where $\lambda(x)$ and $S_k(x)$ satisfy a first order partial differential equation in which $a_k(x)$ is also involved as a coefficient function. If $\lambda =$ const., then S_k is the gradient of a scalar function S, and the difference $F^*(x, \dot{x}) - \lambda F(x, \dot{x})$ is a total differential of S with respect to $\dot{x}$. He solved similar problems for (14,b) (also in the case of rank $\|\lambda_i^k\| < n$) and also for variational problems in several variables: $F\big(x^i(u), \frac{\partial x^i}{\partial u^\alpha}\big)$ $i = 1, \ldots, n$, $\alpha = 1, \ldots, m < n$, and with multiple integrals $\int \underset{U}{\cdots} \int F\big(x, \frac{\partial x}{\partial u}\big)\, du$, $u \in U$, see {32}.

Hungarian mathematicians performed successful investigations in the 20^{th} century not only on Finsler-, but also on Riemannian-geometry. One of them was Pál Dienes.

In 1917 Tullio Levi-Civitá created a suitable notion of parallelism of vectors in Riemannian spaces. He considered a surface $\phi : x^i = x^i(u^1, u^2)$, $i = 1, 2, 3$, of a Euclidean 3-space E^3, a curve $C(t) \subset \phi : x^i(t) = x^i\big(u^\alpha(t)\big)$, and a tangent vector $\xi_0 \in T_{C(t_0)}\phi$. Then he translated ξ_0 parallel in the E^3 to the nearby point $C(t_0 + \Delta t)$ of the curve obtaining in E^3 a vector $\hat{\xi}(t_0 + \Delta t)$ (with components $\hat{\xi}^i = \xi_0^i$, $i = 1, 2, 3$), and he called the perpendicular projection $\bar{\xi}$ of $\hat{\xi}$ on $T_{C(t_0 + \Delta t)}\phi$ parallel to ξ_0. Starting this construction again with $\bar{\xi}$, then with $\bar{\bar{\xi}}$, etc., and performing a limit process $\Delta t \to 0$, he obtained a vector field $\xi(t)$ tangent to ϕ which he called parallel along

$C \subset \phi$. This construction leads analytically to the differential equation system

$$\frac{d\xi^\alpha}{dt} + \sum_{\beta,\gamma} \Gamma_\beta{}^\alpha{}_\gamma\big(x(t)\big)\,\xi^\beta \frac{dx^\gamma}{dt} = 0, \quad \alpha,\beta,\gamma = 1,2 \tag{15}$$

where $\Gamma_\beta{}^\alpha{}_\gamma$ is the Christoffel symbol of the second kind of ϕ. This notion of parallelism opened new paths and gave a considerable impulse to the investigation of Riemannian geometry.

Pál Dienes graduated at and took his doctor degree from the Péter Pázmány University in Budapest. At the beginning of his mathematical career in Hungary he dealt with the theory of functions. In 1920 he left Hungary, and at this fortunate moment of the development of Riemannian geometry he started investigations on this geometry in Wales (U.K.).

Since any n-dimensional Riemannian manifold V^n has a local isometric embedding into E^{n+k}, the idea of parallelism just described can easily be extended to V^n. In this case (15) takes the form

$$\frac{d\xi^i}{dt} + \sum_{j,k} \Gamma_j{}^i{}_k\big(x(t)\big)\,\xi^j \frac{dx^k}{dt} \equiv \frac{D\xi^i}{dt} = 0, \quad i,j,k = 1,\dots,n. \tag{15$'$}$$

The equation (15′) has two important consequences. Since it contains as data the functions $\Gamma_j{}^i{}_k(x)$ only, by giving them arbitrarily in a coordinate space $X^n(x)$, we can define parallelism in it, and thus make X^n into an affinely connected space L^n (a differential geometric space with parallelism, but without metric). On the other hand, the left hand side of (15′) behaves as a vector, hence it can be considered as a derivative (the absolute derivative $D\xi^i/dt$) of the vector field $\xi^i\big(x(t)\big)$. Thus the parallelism of a vector field $\xi\big(x(t)\big)$ in an L^n can be defined by $\frac{D\xi^i}{dt} = 0$, similarly as in E^n by $\frac{d\xi^i}{dt} = 0$. Moreover, $\frac{D}{dt}$ can be extended to tensor algebra, and this leads to the absolute differential calculus (Ricci-calculus) of basic importance in differential geometry.

Parallelism, affinely connected spaces and absolute differential calculus raised a number of new questions at the beginning of the 20s of the 20th century. Dienes rendered important contributions to Riemannian geometry and its nonmetrical counterpart, the theory of affinely connected spaces. He found new possible solutions for the main problems raised by the rapid advance of the differential geometry of his time. He had original ideas and realized interesting investigations. His papers appeared in well-known leading journals. His activity on differential geometry can be divided into two well separable periods. In the first of these, mainly between 1922 and 1926, he was interested in the connection theory of tensors and vectors with or without metric (C.R. Acad. Sci. Paris, **174** (1922), 1167–1170; **175** (1922), 209–211; **176** (1923), 370–372) and their application to electromagnetics (C.R. Paris, **176** (1923), 238–241). He developed a generalization of the absolute tensor calculus of Gregorio Ricci–Curbastro and of the parallel translation of Tullio Levi-Cività. He represented a tensor A by the aid of certain elementary tensors $e^1, \dots, e^n$ in the form

$$A_0(x) + \sum_i A_i(x)e^i + \sum_{i,j} A_{i,j}(x)e^ie^j + \dots \ .$$

Then he investigated their addition, multiplication, contraction and derivation, and found conditions for the associativity, distributivity and especially commutativity of these operators. He also discussed a new metric compatible with his parallel translation, and the integration of the differential equation of that general parallel translation {4}.

He also gave another generalization of the parallel translation of a line-element δx in {5} by the relation

$$d\delta x^k = f^k(x, dx, d^2x, \dots, d^mx; \delta x) = 0, \tag{16}$$

where $(x, dx, \dots, d^mx)$ is an m-th order element of a curve and the f^k are first order homogeneous in δx and satisfy a certain homogeneity condition also in $dx, \dots, d^mx$. This is a very general definition of parallel translation having some relation to Finsler and also to Kawaguchi geometry. (In Kawaguchi geometry the arc length of a curve is given by such an integral $ds = \int_a^b \mathcal{L}\big(x, \dot{x}, \ddot{x}, \dots, x^{(m)}\big)\, dt$, where the fundamental function $\mathcal{L}$ depends not only on the first, but also on higher derivatives up to the m-th.) He gave also an intermediate value theorem in relation with this parallel translation. Let $A^k(t)$ be a vector field along a curve $x(t)$ and $A^k(t, t_0)$ the parallel translated according to the new parallel displacement of $A^k(t)$ along the curve to $x(t_0)$. Then this intermediate value theorem has the form

$$A^k(t, t_0) - A^k(t_0) = \frac{DA^k}{Dt}\bigg|_{t=t_0} (t - t_0) + \mu(t - t_0)^2,$$

where $\frac{D}{Dt}$ is the operator of the absolute derivation related to (16). This yields a Taylor formula too if f^k is linear in δx, His paper {6} is also connected to parallel displacement. With the aid of the parallel displacement of Levi-Cività he constructs osculating p-vectors and successive curvatures of a curve and finds explicit expressions for these curvatures. The first curvature coincides with that of Luigi Bianchi.

It is well known that in a V^n the square of the norm of a vector a^i is $|a|^2 = \sum a^i a_i$. In {7} Dienes transfers and extends this important notion on tensors in a very natural and simple way. For a tensor v^{ij} he defines

$$|v|^2 := \sum_{i,j} v^{ij} v_{ij} \equiv (vv) \quad \text{and} \quad \cos(vw) := \frac{(vw)}{|v|\,|w|}.$$

If we are given a v^{ij} and we have no metric tensor, then, in order to obtain $v_{rs} = \sum_{i,j} v^{ij} g_{ir} g_{js}$ a tensor $g_{k\ell}$ is needed, and similarly, starting with a v_{ij} a tensor $g^{k\ell}$ is necessary which may have no relation to $g_{k\ell}$. Then he describes and investigates the most general parallel translation of tensors. He requires only that the parallel translated of a tensor along a curve C from its point p to another point $q \in C$ be a tensor of the same kind depending on C alone. This yields the differential equation system $\frac{dA}{dt} + f(x, \dot{x}, \ddot{x}, \dots; A) = 0$ (suppressing the indices of the tensor A and the corresponding complicated notation at f) (cf. (16)). Then derivation of tensors is obtained in the form $\frac{\Delta A}{\Delta t} = \frac{dA}{dt} + f(x, \dot{x}, \ddot{x}, \dots; A)$. However, the usual good properties of the tensor derivation are assured only under further requirements. In an isotropic space he defines a displacement of tensors which operator can be split into a parallel translation along the curve, followed by a rotation around the endpoint.

His paper {8} represents a transition from his first period to the second one. In this paper he considers an affinely connected space L^n and an m-dimensional subspace X^n_m of it. Then the n-dimensional tangent space splits into an m-dimensional and an $n-m$-dimensional one giving rise to another subspace X^n_{n-m}. Vectors of L^n also split according to X^n_m and X^n_{n-m}. Using non-holonomic coordinate systems in them, four new connections can be derived from the connection of L^n by the aid of the split vectors of the X^n, and also the splitting of the curvature and torsion tensors of the L^n will be obtained.

In the second period of his activity in differential geometry he investigated the infinitesimal deformations of spaces and connections. He considered the infinitesimal deformation

$$'x^i = x^i + \varepsilon v^i \tag{17}$$

of an affinely connected space L^n related to the coordinate system (x). (17) can be considered as a point transformation and, at the same time, as a coordinate transformation giving raise to new components $\overline{T}^i$ of a tensor $T(x)$. Denoting by $'T$ the element of the tensor field $T(x)$ at $'x$ (i.e. $'T = T('x)$) and by T^* the parallel displaced of $T(x)$ to $'x$, he obtained three types of differentials:

$$\delta T \equiv {}'T - \overline{T}, \quad \Delta T = T^* - \overline{T}, \quad DT = T^* - {}'T$$

corresponding to three types of derivations. It is noteworthy that δ can also be applied to quantities which are not tensors, e.g. to connection coefficients. He applied these deformations on L^n and V^n, and also on submanifolds X_m^n of these. He described the effect of these operations on the curvature tensors and connection coefficients using, in the case of a submanifold X_m^n, a splitting into tangential and transversal components. The operator δ was used also by W. Slebodzinki and D. van Danzig, and applied to Lie derivation. It turned out to be a very good tool for the study and characterization of affine and metrical motions.

His papers reporting on these investigations (Sur la déformation des espaces à connexion linéaire générale. C.R. Acad. Sci. Paris, **197** (1933), 1084–1087; Sur la déformation des sous-espaces dans un espace à connexion linéaire générale. C.R. Acad. Sci. Paris, **197** (1933), 1167–1169; On the deformation of tensor manifolds. Proc. London Math. Soc. (2), **37** (1934), 512–519) are closely related to each other and crowned in his most comprehensive paper (On the infinitesimal deformations of tensor submanifolds. J. Math. Pures Appl. (9), **16** (1937), 111–150) written together with E. T. Davies from Southampton who can be considered as his pupil. The problems treated in these papers were problems of his time, and other mathematicians too showed interest in them e.g. J. A. Schouten and E. R. van Kampen.

István Fáry who worked mainly in Berkeley, dealt with convex geometry, cell complexes, topology, etc. However, he has a very nice, interesting result on the global differential geometry of the E^3.

One of the first results of global differential geometry states that the total curvature of a closed curve of the Euclidean two space E^2 is at least 2π. This result was extended by W. Fenchel to closed space curves of the E^3 in 1929, and to those of the E^n by K. Borsuk in 1948. Borsuk also raised the question whether the total curvature of a knot (i.e. a curve of the E^3 which is homeomorphic to the circle, but is not isotopic to it) is always

$\geqq 4\pi$. This interesting question was answered by Fáry in an affirmative way in {10}. His proof is based on the interesting observation that the total curvature of a curve of the E^3 equals the mean of the total curvatures of its orthogonal projections on the 2-dimensional linear subspaces of the E^3. He found this result in France before he settled in the USA. Another positive answer on Borsuk's problem was published by J. W. Milnor in {14} a little later, in 1950 (the two papers were accomplished independently from each other).

Another differential geometrical result is due to István Vincze who worked mainly in statistics and probability theory. Let $L \subset E^3$ be a curve related to the arc length s; $Q_1(s_1)$, $Q_2(s_2)$ points of L, and $S(Q_1, Q_2)$ the center of mass of the arc $\widehat{Q_1Q_2}$ in case of a homogeneous load-distribution. If $Q_1 \to P_0 \in L$, $Q_2 \to P_0$ independently of each other, then S runs over a two parameter family of points, i.e. a surface $F \subset E^3$. Let us denote by $K(Q_1, Q_2)$ the Gauss curvature and by $H(Q_1, Q_2)$ the mean curvature of F at $S(Q_1, Q_2)$. Then, according to his result {58},

$$\lim_{Q_1,Q_2 \to P_0} K(Q_1, Q_2) = -\tau^2, \quad \text{and} \quad \lim_{Q_1,Q_2 \to P_0} H(Q_1, Q_2) = -\frac{3}{5}\frac{1}{\rho^2}\frac{d}{ds}(\varrho^2\tau),$$

where ϱ is the curvature and τ the torsion of L. He also investigated the case of a closed curve L loaded with a density $\mu(s)$ and he determined, among other things, the surface area of F. His investigation is related to a problem of Alfréd Rényi connected to the cosmological theory of O. Yu. Schmidt (cf. I. Vincze, {57}).

Jenő Egerváry was interested mainly in matrix theory and differential equations, but as professor of mathematics at Budapest Technical University he also dealt with different problems of algebra, geometry of E^3, analysis, and certain physical and technical problems. However, Egerváry has interesting results in the differential geometry of the Euclidean n-space E^n too.

Let $L(s)$ be a smooth curve of the E^n given by $x^i = x^i(s)$ or, in vectorial form, by $\boldsymbol{x}(s) = x^i(s)\boldsymbol{e}_i$ with s as arc length, and $\boldsymbol{e}_i$ as an orthonormal base of E^n. It is an elementary fact that the (first) curvature $\frac{1}{\varrho_1(s)} = \kappa_1(s)$ of L is the limit of the ratio of the angle of the tangents at s_0 and s and of $|s - s_0|$ in case of $s \to s_0$; i.e. $\kappa_1(s)$ is the angular velocity of the tangent of the curves. Similarly, the second curvature κ_2 (the torsion τ) is the angular velocity of the second normal. Further curvatures $\kappa_3, \dots, \kappa_{n-1}$ of the L in E^n or even in a Riemannian space V^n (see W. Blaschke, {2}) are defined as the coefficients appearing in the Frenet formulas expressing the derivatives of the vectors of the moving frame linearly by the vectors of the frame itself.

Egerváry considered the Gram determinants

$$G_k = \det |h^{ab}|_k, \quad h^{ab} = \left\langle \boldsymbol{x}^{(a)}, \boldsymbol{x}^{(b)} \right\rangle, \quad \begin{matrix} a, b = 1, \dots k \\ k = 1, 2, \dots, n, \end{matrix} \quad G_0 := 1$$

of order k, where $\boldsymbol{x}^{(a)} = \frac{d^a \boldsymbol{x}}{ds^a}$, and $\langle\ ,\ \rangle$ denotes the Euclidean scalar product, and proved that

$$\frac{d\eta_k}{ds} = \lim_{s \to s_0} \frac{\eta_k}{|s - s_0|} = \frac{\sqrt{G_{k+1} G_{k-1}}}{G_k}(s_0), \quad k = 1, \dots, n-1,$$

where η_k is the angle of the osculating k-planes taken at s and s_0. It turns out that these $\frac{d\eta_k}{ds}$ coincide with the curvatures $\kappa_1, \dots, \kappa_{n-1}$ of the curve L. This can easily be seen in the simple case of $k = 1$. Indeed

$$G_1 = \langle \boldsymbol{x}', \boldsymbol{x}' \rangle = |\boldsymbol{x}'|^2 = 1, \quad G_2 = \begin{vmatrix} \langle \boldsymbol{x}', \boldsymbol{x}' \rangle & \langle \boldsymbol{x}', \boldsymbol{x}'' \rangle \\ \langle \boldsymbol{x}'', \boldsymbol{x}' \rangle & \langle \boldsymbol{x}'', \boldsymbol{x}'' \rangle \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & |\boldsymbol{x}''|^2 \end{vmatrix} = |\boldsymbol{x}''|^2,$$

for the parameter is the arc length, and because of $\boldsymbol{x}' \perp \boldsymbol{x}''$. Thus, taking into account $G_0 = 1$, we obtain $\frac{d\eta_1}{ds} = |\boldsymbol{x}''| = \kappa_1$. These $\frac{d\eta_k}{ds}$ also satisfy the Frenet formulas. So Egerváry's result revealed the geometric meaning of the formally defined curvatures κ_k both in E^n and V^n. Moreover, he could express $\frac{d\eta_k}{ds}$ also by volumes of simplexes with vertices on L and by distances of the vertices. In this form curvatures do not need the differentiability of the curve. Egerváry's results and ideas found applications in the book of L. M. Blumenthal {3}), and are also related to some works of G. Alexits.

One can consider a curve L as the set of points represented by $x^i = x^i(t) \in C^0$, $t \in I = (a, b)$. After Peano's investigation it became clear that this set may be a cube. This can not happen if one requires the mapping $I \to L$ to be $1:1$. Nevertheless this excludes so simple and important cases as multiple points. A new set theoretical and topological theory of curves in metrical (distance) space without any differentiability was initiated by K. Menger {12}, {13} and P. Urysohn {42}. These investigations were presented to the Hungarian mathematical community by György Alexits {1}.

György Alexits, known for his works on approximation theory and orthogonal series, joined these investigations in the 30s. He investigated the curvatures of a curve in the general distance and semi-distance spaces. He called them linear curvatures in order to distinguish them from the curvature of a surface or of a space used in Riemannian geometry. A distance space is a metric space with its well known three axioms. The space is a semi-distance space if the triangle axiom is not required (e.g. in a Minkowski space with non convex indicatrix). It was K. Menger who first started investigations on the (first) curvature of a curve in distance spaces. Alexits introduced the notion of the k-th linear curvature, and devoted several papers, first alone, and then together with Egerváry, to their investigation (La torsion des espaces distanciés. Comp. Math., **6** (1939), 471–477; Der Torsionsbegriff in metrischen Räumen. Mat. Fiz. Lapok, **46** (1939), 13–28 in Hungarian with German summary). It turned out that in the case of a Euclidean space his linear curvatures reduce to Egerváry's $\frac{d\eta_k}{ds}$ and in a Riemannian space to Blaschke's curvatures, see {9}.

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