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Classical (Unweighted) and Weighted Interpolation

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1. Introduction

What is interpolation? "Perhaps it would be interesting to dig to the roots of the theory and to indicate its historical origin. Newton, who wanted to draw conclusions from the observed location of comets at equidistant times as to their location at arbitrary times arrived at the problem of determining a 'geometric' curve passing through arbitrarily many given points. He solved this problem by the interpolation polynomial bearing his name "(*Pál Turán* {128, p. 23}.)

Interpolation theory has been one of the favorite subjects of the twentieth century's Hungarian approximators. The backbone (mainly of classical interpolation) is the theory developed by *Lipót Fejér*, *Ervin Feldheim*, *Géza Grünwald*, *Pál Erdős* and *Pál Turán*.

One can find hundreds of papers dealing with different interpolatory processes (Lagrange-, Birkhoff (lacunary)-, Hermite–Fejér interpolation, etc.).

In the last 40 years or so there has developed a new branch in approximation theory: the so called *weighted approximation.*

During those years, even in this relatively new area, many interpolatory results were proved by the Hungarian school.

In Part A we quote the *classical* results while Part B considers the *weighted* ones. Since weighted approximation is relatively new, Part B is much shorter.

The interested readers may find many other details and results in the booklet of *Ervin Feldheim* {49} and in the book of *József Szabados* and

Péter Vértesi, Interpolation of functions {111}. Generally, we concentrate on the Lagrange interpolation. Analogous results may be proved for the trigonometric and the complex case (see {111} again).

A
CLASSICAL CASE

2. Lagrange Interpolation. Lebesgue Function. Lebesgue Constant. Optimal Lebesgue Constant. Divergence of Interpolation

2.1. Let us begin with some definitions and notation. Let $C = C(I)$ denote the space of continuous functions on the interval $I := [-1, 1]$, and let $\mathcal{P}_n$ denote the set of algebraic polynomials of degree at most n. $\|\cdot\|$ stands for the usual maximum norm on C. Let X be an *interpolatory matrix (array)*, i.e.,

$$X = \big\{ x_{kn} = \cos\vartheta_{kn}; \qquad k = 1, \dots, n; \qquad n = 0, 1, 2, \dots \big\},$$

with

$$-1 \le x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} \le 1 \tag{2.1}$$

and $0 \le \vartheta_{kn} \le \pi$, and consider the corresponding *Lagrange interpolation polynomial*

$$L_n(f, X, x) := \sum_{k=1}^{n} f(x_{kn}) \ell_{kn}(X, x), \qquad n \in \mathbb{N}. \tag{2.2}$$

Here, for $n \in \mathbb{N}$,

$$\ell_{kn}(X, x) := \frac{\omega_n(X, x)}{\omega_n'(X, x_{kn})(x - x_{kn})}, \qquad 1 \le k \le n,$$

with

$$\omega_n(X, x) := \prod_{k=1}^{n} (x - x_{kn}),$$

are polynomials of exact degree $n-1$. They are called the *fundamental polynomials* associated with the *nodes* $\{x_{kn} : k = 1, \dots, n\}$ obeying the relations $\ell_{kn}(X, x_{jn}) = \delta_{kj}$, $1 \le k, j \le n$.

The main question is: For what choices of the interpolation array X we can expect that (uniformly, pointwise, etc.) $L_n(f, X) \to f \quad (n \to \infty)$?

Since, by the Čebishov alternation theorem ({95} Chap. 2, Theorem 9), the best uniform approximation, $P_{n-1}(f)$, to $f \in C$ from $\mathcal{P}_{n-1}$ interpolates f in at least n points, there exists, for each $f \in C$, an interpolation matrix, Y, for which

$$\big\| L_n(f, Y) - f \big\| = E_{n-1}(f) := \min_{P \in \mathcal{P}_{n-1}} \|f - P\|$$

goes to 0 as $n \to \infty$. However, for the *whole class* C, the situation is different.

To formulate the corresponding negative result, we quote some estimates and introduce further definitions.

By the classical Lebesgue estimate,

(2.3)

$$\begin{aligned} \big| L_n(f, X, x) - f(x) \big| &\le \big| L_n(f, X, x) - P_{n-1}(f, x) \big| + \big| P_{n-1}(f, x) - f(x) \big| \\ &\le \big| L_n(f - P_{n-1}, X, x) \big| + E_{n-1}(f) \\ &\le \left(\sum_{k=1}^{n} \big| \ell_{k,n}(X, x) \big| + 1 \right) E_{n-1}(f), \end{aligned}$$

therefore, with the notations

$$\lambda_n(X, x) := \sum_{k=1}^{n} \big| \ell_{kn}(X, x) \big|, \qquad n \in \mathbb{N}, \tag{2.4}$$

$$\Lambda_n(X) := \big\| \lambda_n(X, x) \big\|, \qquad n \in \mathbb{N}, \tag{2.5}$$

(*Lebesgue function* and *Lebesgue constant* (of Lagrange interpolation), respectively,) we have for $n \in \mathbb{N}$

$$\big| L_n(f, X, x) - f(x) \big| \le \big\{ \lambda_n(X, x) + 1 \big\} E_{n-1}(f) \tag{2.6}$$

and

$$\big\| L_n(f, X) - f \big\| \le \big\{ \Lambda_n(X) + 1 \big\} E_{n-1}(f). \tag{2.7}$$

"After ... the approximation theorem of *Karl Weierstrass*, it was hoped that there exists a (non-equidistant) system of nodes for which the Lagrange interpolation polynomials converge uniformly for every function continuous in $[-1,1]$. The mathematical world was awakened from this dream in 1914 by *Georg Faber* who showed that there is *no* such system." (*Turán* {128, p. 25})

Namely, he proved the then rather surprising lower bound

$$\Lambda_n(X) \geq \frac{1}{12}\log n, \qquad n \geq 1, \tag{2.8}$$

for any interpolation array X. Based on this result he obtained

Theorem 2.1 (Faber {40}). *For any fixed interpolation array X there exists a function $f \in C$ for which*

$$\overline{\lim_{n\to\infty}} \big\| L_n(f,X) \big\| = \infty. \tag{2.9}$$

2.2. The previous estimates show clearly the importance of the Lebesgue function, $\lambda_n(X,x)$, and the Lebesgue constant, $\Lambda_n(X)$. During the last 90 years, very general relations concerning their behaviour were proved and applied to obtain divergence theorems for $L_n(f,X)$.

First, we state the counterpart of (2.8). Namely, using an estimate of *L. Fejér* {45} (cf. {127, Section 4.12.6})

$$\Lambda_n(T) = \frac{2}{\pi}\log n + O(1),$$

one can see that the order $\log n$ in (2.8) is best possible (here T is the Čebishov matrix, i.e. $x_{kn} = \cos\frac{2k-1}{2n}\pi$).

A very natural problem, raised and answered in 1958 by *Erdős*, says that $\lambda_n(X,x)$ is "big" on a "large" set.

Theorem 2.2 (Erdős {36}). *For any fixed interpolation matrix $X \subset [-1,1]$, real $\varepsilon > 0$, and $A > 0$, there exists $n_0 = n_0(A,\varepsilon)$ so that the set*

$$\big\{ x \in \mathbb{R} : \lambda_n(X,x) \leq A \text{ for all } n \geq n_0(A,\varepsilon) \big\}$$

has measure less than ε.

In 1978, *P. Erdős* and *J. Szabados* obtained a "best possible result in order" for $\lambda_n(X,x)$. Namely one has

Theorem 2.3 ({23}). *For any interpolatory matrix X and subinterval $[a,b]\subset[-1,1]$ there exists $c>0$ such that*

$$\int_a^b \lambda_n(X,x)\,dx \ge c(b-a)\log n, \qquad n\ge n_0(a,b).$$

The next statement, the more or less complete pointwise estimation is due to *P. Erdős* and *P. Vértesi* {31} from 1981.

Theorem 2.4. *Let $\varepsilon>0$ be given. Then, for any fixed interpolation matrix $X\subset[-1,1]$ there exist sets $H_n=H_n(\varepsilon,X)$ of measure $\le\varepsilon$ and a number $\eta=\eta(\varepsilon)>0$ such that*

$$\lambda_n(X,x)>\eta\log n \tag{2.10}$$

if $x\in[-1,1]\setminus H_n$ and $n\ge 1$.

Closer investigation shows that (instead of the original $\eta=c\varepsilon^3$) $\eta=c\varepsilon$ can be attained ({134}). The behaviour of the Čebishov matrix, T, shows that (2.10) is the best possible in order.

2.3. Let us say some words about the *optimal Lebesgue constant.* In 1961, *P. Erdős*, improving a previous result of *P. Turán* and himself (see {24} and {38}), proved that

$$\left|\Lambda_n^* - \frac{2}{\pi}\log n\right| \le c,$$

where

$$\Lambda_n^* := \min_{X\subset I}\Lambda_n(X), \qquad n\ge 1,$$

is the *optimal Lebesgue constant.* As a consequence of this result, the closer investigation of Λ_n^* attracted the attention of many mathematicians.

In 1978, *Ted Kilgore*, *Carl de Boor* and *Alan Pinkus* proved the so–called Bernstein-Erdős conjectures concerning the optimal interpolation array X (cf. {8}, {9}, {14} and {62}).

Using this result *P. Vértesi* {139} obtained the value of Λ_n^* *within the error* $o(1)$. Namely,

$$\Lambda_n^* = \frac{2}{\pi}\log n + \chi + O\left(\left(\frac{\log\log n}{\log n}\right)^2\right) \tag{2.11}$$

where $\chi = \frac{2}{\pi}(\gamma + \log \frac{4}{\pi}) = 0.521251\ldots$ and $\gamma = 0.577215\ldots$ is the Euler–Mascheroni constant (cf. 2.8.3).

2.4. One of the most talented approximators, *Géza Grünwald*, was a holocaust victim; he was killed in 1942 at the age 32. He was about 25 when, in two fundamental papers, he proved that the Lagrange interpolation can be very bad even for the good matrix $T = \left\{\cos \frac{2k-1}{2n}\pi\right\}$ (see {53}, {54}, {81}).

Theorem 2.5 (*Grünwald–Marcinkiewicz*)*. *There exists a function $f \in C$ for which*

$$\overline{\lim_{n\to\infty}} \left| L_n(f, T, x) \right| = \infty$$

for every $x \in [-1, 1]$.

In their second joint paper, {21} *Erdős* and *Grünwald* sharpen this result. They construct a function $f \in C$ satisfying Theorem 2.5, where at the same time, the even function $f(\cos\vartheta)$ has a uniformly convergent Fourier series on $[0, \pi]$.

Marcinkiewicz {81} showed that for every x_0 there exists a continuous f for which

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} L_k(f, T, x_0) = \infty. \tag{2.12}$$

In other words, the arithmetic means of Lagrange interpolating polynomials of a continuous function can diverge at a given point. This is in marked contrast to the celebrated theorem of Fejér {47} for Fourier series.

In their third joint paper, {20} *Erdős* and *Grünwald* claimed to prove a far-reaching generalization of (2.12), namely the existence of an $f \in C$ for which

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \left| \sum_{k=1}^{n} L_k(f, T, x) \right| = \infty, \tag{2.13}$$

*At the same time the same statement was proved by the Polish mathematician, *Józef Marcinkiewicz.* We must note some other similarities between them. Both were born in 1910; both included the above theorem into their PhD dissertations; they were submitted in 1935; moreover *Marcinkiewicz* was also a victim of the war: as his teacher *Antoni Zygmund* writes: "On September 2 [1939], the second day of the war I came across him accidentally in the street in Wilno [Vilnius], already in military uniform… A few months later came the news that he was a prisoner of war and was asking for mathematical books. It seems that this was the last news about *Marcinkiewicz*" ({144, p. 4}).

for all $x \in [-1, 1]$. However, as it was discovered later by *Erdős* himself, there is an oversight in the proof and the method only gives the result with the modulus sign *inside* the summation.

Only in {22}, where *Erdős* and *Gábor Halász* (who was born four years after the Erdős–Grünwald paper) were able to complete the proof and obtained the following.

Theorem 2.6. *Given a positive sequence $\{\varepsilon_n\}$ converging to zero however slowly, one can construct a function $f \in C$ such that for almost all $x \in [-1, 1]$*

$$\frac{1}{n}\left|\sum_{k=1}^{n} L_k(f, T, x)\right| \geq \varepsilon_n \log\log n \tag{2.14}$$

for infinitely many n.

The right-hand side is optimal, for in the paper {39} *Erdős* has proved

Theorem 2.7.

$$\frac{1}{n}\left|\sum_{k=1}^{n} L_k(f, T, x)\right| = o(\log\log n)$$

for almost all x, whenever $f \in C$.

The proof was an ingenious combination of ideas from number theory, probability and interpolation; it is not by chance that the authors are *Erdős* and *Halász*!

2.5. After the result of *Grünwald* and *Marcinkiewicz* a natural problem was to obtain an analogous result for an *arbitrary* array X. In {37, p. 384}, *Erdős* wrote: "In a subsequent paper I hope to prove the following result:

Let $X \subset [-1, 1]$ be any point group [interpolatory array]. Then there exists a continuous function $f(x)$ so that for almost all x

$$\overline{\lim_{n\to\infty}} \left|L_n(f, X, x)\right| = \infty."$$

After 4 years of work, *Erdős* and *Vértesi* proved the above result ({28}–{30}). *Erdős* writes in {29}: "[Here we prove the above] statement in full detail. The detailed proof turns out to be quite complicated and several unexpected difficulties had to be overcome."*

2.6. Another significant contribution of the Hungarian approximators to interpolation is the so called "fine and rough theory" (a name coined by *Erdős* and *Turán* in their basic joint paper {27} dedicated to *L. Fejér* on his 75th birthday in 1955).

In the class $\operatorname{Lip}\alpha$ $(0 < \alpha < 1)$ (we give the exact definitions a little later), a natural error estimate for Lagrange interpolation is

$$\left\| L_n(f, X) - f \right\| \leq cn^{-\alpha}\Lambda_n(X)$$

(cf. (2.7)). *Erdős* and *Turán* raised the obvious question: *How sharp is this estimate in terms of the order of the Lebesgue constant as* $n \to \infty$? They themselves considered interpolatory arrays X where

$$\Lambda_n(X) \sim n^\beta \qquad (\beta > 0).$$

(In the class $\operatorname{Lip}\alpha$ this is the natural setting.) In the above paper {27} they prove essentially

Theorem 2.8. *Let* X *be as above. If* $\alpha > \beta$, *then we have uniform convergence in* $\operatorname{Lip}\alpha$. *If* $\alpha \leq \beta/(\beta + 2)$, *then for some* $f \in \operatorname{Lip}\alpha$, *Lagrange interpolation is divergent.*

These two cases comprise what is called the "rough theory", since *solely on the basis of the order* of $\Lambda_n(X)$ one can decide the convergence-divergence behavior. However,

Theorem 2.9. *If* $\beta/(\beta + 2) < \alpha \leq \beta$ *then anything can happen. That is, there is an interpolatory array* Y_1 *with* $\Lambda_n(Y_1) \sim n^\beta$ *and a function* $f_1 \in \operatorname{Lip}\alpha$ *such that* $\overline{\lim}_{n\to\infty} \left\| L_n(f_1, Y_1) \right\| = \infty$, *and another interpolation array* Y_2 *with* $\Lambda_n(Y_2) \sim n^\beta$, *such that* $\lim_{n\to\infty} \left\| L_n(f, Y_2) - f \right\| = 0$ *for every* $f \in \operatorname{Lip}\alpha$.

*In a personal letter *Erdős* wrote about the *main idea of the proof*: [First] "we should prove that for every fixed A and $\eta > 0$ there exists an M $(M = M(A, \eta))$ such that if we divide the interval $[-1, 1]$ into M equal parts $I_1, \dots, I_M$ then

$$\sum\nolimits_k' |\ell_{k,n}(X, x)| > A, \qquad x \in I_r,$$

apart from a set of measure $\leq \eta$. Here $\sum'$ means that k takes those values for which $x \notin I_r$".

That is, to decide the convergence-divergence behavior *we need more information than just the order of the Lebesgue constant.* The corresponding situation is called "fine theory".

This paper of *Erdős* and *Turán* has been very influential. It left open a number of problems and attracted the attention not only of the Hungarian school of interpolation (*Géza Freud*, *Ottó Kis*, *Melania Sallay*, *József Szabados*, *Péter Vértesi*), but also of others (including *R. J. Nessel*, *W. Dickmeis*, *E. van Wickeren*).

We mention three generalizations. Let $\omega_m(t)$ be an increasing continuous function for $t \geq 0$ with $\omega_m(0) = 0$, $\omega_m(t) > 0$ $(t > 0)$, $t^m/\omega_m(t) \leq T^m/\omega_m(T)$ $(t \leq T)$; $m \geq 1$ is a fixed integer. The function ω_m is *an* m-th *modulus of smoothness.* If $m = 1$, we write $\omega(t)$ (modulus of continuity).

With $\omega_m(t)$ we define the function-class $C(\omega_m)$ as

$$C(\omega_m) = \big\{ f \in C \text{ and } \omega_m(f,t) \leq c_m(f)\omega_m(t) \big\},$$

where, with $\Delta_h^m f(x) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+kh)$,

$$\omega_m(f,t) = \sup_{\substack{x,x+mh \in [-1,1] \\ |h| \leq t}} \big| \Delta_h^m f(x) \big|,$$

is the m-th *modulus of smoothness* of f; if $m = 1$, $\omega(f,t)$ is the *modulus of continuity* of f. If $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$, then by definition $C(\omega) \equiv \operatorname{Lip} \alpha$.

In his paper {69} *O. Kis* proved the following

Theorem 2.10. *For an arbitrary fixed interpolatory matrix X one can find an $f \in C(\omega_m)$ with*

$$\overline{\lim_{n\to\infty}} \frac{\big\| L_n(f,X,x) - f(x) \big\|}{\Lambda_n(X)\omega_m\big(d_n(X)\big)} > 1,$$

provided that

$$\lim_{t\to 0} \omega_m(t) t^{-m} = \infty \tag{2.15}$$

Here

$$d_n(X) = \min_{1 \leq k \leq n} (x_{kn} - x_{k+1,n}), \qquad n \geq 2.$$

Here is another generalization, a strong *pointwise-type* divergence result of *P. Vértesi* {111, Theorem 4.20}

Theorem 2.11. *Let* X *and* $\omega(t)$ *be given. If*

$$\underline{\lim}_{t\to 0}\,\omega(t)\big|\log t\big| > 0$$

then with an appropriate $f \in C(\omega)$

$$\overline{\lim}_{n\to\infty}\big|L_n(f,X,x) - f(x)\big| > 1$$

on a dense set of second category in $[-1,1]$.

To state a new and rather deep theorem of *G. Halász* {57}, we define, deviating somewhat from its previous meaning, the Lipschitz class Lip A of exponent $A = r + \alpha$ $(0 \le \alpha < 1)$, $r = [A]$, as the space of functions f for which $f^{(r)}$ exists everywhere. Moreover

$$\sup_{t_1\ne t_2}\big|f^{(r)}(t_1) - f^{(r)}(t_2)\big|/|t_1 - t_2|^{\alpha} < \infty,$$

in particular, Lip 0 consists of the bounded functions. We define the characteristic $D(A) = D(A,X)$ by

$$D(A,X) = \sup_{f\in \mathrm{Lip}\,A}\left\{a;\big|L_n(f,X,x) - f(x)\big| \le c\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{a}\right\}$$

where $c > 0$ may depend on f and α but not on x and n.

It is clear that $-\infty \le D(A) \le A$. Moreover (cf. {57, Part 2}) we have

Theorem 2.12. *Let* X *be given. Then*

(i) $D(A)$ *is concave from below.*

(ii) $D'(A) \ge \frac{1}{2}$ *whenever* $D(A)$ *is finite.*

(iii) $2D(A_1) + D(A_2) \le A_1 + 2A_2 + 2$ *for any* $A_1, A_2 \ge 0$.

The trigonometric version of the above considerations is {57, Part 1.}. However, as it was proved by *J. Szabados* {109, Theorem}, the corresponding *trigonometric characteristic of* D *is fully described by the corresponding properties.*

2.7. The Faber-theorem (2.9) is a special case of a general statement proved by *S. M. Losinskii* and *F. I. Harsiladze* on *(linear) projection operators* (p.o.). (That means $\mathcal{L}_n$ is a linear bounded operator with $\mathcal{L}_n : C \to \mathcal{P}_{n-1}$ and $\mathcal{L}_n(f) \equiv f$ iff $f \in \mathcal{P}_{n-1}$). Namely, they proved that if

$$|||\mathcal{L}_n||| := \sup_{\|f\| \le 1} \big\| \mathcal{L}_n(f,x) \big\|, \qquad f \in C,$$

then

$$|||\mathcal{L}_n||| \ge \frac{\log n}{8\sqrt{\pi}}$$

($\mathcal{L}_n$ is a p.o.). If $\mathcal{L}_n = L_n(X)$ (Lagrange interpolation), then, obviously $\Lambda_n(X) = |||\mathcal{L}_n|||$.

In his paper {56}, *G. Halász* formulated some results on

$$\mathcal{L}_n(x) := \sup_{\|f\| \le 1} \big| \mathcal{L}_n(f,x) \big|, \qquad f \in C$$

(it generalizes the Lebesgue function $\lambda_n(X,x)$). Among others he states

Theorem 2.13. *For any sequence of projections* $\mathcal{L}_n$

(i) $\overline{\lim_{n\to\infty}} \mathcal{L}_n(x) = \infty$ *on a set of positive measure in* $[-1,1]$;

(ii) $\lim_{n\to\infty} \int_{-1}^{1} h\big(\log \mathcal{L}_n(x)\big) \log \mathcal{L}_n(x)\, dx = \infty$ *whenever*

$$I := \int_2^\infty \frac{h(x)}{x \log x}\, dx = \infty.$$

(iii) *If* $I < \infty$ *then there exists a sequence* $\mathcal{L}_n$ *such that*

$$\sup_n \int_{-1}^{1} h\big(\log \mathcal{L}_n(x)\big) \log \mathcal{L}_n(x)\, dx < \infty.$$

Let r be an integer, $r \ge 0$. We will be concerned with the investigation of the norm of the derivative $\mathcal{L}_n^{(r)}$ of the p.o. $\mathcal{L}_n$, i.e. $\mathcal{L}_n^{(r)}(f,x) = \frac{d^r}{dx^r}\mathcal{L}_n(f,x)$. If

$$\Big|\Big|\Big|\mathcal{L}_n^{[r]}\Big|\Big|\Big| := \sup_{\|f\| \le 1} \big\|\mathcal{L}_n^{(r)}(f,x)\big\|, \qquad f \in C$$

then, according *D. L. Berman* {7}, $\Big|\Big|\Big|\mathcal{L}_n^{[r]}\Big|\Big|\Big| \ge c_r n^{2r}$ $(r \ge 1)$. However, we can do better.

Motivated by the Nikolskii–Timan–Gopengauz phenomenon in polynomial approximation ({88}, say), let

$$\mathcal{L}_{n\mu}^{[r]}(x) := \sup_{|f(x)|\le(1-x^2)^\mu} \left|\mathcal{L}_n^{(r)}(f,x)\right|, \qquad f\in C, \qquad \mu\ge 0;$$

(see *N. S. Baiguzov* {2}, *L. Neckermann*, *P. O. Runck* {88} and for arbitrary $r\ge 3$ *J. Szabados* {115}). We can prove (see {115}, {130}):

Theorem 2.14. *For an arbitrary projection operator* $\mathcal{L}_n$ *and fixed* $\mu\ge 0$, $r=0,1,2,\dots,$

$$\int_{-1}^{1} \mathcal{L}_{n\mu}^{[r]}(x)\,dx \ge c(r,\mu)n^r \log n, \qquad n\ge 1.$$

By Theorem 2.13

$$\text{(2.16)}\qquad \Big(\big|\big|\big|\mathcal{L}_n^{[r]}\big|\big|\big| \ge\Big)\big|\big|\big|\mathcal{L}_{n\mu}^{[r]}\big|\big|\big| := \big|\big|\big|\mathcal{L}_{n\mu}^{[r]}(x)\big|\big|\big| \ge c_1(r,\mu)n^r\log n.$$

Moreover as a nice application of the "additional points method", one can prove that the estimation (2.16) is the best possible in order (see {115}). Actually, the so called "additional point method" has a long history. Perhaps *Fejér* was the first who noticed that restricting the interpolation at the endpoints may improve its convergence behaviour ({44}, {43}). After some other initial results due to *E. Egerváry, P. Turán, P. Szász, G. Freud, N. S. Baiguzov* and others, *J. Szabados* was the first who *systematically applied the method of adding some new points to the original interpolatory matrix X to improve the behaviour of the interpolation.*

2.8. Remarks. 1. Let us mention two basic relations concerning the estimation of the Lebesgue function (see *P. Erdős*, *P. Turán* {26, p. 529} and *P. Erdős*, {36, p. 387}).

(a) *For an arbitrary interpolatory matrix* $X\subset[-1,1]$

$$\ell_{kn}(X,x)+\ell_{k+1,n}(X,x)\ge 1 \quad \text{if} \quad x\in[x_{k+1,n},x_{kn}], \qquad 1\le k\le n-1.$$

(b) *Let* $y_1,y_2,\dots,y_t$ *be any* t $(t>t_0)$ *distinct numbers in* $[-1,1]$ *not necessarily in increasing order. Then for at least one* u $(1\le u\le t)$

$$\sum_{i=1}^{u-1}\frac{1}{|y_i-y_u|} > \frac{t\log t}{8}.$$

(The half-page proof is based on the inequality between the arithmetic and harmonic means.)

2. An improvement of Theorem 2.3 that settles "small" intervals whose lengths may depend on n is in {33} (see also {34}).

3. It may be instructive to compare some values of Λ_n^* with the Lebesgue constants $\Lambda_n(S)$ and $\Lambda_n(\widehat{T})$, respectively (see *Lev Brutman* {10, p. 122}, the values are of 7-digit precision) Here S is the matrix by which the estimations (2.11) were obtained; $\hat{T} = \left\{\cos\frac{2k-1}{2n}\pi/\cos\frac{\pi}{2n}\right\}$ is an extended Čebishov matrix.

n	Λ_n^*	$\Lambda_n(S)$	$\Lambda_n(\widehat{T})$
3	1.422 920	1.448 083	1.429 873
4	1.559 490	1.575 680	1.570 167
5	1.672 210	1.683 646	1.685 140
6	1.768 135	1.776 834	1.782 530
7	1.851 599	1.858 521	1.866 999
8	1.925 458	1.931 112	1.941 573
9	1.991 685	1.996 560	2.008 327
10	2.051 706	2.056 087	2.068 744
20	2.460 788	2.463 129	2.479 193
30	2.708 082	2.709 645	2.726 693
40	2.885 809	2.887 067	2.904 441
50	3.024 619	3.025 651	3.043 229
60	3.138 527	3.139 389	3.157 102
70	3.235 120	3.235 887	3.253 659
80	3.318 973	3.319 660	3.337 477
90	3.393 058	3.393 677	3.411 530
100	3.459 415	3.459 973	3.477 858
150	3.715 393	3.715 787	3.733 720
200	3.897 466	3.897 772	3.915 713

The above table shows that even for relatively small values of n, $\Lambda_n(S)$ is quite close to Λ_n^*; much closer than the corresponding values of $\Lambda_n(\widehat{T})$.

4. The optimal matrix and the corresponding Lebesgue constants are well-known in the trigonometric and complex cases (see {14}, {9}). For other generalizations see {111}, Chapters III and IV.

5. As *O. Kis* remarked in his paper {64}, there is a predecessor of the fundamental work {27}. Namely *S. M. Losinskii* in 1948 stated some analogous results in his short *Dokladi* paper {79}, but he never published the proofs. On the other hand, their verifications are in the exhausting paper {63}; for other developments see {111, Chapter I}.

6. The characterization of the "trigonometric $D(A)$" can be found in the papers {57} and *J. Szabados* {109}.

3. On the Convergence of the Interpolatory Processes

3.1. There are at least 4 simple possibilities to ensure convergence:

(a) raising the degree of the interpolatory polynomials (see Sections 3 and 4);

(b) using mean convergence (instead of the uniform one (see Section 5);

(c) restricting ourselves to a part (subclass) of C (see Sections 3.2–3.3);

(d) applying a combination of the fundamental functions $\ell_{kn}(X,x)$ (see Section 3.4).

3.2. Our first statement on "good" functions goes back to *L. Fejér* {42} and *László Kalmár* {61}.

Theorem 3.1. *Let f be analytic on $[-1,1]$ ($f \in A$, shortly). Then*

$$\lim_{n\to\infty} \left\| L_n(f,X,x) - f(x) \right\| = 0 \qquad \forall f \in A$$

iff the nodes $\{x_{kn}\}$ are uniformly distributed on $[-1,1]$

We say that the nodes $x_{kn} = \cos\vartheta_{kn}$, $1 \le k \le n$, $n \in \mathbb{N}$, are *uniformly distributed* on $[-1,1]$ if for every subinterval $I \subset [0,\pi]$,

$$\lim_{n\to\infty} \frac{N_n(I)}{n} = \frac{|I|}{\pi},$$

where $N_n(I)$ is the number of ϑ_{kn} in I (cf. (4.4)).

Exactly 30 years after Kalmár in his Ph.D. dissertation *O. Kis* {66, Theorem 5} proved the *trigonometric* version. Here is another statement on the convergence of the trigonometric interpolatory polynomials $T_n(g,T,t)$

belonging to the set T_n of trigonometric polynomials of order n, based on the interpolatory matrix $T = \{t_{kn},\ 0 \le k \le 2n,\ n \in \mathbb{N}\} \subset [0, 2\pi]$ for $g \in \widetilde{C}$ (g is 2π-periodic and continuous) (see {66, Theorem 6}). Let $B = \Big\{g : g$ is 2π-periodic and analytic if $|\operatorname{Im} z| \le 2\log\big(1+\sqrt{2}\,\big)\Big\}$.

Theorem 3.2. *For an arbitrary interpolatory $T \subset [0, 2\pi)$*

$$\lim_{n\to\infty} \big\| T_n(g, T, t) - g(t) \big\| = 0 \qquad g(t) \in \widetilde{C}$$

iff $g \in B$, where $\|\cdot\| = \max_{t\in\mathbb{R}} |\cdot|$.

3.3. Using the Lebesgue estimation (2.7), we obtained a convergence result if $f \in \operatorname{Lip}\alpha$ and $E_n(f) = o(n^{-\alpha})$ (see Section 2.6, too). Another, in a sense analogous statement (see the proof), is as follows. Let $f \in CBV$ (f is continuous and of bounded variation). Then (see {138}):

Theorem 3.3. *Let $-1 < \gamma = \max(\alpha, \beta) < 1/2$ be fixed. Then*

$$\lim_{n\to\infty} \big\| L_n^{(\alpha,\beta)}(f) - f \big\| = 0 \quad \textit{if} \quad f \in CBV.$$

The result, in a sense, is the best possible.

Above, $L_n^{(\alpha,\beta)}$ is the Lagrange interpolation (2.2) based on the n roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$ (see [174, Chapter 4]).

3.4. In a series of paper *O. Kis* generalized some convergent processes of *G. Grünwald*, *S. N. Bernstein* and others. Using fairly delicate considerations, he obtained some "best possible" estimates (see {65}).

Let $g \in \widetilde{C}$. Then for a fixed integer $k \ge 0$ the trigonometric polynomials

$$S_{nk}(g, x) := a_0 + \sum_{j=1}^{n-1} (a_j \cos jx + b_j \sin jx) + b_n \sin nx, \qquad n \ge 1$$

are *uniquely determined* by

$$S_{nk}\left(g, \frac{2i-1}{2n}\right) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} g\left(\frac{2i-2j+k-1}{2n}\pi\right), \qquad 1 \le i \le 2n.$$

Let us see some examples.

$$S_{n0}\left(g,\frac{2i-1}{2n}\pi\right)=g\left(\frac{2i-1}{2n}\pi\right),$$

$$S_{n1}\left(g,\frac{2i-1}{2n}\pi\right)=\frac{1}{2}\left\{g\left(\frac{i}{n}\pi\right)+g\left(\frac{i-1}{n}\pi\right)\right\},$$

$$S_{n2}\left(g,\frac{2i-1}{2n}\pi\right)=\frac{1}{4}\left\{g\left(\frac{2i+1}{2n}\pi\right)+2g\left(\frac{2i-1}{2n}\pi\right)+g\left(\frac{2i-3}{2n}\pi\right)\right\},$$

(usual trigonometric interpolation, Grünwald-type process and Bernstein process, respectively).

A natural setting is the investigation of

$$\lambda_{kn}(x):=\sup_{\substack{g\in\widetilde{C}\\ g\neq\text{const}}}\frac{\big|S_{kn}(g,x)-g(x)\big|}{\omega\big(g,\frac{\pi}{2n}\big)},\qquad \Lambda_{kn}:=\big\|\lambda_{kn}(x)\big\|.$$

The results are as follows:

Theorem 3.4. *We have*

$$\Lambda_{0n}=\frac{1}{2}+\frac{1}{2n}\sum_{i=1}^{n}\operatorname{ctg}\frac{2i-1}{4n}\qquad (\textit{H. Ehlich, K. Zeller});$$

$$\Lambda_{1n}=\begin{cases}1+\dfrac{1}{2n}\dfrac{1}{\sin\dfrac{\pi}{2n}}, & n=2,4,6\dots;\\[2ex] 1+\dfrac{1}{2n}\operatorname{ctg}\dfrac{\pi}{2n}, & n=1,3,5\dots;\end{cases}$$

$$\Lambda_{2n}=\frac{5}{4},\qquad n\ge 2;$$

$$\Lambda_{3n}\le\frac{5}{4}+\frac{2}{3\pi},\qquad n=2,4,6\dots;$$

$$\Lambda_{4n}=\frac{23}{16},\qquad n\ge 3.$$

3.5. Remarks. 1. The results in {42}, {61}, {66} say much more than the quoted theorems. The interested reader may consult the original paper or the book of *Dieter Gaier* {51}.

2. The statement of Theorem 3.3 is valid for functions satisfying the so-called one-sided Lip δ conditions. A new theorem deals with nodes corresponding to generalized Jacobi weights (see {16}).

3. There are many applications and generalizations of the idea in {65} including algebraic, de la Vallée Poussin type procedures and saturation problems.

4. This part is restricted to the investigation of *uniform convergence*. There are, of course, many papers dealing with *pointwise* convergence. Most of them use tools closely connected to the theory of the orthogonal polynomials. The interested reader may consult the book of *G. Freud* [47] and the monograph of *Paul Nevai* (*Pál Névai*) {95}.

4. Hermite–Fejér Type and Other Convergent Interpolatory Processes

4.1. "After the discovery of *Faber* [cf. Theorem 2.1] the following question naturally arose. Does there exist a procedure different from Lagrange's interpolation which is efficient for the class C? Immediately after Faber's proof of his theorem *Fejér* discovered that the situation changes if we consider the Hermite interpolation that is the polynomial $\mathcal{H}_n(f, X, x)$ of degree at most $2n-1$ characterized by the properties

$$\left.\begin{aligned} &\mathcal{H}_n(f, X, x_{kn}) = f(x_{kn}), \qquad 1 \le k \le n, \\ &\mathcal{H}_n'(f, X, x_{kn}) = y_{kn}, \qquad 1 \le k \le n. \end{aligned}\right\} \tag{4.1}$$

These polynomials can be written as

$$\mathcal{H}_n(f, X, x) = \sum_{k=1}^{n} f(x_{kn}) h_{kn}(X, x) + \sum_{k=1}^{n} y_{kn} \mathfrak{h}_{kn}(X, x). \tag{4.2}$$

Here the fundamental functions of the first and second type satisfy the conditions

$$h_{kn}^{(i)}(x_{jn}) = \delta_{kj}\delta_{0i},\ \mathfrak{h}_{kn}^{(i)}(x_{jn}) = \delta_{kj}\delta_{1i}\ (1 \le k, j \le n), \quad 0 \le i \le 1.$$

Fejér found the relations

$$h_{kn}(X,x) := v_{kn}(X,x)l_{kn}^2(X,x) \equiv$$

$$\equiv \left(1 - \frac{\omega_n''(X,x_{kn})}{\omega_n'(X,x_{kn})}(x-x_{kn})\right) l_{kn}^2(X,x), \qquad 1 \le k \le n$$

$$\mathfrak{h}_{kn}(X,x) = (x-x_{kn})l_{kn}^2(X,x), \qquad 1 \le k \le n.\text{”}$$

(*P. Turán*, {128, p. 39}.)

In his fundamental paper {41, Theorem XI} *L. Fejér* proved for the matrix $T = \left\{\cos\frac{2k-1}{2n}\pi\right\}$:

Theorem 4.1. *Let $f \in C$. Then*

$$\lim_{n\to\infty} \left\| H_n(f,T,x) - f(x) \right\| = 0 \tag{4.3}$$

Here and later $H_n(f,X,x) = \sum_{k=1}^n f(x_{kn})h_{kn}(X,x)$ (i.e. $y_{kn} = 0$); this is the *classical Hermite–Fejér (HF) step-parabola* reminding us that the tangent lines to H_n at x_{kn} are parallel to the x-axis; if $y_{kn} = f'(x_{kn})$, then we write $\mathcal{H}_n(f,f',X,x)$ (above $1 \le k \le n$).

In 1932 *Gábor Szegő* [174, Theorem 14.6] generalized the previous result:

Theorem 4.2. *Supposing that $-1 < \alpha, \beta < 0$,*

$$\lim_{n\to\infty} \left\| H_n^{(\alpha,\beta)}(f,x) - f(x) \right\| = 0 \qquad \textit{whenever} \qquad f \in C.$$

Moreover, if $\gamma := \max(\alpha,\beta) \ge 0$, the result does not hold.

(Above, $H_n^{(\alpha,\beta)}$ stands for the H_n process based on the roots $x_{kn}^{(\alpha,\beta)}$ $(1 \le k \le n)$ of $P_n^{(\alpha,\beta)}(x)$.)

4.2. Let $\rho \ge 0$. If (the linear function) $v_{kn}(X,x) \ge \rho$ $(1 \le k \le n,\ n \in \mathbb{N},\ x \in [-1,1])$, then X is said to be ρ-*normal* if $\rho > 0$; when $\rho = 0$, X is *normal*. An easy calculation shows that $X^{(\alpha,\beta)} = \left\{x_{kn}^{(\alpha,\beta)}\right\}$ is $\rho = \min(-\alpha,-\beta)$ *normal* if $\alpha, \beta < 0$; the $X^{(0,0)}$ matrix (roots of the Legendre polynomials) forms a *normal* point-system (see [174, §14.5]).

The name ρ-normal (or normal) point-system was coined by *L. Fejér* {46, Part 5}. However, its real significance was revealed by *G. Grünwald* {55} in 1942:

Theorem 4.3. *Let X be ρ-normal. Then*

$$\lim_{n\to\infty} \big\| H_n(f,X,x) - f(x) \big\| = 0 \qquad \textit{if} \qquad f \in C.$$

But, even in mathematics, there is *no* "free lunch": The price of the good convergence behaviour is *the saturation* of the process H_n. In {113} *J. Szabados* proved that $\big\| f - H_n(f,T) \big\| = o(n^{-1})$ iff $f = \text{const}$ (at the same time, he gives the saturation class, too); a more general result of *Y. G. Shi* {104} says the following:

Let $f_k(x) = x^k$. Then we have

Theorem 4.4. *For an arbitrary interpolatory X*

$$\max_{k=1,2} \Big\{ \big\| H_n(f_k,X,x) - f_k(x) \big\| \Big\} \neq o(n^{-1}).$$

4.3. The next natural problem was raised in {128, Problems XIX, XX}. Do the Hermite–Fejér step-parabolas have a rough convergence theory? (cf. Part 2.6). Now, if we use $\Lambda_{n2}(X) := \big\| \sum_{k=1}^{n} \big| h_{kn}(X,x) \big| \big\|$, the next surprising result can be proved (see *P. Vértesi* {111, Corollary 6.18}).

Theorem 4.5. *Using $\Lambda_{n2}(X)$ and $\operatorname{Lip}\alpha$, there is no rough convergence theory for the Hermite–Fejér step-parabolas either on the whole interval $[-1,1]$ or on a closed subinterval $[a,b]$.*

It is worthwhile to compare this result with Theorems 2.8 and 2.9.

4.4. *Fejér*'s result (4.3) shows that if the degree of the interpolation polynomial is about two times bigger than the number of interpolation points, then we can get convergence. *Erdős* raised the following question. Given $\varepsilon > 0$, suppose we interpolate at n nodes, but allow polynomials of degree at most $n(1+\varepsilon)$. Under what conditions will they converge for all continuous function?

The first answer was given by himself in {35}. Namely, he proved:

Theorem 4.6. *If the absolute values of the fundamental polynomials $\ell_{kn}(X,x)$ are uniformly bounded in $x \in [-1,1]$, k $(1 \le k \le n)$ and $n \in \mathbb{N}$, then for every $\varepsilon > 0$ and $f \in C$ there exists a sequence of polynomials $\varphi_n = \varphi_n(x) = \varphi_n(f,\varepsilon,x)$ with*

(i) $\deg \varphi_n \le n(1+\varepsilon)$,

(ii) $\varphi_n(x_{kn}) = f(x_{kn})$, $1 \le k \le n$, $n \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} \|\varphi_n - f\| = 0$.

The answer for a more general system was given in the same paper and {32}.

The story is typically Erdősian. In {35}, *Erdős* stated an answer to the above problem, but instead of proving it, he just gave an indication that "the proof is a simple modification of Theorem 3". After some 45 years, as a result of the joint effort of *Erdős*, *András Kroó* and *Szabados*, the original statement concerning the above problem was completed, even in a slightly stronger form. The result is the following {32}:

Theorem 4.7. *For every $f \in C$ and $\varepsilon > 0$, there exists a sequence of polynomials $p_n(f)$ of degree at most $n(1+\varepsilon)$ such that*

$$p_n(f, x_{k,n}) = f(x_{k,n}), \qquad 1 \le k \le n,$$

and that

$$\big\| f - p_n(f) \big\| \le cE_{[n(1+\varepsilon)]}(f)$$

holds for some $c > 0$, if and only if

$$\limsup_{n\to\infty} \frac{N_n(I_n)}{n|I_n|} \le \frac{1}{\pi} \tag{4.4}$$

whenever I_n is a sequence of subintervals of I such that $\lim\limits_{n\to\infty} n|I_n| = \infty$ and

$$\lim_{n\to\infty} \Big\{ n \min_{1\le k\le n-1} (\vartheta_{k+1,n} - \vartheta_{n,k}) \Big\} > 0. \tag{4.5}$$

Here $N_n(I_n)$ is the number of the $\vartheta_{k,n}$ in $I_n \subset I$. Condition (4.4) ensures that the nodes are not too dense, and condition (4.5) says that adjacent nodes should not be too close.

4.5. Remarks. 1. First we call the reader's attention to the comprehensive bibliography on HF interpolation compiled by *H. H. Gonska* and *H-B. Knoop* {52} containing about 400 entries from the period 1914–1987. (Of course, dozens of new papers were (and will be) written after 1987.)

2. It is appropriate to make some historical remarks. In his paper {58} *Dunham Jackson* considered the discrete analogy of the famous Fejér means and proved that $J_n(g, t_{kn}) = g(t_{kn})$, $t_{kn} = 2k\pi/n$, $1 \le k \le n$, moreover

$$\lim_{n\to\infty} \big\| J_n(g) - g \big\| = 0 \qquad \text{for every} \qquad g \in \widetilde{C},$$

where

$$J_n(g,t) = \sum_{k=1}^{n} g(t_{kn}) \left(\frac{\sin n \dfrac{t - t_{kn}}{2}}{n \sin \dfrac{t - t_{kn}}{2}} \right)^2$$

(today they are called "Jackson polynomials").

As we know, *Bernstein* and *Fejér* {41, (85)} were the first to point out the property $J_n'(g, t_{kn}) = 0$, $1 \le k \le n$. For other details, see {52, p. 148} and {143, p. 21}.

3. The almost unbelievable popularity of the HF interpolation lies at least in 3 facts:

— simple form,

— easy to compute and (last but not least),

— it serves in many textbooks as a transparent proof of the Weierstrass approximation theorem.

4. The behaviour of $H_n^{(\alpha,\beta)}$ near at the endpoints ± 1 if $\max(\alpha, \beta) \ge 0$, was first investigated by *L. Fejér* {44}, {43}. Actually, in the previous, Hungarian version of {44}, he considered the Legendre roots only (i.e. when $\alpha = \beta = 0$), where the uniform convergence for the *whole interval* was ensured by the additional conditions $f(1) = f(-1) = \frac{1}{2}\int_{-1}^{1} f(x)\,dx$ (see {43}). A solution for arbitrary $\max(\alpha, \beta) \ge 0$ is given in *Szabados* {112}, {113}.

Here we quote a simple special case:

If $f' \in C$, *then* $\lim_{n\to\infty} \left\| H_n^{(1/2,1/2)}(f) - f \right\| = 0$ *whenever*

$$\int_{-1}^{1} \frac{x f(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \frac{(2x-1) f(x)}{\sqrt{1-x^2}}\,dx = 0.$$

Another approach is given by *Vértesi* {132} and *P. Nevai*, *P. Vértesi* {89}. Namely,

if $\alpha \ge 0$, $\beta > -1$, $f \in C$, *then*

$$\lim_{n\to\infty} \left\| H_n^{(\alpha,\beta)}(f) - f \right\|_{[-1+\varepsilon,1]} = 0 \quad (0 < \varepsilon < 2) \quad \textit{iff}$$

$$\begin{cases} \lim\limits_{n\to\infty} H_n^{(\alpha,\beta)}(f,1) = f(1) & \textit{and} \quad (\textit{if } \alpha \ge 1) \\ \left(H_n^{(\alpha,\beta)}(f,x)\right)_{x=1}^{(r)} = o(n^{2r}), & r = 1, 2, \ldots, [\alpha]. \end{cases}$$

For other details see {111, Ch. V/3}.

5. The previous theorem was proved using the idea of the so called quasi-Hermite–Fejér interpolation (qHFi): In their paper {17} *Jenő Egerváry* and *P. Turán* observed that if the HF step-parabolas are replaced by the polynomials of degree $2n+1$ (sic!) taking the values and zero derivatives at the Legendre nodes *and the values of the function* at ± 1, then the convergence of this so called qHF polynomials *becomes uniform in* $[-1,1]$ ($f \in C$)! I.e., adding two points with multiplicity one, we improve the convergence behaviour. This idea has many natural generalization (see the papers of *A. Schönhage*, *G. Freud* and others in {111, p. 199–200}.

6. Another generalization is the so called HF-type interpolation. Let $m \in \mathbb{N}$, $X \subset [-1,1]$ be given. If $f \in C$, then $I_{nm}(f,X,x) \in \mathcal{P}_{nm-1}$ is defined by

$$I_{nm}^{(t)}(f,X,x_{kn}) = f(x_{kn})\delta_{0t}, \qquad 1 \le k \le n,\ 0 \le t \le m-1.$$

(a) *If m is odd,* the processes will be denoted by L_{nm} (obviously $L_{n1} \equiv L_n$ (Lagrange interpolation)). As it turns out, they behave similarly to the Lagrange interpolation: One can prove a Faber type result if $n \to \infty$ (m is fixed); the corresponding Lebesgue constant, $\Lambda_{nm}(X) \ge c \log n$ for any X (see *Szabados* {116}; actually in this sophisticated paper the exact lower bound for other fundamental functions are given, too); the Lebesgue function $\lambda_{nm}(X,x) \ge c \log n$ on a "big" set (*Vértesi* {133}). Results on $I_{nm}(f,T,x)$ are in the papers *Terry M. Mills*, *P. Vértesi* {80} and *Simon J. Smith* {105}.

(b) *If m is even,* we use the notation H_{nm} (clearly, $H_{n2} = H_n$ (HF interpolation)) because the behaviour is similar to the HF process. Here is a convergence result: Using obvious notations, one can prove that *the following statements are equivalent.*

  (i) $\lim_{n\to\infty} \left\| H_{nm}^{(\alpha,\beta)}(f) - f \right\| = 0 \qquad \forall f \in C$,

  (ii) $-\frac{1}{2} - \frac{2}{m} \le \alpha, \beta < -\frac{1}{2} + \frac{1}{m}$ *and* $|\alpha - \beta| \le \frac{2}{m}$.

(see the survey paper *P. Vértesi* {135} and its references).

7. Let us say some words about *Grünwald*'s celebrated results (Theorem 4.3). First, today it may be proved using the result of *Pavel P. Korovkin* on positive linear operators (obviously, if X is ρ-normal, then $H_n(f,X)$ is a positive linear operator; cf. {70}). Secondly, as it turned out from the paper *János Balázs* {5}, the ρ-normality *is not necessary* to the good behaviour of $H_n(f,x)$ (cf. {44} and {131}).

8. There is a wide variety of the form of the error estimations. We refer to {111, Ch. V/2 Corr. 7.16}; another interesting question is the comparison of the process $L_n(f,x)$ and $H_n(f,X)$ (cf. {111, Ch. VI.}).

9. Around 1960, *Paul Butzer* raised the problem of proving the Jackson theorem by interpolatory processes. Many interesting papers were written by *G. Freud*, *O. Kis*, *M. Sallay* and others (see {111, Ch. II./ 5,6}).

10. After 1975, following a short paper of *J. Balázs*, the mathematical world rediscovered the simple but efficient rational interpolatory Shepard type operators. In the last 25 years dozens of papers were completed. Here we quote a nice result of the very talented young Hungarian mathematician *Gábor Somorjai**:

Let $f \in C[0,1]$, $\alpha > 2$ *real, and* let

$$S_n(f,x) := \frac{\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\left|x-\frac{k}{n}\right|^{-\alpha}}{\sum_{k=0}^{n}\left|x-\frac{k}{n}\right|^{-\alpha}}.$$

We have

(i) $\left\| S_n(f)-f \right\|_{[0,1]} = o\left(\frac{1}{n}\right)$ *iff* $f = \text{const}$,

(ii) $\left\| S_n(f)-f \right\|_{[0,1]} = O\left(\frac{1}{n}\right)$ *iff* $f \in \text{Lip}\,1$.

(Cf. {106}; other relevant results are in the survey paper of *Bianca Della Vecchia* {15} and in {140}.)

5. Lacunary or Birkhoff Interpolation

5.1. In the classical Hermite interpolation we prescribe the consecutive derivatives of the interpolatory polynomials. Dropping this, we arrive at *lacunary* interpolation.

"While polynomials of the previous kind always exist, in Birkhoff's case, polynomials satisfying his conditions do not necessarily exist. Hence, we have the basic question:

*He died at the age 26, in 1978.

(a) existence,
(b) uniqueness,
(c) possibly, explicit representation,
(d) convergence,
(e) applications.”

(*P. Turán* {128, p. 48}).

P. Turán and his collaborators, *János Surányi*, *J. Balázs* then *G. Freud* in a series of papers investigated the so-called $(0,2)$ interpolation on the roots of $P_n^{(-1,-1)}(x)$ (see {3}, {4}, {50} and {108}).

It turned out that for $n =$*even*, the $(0,2)$ interpolatory polynomial $R_n(f,x) = \sum_{k=1}^{n} f(x_{kn}) r_{kn}(x) \in \mathcal{P}_{2n-1}$ satisfying

$$R_n(f,x_{kn}) = f(x_{kn}), \qquad R_n''(f,x_{kn}) = 0, \qquad 1 \le k \le n,$$

is uniquely defined ($f \in C$, the x_{kn} are the roots of $P_n^{(-1,-1)}$), moreover

Theorem 5.1. *If* $\omega_2(f,t) = o(t)$, *then*

$$\lim_{n\to\infty} \big\| R_n(f,x) - f(x) \big\| = 0, \qquad n = 2,4,6\dots.$$

Furthermore, one can see that the condition $\omega_2(f,t) = o(t)$ is the best possible using that the corresponding $(0,2)$-Lebesgue constants satisfy $\big\| \sum_{k=1}^{n} \big| r_{kn}(x) \big| \big\| \ge cn$.

5.2. The theory of lacunary interpolation became very popular (again) not only in Hungary, but everywhere on the world:

This popularity resulted in the monograph of *G. G. Lorentz*, *Kurt Jetter* and *Sherman D. RiemenSchneider* {75}.

During the years many questions concerning the existence of the lacunary polynomials were answered. Among them we mention the relatively new paper of *A. A. Chak*, *A. Sharma* and *J. Szabados* {12}, solving the existence, uniqueness and representation on the roots of $P_n^{(\alpha,\beta)}(x)$, $\alpha,\beta \ge -1$.

5.3. The investigation of the $(0,m)$ *trigonometric* interpolation on equidistant nodes ($m \ge 2$) was initiated by *O. Kis* {67}. While *O. Kis* investigated the case $m = 2$, soon after, *A. Sharma* and *Arun K. Varma* settled the other values of m ({102}).

Their significant achievements were the simple explicit forms of the fundamental polynomials. Using these formulas and *J. Szabados* {111, Theorem 7.12}, we state the following:

Let $t_{kn} = \frac{2k\pi}{n}$, $k = 0, \pm 1, \pm 2, \ldots$. We are looking for a trigonometric polynomial $T_n(g, (0,m), t) \in \mathcal{T}_{n-1}$ ($g \in \widetilde{C}$, say) satisfying

$$T_n\big(g, (0,m), t_{kn}\big) = g(t_{kn}) \qquad T_n^{(m)}\big(g, (0,m), t_{kn}\big) = 0 \qquad k = 0, \pm 1, \ldots$$

Theorem 5.2. *The above $(0,m)$ problem is uniquely solvable iff*

(i) *m=odd and n is arbitrary; or*

(ii) *m =even and n =odd.*

Moreover,

$$\big\| g(t) - T_n\big(g, (0,m), t\big) \big\| \le \frac{c}{n^m} \sum_{k=0}^{n} \frac{\widetilde{E}_k(g)}{(k+1)^{1-m}}$$

$$+ c\big\{1 + (-1)^m\big\}^n \widetilde{E}_{[\frac{n}{4}]}(g).$$

In general the above conditions are also necessary (see {111, p. 252}*).*

Above, $\tilde{E}_n(g) := \min_{\tau \in \mathcal{T}_n} \|g - \tau\|$.

Let us note that when m is *odd* the second term does not appear and since the first term tends to 0 (as $n \to \infty$) the procedure is convergent for all $g \in \widetilde{C}$. When m is *even*, the condition, $\omega_2\big(g, \frac{1}{n}\big) = o\big(\frac{1}{n}\big)$ ensures uniform convergence.

5.4. As *Turán* suggested to him, *O. Kis* investigated the $(0,2)$ complex interpolation *at the roots of unity*. It turned out that existence and unicity always hold. Moreover, he proved the following (cf. {68}):

Theorem 5.3. *The corresponding Lebesgue constants (operator-norm) has the exact order* $\log n$.

The result is in striking contrast to Theorems 5.1 and 5.2, where the exact orders were n.*

The result was generalized by *A. S. Cavaretta*, *A. Sharma* and *R. S. Varga* {11} proving that in any case the $(0, m_1, m_2, \ldots, m_q)$ interpolation

*As *O. Kis* frequently told us, first *Turán* did not believe him: they sat down and *Turán* checked every small detail. And after some busy hours *Turán* was convinced…

on the roots of unity exists uniquely ($n \geq qm_q$) and the exact order of the corresponding operator norm is again $\log n$.

5.5. The next nice and surprising statement of *G. Somorjai* {107} shows that the above order $\log n$ is optimal.

Let $\Gamma = \{z : |z| = 1\}$ be the unit circle and let $C(T)$ be the Banach space of continuous functions on Γ endowed with the supremum norm $\|\cdot\|$. The closed subspace $AC \subset C(\Gamma)$ consists of the restriction to $C(\Gamma)$ of those functions which are analytic in $|z| < 1$ and continuous on $|z| \leq 1$. $B(C, A)$ will denote the space of bounded linear operators mapping $C(\Gamma)$ into AC endowed with the usual norm $\|\mathcal{L}\|_C = \sup_{f \in C(\Gamma), \|f\| \leq 1} \|\mathcal{L}(f, z)\|$ $(\mathcal{L} \in B(C, A))$. We shall say that *the operator $\mathcal{L} \in B(C, A)$ is determined on the set* $H \subset \Gamma$ if $f \in C(\Gamma)$, $f|_H \equiv 0$ (i.e., the restriction of f to H is identically zero) implies $\mathcal{L}(f, z) \equiv 0$.

Theorem 5.4. *Let $H_n \subset \Gamma$ be closed sets of angular Lebesgue measure zero, and suppose that $\mathcal{L}_n \in B(C, A)$ are determined on H_n $(n = 1, 2, \dots)$. Then there is an $f \in AC$ for which*

$$\overline{\lim_{n \to \infty}} \|f(z) - \mathcal{L}_n(f, z)\| > 0.$$

In particular whatever is the set $\{z_{kn}\}_{k=1}^n \subset \Gamma$, *there do not exist discrete linear operators of the form*

$$\mathcal{L}_n(f, z) = \sum_{k=1}^{n} f(z_{kn}) a_{kn}(z), \qquad a_{kn}(z) \in AC, \qquad k = 1, \dots, n,$$

which would uniformly converge to every $f \in AC$.

5.6. In 1975 L. Pál {98} investigated a special Birkhoff interpolation which today called as *Pál-type interpolation.* His main idea was to prescribe the derivatives of the corresponding interpolation at points which are *different* from the nodes where the function values were given.

These simple idea was very fruitful in many cases. After getting the first convergence result in 1983 L. Szili {124}, in the last 20 years or so more than 50 papers have been written in this topic. A part of them consider regularity problem on quite general point-system,the other ones prove convergence theorems using special nodes-system.

While generally the “Lebesgue constants” tend to infinity,in their paper J. Szabados and A. K. Varma {118} obtained a process which converge for *arbitrary* continuous function.

Theorem 5.5. *For $n = 1, \ldots, n$ let x_{kn} $(k = 1, \ldots, n)$ resp. x^*_{jn} $(j = 1, \ldots, n-1)$ be the roots of $P_n^{(-1,-1)}$ resp. $P_n^{(1,1)}$. If $f \in C[-1,1]$ then there exists a unique polinomial $R_n(f, \cdot)$ of degree $\leq 2n$ which satisfies*

$$R_n(f, x_{kn}) = f(x_{kn}) \quad (k = 1, \ldots, n),$$

$$R'_n(f, \pm) = R''_n(f, x^*_{jn}) = 0 \quad (j = 1, \ldots, n-1),$$

moreover we have

$$\lim_{n\to\infty} \big\| R_n(f,x) - f(x) \big\| = 0.$$

Other interesting results are in the short survey paper {119}.

5.7. Remarks. 1. If somebody uses a more systematic and detailed treatment of lacunary interpolation and tries to dig to the roots, the name of *György Pólya* must be mentioned (see the book {11}). Also, {111, Ch. VII} is a good source of some further results and proofs.

2. The $T_n\big((0,m)\big)$ process is *saturated with the order* n^{-m} (see {111, Theorem 7.14 and 7.15}).

3. One can consider $(0,2)$ interpolation on the infinite interval or weighted $(0,2)$ polynomials ({111, p. 234} and *J. Balázs* {11}).

6. On the Mean Convergence of Interpolation

6.1. The negative results in Part 2 (cf. Theorems 2.1, 2.5 and Part 2.5) motivate the fact that the attention turned to the *mean convergence* of interpolation. The first such result is due to *P. Erdős* and *P. Turán* {25} from 1937.

Theorem 6.1. *For an arbitrary weight w and $f \in C$,*

$$\lim_{n\to\infty} \int_{-1}^{1} \big\{ L_n(f,w,x) - f(x) \big\}^2 w(x)\, dx = 0. \tag{6.1}$$

Here and later w is a weight if $w \ge 0$ and $0 < \int_{-1}^{1} w < \infty$; $L_n(f,w)$ is the Lagrange interpolation with nodes at on the roots of the corresponding orthogonal polynomials (ONP) $p_n(w)$ (see [47] or [174]).

Using the Čebishov roots, *P. Erdős* and *Ervin Feldheim* proved much more {19}:

Theorem 6.2. *Let $f \in C$ and $p > 1$. Then*

$$\lim_{n=\to\infty} \int_{-1}^{1} \left| f(x) - L_n(f,T,x) \right|^p \frac{1}{\sqrt{1-x^2}}\, dx = 0.$$

6.2. However, as *E. Feldheim* {48} showed, one can have divergence type results in the L_p-metrics, too:

Namely, *for a suitable* $f_1 \in C$,

$$\overline{\lim_{n\to\infty}} \int_{-1}^{1} \left| L_n\big(f_1, X^{(1/2,1/2)}, x\big) - f_1(x) \right|^4 \sqrt{1-x^2}\, dx > 0. \tag{6.2}$$

The results (6.1) and (6.2), justify the problem of *Erdős*, *Turán* and *Freud* (see {25}, [47]): Investigate the expression

$$\left\| f - L_n(f,w) \right\|_{p,u} := \int_{-1}^{1} \left| f(x) - L_n(f,w,x) \right|^p u(x) dx.$$

Here $p > 0$, u and w are weights.

After the initial results of *Richard Askey* and *V. M. Badkov*, *P. Nevai* {94} proved the next fairly general

Theorem 6.3. *Assume that $w \in GJ$, $0 < p < \infty$, $u \ge 0$, $0 < \int_{-1}^{1} u(x) \log^+ u(x) dx < \infty$. Then*

$$\lim_{n\to\infty} \left\| \left| f - L_n(f,w) \right| u \right\|_p = 0 \qquad \forall f \in C$$

iff

$$\frac{u(x)}{\sqrt{w(x)\sqrt{1-x^2}}} \in L_p.$$

Here $w \in GJ$ if $w(x) = \prod_{k=0}^{v+1} |x - t_k|^{\Gamma_k}$, $\Gamma_k > -1$, $0 \le k \le v+1$ and $-1 \equiv t_{v+1} < t_v < \cdots < t_1 < t_0 \equiv 1$ are fixed.

6.3. Let us say some words about the proof. First we mention a result of *J. Marcinkiewicz* {82} on the trigonometric case. Namely, using the so called Marcinkiewicz–Zygmund inequalities

$$\text{(i)}\quad \left(\frac{2\pi}{2n+1}\sum_{k=0}^{2n}\left|T_n(t_{kn})\right|^p\right)^{1/p} \le c\|T_n\|_p, \quad 1 \le p < \infty,$$

$$\text{(ii)}\quad \|T_n\|_p \le c_p\left(\frac{2\pi}{2n+1}\sum_{k=0}^{2n}\left|T_n(t_{kn})\right|^p\right)^{1/p},\ 1 < p < \infty$$

$\left(t_{kn} = \frac{2k\pi}{2n+1}, T_n \in \mathcal{T}_n\right)$, *one can prove that* for $p > 1$ $\lim_{n\to\infty}\left\|I_n(g) - g\right\|_p = 0$ *for every* $g \in C$.

Here I_n is the trigonometric interpolatory polynomial based on $\{t_{kn}\}$, $0 \le k \le 2n$ (compare with Theorem 6.2).

So if we try to prove an "algebraic" mean convergence result, first, as did *Richard Askey*, we may try to find the analogue of (i) and (ii). The method works if both u and $w \in GJ$. On the other hand, for an *arbitrary weight* u, *Nevai*, using a different approach, considered Lagrange interpolation as a mapping from the space of bounded functions into the appropriate weighted L^p spaces (and *not* as a mapping from L^p into L^p). Using this philosophy and some delicate arguments, he proved the above result.

6.4. Theorem 6.1 is a reasonable motivation of the problem raised by *Turán* {128, Problem VIII}:

Does there exists a weight w *and* $f \in C$ *such that*

$$\overline{\lim_{n\to\infty}}\left\|f - L_n(f,w)\right\|_{p,w} = \infty$$

for every $p > 2$?

The "yes" answer was conjectured by *R. Askey*, however the rather complicated proof solving a more general problem is due to *P. Nevai* from 1985 (see {1} and {97}).

Nevai's proof requires a lot of difficult and far-reaching statements on orthogonal polynomials. But as it turned out from a paper of *Y. G. Shi*, using a new approach, many considerations can be saved and at the same time more general results can be obtained. Among others he proves in {103, Corr. 14}

Theorem 6.4. *Let u and w be weights. If with a fixed $p_0 \geq 2$*

$$\left\| \frac{1}{\sqrt{w\sqrt{1-x^2}}} \right\|_{p,u} = \infty \qquad \text{for every} \qquad p > p_0,$$

then there exists an $f \in C$ satisfying

$$\overline{\lim_{n\to\infty}} \left\| L_n(f,w) \right\|_{p,u} = \infty \qquad \text{whenever} \qquad p > p_0.$$

6.5. Because of the good uniform convergence behaviour of the HFi, the investigation of its mean convergence is relatively new. In 1985, *Nevai* and *Vértesi* proved ({90})

Theorem 6.5. *Let $u, w \in J$. Then*

$$\lim_{n\to\infty} \left\| H_n(f,w) - f \right\|_{p,u} = 0 \qquad \text{for all} \qquad f \in C \tag{6.3}$$

iff $w^{-p}u \in L_1$.

Let $u = w = 1$. By (6.3), $\int_{-1}^{1} \left| H_n^{(0,0)}(f) - f \right| dx \to 0$ for all $f \in C$; on the other hand if $f_1(x) = 1 - x$, then $\lim_{n\to\infty} \left\| f_1 - H_n^{(0,0)}(f_1) \right\| > 0$ ([174, (14.6.17)]), i.e. mean convergence *may improve the convergence behaviour of HFi, too.*

6.6. Remarks. 1. In paper {85} "very general" Jacobi weights have been considered and statements connected to Theorem 6.3 have been proved.

2. The main formal distinction between Theorem 6.4 and *Nevai*'s result in {97} is that *Nevai* must suppose $\log w(\cos t) \in L_1$; however *the ideas of the proofs are totally different:*

Shi uses some ideas of the *Erdős-Vértesi*'s paper, where $\lambda_n(X,x)$ was estimated ({31}, {134}). For other related results., see {84} and {141}.

3. Finally we mention two results: the first is due to *J. Prasad* and *A. K. Varma* {99}. *We have with $w(x) = 1/\sqrt{1-x^2}$*

$$\left\| H_n(f,w) - f \right\|_{p,w} \leq c\omega\left(f, \frac{1}{n}\right) \qquad \text{if} \qquad f \in C;$$

i.e., the result is better than the well-known uniform estimation

$$\| H_n(f_1, w) - f_1 \| \le c \frac{\log n}{n}, \qquad f_1(x) = |x|.$$

The second one is due to *G. Mastroianni* and *P. Nevai* {83}, Theorem 3.2.

Let $u, w \in GJ$, $r \ge 0$, $0 < p < \infty$. *If* $X(w, r)$ *is an interpolatory matrix obtained by adding an appropriate number of points* to $X(w)$ *near* ± 1, *then under certain conditions*

$$\| f^{(l)} - L_n^{(l)} (f, X(w, r)) \|_{p,u} = O(n^{l-r}) \omega \left(f^{(r)}, \frac{1}{n} \right), \qquad 0 \le l \le r,$$

whenever $f^{(r)} \in C$.

I.e., mean convergence *eliminates the "*$\log n$*" factor in the above two theorems!*

The proof of the last statement heavily uses the additional points method.

4. For other related results you may consult {110}.

B
WEIGHTED CASE

7. Weighted Lagrange Interpolation, Weighted Lebesgue Function, Weighted Lebesgue Constant

7.1. Let f be a continuous function, say. If, instead of the interval $[-1,1]$, we try to approximate it on $\mathbb{R}=(-\infty,\infty)$, we have to deal with the obvious fact that polynomials (of degree ≥ 1) tend to infinity if $|x|\to\infty$. So to get a suitable approximation tool, we may try to *moderate their growth applying proper weights.*

If the weight $w(x)=e^{-Q(x)}$, $x\in\mathbb{R}$, satisfies

$$\lim_{|x|=\infty}\frac{Q(x)}{\log|x|}=\infty,$$

as well as some other mild restrictions and the Akhiezer–Babenko–Carleson–Dzrbasjan relation

$$\int_{-\infty}^{\infty}\frac{Q(x)}{1+x^2}\,dx=\infty,$$

then for $f\in C(w,\mathbb{R})$, where

$$C(w,\mathbb{R}):=\Big\{f;\ f \text{ is continuous on } \mathbb{R} \text{ and } \lim_{|x|\to\infty}f(x)w(x)=0\Big\},$$

we have, if $\|\cdot\|$ denotes now the supnorm on $\mathbb{R}$,

$$E_n(f,w):=\inf_{p\in\mathcal{P}_n}\big\|(f-p)w\big\|\equiv\inf_{p\in\mathcal{P}_n}\|fw-pw\|\to 0\qquad\text{as } n\to\infty.$$

So, instead of approximating $f\in C$ by $L_n(f,X)$ on $[-1,1]$, *we may estimate* $\big\{f(x)w(x)-L_n(f,w,X,x)\big\}$ *on the real line* $\mathbb{R}$ *for* $f\in C(w,\mathbb{R})$. Here $X\subset\mathbb{R}$,

$$t_k(x):=t_{kn}(w,X,x):=\frac{w(x)\omega_n(X,x)}{w(x_k)\omega_n'(X,x_k)(x-x_k)},\qquad 1\leq k\leq n,$$

and

$$L_n(f,w,X,x) := \sum_{k=1}^{n} \big\{ f(x_k)w(x_k) \big\} t_k(x), \qquad n \in \mathbb{N}.$$

The Lebesgue estimate now has the form

$$\big| L_n(f,w,X,x) - f(x)w(x) \big| \le \big\{ \lambda_n(w,X,x) + 1 \big\} E_{n-1}(f,w) \tag{7.1}$$

where the *(weighted) Lebesgue function* is defined by

$$\lambda_n(w,X,x) := \sum_{k=1}^{n} \big| t_k(w,X,x) \big|, \ x \in \mathbb{R}, \ n \in \mathbb{N} \tag{7.2}$$

(cf. Part 2.1); the existence of $r_{n-1}(f,w)$ for which $E_{n-1}(f,w) = \big\| (f - r_{n-1})w \big\|$ is well-known.

Formula (7.2) implies the natural definition of the *(weighted) Lebesgue constant*

$$\Lambda_n(w,X) := \big\| \lambda_n(w,X,x) \big\|, \qquad n \in \mathbb{N}. \tag{7.3}$$

Estimation (7.1) and its immediate consequence

$$\big\| L_n(f,w,X) - fw \big\| \le \big\{ \Lambda_n(w,X) + 1 \big\} E_{n-1}(f,w), \qquad n \in \mathbb{N},$$

show that, analogous to the classical case, the investigation of $\lambda_n(w,X,x)$ and $\Lambda_n(w,X)$ is of fundamental importance to get convergence-divergence results for the weighted Lagrange interpolation.

To expect reasonable estimations, as it turns out, we need a considerable knowledge about the weight $w(x)$ and on the behaviour of the ONP $p_n(w^2,x)$ corresponding to the weight w^2.

7.2. As *Nevai* writes in his instructive monograph {92, Part 4.15}, about 40 years ago there was a great amount of information on orthogonal polynomials on infinite intervals, however as *G. Freud* realized in the sixties, there had been a complete lack of *systematic treatment* of the general theory; the results were of mostly ad hoc nature. And *G. Freud*, in the last 10 years of his life, laid down the basic tools of the systematic investigation.

During the years a great number from the approximators and/or orthogonalists joined *G. Freud* and his work, including many Hungarians. As a result, today our knowledge is more comprehensive and more solid than before. On the other hand, this branch is relatively young; a lot of new, exciting problems remain to be solved!

Let us return to *Freud*'s work. His first natural step was taking the Hermite polynomials as a prototype of ONP with weights whose support is noncompact. Later, he introduced $Q(x)$ (instead of the Hermitian $x^2/2$) and q_n. (It corresponds to the MRS number — see 7.3.)

Nowadays these "Hermite type" weights bear *Freud*'s name. To be more precise, here is a quite general definition of the so called *Freud-type weights.*

We say that $w(x) = e^{-Q(x)} \in \mathcal{F}$ if $Q : \mathbb{R} \to \mathbb{R}$ is even and differentiable in $\mathbb{R}$, Q'' is continuous in $(0,\infty)$, $Q' > 0$ in $(0,\infty)$ and for some $A, B > 1$

$$A \leq \frac{\big(xQ'(x)\big)'}{Q'(x)} \leq B, \qquad x \in (0,\infty).$$

Clearly, if $w \in \mathcal{F}$ then $w^2 \in \mathcal{F}$, too (see *A. L. Levin, Doron S. Lubinsky* {137}, say).

The simplest cases are the so-called *Freud weights* $w(x) = e^{-|x|^\alpha}$, $\alpha > 1$. Here $w \in \mathcal{F}$ with $A = B = \alpha$.

Let $w \in \mathcal{F}$ and denote by $\big\{y_k = y_{kn}(w^2)\big\}_{k=1}^{n}$ the n different roots of the ONP $\big\{p_n(w^2,x)\big\}_{n=0}^{\infty}$ (with respect to the weight $w^2 \in \mathcal{F}$). We index them in decreasing order as

$$-\infty < y_{nn} < y_{n-1,n} < \ldots < y_{2n} < y_{1n} < \infty. \tag{7.4}$$

If $w \in \mathcal{F}$

$$\Lambda_n\big(w, Y(w^2)\big) \sim n^{1/6} \tag{7.5}$$

where $Y(w^2) = \{y_{kn}(w^2);\ 1 \leq k \leq n,\ n \in \mathbb{N}\}$ (see *D. M. Matjila* and *J. Szabados* {86}, {117}).

However, one can do better (see {117, Theorem 1}).

Applying the "additional points method", *J. Szabados* {117, Theorem 1} improved (7.5) as follows.

Let $y_0 = y_{0n} > 0$ denote a point such that

$$\big|p_n(w^2,y_0)w(y_0)\big| = \big\|p_n(w^2)w\big\|.$$

Now if

$$V(w^2) = \Big\{\big\{y_{kn}(w^2), 1 \leq k \leq n\big\} \cup \{y_{0n}, -y_{0n}\}, n \in \mathbb{N}\Big\}, \tag{7.6}$$

one can prove the following:

Let $w \in \mathcal{F}$*. Then*

$$\Lambda_n\big(w, V(w^2)\big) \sim \log n. \tag{7.7}$$

7.3. To make the choice of $\pm y_0$ more clear, we introduce the so–called *Mhaskar–Rahmanov–Saff* (MRS) *number.* From now on let $I = (-1, 1)$ or $\mathbb{R}$ and $w = e^{-Q}$ where $Q : I \to \mathbb{R}$ is even and convex in I and has limit ∞ at the endpoints of I.

Let $u > 0$ be fixed; then the (unique) a satisfying

$$u = \frac{2}{\pi} \int_0^1 \frac{atQ'(at)}{\sqrt{1-t^2}}\, dt \tag{7.8}$$

is by definition the MRS *number.* It is denoted by $a_u(w)$. A very important and useful property of $a_n(w)$ is that

$$\begin{cases} \|r_n w\| = \max_{|x| \le a_n(w)} \big| r_n(x) w(x) \big|, \\ \|r_n w\| > \big| r_n(x) w(x) \big| \quad \text{for } |x| > a_n(w) \end{cases} \tag{7.9}$$

if $r_n \in \mathcal{P}_n$ $(r_n \not\equiv 0;\ \|\cdot\|$ is the supnorm on $I)$ and that asymptotically (as $n \to \infty$) $a_n(w)$ is the smallest such number. Relation (7.9) may be formulated such that $r_n w$ "lives" on $[-a_n, a_n]$.

As *an example*, let $Q(x) = |x|^\alpha$. Then

$$a_n(w) = c(\alpha)\, n^{1/\alpha}, \qquad \alpha > 1,$$

and as one can see, y_{1n} and y_{0n} are "close" to a_n: namely, $|y_{0n} - y_{1n}| \le c\frac{a_n(w)}{n^{2/3}}$.

7.4. A very natural question is whether the order $\log n$ in (7.7) is optimal (see the Faber result (2.8)). *J. Szabados* {117, Theorem 2} verified this hint for the special Hermite weight $w(x) = e^{-x^2/2}$ (actually, he proved it also for other projection operators).

Generalizing the method and ideas of our common paper with *Erdős*, one can prove a statement on the weighted Lebesgue function $\lambda_n(w, X, x)$ (see *P. Vértesi* {136}).

Theorem 7.1. *Let $w \in \mathcal{F}$. If $\varepsilon > 0$ is an arbitrary fixed number, then for any interpolatory matrix $X \subset \mathbb{R}$ there exist sets $H_n = H_n(w, \varepsilon, X)$ with $|H_n| \le 2a_n(w)\varepsilon$ such that*

$$\lambda_n(w, X, x) \ge \frac{\varepsilon}{3840} \log n \tag{7.10}$$

if $x \in \big[-a_n(w), a_n(w)\big] \setminus H_n,\ n \ge n_1(\varepsilon)$.

This statement is a complete analogue of Theorem 2.4. Roughly speaking, it says that the weighted Lebesgue function is at least $c \log n$ on a "big part" of $[-a_n, a_n]$ for arbitrary fixed $X \subset (-\infty, \infty)$ and $w \in \mathcal{F}$.

7.5. The previous consideration can be developed for other weights. We mention some relations without going into the details.

We say that $w = e^{-Q}$ is an *Erdős weight* ($w \in \mathcal{E}$) if $Q(x) \approx \exp_k\big(|x|^\alpha\big)$, $\alpha > 1$ (see *A. L. Levin*, *D. S. Lubinsky* and *T. Z. Mthembu* {74} for the meaning of "$\approx$"). Using definitions and notations analogous the previous parts we have the following:

Let $w \in \mathcal{E}$. Then

$$a_n(w) = \{\log_k n\}^{1/\alpha}\big(1 + o(1)\big) \qquad \text{if} \qquad Q(x) = \exp_k |x|;$$

$$\begin{cases} \Lambda_n\big(w, Y(w^2)\big) \sim (nT_n)^{1/6}, & \text{where} \quad T_n \nearrow \infty, \qquad T_n = o(n^2), \\ \Lambda_n\big(w, V(w^2)\big) \sim \log n \end{cases}$$

(see {74} and *Steve Damelin* {13}).

7.6. Analogous theorems can be proved if $w \in \mathcal{L}$ ($w(x) \approx x^\beta e^{-x^\alpha}$, $\beta > -1$, $\alpha > 1$, $x \in (0, \infty)$ (Laguerre type weights)); $w \in \mathcal{EXP}$ $\Big(Q(x) \approx \exp_k\big((1 - x^2)^{-\alpha}\big),\ x \in (-1, 1)\Big)$; $w \in GSJ$ ($w(x) \approx \prod |x - t_k|$, where $T_n \sim n^2$, $x \in (-1, 1)$) or $w \in M$ ($w = uv$ where $u \in \mathcal{F} \cup \mathcal{E} \cup \mathcal{L}$ and $v \in GSJ$). Further, we have {137}.

Theorem 7.2. *Let $w \in \mathcal{E} \bigcup \mathcal{L} \bigcup \mathcal{EXP} \bigcup GSJ \bigcup M$. Then the estimation analogous to* (7.10) *can be proved.*

7.7. Remarks. 1. First we formulate a useful relation analogous to 2.8.1

Let $(a,b) \subset \mathbb{R}$ *and* $w = e^{-Q} : (a,b) \to (0,\infty)$. *Assume that* Q' *exists and increasing in* (a,b). *Then for* $1 \le k \le n-1$,

$$t_{kn}(w,X,x) + t_{k+1,n}(w,X,x) \ge 1 \qquad \text{if} \qquad x \in [x_{k+1,n}, x_{kn}]$$

for arbitrary interpolatory $X \subset (a,b)$ (see *D. S. Lubinsky* {130}).

2. There are a lot of problems to be solved: the weighted version of the *Grünwald–Marcinkiewicz*, *Erdős–Halász*, *Erdős–Vértesi*, *Erdős–Kroó–Szabados* results. Other ones can be formulated according to Part A.

8. Getting Convergence by Raising the Degree

8.1. Using Theorems 7.1 and 7.2, one can easily get Faber type results. To improve the behaviour of the weighted interpolation, one may try to apply the analogue of the HFi (cf. Part 4). However, as it turned out from the papers of *D. S. Lubinsky*, *P. Rabinowitz*, *J. Szabados* and others, the corresponding results are not quite satisfactory: one has to take the function class $C(w_2,\mathbb{R})$ to get convergence for $\big\| w_1\big(f - H_n\big(f, Y(w^2)\big)\big)\big\|$ where $w_1(x) = o\big(w^2(x)\big)$, $w^2(x) = o\big(w_2(x)\big)$ $\big(|x| \to \infty,\ w \in \mathcal{F}\big)$ instead of the "natural" $w_1 = w_2 = w^2$ (see {123} and its references).

8.2. Very recently *V. E. Sándor Szabó* {123} realized this "natural" settlement by taking a *Grünwald* type process. He proved

Theorem 8.1. *Let* $w \in \mathcal{F}$ *and* $f \in C(w^2,\mathbb{R})$. *Then with* $G_n(f,x) = \sum_{k=1}^{n} f\big(y_{kn}(w^2)\big)\, \ell_{kn}^2(Y(w^2),x) \in \mathcal{P}_{2n-2}$

$$\lim_{n\to\infty} \big\| w^2\big(f - G(f)\big)\big\| = 0.$$

Vértesi {129} refers to an *arbitrary matrix* X and proves the analogue of the Erdős theorem (see Theorem 4.6).

Theorem 8.2. *Let* $w \in \mathcal{F}$. *If* $\big|t_{kn}(w,X,x)\big| \le A$ *uniformly in* $x \in \mathbb{R}$, k *and* n, *then for every* $\varepsilon > 0$ *and to every* $f \in C(w^{1+\varepsilon},\mathbb{R})$, *there exists a sequence of polynomials* $\varphi_\Delta(x) = \varphi_\Delta(f,\varepsilon,x) \in \mathcal{P}_\Delta$ *such that*

(i) $\Delta \le n(1 + \varepsilon + c\,\varepsilon n^{-2/3})$,

(ii) $\varphi_\Delta(x_{kn}) = f(x_{kn})$, $1 \le k \le n$, $n \in \mathbb{N}$,

(iii) $\big\| w^{1+\varepsilon}(f - \varphi_\Delta)\big\| \le cE_\Delta(f, w^{1+\varepsilon})$.

8.3. Remarks. 1. The main problem in proving Theorem 8.2 (which is a far-reaching generalization of Theorem 8.1) is that now $\| t_{kn}(w,X,x) \| \le A$ *does not* involve the nice property $\vartheta_{k-1,n} - \vartheta_{kn} \sim \frac{1}{n}$ (which holds true at the classical case whenever $\| \ell_{kn}(X,x) \| \le A$).

2. Statements analogous to Theorem 8.2 for other weights have been proved in {126}.

9. Mean Convergence

9.1. The results of this part originally were formulated for the classical $L_n(f,X,x) = \sum f(x_{kn})\ell_{kn}(X,x)$ Lagrange interpolatory case. However by

$$\big| f(x) - L_n(f,X,x) \big|^p \; w^p(x) = \big| f(x)w(x) - L_n(f,w,X,x) \big|^p$$

they can be transformed into the weighted case. We shall use both formulations. First a counterpart of Theorem 6.1 due to *Shohat* ({77, ref. {65}}):

Theorem 9.1. *Let* $f^2 \in C(w^2, \mathbb{R})$. *Then*

$$\lim_{n\to\infty} \big\| fw - L_n\big(f,w,Y(w^2)\big) \big\|_{L_2(\mathbb{R})} = 0.$$

Here and later $Y(w^2)$ (or $V(w^2)$) is analogous to (7.4) (or (7.6)); $\|h\|_{L_p(\mathbb{R})} = \big(\int_{\mathbb{R}} |h|^p \big)^{1/p}$.

9.2. In many respects the Hermite weight on $\mathbb{R}$ corresponds to the Čebishov weight $1/\sqrt{1-x^2}$ on $(-1,1)$. In spite of this, a result similar to (6.2) does not hold. For simplicity, we quote a rather special case of a paper of *Lubinsky* and *Matjila*, from 1995 {77, ref. {63}}.

Theorem 9.2. *Let* $w \in \mathcal{F}$, $1 < p \le 4$, $0 < \alpha \le 1$. *Then*

$$\lim_{n\to\infty} \Big\| \big(f - L_n\big(f, Y(w^2)\big)\, w_{-\Delta} \big) \Big\|_{L_p(\mathbb{R})} = 0$$

for every $f \in C(w_\alpha, \mathbb{R})$ *iff* $\Delta > -\alpha + 1/p$. *Here* $w_\delta(x) = w(x)\big(1 + |x|\big)^{\delta}$.

9.3. The situation is more satisfactory if we use the matrix $V(w^2)$ instead of $Y(w^2)$. We quote two recent results of *D. S. Lubinsky* and *G. Mastroianni* {76}, {78}.

Theorem 9.3. *Let $w \in \mathcal{F}$ and $1 < p < \infty$. Then for every $f \in C(w, \mathbb{R})$,*

$$\lim_{n\to\infty} \left\| fw - L_n\big(f, w, V(w^2)\big) \right\|_{L_p(\mathbb{R})} = 0.$$

The analogous statement holds true if $w \in \mathcal{EXP}$.

The basic ideas are analogous to the ones in the previous theorem. However, the main emphasis was to get a general Marcinkiewicz–Zygmund inequality using the fairly sophisticated König method.

9.4. Finally here is a result of *Nevai* {91} analogous to Theorem 6.4.

Theorem 9.4. *Let $w(x) = \exp\big(-|x|^m\big)$, $m > 0$, even. Let u (≥ 0) and $\int_{\mathbb{R}} u < \infty$. If $0 < p < \infty$ and*

$$\int_{\mathbb{R}} \Big[\big(w(t)\big)^{1/2}\big(1 + |t|\big)\Big]^{-p} u(t)\, dt = \infty,$$

then there exists a function f supported on a finite interval such that

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}} \big|L_n\big(f, X(w), t\big)\big|^p \, u(t)\, dt = \infty.$$

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