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ON INVARIANT CONVEX SUBSETS IN ALGEBRAS DEFINED ON A LOCALLY COMPACT GROUP G

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Abstract

Suppose that A is either the Banach algebra $L^1(G)$ of a locally compact group G, or measure algebra $M(G)$, or other algebras (usually larger than $L^1(G)$ and $M(G)$) such as the second dual, $L^1(G)^{**}$, of $L^1(G)$ with an Arens product, or $LUC(G)^*$ with an Arens-type product. The left translation invariant closed convex subsets of A are studied. Finally, we obtain necessary and sufficient conditions for $LUC(G)^*$ to have 1-dimensional left ideals.

1. Introduction

Let G be a locally compact group, and let λ denote the left invariant Haar measure on G. Let $M(G)$ be the space of complex Radon measures on G. We define the convolution of two measures $\mu, \nu \in M(G)$ as follows:

$$\int \varphi(z)\, d\mu * \nu(z) = \int\int \varphi(xy)\, d\mu(x)\, d\nu(y) = \int\int \varphi(xy)\, d\nu(y)\, d\mu(x)$$

where $\varphi \in C_0(G)$ ($C_0(G)$ is the set of all continuous functions that vanish at infinity). With the convolution product $M(G)$ becomes a Banach algebra, and convolution is commutative if and only if G is abelian [16]. The Dirac measure $\delta_e \in M(G)$ is the unit of $M(G)$.

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In many respects, the Banach algebra $M(G)$ is too big and complicated to work with easily, and it is preferable to restrict attention to the space $L^1(G)$, which is a subspace of $M(G)$ if we identify the function f with the measure $f(x)\,dx$. If $f, g \in L^1(G)$, the convolution of f and g is the function defined by $f * g(x) = \int f(y)g(y^{-1}x)\,dy$ [8]. We know that $L^1(G)$ is a closed two-sided ideal in the algebra $M(G)$, and $\delta_e \in L^1(G)$ if and only if G is discrete [16].

The closed convex subset of $L^p(G)$ $(1 \leqq p < \infty)$ has been studied in a series of papers recently. Most complete information has been obtained for the hypergroup case [24]. This paper continues these investigations. In this paper, we study the closed subspaces $\mathcal{I}$ of $LUC(G)^*$, $L^\infty(G)$ and $\mathrm{Wap}\,(G)$ that are invariant under all left translations. We proved that if G is a compact abelian group, then each τ_c-closed translation invariant subspace $\mathcal{X}$ of $L^\infty(G)$ is introverted.

2. Notation and preliminary results

Let $C_b(G)$ be the space of continuous, bounded, complex-valued functions on G with the sup norm. $C_b(G)$ is a Banach algebra with point wise operations. Let $LUC(G)$ denote the closed subspace of bounded left uniformly continuous functions on G, i.e., all $f \in C_b(G)$ such that the map $x \mapsto L_x f$ from G into $C_b(G)$ is continuous, where $L_x f(y) = f\big(x^{-1}y\big)$ for $y \in G$ [16]. This is the space of bounded functions on G which are uniformly continuous with respect to the right uniformity on G, that is, for every $\varepsilon > 0$, there is a neighborhood U of the identity in G such that $\big|f(x) - f(y)\big| < \varepsilon$ whenever $xy^{-1} \in U$. The other algebras which we shall consider are defined in the following way. Let $L^\infty(G)$ denote the algebra of essentially bounded Haar measurable complex-valued functions on G with point wise operations. It is known that $L^\infty(G) = L^1(G)^*$. The second dual $L^1(G)^{**}$ of $L^1(G)$ is a Banach algebra with the first Arens product (see [1], [5]). This product is obtained by letting first

$$\langle f_F, \varphi\rangle = \langle F, \hat{\varphi} * f\rangle \quad \text{for all} \quad F \in L^1(G)^{**}, \quad f \in L^\infty(G) \quad \text{and} \quad \varphi \in L^1(G),$$

where $\hat{\varphi}(x) = \Delta\big(x^{-1}\big)\,\varphi\big(x^{-1}\big)$ for $x \in G$. Then, for F and G in $L^1(G)^{**}$,

$$\langle GF, f\rangle = \langle G, f_F\rangle \quad \text{for all} \quad f \in L^\infty(G).$$

If E is a Banach space, E^* will denote the collection of all continuous linear mapping of E into $\mathbb{C}$. Then $LUC(G)^*$ with the product inherited from $L^1(G)^{**}$ (by restriction) is also a Banach algebra (see [1] and Lemma 3 in [19]). Among the elements of $LUC(G)^*$ are the point masses δ_x for $x \in G$.

These do not appear in $L^1(G)^{**}$. Moreover, δ_e is an identity in $LUC(G)^*$, and $L^1(G)^{**}$ has a right identity [1].

The space $L^\infty(G)$ may be embedded into $\mathcal{B}\big(L^1(G), L^\infty(G)\big)$ by the linear map T such that $T(f)(\varphi) = \varphi * f$ where $f \in L^\infty(G)$ and $\varphi \in L^1(G)$. Since $\mathcal{B}\big(L^1(G), L^\infty(G)\big)$ carries naturally the strong operator topology, T allows us to consider the induced topology on $L^\infty(G)$, which we denote by τ_c. In fact, a net $\{f_\alpha\}$ in $L^\infty(G)$ converges to f in the τ_c-topology if and only if $\|\varphi * f_\alpha - \varphi * f\| \to 0$ for all $\varphi \in L^1(G)$ (for more information, see [3], [4] or [11] and [12]). For a locally compact group G the τ_c-topology is not weaker than the weak* topology and not stronger than the norm topology on $L^\infty(G)$. It is known that the τ_c-topology is different from the weak* topology whenever G is infinite [3]. Crombez and Govaerts [4] have proved that the τ_c-topology coincides with the norm topology if and only if G is discrete.

Put $P^1(G) = \big\{\varphi \in L^1(G);\ \varphi \geqq 0,\ \|\varphi\|_1 = 1\big\}$. If $f : G \to \mathbb{C}$ and $a \in G$, we put ${}_af(x) = f(ax)$, $f_a(x) = f(xa)$ where $x \in G$. If $f : G \to \mathbb{C}$ and $a \in G$, we also consider ${}_af_a(x) = f\big(a^{-1}xa\big)$, $x \in G$ [22]. As far as possible, we follow [16] in our notation and refer to [26] for basic functional analysis and to [8] for basic harmonic analysis (see also [16]).

3. Invariant subsets of group algebras

Our starting point of this section is the following lemma whose proof is straightforward.

Lemma 1. *Let G be a locally compact group.*

(1) *For $f \in L^\infty(G)$ and $\varphi, \psi \in L^1(G)$, we have $\langle \varphi * f, \psi\rangle = \langle f, \hat{\varphi} * \psi\rangle$.*

(2) *Let $\mathcal{U}$ be the family of all neighborhoods of e, regarded as directed set in the usual way: $U \succeq V$ if $U \subseteqq V$. For each $U \in \mathcal{U}$ choose a non-negative function $\varphi_U \in L^1(G)$ such that φ_U vanishes outside of U and $\int \varphi_U(x)\,dx = 1$. If $f \in L^\infty(G)$ and $x \in G$, then $\{\delta_x * \varphi_U * f\}_{U \in \mathcal{U}}$ converges to $\delta_x * f$ in the weak* topology.*

Definition 1. A subset $\mathcal{X}$ of $L^\infty(G)$ is said to be topologically invariant if $\varphi * f \in \mathcal{X}$ for all $\varphi \in P^1(G)$ and $f \in \mathcal{X}$. We say that $\mathcal{X}$ is left translation invariant if $L_xf \in \mathcal{X}$ whenever $f \in \mathcal{X}$ and $x \in G$.

It is well known that a closed convex subset $\mathcal{X}$ of $L^p(G)$, $1 \leqq p < \infty$ is left invariant if and only if it is topologically invariant (see Theorem 4.1 in [17]). For $p = \infty$, Lau [17] proved that if $\mathcal{X}$ is a weak* closed convex subset of $L^\infty(G)$, then $\mathcal{X}$ is left invariant if and only if $\mathcal{X}$ is topologically invariant. Our first result is a generalization of this fact to τ_c-closed convex subsets of $L^\infty(G)$.

Theorem 1. *Let G be a compact group. Let $\mathcal{X}$ be a τ_c-closed convex subset of $L^\infty(G)$. Then $\mathcal{X}$ is left translation invariant if and only if it is topologically invariant.*

Proof. Assume that there exist $f \in \mathcal{X}$ and $\varphi \in P^1(G)$ such that $\varphi * f \notin \mathcal{X}$. By the Hahn–Banach Theorem [26], there exists $F \in \big(L^\infty(G), \tau_c\big)^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\mathrm{Re}\,\langle F, L_x f\rangle < \gamma_1 < \gamma_2 < \langle F, \varphi * f\rangle,$$

where $x \in G$. Since $L^1(G)$ is the dual of $(L^\infty(G), \tau_c)$ (see Corollary 2 in [4]), so $F = \psi$ for some $\psi \in L^1(G)$. Hence $\langle F, L_x f\rangle = \langle \psi, L_x f\rangle = \langle f, \delta_{x^{-1}} * \psi\rangle$ for all $x \in G$. By Lemma 3.4 in [12],

$$\begin{aligned}\mathrm{Re}\,\langle \varphi * f, \psi\rangle = \mathrm{Re}\,\langle f, \hat{\varphi} * \psi\rangle = \mathrm{Re} \int \langle f, \delta_x * \psi\rangle d\hat{\varphi}(x)\\ \leqq \gamma_1 < \gamma_2 < \mathrm{Re}\,\langle \varphi * f, \psi\rangle.\end{aligned}$$

We would come to a contradiction. Therefore $\varphi * f \in \mathcal{X}$ for all $f \in \mathcal{X}$ and $\varphi \in P^1(G)$.

To prove the converse, let $f \in \mathcal{X}$ and $x \in G$. Let $\{e_\alpha\}$ be an approximate identity in $L^1(G)$ such that each e_α belongs to $P^1(G)$ [8]. Let

$$V := \big\{ h \in L^\infty(G);\ \|\varphi_i * L_x f - \varphi_i * h\| < \varepsilon,\ 1 \leqq i \leqq n \big\}$$

be a τ_c-neighborhood of $L_x f$. There exists α such that

$$\|\varphi_i * e_\alpha * L_x f - \varphi_i * L_x f\| = \|\varphi_i * e_\alpha * \delta_x * f - \varphi_i * \delta_x * f\| < \varepsilon$$

for all $1 \leqq i \leqq n$. By assumption $e_\alpha * \delta_x * f \in \mathcal{X}$. It follows that $L_x f$ is in the closure of $\mathcal{X}$ in the τ_c-topology. On the other hand, $\overline{\mathcal{X}} = \mathcal{X}$, and so $L_x f \in \mathcal{X}$. This completes our proof. □

Theorem 2. *If G is a compact abelian group, let $\mathcal{X}$ be a τ_c-closed subspace of $L^\infty(G)$ that is left translation invariant. Then $\mathcal{X}$ is introverted, that is, $f_F \in \mathcal{X}$ whenever $F \in L^\infty(G)^*$ and $f \in \mathcal{X}$.*

Proof. Let $F \in L^\infty(G)^*$ and $f \in \mathcal{X}$. As $L^1(G)$ is weak* dense in $L^\infty(G)^*$ by Goldstine's Theorem [26], there exist a bounded net $\{\varphi_\alpha\}_{\alpha\in I}$ bounded by $\|F\|$ in $L^1(G)$ such that $\{\varphi_\alpha\}_{\alpha\in I}$ converges to F in the weak* topology. It is easy to see that $\{\hat{\varphi_\alpha} * f\}_{\alpha\in I}$ converges to f_F in the weak* topology. We first show that $\{\hat{\varphi}_\alpha * f\}_{\alpha\in I}$ converges to f_F in the τ_c-topology. To see this, let

$$V := \big\{ h \in L^\infty(G);\ \|\varphi_i * f_F - \varphi_i * h\| < \varepsilon,\ 1 \leqq i \leqq n \big\}$$

be the τ_c-neighborhood of f_F determined by $\varphi_1, \ldots, \varphi_n$ in $L^1(G)$ and $\varepsilon > 0$. Since the mapping $x \mapsto L_x\hat{\varphi}_i$ $(1 \leqq i \leqq n)$ is continuous [16], for every $x \in G$, there exists an open neighborhood U_x of x such that

$$\|L_x\hat{\varphi}_i - L_y\hat{\varphi}_i\|_1 < \frac{\varepsilon}{4\|F\|\,\|f\| + 1}$$

whenever $y \in U_x$ and $1 \leqq i \leqq n$. Since G is compact, we can choose a finite subset $\{x_1, \ldots, x_k\}$ in G such that $G \subseteqq \bigcup_{j=1}^{k} U_{x_j}$ and $\|L_{x_j}\hat{\varphi}_i - L_y\hat{\varphi}_i\|_1 < \frac{\varepsilon}{4\|F\|\,\|f\|+1}$ whenever $y \in U_{x_j}$, $1 \leqq j \leqq k$ and $1 \leqq i \leqq n$. Since $\big\{h \in L^\infty(G);$ $\big|\langle h - f_F, L_{x_j}\hat{\varphi}_i\rangle\big| < \frac{\varepsilon}{4}$ for all i, $j\big\}$ is a weak* neighborhood of f_F, there exists $\alpha_0 \in I$ such that $\big|\langle\hat{\varphi_\alpha} * f - f_F, L_{x_j}\hat{\varphi}_i\rangle\big| < \frac{\varepsilon}{4}$ for $1 \leqq i \leqq n$, $1 \leqq j \leqq k$ and $\alpha \in I$ with $\alpha \succeq \alpha_0$. Let $x \in G$ and $\alpha \succeq \alpha_0$. Let x_j be chosen such that $x \in U_{x_j}$. For every $1 \leqq i \leqq n$,

$$\begin{aligned}\big|\langle\hat{\varphi_\alpha} * f - f_F, L_x\hat{\varphi}_i\rangle\big| &\leqq \big|\langle\hat{\varphi_\alpha} * f - f_F, L_x\hat{\varphi}_i - L_{x_j}\hat{\varphi}_i\rangle\big| \\ &\quad + \big|\langle\hat{\varphi_\alpha} * f - f_F, L_{x_j}\hat{\varphi}_i\rangle\big| < \frac{\varepsilon}{2}.\end{aligned}$$

Hence for all $\alpha \in I$ with $\alpha \succeq \alpha_0$, $x \in G$ and $1 \leqq i \leqq n$, $\big|\langle\hat{\varphi_\alpha} * f - f_F,$ $\delta_x * \hat{\varphi}_i\rangle\big| < \frac{\varepsilon}{2}$. For every $\psi \in C_C(G)$, by Lemma 3.4 in [12] we have

$$\begin{aligned}\big|\langle\varphi_i * \hat{\varphi_\alpha} * f - \varphi_i * f_F, \psi\rangle\big| &= \big|\langle\hat{\varphi_\alpha} * f - f_F, \hat{\varphi}_i * \psi\rangle\big| \\ &\leqq \int \big|\langle\hat{\varphi_\alpha} * f - f_F, \delta_x * \hat{\varphi}_i\rangle\big|\, d|\psi|(x) \\ &\leqq \frac{\varepsilon}{2}\|\psi\|_1\end{aligned}$$

whenever $\alpha \succeq \alpha_0$ and $i \in \{1, \ldots, n\}$. Therefore $\|\varphi_i * \hat{\varphi_\alpha} * f - \varphi_i * f_F\| < \varepsilon$ for each $\alpha \succeq \alpha_0$ and $i \in \{1, \ldots, n\}$. This shows that $\{\hat{\varphi_\alpha} * f\}_{\alpha \in I}$ converges to f_F in the τ_c-topology. Since $\mathcal{X}$ is left translation invariant, by Theorem 1, $\mathcal{X}$ is topologically invariant. Hence $\hat{\varphi_\alpha} * f \in \mathcal{X}$ for every $\alpha \in I$. Thus $f_F \in \overline{\mathcal{X}} = \mathcal{X}$. □

Note that the preceding two Theorems give the nice result that a τ_c-closed subspace of $L^\infty(G)$ is left translation invariant if and only if it is introverted.

Remark 1. It will be interesting to consider the analogue of Theorem 2 for the group von Neumann algebra $VN(G)$ of a locally compact group (see Section 7 of [20]). Note that in this case G is abelian, $VN(G)$ is isometrically isomorphic to $L^\infty(\hat{G})$.

Definition 2. A bounded linear operator T from $L^\infty(G)$ to $L^\infty(G)$ is called a left multiplier if $L_xT(f)=T(L_xf)$, for all $f\in L^\infty(G)$ and $x\in G$. The set of the left multipliers will be denoted by $\mathcal{M}\big(L^\infty(G),L^\infty(G)\big)$.

For more on multipliers, the reader is referred to [17] and [24]. The affine mappings which commute with translations have been studied by Lau [17]. He proved that an affine continuous mapping T from $L^p(G)$ $(1\leqq p<\infty)$ into $L^q(G)$ commutes with left translation if and only if $T(\varphi*f)=\varphi*T(f)$ for each $\varphi\in P^1(G)$ and $f\in L^p(G)$.

Theorem 3. *Let G be a compact abelian group. Let T be a bounded linear operator on $L^\infty(G)$, that is τ_c-τ_c continuous. Then T is in $\mathcal{M}\big(L^\infty(G),L^\infty(G)\big)$ if and only if $T(f_F)=T(f)_F$ for all $f\in L^\infty(G)$, $F\in L^\infty(G)^*$.*

Proof. See Theorem 2 and its proof. □

Recall that L_x, $x\in G$, is the translation operator in $L^1(G)$, that is, for $f\in L^1(G)$, $L_xf(y)=f\big(x^{-1}y\big)$. It is a well-known phenomenon that a closed subspace in $L^1(G)$ is an ideal if and only if it is invariant under each L_x, $x\in G$. Moreover, if G is not discrete, there exists a closed subspace of $M(G)$ which is invariant under translation and which is not an ideal in $M(G)$ [16].

Theorem 4 (Civin [2]). *Let G be a locally compact abelian group. There exists a closed subspace of $L^1(G)^{**}$ which is invariant under L_x^{**} for all $x\in G$ and which is neither a left nor a right ideal.*

The following theorem is to discuss the similarities and differences in this type of behavior where one considers translations in $LUC(G)^*$.

Theorem 5. *Let G be a locally compact group. A weak* closed subspace $\mathcal{I}$ of $LUC(G)^*$ is a left ideal if and only if it is invariant under L_x^{**}, $x\in G$.*

Proof. We only prove that $\mathcal{I}$ is invariant under L_x^{**}. Let $\mathcal{I}$ be a left ideal in $LUC(G)^*$ and $F\in\mathcal{I}$, $x\in G$. Then $\mathcal{U}$, the collection of all symmetric compact neighborhoods of e, ordered by inclusion (i.e., for $U_1,U_2\in\mathcal{U}$, we write $U_2\succeq U_1$ if and only if $U_2\subseteqq U_1$) form a directed set. For each $U\in\mathcal{U}$, let ψ_U be a function such that *supp*ψ_U is compact and contained in U, $\psi_U\geqq 0$, $\psi_U\big(t^{-1}\big)=\psi_U(t)$, and $\int\psi_U(t)\,dt=1$. For $f\in LUC(G)$ and $x\in G$, we have

$$\begin{aligned}\big|\,\delta_{x^{-1}}*\psi_U*f(t)-\delta_{x^{-1}}*f(t)\big| &= \left|\int f\big(y^{-1}t\big)\,\delta_{x^{-1}}*\psi_U(y)\,dy-f(xt)\right|\\ &= \left|\int f\big(y^{-1}t\big)\,\psi_U(xy)-f(xt)\psi_U(y)\,dy\right|\\ &= \left|\int \big(f\big(y^{-1}xt\big)\,-f(xt)\big)\psi_U(y)\,dy\right|\end{aligned}$$

$$= \left| \int \left(_{y^{-1}}f(xt) - f(xt) \right) \psi_U(y)\, dy \right|$$
$$\leqq \int \|_{y^{-1}}f - f\| \psi_U(y)\, dy.$$

This shows that $\{\delta_{x^{-1}} * \psi_U * f\}$ converges to $\delta_{x^{-1}} * f$ in the norm topology. Hence $\langle \widehat{\delta_{x^{-1}} * \psi_U} F, f \rangle = \langle F, \delta_{x^{-1}} * \psi_U * f \rangle$ converges to $\langle F, \delta_{x^{-1}} * f \rangle = \left\langle L_x^{**}(F), f \right\rangle$. Consequently, $\{\widehat{\delta_{x^{-1}} * \psi_U} F\}$ converges to $L_x^{**}F$ in the weak* topology. On the other hand, $\widehat{\delta_{x^{-1}} * \psi_U} F \in \mathcal{I}$. Since $\mathcal{I}$ is closed, thus $L_x^{**}F \in \mathcal{I}$. This shows that $\mathcal{I}$ is invariant under L_x^{**}. □

It is known that (see [18]) for any locally compact group G,

$$LUC(G)^* = M(G) \oplus C_0(G)^{\perp}.$$

Suppose that G is a compact group. Then $M(G) = C_b(G)^* = LUC(G)^*$. So $L^1(G)$ is a two-sided ideal in $LUC(G)^*$. If $L^1(G)$ is a right (or left) ideal in $LUC(G)^*$ then it is a right (resp. left) ideal in $L^1(G)^{**}$. Then G will be a compact group [15]. Let G be a non compact group. $L^1(G)$ is clearly invariant under translations and it is neither a left nor a right ideal in $LUC(G)^*$. Now, we may suppose that G is a non discrete compact group. Then

$$\left\{ \sum_{k=1}^{\infty} \alpha_k \delta_{x_k};\ x_k \in G,\ \alpha_k \in \mathbb{C},\ \sum_{k=1}^{\infty} |\alpha_k| < \infty \right\}$$

is a closed subalgebra of $M(G)$ invariant under translations. This set is neither a right nor a left ideal in $LUC(G)^*$.

Definition 3. A function $f \in C_b(G)$ is said to be weakly almost periodic if $\{L_x f;\ x \in G\}$ is relatively weakly compact in $C_b(G)$. The set of all weakly almost periodic functions on G is denoted by $\mathrm{Wap}\,(G)$ [11].

Theorem 6. *Let G be a locally compact group. A closed convex subset $\mathcal{I}$ of* $\mathrm{Wap}\,(G)$ *is left translation invariant if and only if it is topologically invariant.*

Proof. Let $\mathcal{I}$ be a closed convex subset in $\mathrm{Wap}\,(G)$. Let f be in $\mathcal{I}$ and φ in $P^1(G)$. By assumption and by the Krein–Smulian Theorem, the relative weak compactness of $\{L_x f;\ x \in G\}$ implies the relative weak compactness of $\mathrm{co}\,\{L_x f;\ x \in G\}$. We know that $\overline{\mathrm{co}}\,\{L_x f;\ x \in G\}$ inherits two topologies: one weak topology and the other weak* topology from $L^{\infty}(G)^*$. It is easy to see that these two topologies coincide on $\overline{\mathrm{co}}\,\{L_x f;\ x \in G\}$ (or see [27]).

If $\varphi * f \notin \overline{\text{co}}\,\{L_x f;\ x \in G\}$, then Theorem 3.4 in [26] implies that there is a ψ in $L^1(G)$, an α in $\mathbb{R}$, and an $\varepsilon > 0$ such that

$$\operatorname{Re}\langle L_x f, \psi\rangle \leqq \alpha < \alpha + \varepsilon \leqq \operatorname{Re}\langle \varphi * f, \psi\rangle$$

for all x in G. On the other hand,

$$\begin{aligned}
\operatorname{Re}\langle \varphi * f, \psi\rangle &= \operatorname{Re}\int \varphi * f(t)\psi(t)\,dt \\
&= \operatorname{Re}\int\int L_x f(t)\varphi(x)\psi(t)\,dx\,dt \\
&= \operatorname{Re}\int\int L_x f(t)\psi(t)\varphi(x)\,dt\,dx \\
&= \operatorname{Re}\int \langle L_x f, \psi\rangle\varphi(x)\,dx < \alpha + \varepsilon.
\end{aligned}$$

This is a contradiction. This clearly implies

$$\overline{\{\varphi * f;\ \varphi \in P^1(G)\}} \subseteqq \overline{\text{co}}\,\{L_x f;\ x \in G\}.$$

If $\mathcal{I}$ is a left translation invariant closed convex subset of $\operatorname{Wap}(G)$, then $\mathcal{I}$ is topologically invariant.

Conversely, Let $\mathcal{I}$ be a topologically invariant closed convex subset in $\operatorname{Wap}(G)$. Let $\mathcal{U}$ denote the family of symmetric compact neighborhoods of e and regard $\mathcal{U}$ as a directed set in the usual way: $U \succeq V$ if $U \subseteqq V$. For each $U \in \mathcal{U}$, choose a function $\varphi_U \in L^1(G)$ such that $\int \varphi_U(t)\,dt = 1$, $\varphi_U \geqq 0$, $\varphi_U(x) = \varphi_U(x^{-1})$ and $\|\varphi_U\|_1 = 1$. Let f be in $\mathcal{I}$ and x in G. For every $U \in \mathcal{U}$ and $\psi \in L^1(G)$,

$$\begin{aligned}
\big\|\,(\widehat{\delta_x * \varphi_U}) * \psi - {}_x\psi\big\|_1 &= \int \big|\,(\widehat{\delta_x * \varphi_U}) * \psi(t) - {}_x\psi(t)\big|\,dt \\
&= \int \left|\int \psi(y^{-1}t)\,\varphi_U(x^{-1}y^{-1})\,\Delta(y^{-1})\,dy - {}_x\psi(t)\right|\,dt \\
&= \int \left|\int \psi(y^{-1}t)\,\varphi_U(yx)\Delta(y^{-1}) - {}_x\psi(t)\varphi_U(y)\,dy\right|\,dt \\
&\leqq \int \varphi_U(y)\int \big|\psi(xy^{-1}t)\,\Delta(y^{-1}) - {}_x\psi(t)\big|\,dt\,dy \\
&= \int \varphi_U(y)\big\|{}_{xy^{-1}}\psi\Delta(y^{-1}) - {}_x\psi\big\|_1\,dy.
\end{aligned}$$

As $\psi \in L^1(G)$, the mapping $x \mapsto {}_x\psi$ is continuous (see theorem 20.4 in [16]). Let $\varepsilon > 0$ be given. There exists an open neighborhood U of e in G such that for all $y \in U$,

$$\left\|{}_{xy^{-1}}\psi\Delta\left(y^{-1}\right) - {}_x\psi\right\|_1 < \varepsilon.$$

This shows that $\left\{(\widehat{\delta_x * \varphi_U}) * \psi\right\}_{U\in\mathcal{U}}$ converges to $\{{}_x\psi\}$ in the norm topology. On the other hand, $\langle \delta_x * \varphi_U * f, \psi\rangle = \left\langle f, (\widehat{\delta_x * \varphi_U}) * \psi\right\rangle$ for all $\psi \in L^1(G)$ and U. So $\{\delta_x * \varphi_U * f\}_{U\in\mathcal{U}}$ converges to $\{L_x f\}$ in the weak* topology. Clearly $\{\delta_x * \varphi_U * f\}_{U\in\mathcal{U}}$ converges to $\{L_x f\}$ in the weak topology. Since $\mathcal{I}$ is a closed convex topologically invariant subset of Wap(G), so $\mathcal{I}$ is a closed left invariant subset of Wap(G). □

Recall that for $\varphi, \psi \in L^1(G)$, $\varphi * \psi(x) = \int \psi\left(y^{-1}x\right)\varphi(y)\,dy$. Also $L^1(G)$, equipped with the convolution product, is a Banach subalgebra of $M(G)$, called the group algebra of G. It is known that a closed subspace $\mathcal{I}$ of $L^1(G)$ is a left ideal if and only if ${}_x\varphi \in \mathcal{I}$ for every $\varphi \in \mathcal{I}$ and $x \in G$. Following Li and Pier [21], we define

$$\varphi \circledast \psi(x) = \int \psi(y^{-1}xy)\varphi(y)\Delta(y)^{\frac{1}{p}}\,dy$$

for $\varphi \in L^1(G)$, $\psi \in L^p(G)$ and $x \in G$, where $1 \leqq p < \infty$. With this product $L^1(G)$ becomes a Banach algebra.

Theorem 7. *Let G be a locally compact group. A closed linear subspace $\mathcal{I}$ of $L^1(G)$ is a left ideal of $L^1(G)$ if and only if $\varphi \in \mathcal{I}$ and $x \in G$ imply that ${}_x\varphi_x \in \mathcal{I}$.*

Proof. Let $\mathcal{I}$ be a closed left ideal in $L^1(G)$ and let φ be in $\mathcal{I}$ and $x \in G$. Let ε be any positive number, and choose a compact neighborhood U of the identity in G such that

$$\left\|{}_y({}_x\varphi_x)_y - {}_x\varphi_x\right\|_1\Delta(y) < \frac{\varepsilon}{2}, \quad \left|\Delta(y) - 1\right| < \frac{\varepsilon}{2\left(\|{}_x\varphi_x\|_1 + 1\right)}$$

for any $y \in U$ (see Theorem 20.4 in [16]). Put $\xi_U = \frac{\chi_U}{\lambda(U)}$ and $\phi = {}_x\varphi_x$. For every $\psi \in C_C(G)$, we have

$$\begin{aligned}\left|\langle\psi, \xi_U \circledast \phi - \phi\rangle\right| &= \left|\int \psi(z)\xi_U \circledast \phi(z)\,dz - \int \psi(z)\phi(z)\,dz\right| \\ &= \left|\int\int \psi(z)\xi_U(y)\phi(y^{-1}zy)\Delta(y)\,dy\,dz - \int \psi(z)\phi(z)\,dz\right|\end{aligned}$$

$$\leqq \frac{1}{\lambda(U)} \int |\psi(z)| \int_U \left|\phi\left(y^{-1}zy\right)\Delta(y) - \phi(z)\right| dy\, dz$$

$$\leqq \frac{1}{\lambda(U)} \int_U \int_G \left|\phi\left(y^{-1}zy\right)\Delta(y) - \phi(z)\right|\left|\psi(z)\right| dz\, dy$$

$$\leqq \frac{1}{\lambda(U)} \int_U \left\|{}_y\phi_y\Delta(y) - \phi\Delta(y)\right\|_1 \|\psi\|_\infty\, dy$$

$$+ \frac{1}{\lambda(U)} \int_U \left\|\phi\Delta(y) - \phi\right\|_1 \|\psi\|_\infty\, dy.$$

It follows that $\|\xi_U \circledast {}_x\varphi_x - {}_x\varphi_x\|_1 < \varepsilon$ (see Theorem 14.5 in [16]). On the other hand,

$$\xi_{Ux^{-1}} \circledast \varphi(y) = \frac{1}{\lambda(U)} \int \varphi(z^{-1}yz)\chi_U\left(zx^{-1}\right)\Delta(z)\, dz$$

$$= \frac{\Delta(x)^2}{\lambda(U)} \int_U \varphi\left(x^{-1}z^{-1}yzx\right)\Delta(z)\, dz$$

$$= \frac{\Delta(x)^2}{\lambda(U)} \int_U {}_x\varphi_x\left(z^{-1}yz\right)\Delta(z)\, dz = \Delta(x)^2\xi_U \circledast {}_x\varphi_x(y).$$

Therefore $\left\|\Delta\left(x^{-1}\right)^2\xi_{Ux^{-1}} \circledast \varphi - {}_x\varphi_x\right\|_1 < \varepsilon$. By assumption,

$$\Delta\left(x^{-1}\right)^2\xi_{Ux^{-1}} \circledast \varphi \in \mathcal{I}.$$

It follows that ${}_x\varphi_x \in \mathcal{I}$.

Suppose conversely that $\varphi \in \mathcal{I}$ and $x \in G$ imply ${}_x\varphi_x \in \mathcal{I}$. Let φ be in $\mathcal{I}$ and ψ in $L^1(G)$. If every bounded linear functional Λ on $L^\infty(G)$ for which $\Lambda(\mathcal{I}) = 0$ also satisfies $\Lambda(\psi \circledast \varphi) = 0$, then by the Hahn–Banach Theorem [26], $\psi \circledast \varphi$ also belongs to $\mathcal{I}$. Let Λ be as above. There is a function $f \in L^\infty(G)$ such that

$$\Lambda(\phi) = \int_G \phi(x)f(x)\, dx \quad \text{for} \quad \phi \in L^1(G).$$

We then have

$$\Lambda(\psi \circledast \varphi) = \int f(x)\psi \circledast \varphi(x)\,dx = \int\int \varphi\big(z^{-1}xz\big)\,\psi(z)\Delta(z)f(x)\,dz\,dx$$
$$= \int \Delta(z)\psi(z)\int f(x)\varphi(z^{-1}xz)\,dx\,dz = \int \psi(z)\Delta(z)\Lambda({}_z\varphi_z)\,dz = 0,$$

since ${}_z\varphi_z \in \mathcal{I}$ for $z \in G$ and $\Lambda(\mathcal{I}) = 0$. □

Completely analogously to Theorem 5 and Theorem 6, we also have

THEOREM 8. *Let G be a locally compact group.*

(1) *If $\mathcal{X}$ is a closed subspace of $L^p(G)$, $1 \leqq p < \infty$, then $\mathcal{X}$ is conjugate invariant, that is, ${}_xf_x \in \mathcal{X}$ for every $f \in \mathcal{X}$ and $x \in G$ if and only if $f \in \mathcal{X}$ and $\varphi \in P^1(G)$ imply that $\varphi \circledast f \in \mathcal{X}$.*

(2) *If $\mathcal{X}$ is a weak* closed subspace of $L^\infty(G)$, then $\mathcal{X}$ is conjugate invariant if and only if $f \in \mathcal{X}$ and $\varphi \in P^1(G)$ imply that $\varphi \circledast f \in \mathcal{X}$.*

THEOREM 9. *Let G be any locally compact group. If $f \in L^\infty(G)$ such that $\{{}_xf_x;\, x \in G\}$ is relatively compact in the norm topology (weak topology) of $L^\infty(G)$, then the map $h \mapsto h \circledast f$ from $L^1(G)$ into $L^\infty(G)$ is a compact (weakly compact) linear operator.*

PROOF. If $\{{}_xf_x;\ x \in G\}$ is relatively compact in the weak topology of $L^\infty(G)$, then the set $K = \overline{\mathrm{co}}\,\{{}_xf_x;\ x \in G\}$ is weakly compact subset of $L^\infty(G)$. It is easy to see that

$$\overline{\mathrm{co}}^{w^*}\{{}_xf_x;\ x \in G\} = \overline{\big\{\varphi \circledast f;\ \varphi \in P^1(G)\big\}}^{w^*}.$$

Since the weak* topology is Hausdorff on K, it follows that the weak* and weak topologies agree on K [27]. Consequently $K = \overline{\big\{\varphi \circledast f;\ \varphi \in P^1(G)\big\}}$ is a weakly compact subset of $L^\infty(G)$ and also $K_1 = \{\lambda k;\, k \in K,\, 0 \leqq \lambda \leqq 1\}$ is weakly compact. Now if $h \in L^1(G)$ and $\|h\|_1 \leqq 1$, then $h = (h_1 - h_2) + i(h_3 - h_4)$ where each h_i is positive, and $\|h_i\|_1 \leqq 1$. It follows that $h \circledast f \in (K_1 - K_1) + i(K_1 - K_1)$. Hence the map $h \mapsto h \circledast f$ is compact.

The proof for the norm compact case is similar. □

DEFINITION 4. The group G is said to be amenable if there exists a positive functional M on $LUC(G)$ with norm one such that $\langle M, \varphi * f\rangle = \langle M, f\rangle$ for each $f \in LUC(G)$ and $\varphi \in P^1(G)$ (see [23], [25], [9] and [10]).

Let $UC(G)$ be the space of bounded, uniformly continuous, complex-valued functions on G. Filali proved that finite-dimensional right ideals exist in $U(G)^*$ if and only if G is compact (see Theorem 3 in [7]). We proved that if there exists a measure $\mu \in \mathrm{soc}\big(M(G)\big)$ such that $\mu(G) \neq 0$, then G is compact [13]. In [6], Filali proved that finite dimensional left ideals exist in $LUC(G)^*$ if and only if G is amenable. In the following Theorem, a simple proof of the above result is given.

THEOREM 10. *Let G be a locally compact group. Then*

(1) 1-*dimensional left ideals exist in $LUC(G)^*$ if and only if G amenable*

(2) 1-*dimensional left ideals exist in $L^1(G)^{**}$ if and only if G amenable.*

PROOF. Let $LUC(G)^*F$ be a left ideal of $LUC(G)^*$ of dimension 1. Then the mapping $T_F : E \mapsto EF$, is a rank one operator and hence compact. Since every compact right multiplier on $LUC(G)^*$ can be described as a linear combination of four compact positive right multipliers [14], it is easy to see that G is amenable (or see Theorem 2.1 in [14]).

To prove the converse, let M be a topologically left invariant mean on $LUC(G)^*$. Then $M(\varphi * f) = M(f)$ for all $f \in LUC(G)$ and $\varphi \in P^1(G)$. Since $L^1(G)$ is weak* dense in $LUC(G)^*$, so that for any F we can find a bounded net $\{\varphi_\alpha\}_{\alpha\in I}$ in $L^1(G)$ with weak* limit $\varphi_\alpha = F$. For any $\alpha \in I$, there exist $\varphi_{\alpha 1}, \varphi_{\alpha 2}, \varphi_{\alpha 3}, \varphi_{\alpha 4} \in P^1(G)$ and $\lambda_{\alpha 1}, \lambda_{\alpha 2}, \lambda_{\alpha 3}, \lambda_{\alpha 4} \in \mathbb{R}$ such that $\hat{\varphi_\alpha} = \lambda_{\alpha 1}\varphi_{\alpha 1} - \lambda_{\alpha 2}\varphi_{\alpha 2} + i(\lambda_{\alpha 3}\varphi_{\alpha 3} - \lambda_{\alpha 4}\varphi_{\alpha 4})$. Since $\{\hat{\varphi_\alpha}\}$ is a bounded subset in $L^1(G)$, passing to a subnet if necessary, we may assume that $\{\hat{\varphi_\alpha}\}$ converges to some $\hat{F}$ in $LUC(G)^*$ in the weak* topology. Clearly $\lambda_{\alpha 1} - \lambda_{\alpha 2} + i(\lambda_{\alpha 3} - \lambda_{\alpha 4}) = 1(\hat{\varphi_\alpha}) = 1_M(\hat{\varphi_\alpha})$ converges to $\hat{F}(1)$. It follows that

$$\langle FM, f\rangle = \lim_\alpha \langle \varphi_\alpha, f_M\rangle = \lim_\alpha \langle M, \hat{\varphi}_\alpha * f\rangle = \hat{F}(1)\langle M, f\rangle$$

for any $f \in LUC(G)$. Hence $FM = \hat{F}(1)M$. This shows that $LUC(G)^*M$ is 1-dimensional.

Similar arguments apply in $L^\infty(G)^*$. This completes the proof. □

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