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THE LARGEST FAMILY OF SUBSETS SATISFYING SEQUENTIAL-EVALUATION CONVERGENCE

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Abstract

Suppose X is a locally convex space, Y is a topological vector space and $\lambda(X)^{\beta Y}$ is the β-dual of some X valued sequence space $\lambda(X)$. When $\lambda(X)$ is $c_0(X)$ or $l_\infty(X)$, we have found the largest $\mathcal{M} \subset 2^{\lambda(X)}$ for which $(A_j) \in \lambda(X)^{\beta Y}$ if and only if $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly with respect to (x_j) in any $M \in \mathcal{M}$. Also, a remark is given when $\lambda(X)$ is $l_p(X)$ for $0 < p < +\infty$.

1. Introduction

If X, Y are Banach spaces, for classical Banach sequence spaces $c_0(X)$, $l_\infty(X)$ and $l_p(X)$ with $0 < p < +\infty$, R. Li et al. [3] have determined the largest $\mathcal{M} \subset 2^{\lambda(X)}$ for which $(A_j) \in \lambda(X)^{\beta Y}$ if and only if $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly with respect to (x_j) in any $M \in \mathcal{M}$. In this paper, for $c_0(X)$ and $l_\infty(X)$, we generalize the result to the case when X is a locally convex space and Y is a topological vector space. For $l_p(X)$ with $0 < p < +\infty$, we give a remark in the last section.

We adopt the following notation which is the same as in [9, p. 229]. Let X be a locally convex space generated by the family of semi-norms $\mathcal{P}$ and $\lambda(X)$ be some X valued sequence space. Define

$$c_0(X) = \big\{ (x_j) \in X^{\mathbb{N}} : q(x_j) \to 0 \text{ for all } q \in \mathcal{P} \big\}$$

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and

$$l_\infty(X) = \Big\{(x_j) \in X^{\mathbb{N}} : \sup_j q(x_j) < +\infty \text{ for all } q \in \mathcal{P}\Big\}.$$

The natural topology on $c_0(X)$ or $l_\infty(X)$ is generated by the semi-norms:

$$q_\infty\big[(x_j)\big] = \sup_j q(x_j),\ \ q \in \mathcal{P}$$

For $0 < p < +\infty$, define

$$l_p(X) = \Big\{(x_j) \in X^{\mathbb{N}} : \sum_{j=1}^{\infty} q(x_j)^p < \infty \text{ for all } q \in \mathcal{P}\Big\}.$$

If $1 \leqq p < +\infty$, the topology of $l_p(X)$ is generated by the semi-norms:

$$q_p\big[(x_j)\big] = \Big(\sum_{j=1}^{\infty} q(x_j)^p\Big)^{1/p}, \quad q \in \mathcal{P}.$$

If $0 < p < 1$, the topology of $l_p(X)$ is generated by the paranorms:

$$q_p\big[(x_j)\big] = \Big(\sum_{j=1}^{\infty} q(x_j)^p\Big), \quad q \in \mathcal{P}.$$

Let Y be a topological vector space. For a sequence (A_j) of linear operators from X into Y, we say that the series $\sum A_j$ is $\lambda(X)$-evaluation convergent if $\sum_{j=1}^{\infty} A_j(x_j)$ converges for all $(x_j) \in \lambda(X)$ [10, 6, 2]. The original definition for the generalized Köthe–Toeplitz β-dual of $\lambda(X)$ is due to I. Maddox [4, p. 19], which is given by

$$\lambda^\beta(X) = \Big\{(A_j) : \text{ each } A_j \in Y^X \text{ is linear,} \\ \sum_{j=1}^{\infty} A_j(x_j) \text{ is } \lambda(X)\text{-evaluation convergent}\Big\},$$

In this paper, we adopt the notation $\lambda(X)^{\beta Y}$ as in [8, p. 153], especially, we drop the linearity restriction forced on the mappings as in [3], and let Y^X

be the family of all Y-valued Mappings on X, then the β-dual of $\lambda(X)$ is defined by

$$\lambda(X)^{\beta Y} = \left\{(A_j) \subset Y^X : \sum_{j=1}^{\infty} A_j(x_j) \text{ converges for all } (x_j) \in \lambda(X)\right\}.$$

Henceforth, let X be a locally convex space and Y be a topological vector space if there is no special indication.

2. Uniformly strong gliding hump property

Let χ_σ denote the characteristic function of σ. For $x = (x_j) \subset X$ and $\sigma \subset \mathbb{N}$, $\chi_\sigma x$ will denote the coordinatewise product of χ_σ and x, that is, if we let $(u_l) = \chi_\sigma x$, then

$$u_l = \begin{cases} x_l, & \text{if } \ l \in \sigma \\ 0, & \text{otherwise.} \end{cases}$$

$\lambda(X)$ is said to be monotone if $\chi_\sigma x \in \lambda(X)$ for every $\sigma \subset \mathbb{N}$ and $x \in \lambda(X)$. For example, $c_0(X)$, $l_p(X)$ and $l_\infty(X)$ are all monotone [9, p. 233]. An interval in $\mathbb{N}$ is a subset of the form

$$[m, n] = \{j \in \mathbb{N} : m \leqq j \leqq n\},$$

where $m, n \in \mathbb{N}$ with $m \leqq n$. A sequence of intervals $\{I_k\}$ is increasing if $\max I_k < \min I_{k+1}$ for every k. $\lambda(X)$ is said to have the strong gliding hump property [7] if for every bounded sequence $\{x^k\} \subset \lambda(X)$ and every increasing sequence of intervals $\{I_k\}$, there exists a subsequence $\{k_j\} \subset \{k\}$ such that $\sum_{j=1}^{\infty} \chi_{I_{k_j}} x^{k_j} \in \lambda(X)$. In this paper, we do not use the definition of strong gliding hump property, but introduce a new definition called uniformly strong gliding hump property, denoted by USGHP for short.

Definition 2.1. For $M \subset \lambda(X)$, we say M has the USGHP, if for every sequence $\{x^k\} \subset M$ and every increasing sequence of intervals $\{I_k\}$, there exists a subsequence $\{k_j\} \subset \{k\}$ such that $\sum_{j=1}^{\infty} \chi_{I_{k_j}} x^{k_j} \in \lambda(X)$.

Throughout this paper, we will use the following statement:

(T1) $M \subset \lambda(X)$ has the USGHP;

(T2) For every topological vector space Y and $(A_j) \in \lambda(X)^{\beta Y}$, $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly for $(x_j) \in M$.

Theorem 2.2. (T1) *implies* (T2).

Proof. Assume that M has the USGHP but (T2) fails to hold, and hence there exists a topological vector space Y and $(A_j) \in \lambda(X)^{\beta Y}$ such that the convergence of $\sum_{j=1}^{\infty} A_j(x_j)$ is not uniform for $(x_j) \in M$. Then there exists a neighborhood $W \subset Y$ of 0, such that for each $m_0 \in \mathbb{N}$, we have an $m > m_0$ and $x^{m_0} \in M$ for which $\sum_{j=m}^{\infty} A_j\big(x_j^{m_0}\big) \notin W$. Since Y is a topological vector space, for W, there exists a neighborhood $V \subset Y$ of 0, such that $V + V \subset W$.

For $m_0 = 1$, we have $m_1 > 1$ and $x^1 \in M$ such that $\sum_{j=m_1}^{\infty} A_j\big(x_j^1\big) \notin W$. Since $(A_j) \in \lambda(X)^{\beta Y}$, $\sum_{j=1}^{\infty} A_j\big(x_j^1\big)$ converges. For V, there exists $n_1 > m_1$ such that $\sum_{j=n_1+1}^{\infty} A_j\big(x_j^1\big) \in V$. We have $\sum_{j=m_1}^{n_1} A_j\big(x_j^1\big) \notin V$. Otherwise,

$$\sum_{j=m_1}^{\infty} A_j\big(x_j^1\big) = \sum_{j=m_1}^{n_1} A_j\big(x_j^1\big) + \sum_{j=n_1+1}^{\infty} A_j\big(x_j^1\big) \in V + V \subset W$$

which gives a contradiction.

For $m_0 = n_1$, choose $m_2 > n_1$ and $x^2 \in M$ such that $\sum_{j=m_2}^{\infty} A_j\big(x_j^2\big) \notin W$. Then choose $n_2 > m_2$ such that

$$\sum_{j=n_2+1}^{\infty} A_j\big(x_j^2\big) \in V \quad \text{and hence} \quad \sum_{j=m_2}^{n_2} A_j\big(x_j^2\big) \notin V.$$

Consequently, we obtain $\{x^k\} \subset M$ and an increasing sequence of intervals $\{I_k\} = [m_k, n_k]$ such that

$$(\sharp) \qquad \sum_{l \in I_k} A_l(x_l^k) \notin V, \quad \text{for all} \quad k \in \mathbb{N}.$$

By (T1), for $\{x^k\} \subset M$ and $\{I_k\}$, there exists a subsequence $\{k_j\}$ such that $(u_l) = \sum_{j=1}^{\infty} \chi_{I_{k_j}} x^{k_j} \in \lambda(X)$, so $\sum_{l=1}^{\infty} A_l(u_l)$ converges and hence $\sum_{l \in I_{k_j}} A_l(u_l) = \sum_{l \in I_{k_j}} A_l(x_l^{k_j})$ converges to 0. But, this contradicts ($\sharp$). □

Corollary 2.3. *Let X, Y be locally convex spaces and $\lambda(X)$ be monotone, then for $\{f_j\} \subset \big\{f \in Y^X : f(0) = 0\big\}$, the following* (C1) *and* (C2) *are equivalent:*

(C1) *For every $(x_j) \in \lambda(X)$, $\sum_{j=1}^{\infty} f_j(x_j)$ converges weakly;*

(C2) *For every $M \subset \lambda(X)$ satisfying the USGHP, $\sum_{j=1}^{\infty} f_j(x_j)$ converges uniformly for $(x_j) \in M$.*

PROOF. For every $(x_j) \in \lambda(X)$ and integers $j_1 < j_2 < \cdots$ let

$$u_j = \begin{cases} x_{j_k}, & \text{if } j = j_k,\ k = 1, 2, 3, \dots \\ 0, & \text{otherwise.} \end{cases}$$

Since $\lambda(X)$ is monotone, $(u_j) \in \lambda(X)$ and hence $\sum_{j=1}^{\infty} f_j(u_j)$ converges weakly by (C1). Since $f_j(0) = 0$ for all j, it follows from

$$\sum_{k=1}^{n} f_{j_k}(x_{j_k}) = \sum_{j=1}^{j_n} f_j(u_j)$$

that $\sum_{k=1}^{\infty} f_{j_k}(x_{j_k})$ is weakly convergent. By the Orlicz–Pettis theorem [5], $\sum_{j=1}^{\infty} f_j(x_j)$ converges in Y. Hence, (C1) implies that $\{f_j\} \in \lambda(X)^{\beta Y}$ and (C2) holds by Theorem 2.2. □

By the monotone property of $c_0(X)$, $l_\infty(X)$ and $l_p(X)$, Corollary 2.3 holds for these spaces.

3. $c_0(X)$-evaluation convergence

THEOREM 3.1. (T1) *is equivalent to* (T2) *when* $\lambda(X)$ *is* $c_0(X)$.

PROOF. (T1) implies (T2) is obvious by Theorem 2.2. Conversely, assume that (T2) holds but $M \subset c_0(X)$ doesn't have the USGHP. Then there exist $\{x^k\} \subset M$ and an increasing sequence of intervals $\{I_k\}$ such that $(u_j) = \sum_{k=1}^{\infty} \chi_{I_k} x^k \notin c_0(X)$, then

$$u_j = \begin{cases} x_j^k, & \text{if } j \in I_k,\ k = 1, 2, 3, \dots \\ 0, & \text{otherwise.} \end{cases}$$

Hence there exists a $q_0 \in \mathcal{P}$ such that $q_0(u_j) \nrightarrow 0$ as $j \to \infty$, and then there exists a $\delta > 0$, a subsequence $\{k_n\}$ and some $j_n \in I_{k_n}$ such that $q_0\big(x_{j_n}^{k_n}\big) > \delta$ for every $n \in \mathbb{N}$. Define $A_m : X \to c_0(X)$ by

$$A_m(x) = \Big(0, \cdots, 0, \overset{(m)}{x}, 0, \cdots\Big)$$

For $(x_m) \in c_0(X)$, it is obvious that

$$q_\infty\left[(x_m) - \sum_{m=1}^{n} A_m(x_m)\right] = \sup_{m \geqq n+1} q(x_m) \to 0 \quad \text{as} \quad n \to \infty.$$

for every $q \in \mathcal{P}$ and hence $\sum_{m=1}^{\infty} A_m(x_m) = (x_m) \in c_0(X)$. Thus, $(A_m) \in c_0(X)^{\beta c_0(X)}$. However,

$$q_{0\infty}\left(\sum_{l \in I_{k_n}} A_l(x_l^{k_n})\right) \geqq q_0(x_{j_n}^{k_n}) > \delta, \quad \text{for every} \quad n \in \mathbb{N}.$$

This means that the convergence of $\sum_{j=1}^{\infty} A_j(x_j)$ is not uniform for $(x_j) \in M$, which contradicts (T2). □

Let $\mathcal{M}_0 = \{ M \subset c_0(X) : M \text{ has the USGHP} \}$. Theorem 3.1 shows that $\mathcal{M}_0$ is the largest family of subsets for which $(A_j) \in c_0(X)^{\beta Y}$ if and only if $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly with respect to (x_j) in any $M \in \mathcal{M}_0$.

Definition 3.2. $M \subset c_0(X)$ is said to be uniformly vanishing if for each $q \in \mathcal{P}$ $\lim_j q(x_j) = 0$ uniformly for $(x_j) \in M$.

If X is a Banach space, [3] concludes that M is uniformly vanishing if and only if (T2) holds. The next theorem shows that this conclusion holds for the case when X is a locally convex space.

Theorem 3.3. *$M \subset c_0(X)$ has the USGHP if and only if M is uniformly vanishing.*

Proof. Suppose that $M \subset c_0(X)$ has the USGHP but is not uniformly vanishing, then there exists a $q_0 \in \mathcal{P}$ such that $\lim_j q_0(x_j) = 0$ not uniformly for $(x_j) \in M$. For some $\delta > 0$, there exist $\{x^k\} \subset M$ and a subsequence $\{j_k\}$ such that $q_0(x_{j_k}^k) > \delta$. Thus $\{x_{j_k}^k\}_k$ have no subsequence in $c_0(X)$, this contradicts the USGHP of M.

Conversely, suppose that $M \subset c_0(X)$ is uniformly vanishing but doesn't have the USGHP, then there exist $\{x^k\} \subset M$ and an increasing intervals $\{I_k\}$ such that $(u_j) = \sum_{k=1}^{\infty} \chi_{I_k} x^k \notin c_0(X)$. Then there exists a $q_0 \in \mathcal{P}$ such that $q_0(u_j) \nrightarrow 0$ as $j \to \infty$. Hence for some $\delta > 0$, for each $n \in \mathbb{N}$, there exists a $j_n > n$ such that $q_0(u_{j_n}) > \delta$, namely, there exists some k_n such that $u_{j_n} = x_{j_n}^{k_n}$, it follows that $q_0(x_{j_n}^{k_n}) > \delta$, which contradicts the uniformly vanishing property of M. □

4. $l_\infty(X)$-evaluation convergence

THEOREM 4.1. (T1) *is equivalent to* (T2) *when* $\lambda(X)$ *is* $l_\infty(X)$.

PROOF. (T1) implies (T2) is obvious by Theorem 2.2. Conversely, assume that (T2) holds but $M \subset l_\infty(X)$ doesn't have the USGHP. Then there exists $x^k \in M$ and an increasing sequence of intervals $\{I_k\}$ such that $\sum_{k=1}^\infty \chi_{I_k} x^k \notin l_\infty(X)$. Hence there exists $q_0 \in \mathcal{P}$ such that

$$\sup_{j\in I_k,\ k\in\mathbb{N}} q_0\big(x_j^k\big) = +\infty,$$

namely, there exists a subsequence $\{k_n\}$ and some $j_n \in I_{k_n}$ such that $q_0\big(x_{j_n}^{k_n}\big) > n^2$ for every $n \in \mathbb{N}$. Define $A_m : X \to c_0(X)$ by

$$A_m(x) = \begin{cases} \Big(0, \cdots, 0, \overset{(m)}{\frac{1}{n^2}x}, 0, \cdots\Big), & \text{if } m \in I_{k_n} \text{ for some } n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

For $(x_m) \in l_\infty(X)$, let

$$u_m = \begin{cases} \frac{1}{n^2}x_m, & \text{if } m \in I_{k_n} \text{ for some } n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

then $(u_m) \in c_0(X)$ by the boundness of (x_m). It is obvious that

$$q_\infty\bigg((u_m) - \sum_{m=1}^n A_m(x_m)\bigg) = \sup_{m\geqq n+1} q(u_m) \to 0 \quad \text{as} \quad n \to \infty.$$

for every $q \in \mathcal{P}$ and hence $\sum_{m=1}^\infty A_m(x_m) = (u_m) \in c_0(X)$. Thus, $(A_m) \in l_\infty(X)^{\beta c_0(X)}$. However,

$$q_{0\infty}\bigg(\sum_{l\in I_{k_n}} A_l(x_l^{k_n})\bigg) \geqq \frac{1}{n^2} q_0(x_{j_n}^{k_n}) > 1, \quad \text{for every} \quad n \in \mathbb{N}.$$

This means that the convergence of $\sum_{j=1}^\infty A_j(x_j)$ is not uniform for $(x_j) \in M$, which contradicts (T2). □

Let $\mathcal{M}_\infty = \{M \subset l_\infty(X) : M \text{ has the USGHP}\}$. Theorem 4.1 shows that $\mathcal{M}_\infty$ is the largest family of subsets for which $(A_j) \in l_\infty(X)^{\beta Y}$ if and only if $\sum_{j=1}^\infty A_j(x_j)$ converges uniformly with respect to (x_j) in any $M \in \mathcal{M}_\infty$.

Definition 4.2. $M \subset l_\infty(X)$ is said to be essentially bounded if for each $q \in \mathcal{P}$, there exists a $j_0 \in \mathbb{N}$ such that $\sup_{(x_j)\in M,\ j\geqq j_0} q(x_j) < +\infty$.

If X is a Banach space, [3] concludes that M is essentially bounded if and only if (T2) holds. The next theorem shows that this conclusion holds for the case when X is a locally convex space.

Theorem 4.3. *$M \subset l_\infty(X)$ has the USGHP if and only if M is essentially bounded.*

Proof. Suppose that $M \subset l_\infty(X)$ has the USGHP but is not essentially bounded, then there exists a $q_0 \in \mathcal{P}$ such that for each $n \in \mathbb{N}$ we have $\sup_{(x_j)\in M,\ j\geqq n} q_0(x_j) = +\infty$. Then we can find $\{x^k\} \subset M$ and a subsequence $\{j_k\}$ such that $q_0\big(x^k_{j_k}\big)$ is increasing and $q_0\big(x^k_{j_k}\big) > k$. Therefore $\big\{x^k_{j_k}\big\}_k$ has no subsequence in $l_\infty(X)$, which contradicts the USGHP of M.

Conversely, suppose that $M \subset l_\infty(X)$ is essentially bounded but doesn't have the USGHP, then there exist $\{x^k\} \subset M$ and an increasing intervals $\{I_k\}$ such that $(u_j) = \sum_{j=1}^\infty \chi_{I_k} x^k \notin l_\infty(X)$. Then there exists a $q_0 \in \mathcal{P}$ such that $\sup_j q_0(u_j) = +\infty$. Thus for each $n \in \mathbb{N}$, there exists a $j_n > n$ such that $q_0(u_{j_n}) > n$, namely, there exists some k_n such that $q_0(x^{k_n}_{j_n}) = q_0(u_{j_n}) > n$, which contradicts the essentially bounded property of M. □

5. A remark on $l_p(X)$-evaluation convergence

Definition 5.1. $M \subset l_p(X)$ is said to be uniformly exhaustive if for each $q \in \mathcal{P}$ $\lim_n \sum_{j=n}^\infty q(x_j)^p = 0$ uniformly for $(x_j) \in M$.

Let $\mathcal{M}_p = \{M \subset l_p(X) : M \text{ is uniformly exhaustive}\}$. If X is a Banach space, [3] concludes that $\mathcal{M}_p$ is the largest family of subsets for which $(A_j) \in l_p(X)^{\beta Y}$ if and only if $\sum_{j=1}^\infty A_j(x_j)$ converges uniformly with respect to (x_j) in any $M \in \mathcal{M}_p$. However, this conclusion dose not hold for the case when X is a locally convex space.

Theorem 5.2. *If $M \subset l_p(X)$ satisfies* (T2), *then M is uniformly exhaustive.*

PROOF. Define A_j from X to $l_p(X)$ by

$$A_j(x) = \Big(0, \cdots, 0, \overset{(j)}{x}, 0, \cdots\Big)$$

then $(A_j) \in l_p(X)^{\beta l_p(X)}$, and hence $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly for $(x_j) \in M$, which means M is uniformly exhaustive. □

The next example shows that the inverse of Theorem 5.2 is not true.

EXAMPLE 5.3. Let $\mathbb{R}^I$ be the set of all real functions on $[0,1]$, and c_0 be the set of all real sequences converging to zero. For $t \in [0,1]$, let

$$t(f) = \big|f(t)\big|, \quad \text{for each} \quad f \in \mathbb{R}^I,$$

then t is a semi-norm on $\mathbb{R}^I$. It is well known that $\mathbb{R}^I$ is a complete locally convex space endowed with the family of semi-norms $\big\{t : t \in [0,1]\big\}$ [1, Example II.1].

As we know, the cardinality of $[0,1]$ and c_0 is equal. Thus there exists a bijective map which maps $t \in [0,1]$ to $(\alpha_1, \alpha_2, \cdots, \alpha_k, \cdots) \in c_0$. Define $f_k(t) = \alpha_k$, then $f_k \in \mathbb{R}^I$. Let

$$\xi^k = (0, \cdots, 0, f_k, 0, \cdots) \in l^p\big(\mathbb{R}^I\big)$$

and

$$M = \{\xi^k : k \in \mathbb{N}\}.$$

First, $M \subset l_p\big(\mathbb{R}^I\big)$ defined above is uniformly exhaustive. For each $t \in [0,1]$, there exists an unique $(\alpha_k) \in c_0$ mapping to t, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $\big|f_n(t)\big| = |\alpha_n| < \varepsilon$. If $m > N$,

$$\sum_{j=m}^{\infty} \big|t(\xi_j^k)\big|^p = \begin{cases} 0, & \text{if } \ k < m, \\ \big|t(f_k)\big|^p = \big|f_k(t)\big|^p < \varepsilon^p, & \text{if } \ k \geqq m. \end{cases}$$

Therefore, $\lim_n \sum_{j=n}^{\infty} \big|t(\xi_j^k)\big|^p = 0$ uniformly for $\xi^k \in M$, which means that M is uniformly exhaustive.

Next, (T2) dose not hold for $M \subset l_p\big(\mathbb{R}^I\big)$. Define A_j from $\mathbb{R}^I$ to $l_p\big(\mathbb{R}^I\big)$ by

$$A_j(f) = \begin{cases} \Big(0, \cdots, 0, \overset{(j)}{f}, 0, \cdots\Big), & \text{if } \ f \neq f_j \text{ for every } j \in \mathbb{N} \\ \Big(0, \cdots, 0, \overset{(j)}{1_I}, 0, \cdots\Big), & \text{otherwise.} \end{cases}$$

where $1_I \in \mathbb{R}^I$ by defining $1_I(t) \equiv 1$ for $t \in [0,1]$. Then for every subsequence $\{j_k\}$, choose t mapping to $(\alpha_n) \in c_0$ where

$$\alpha_i = \begin{cases} \dfrac{1}{k^{1/p}}, & \text{if } \ i = j_k \text{ for some } k \\ 0, & \text{otherwise.} \end{cases}$$

then $\sum_{k=1}^{\infty} \big| f_{j_k}(t) \big|^p = \infty$, which implies $(f_{j_k}) \notin l^p\big(\mathbb{R}^I\big)$. Thus for each $x = (x_j) \in l^p\big(\mathbb{R}^I\big)$, $\{x_j : j \in \mathbb{N}\}$ contains at most finite elements in $\{f_j : j \in \mathbb{N}\}$. Thus, $(A_j) \in l^p\big(\mathbb{R}^I\big)$. For each t, $A_j(\xi_j^j) = A_j(f_j) = 1_I(t) = 1$, which contradicts (T2).

Above all, $M \subset l_p\big(\mathbb{R}^I\big)$ defined above is uniformly exhaustive but dose not satisfy (T2).

By Theorem 2.2 and Theorem 5.2, if $M \subset l_p(X)$ has the USGHP, then M is uniformly exhaustive. Also, Example 5.3 shows that the inverse is not true. Now, a natural question to ask is whether the inverse of Theorem 2.2 for $l_p(X)$ holds. For a special case when X is a Banach space, the answer is yes.

Theorem 5.4. *If X is a Banach space and $M \subset l_p(X)$ is uniformly exhaustive, then* (T1) *holds.*

Proof. Suppose not, there exist $\{x^k\} \subset M$ and an increasing sequence of intervals $\{I_k\}$ such that $\sum_{j=1}^{\infty} \chi_{I_{k_j}} x^{k_j} \notin l_p(X)$ for every subsequence $\{k_j\}$, namely, $\sum_{j=1}^{\infty} \sum_{l \in I_{k_j}} \big\| x_l^{k_j} \big\|^p = +\infty$. Then we have $\sum_{l \in I_k} \big\| x_l^k \big\|^p \nrightarrow 0$ as $k \to \infty$, namely, there exists a $\delta > 0$ such that $\sum_{l \in I_{k_j}} \big\| x_l^{k_j} \big\|^p > \delta$ for some subsequence $\{k_j\}$, which contradicts the fact that $\lim_n \sum_{j=n}^{\infty} \|x_j\|^p = 0$ uniformly for $(x_j) \in M$. □

REFERENCES

[1] Khaleelulla, S. M., *Counterexamples in Topological Vector Spaces,* Springer-Verlag (Berlin, Heidelberg, 1982). *MR* 0665588 (**84a**:46002)

[2] Li, R., Cui, C. and Cho, M. H., Improverments of Thorp–Rolewicz Theorems on operator series, *Bull. Korean Math. Soc.,* **35(1)** (1998), 75–82. *MR* 1608986 (**98m**:47001)

[3] Li, R., Wang, F. and Zhong, S., The strongest intrinsic meaning of sequential-evaluation convergence, *Topology Appl.,* **154** (2007), 1195–1205. *MR* 2298634 (**2008c**:46013)

[4] MADDOX, I. J., *Infinite Matrices of Operators,* Springer (Berlin, 1980). *MR* 0568707 (**82c**:40012)

[5] MCARTHUR, C. W., On a theorem of Orlicz–Pettis, *Pacific J. Math.,* **22** (1967), 297–302.

[6] ROLEWICZ, S., On unconditional convergence of linear operators, *Demostratio Math.,* **21** (1988), no. 3, 835–842. *MR* 0994316 (**90c**:47030)

[7] SWARTZ, C., The gliding hump property in vector sequence spaces, *Monatsh. Math.,* **116** (1993), no. 2, 147–158. *MR* 1245861 (**94i**:46013)

[8] SWARTZ, C., *Infinite Matrices and the Gliding Hump,* World Scientific (Singapore, 1996). *MR* 1423136 (**98b**:46002)

[9] SWARTZ, C., *Multiplier Convergent Series,* World Scientific (2009). *MR* 2483438 (**2010b**:46002)

[10] THORP, B. L. D., Sequential-evaluation convergence, *J. London Math. Soc.,* **44** (1969), 201–209. *MR* 0236675 (**38**#4970)