Studia Scientiarum Mathematicarum Hungarica 49 (3), 326–337 (2012)
DOI: 10.1556/SScMath.49.2012.3.1205

NILPOTENT ELEMENTS AND MCCOY RINGS

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Communicated by L. Márki

(Received September 12, 2010; accepted February 4, 2011)

Abstract

We introduce the concept of nil-McCoy rings to study the structure of the set of nilpotent elements in McCoy rings. This notion extends the concepts of McCoy rings and nil-Armendariz rings. It is proved that every semicommutative ring is nil-McCoy. We shall give an example to show that nil-McCoy rings need not be semicommutative. Moreover, we show that nil-McCoy rings need not be right linearly McCoy. More examples of nil-McCoy rings are given by various extensions. On the other hand, the properties of α-McCoy rings by considering the polynomials in the skew polynomial ring $R[x;\alpha]$ in place of the ring $R[x]$ are also investigated. For a monomorphism α of a ring R, it is shown that if R is weak α-rigid and α-reversible then R is α-McCoy.

1. Introduction

Throughout this paper, all rings are associative with identity. Given a ring R, we denote by $\operatorname{nil}(R)$ the set of all nilpotent elements of R. In [9], Nielsen introduced the notion of a McCoy ring. A ring R is said to be right McCoy (resp., left McCoy) if for each pair of nonzero polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$, then there exists a nonzero element $r \in R$ with $f(x)r = 0$ (resp., $rg(x) = 0$). A ring R is McCoy if it is both left and right McCoy. The name of the ring was given due to N. H. McCoy who proved in [8] that every commutative ring satisfies this condition. According to [3], a ring R is called right (resp., left) linearly McCoy if for each pair of nonzero polynomials $f(x) = a_0 + a_1x$ and $g(x) = b_0 + b_1x \in R[x]$

2000 *Mathematics Subject Classification.* Primary 16U80, 16S34.
Key words and phrases. Nil-McCoy rings, linearly McCoy rings, α-McCoy rings.

with $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ such that $f(x)r = 0$ (resp., $rg(x) = 0$). A ring R is linearly McCoy if it is both left and right linearly McCoy.

Recently, the reversible property of a ring was extended to a ring endomorphism in [2] as follows: an endomorphism α of a ring R is called right (resp., left) reversible if whenever $ab = 0$ for $a, b \in R$, $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$). A ring R is called right (resp., left) α-reversible if there exists a right (resp., left) reversible endomorphism α of R. R is α-reversible if it is both right and left α-reversible. The notion of an α-reversible ring is a generalization of α-rigid rings as well as an extension of reversible rings. Rege and Chhawchharia [11] introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. Armendariz rings are McCoy, but there exists a McCoy ring which is not Armendariz by [12, Lemma 2.1] and [5, Example 3]. In [1], the nilpotent elements of Armendariz rings were investigated. We shall consider the analogous questions by considering the nilpotent elements in McCoy rings. We first note that every reduced ring (i.e., rings without nonzero nilpotent elements) is McCoy (see [9]). This further motivates the study of the nilpotent elements in McCoy rings.

The aim of this note is to study the structure of the set of nilpotent elements in McCoy rings. In the present paper, we introduce the concept of nil-McCoy rings to study McCoy rings in a general setting. It was shown in [9] that there is a semicommutative ring which is not McCoy. But we shall prove that semicommutative rings are nil-McCoy rings, and we shall give an example to show that nil-McCoy rings need not be semicommutative. Various extensions on nil-McCoy rings are given for constructing more examples of nil-McCoy rings. In particular, among other results, we prove that nil-McCoy rings need not be right linearly McCoy. Moreover, we obtain some characterizations of α-McCoy rings by considering the polynomials in the skew polynomial ring $R[x;\alpha]$ in place of the ring $R[x]$. For a monomorphism α of a ring R, we prove that if R is weak α-rigid and α-reversible then R is α-McCoy.

2. Nil-McCoy rings and examples

Our focus in this section is to introduce the concept of a nil-McCoy ring and consider its properties. Some examples needed in the process are also given. We begin with the following definition.

DEFINITION 2.1. A ring R is said to be right nil-McCoy if $f(x)g(x) \in \operatorname{nil}(R)[x]$, where $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]\backslash\{0\}$, implies that there exists $s \in R\backslash\{0\}$ such that $a_is \in \operatorname{nil}(R)$ for all $0 \leqq i \leqq m$. Left nil-McCoy rings are defined analogously. A ring is nil-McCoy if it is both left and right nil-McCoy.

It is clear that every Armedariz ring, hence every nil-Armendariz ring is nil-McCoy. Therefore, nil-McCoy rings stand as a generalization of McCoy rings and nil-Armendariz rings. We remark that in general the property of being nil-McCoy does not pass to subrings as shown by the following example.

Example 2.2. Let R be a nil-Armendariz ring and let

$$R = \mathbb{F}_2\langle a, b, c, d, e\rangle / I,$$

where I is the ideal generated by the relations: $ac = 0$, $ad + bc = 0$, $bd = 0$, $ea = eb = ec = ed = ee = de = ce = be = ae = 0$. Let S be the subrings generated by elements not involving e. Let $f(x) = a + bx$ and $g(x) = c + dx$, then $f(x)g(x) = 0$. It is straightforward to see that the element as is not nilpotent for any nonzero $s \in S$. However, R is nil-McCoy as $f(x)e, eg(x) \in \operatorname{nil}(R)[x]$ for any polynomials $f(x), g(x) \in R[x]$.

The next example shows that there exists a nil-McCoy ring which is not McCoy.

Example 2.3. Let R be a reduced ring and let

$$T_n(R) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \middle| \, a_{ij} \in R \right\}.$$

For any ring R, the ring of $n \times n$ upper triangular matrices $T_n(R)$ over R is not McCoy by [12, Theorem 2.1]. Note that

$$\operatorname{nil}\big(T_n(R)\big) = \begin{pmatrix} \operatorname{nil}(R) & R & \cdots & R \\ 0 & \operatorname{nil}(R) & \cdots & R \\ 0 & 0 & \ddots & R \\ 0 & 0 & \cdots & \operatorname{nil}(R) \end{pmatrix}.$$

Since R is a reduced ring and every reduced ring is Armendariz, it follows that $T_n(R)$ is nil-McCoy.

Based on Example 2.3, one may suspect that if R is nil-McCoy then $M_n(R)$ is nil-McCoy for $n \geqq 2$. But the following example eliminates the possibility.

EXAMPLE 2.4. Let R be a reduced ring, then R is nil-McCoy. Let $S = M_2(R)$ and let

$$f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x,$$

$$g(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} x$$

be polynomials in $S[x]$ with $f(x)g(x) \in \operatorname{nil}(S)[x]$, then it is straightforward to check that S is not nil-McCoy.

Recall that by [9] reduced rings are McCoy rings. We can prove a similar condition in the case that the set of nilpotent elements forms an ideal.

PROPOSITION 2.5. *Let R be a ring such that* $\operatorname{nil}(R)$ *is an ideal of R. If $f(x)g(x) \in \operatorname{nil}(R)[x]$, then there exists $s \in R\backslash\{0\}$ such that $a_i s \in \operatorname{nil}(R)$ for all $0 \leqq i \leqq m$, where $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]\backslash\{0\}$.*

PROOF. It is easy to see that $R/\operatorname{nil}(R)$ is reduced and hence McCoy. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]\backslash\{0\}$ such that $f(x)g(x) \in \operatorname{nil}(R)[x]$. If we denote by $\bar{f}(x)$, $\bar{g}(x)$ their corresponding polynomials in $R/\operatorname{nil}(R)[x]$, then we have $\bar{f}(x)\bar{g}(x) = \bar{0}$. Since $R/\operatorname{nil}(R)$ is McCoy, there exists $s \in R\backslash\{0\}$ such that $\bar{a}_i\bar{s} = \bar{0}$ for all $0 \leqq i \leqq m$. Hence $a_i s \in \operatorname{nil}(R)$ for all a_i, $0 \leqq i \leqq m$. □

Note that if $\operatorname{nil}(R)$ is an ideal of R, then R is right nil-McCoy by Proposition 2.5. More generally, we obtain the following proposition.

PROPOSITION 2.6. *Let R be a ring and I a nil ideal of R. Then R is nil-McCoy if and only if R/I is nil-McCoy.*

PROOF. Let $\bar{R} = R/I$, then we have $\operatorname{nil}(\bar{R}) = \overline{\operatorname{nil}(R)}$ since I is nil. It follows that $f(x)g(x) \in \operatorname{nil}(R)[x]$ if and only if $\bar{f}(x)\bar{g}(x) \in \operatorname{nil}(\bar{R})[x]$. This implies that if there exists $s \in R\backslash\{0\}$, then $a_i s \in \operatorname{nil}(R)$ if and only if $\bar{a}_i\bar{s} \in \operatorname{nil}(\bar{R})$ for all a_i, $0 \leqq i \leqq m$. Therefore, R is nil-McCoy if and only if $\bar{R}$ is nil-McCoy. □

Recall that a ring R is semicommutative if $ab = 0$ implies $aRb = 0$. To study the semicommutative condition in McCoy rings, we need the following lemma which was proved in [7, Proposition 3.3]. For any $f(x) \in R[x]$, we denote by C_f the set of all coefficients of f.

LEMMA 2.7. *Let R be a semicommutative ring and $f_1(x), f_2(x), \ldots, f_n(x)$ be in $R[x]$. If $C_{f_1 f_2 \cdots f_n} \subseteqq \operatorname{nil}(R)$, then $C_{f_1}C_{f_2}\cdots C_{f_n} \subseteqq \operatorname{nil}(R)$.*

By the main result of [9], there exists a semicommutative ring, which is not right McCoy. But we have the following result which shows that all semicommutative rings are nil-McCoy. We shall give an example to show that in general the inverse of the next proposition is not true.

PROPOSITION 2.8. *Semicommutative rings are nil-McCoy rings.*

PROOF. Let

$$f(x)=\sum_{i=0}^{m}a_ix^i,\quad g(x)=\sum_{j=0}^{n}b_jx^j\in R[x]\backslash\{0\}$$

with $f(x)g(x)\in \operatorname{nil}(R)[x]$, then we have $C_{fg}\in \operatorname{nil}(R)$. It follows that $C_fC_g\in \operatorname{nil}(R)$ by Lemma 2.7. Hence there exists $s=b_j$ for some $0\leqq j\leqq n$ such that $a_is\in \operatorname{nil}(R)$ with $0\leqq i\leqq m$. This implies that R is a right nil-McCoy ring. Similarly, we can show that R is left nil-McCoy. Therefore, R is a nil-McCoy ring. □

We now have the following description of the rings which shows one way to give more nil-McCoy rings from old ones.

PROPOSITION 2.9. *Let* Λ *be an index set and* $\{R_\alpha|_{\alpha\in\Lambda}\}$ *a family of rings. If* $R=\Pi_{\alpha\in\Lambda}R_\alpha$*, then* R *is right nil-McCoy if and only if every* R_α *is right nil-McCoy for each* $\alpha\in\Lambda$.

PROOF. It is straightforward to verify that if R is right nil-McCoy, then every R_α is right nil-McCoy for each α. Conversely, if $f(x)g(x)\in \operatorname{nil}(R)[x]$ for $f(x)=\sum_{i=0}^m a_ix^i$ and $g(x)=\sum_{j=0}^n b_jx^j$ in $R[x]\backslash\{0\}$, where $a_i=(a_{i\alpha})_{\alpha\in\Lambda}$, $b_j=(b_{j\alpha})_{\alpha\in\Lambda}\in R$. For each $\alpha\in\Lambda$, let $f_\alpha(x)=\sum_{i=0}^m a_{i\alpha}x^i$, $g_\alpha(x)=\sum_{j=0}^n b_{j\alpha}x^j\in R_\alpha[x]$, then $f_\alpha g_\alpha\in \operatorname{nil}(R_\alpha)[x]$. Since $g(x)\neq 0$, there exists some index $\beta\in\Lambda$ with $g_\beta(x)\neq 0$. In particular, there exists some nonzero $r_\beta\in R_\beta$ with $a_{i\beta}r_\beta\in \operatorname{nil}(R_\beta)$ for all $0\leqq i\leqq m$ by the nil-McCoy property of R_β. Fix $r_\beta\in R_\beta\backslash\{0\}$, and take r to be the sequence with r_β in the βth coordinate and zero elsewhere. Clearly $a_ir\in \operatorname{nil}(R)$ and $r\neq 0$. □

COROLLARY 2.10. *A finite direct product of right nil-McCoy rings is right nil-McCoy.*

Let R be a ring and $\triangle$ be a multiplicative monoid in R consisting of central regular elements, and let $\triangle^{-1}R=\{u^{-1}a\mid u\in\triangle,\, a\in R\}$, then $\triangle^{-1}R$ is a ring. We have the following result concerning nil-McCoy properties.

PROPOSITION 2.11. *A ring* R *is right nil-McCoy if and only if* $\triangle^{-1}R$ *is right nil-McCoy.*

PROOF. (1) ⇒ (2). Let $S=\triangle^{-1}R$. If $f(x)=\sum_{i=0}^m\alpha_ix^i$ and $g(x)=\sum_{j=0}^n\beta_jx^j$ are in $S[x]\backslash\{0\}$, then we can assume that $\alpha_i=a_iu^{-1}$ and $\beta_j=b_jv^{-1}$ for some $a_i,b_j\in R$, $u,v\in\Delta$ for all i, j. Now suppose that $f(x)g(x)\in \operatorname{nil}(S)[x]$, then we have

$$f(x)g(x)=\sum_{i,j}\alpha_i\beta_jx^{i+j}=\sum_{i,j}a_iu^{-1}b_jv^{-1}x^{i+j}=\sum_{i,j}a_ib_j(vu)^{-1}x^{i+j}.$$

On the other hand, let $f_1(x) = \sum_{i=0}^{m} a_i x^i$ and $g_1(x) = \sum_{j=0}^{n} b_j x^j$, then we obtain $f_1(x)g_1(x) = \sum_{i,j} a_i b_j x^{i+j} \in \operatorname{nil}(R)[x]$ since $(vu)^{-1} \in \Delta$. By the hypothesis, there exists $t \in R \backslash \{0\}$ such that $a_i t \in \operatorname{nil}(R)$ for all $0 \leqq i \leqq m$. Set $s = tv^{-1}$, then $s \in S$. Now it is easy to see that $\alpha_i s = a_i u^{-1} t v^{-1}$ is also a nilpotent element of S for all $0 \leqq i \leqq m$. This shows that S is a right nil-McCoy ring.

(2) $\Rightarrow$ (1). Let $f(x) = \sum_{i=0}^{m} \alpha_i x^i$ and $g(x) = \sum_{j=0}^{n} \beta_j x^j$ be in $R[x]\backslash\{0\}$ such that $f(x)g(x) \in \operatorname{nil}(R)[x]$. Then there exists a nonzero element $\alpha \in S$ such that $a_i \alpha \in \operatorname{nil}(R)$ since S is right McCoy. We can assume $\alpha = au^{-1}$ for some $a \in R\backslash\{0\}$ and central regular element u. Then $a_i a \in \operatorname{nil}(R)$ since u is a central regular element. Therefore, R is right nil-McCoy. □

The ring of Laurent polynomials in x, with coefficients in a ring R, consists of all formal sum $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers. Denote it by $R\left[x; x^{-1}\right]$. As an immediate consequence of this proposition, we have the following result.

Corollary 2.12. *Let R be a ring. If $R[x]$ is a right nil-McCoy ring, then $R\left[x; x^{-1}\right]$ is right nil-McCoy.*

Proof. Let $\triangle = \{1, x, x^2, \dots\}$, then clearly $\triangle$ is a multipicatively closed subset of $R[x]$. Since $R\left[x; x^{-1}\right] \cong \triangle^{-1} R[x]$, it follows directly from Proposition 2.11 that $R\left[x; x^{-1}\right]$ is right nil-McCoy. □

Corollary 2.13. *Let R be a commutative ring. If R is right nil-McCoy, then so is the total quotient ring of R.*

Proof. Let $\triangle$ be the set of all regular elements of R. Then $\triangle$ satisfies the condition of Proposition 2.11 and $\triangle^{-1}R$ is the total quotient ring of R. Thus the total quotient ring of R is right nil-McCoy. □

Now we consider the case of direct limits of direct systems of right nil-McCoy rings.

Proposition 2.14. *The direct limit of a direct system of right nil-McCoy rings is also right nil-McCoy.*

Proof. Let $D = \{R_i, \phi_{ij}\}$ be a direct system of right nil-McCoy rings R_i for $i \in I$ and ring homomorphisms $\phi_{ij} : R_i \to R_j$ for each $i \leqq j$ satisfying $\phi_{ij}(1) = 1$, where I is a direct partially ordered set. Let $R = \varinjlim R_i$ be the direct limit of D with $\iota_i : R_i \to R$ and $\iota_j \phi_{ij} = \iota_i$. We shall prove that R is a right nil-McCoy ring. Take $x, y \in R$. It follows that $x = \iota_i(x_i), y = \iota_j(y_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leqq k$, $j \leqq k$. Now define $x + y = \iota_k\big(\phi_{ik}(x_i) + \phi_{jk}(y_j)\big)$ and $xy = \iota_k\big(\phi_{ik}(x_i)\phi_{jk}(y_j)\big)$, where $\phi_{ik}(x_i)$ and $\phi_{jk}(y_j)$ are in R_k. It is easy to see that R forms a ring with $0 = \iota_i(0)$

and $1 = \iota_i(1)$. Let $f(x)g(x) \in \mathrm{nil}\,(R)[x]$ with $f(x) = \sum_{s=0}^{m} a_s x^s$ and $g(x) = \sum_{t=0}^{n} b_t x^t$ in $R[x]$. Then there are $i_s,\ j_t,\ k \in I$ such that $a_s = \iota_{i_s}(a_{i_s})$, $b_t = \iota_{j_t}(b_{j_t}),\ i_s \leqq k,\ j_t \leqq k$. So we have $a_s b_t = \iota_k\big(\phi_{i_s k}(a_{i_s})\phi_{j_t k}(b_{j_t})\big)$, and from $f(x)g(x) \in \mathrm{nil}\,(R)[x]$ we have

$$f(x)g(x) = \left(\sum_{s=0}^{m} \iota_k\big(\phi_{i_s k}(a_{i_s})\big)\, x^s\right)\left(\sum_{t=0}^{n} \iota_k\big(\phi_{j_t k}(b_{j_t})\big)\, x^t\right) \in \mathrm{nil}\,(R_k)[x].$$

Since R_k is right nil-McCoy, there exists $s_k \in R_k \backslash \{0\}$ such that

$$\iota_k\big(\phi_{i_s k}(a_{i_s})\big)\, s_k \in \mathrm{nil}\,(R_k) \quad \text{for all} \quad 0 \leqq i \leqq m.$$

Let $s = \iota_k(s_k)$, then we have $a_s s \in \mathrm{nil}\,(R)$ and R is right nil-McCoy. □

We have seen that if I is a nil ideal and R/I is nil-McCoy, then R is nil-McCoy. Hence, if R is a nil-McCoy ring, then $S = R[x]/(x^n)$ is nil-McCoy since xS is a nil ideal and $S/xS \simeq R$.

Proposition 2.15. *If R is a nil-McCoy ring and $n \geqq 1$, then $S = R[x]/(x^n)$ is nil-McCoy.*

Let S be a ring and define

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a \end{pmatrix} \middle|\; a, a_{ij} \in S \right\},$$

where $n \geqq 2$ is a positive integer. We denote by e_{ij} the usual matrix unit with 1 in the (i,j) coordinate and zero elsewhere for each i, j. Then we have the following

Proposition 2.16. *Let S be a ring and R_n $(n \geqq 2)$ be the ring defined as above. Then S is a right linearly McCoy ring if and only if R_n is a right linearly McCoy ring.*

Proof. Let

$$F(x) = \begin{pmatrix} a_0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_0 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_0 & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & a_1 & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a_1 & \cdots & a'_{3n} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_1 \end{pmatrix} x,$$

$$G(x) = \begin{pmatrix} b_0 & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b_0 & b_{23} & \cdots & b_{2n} \\ 0 & 0 & b_0 & \cdots & b_{3n} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & b_0 \end{pmatrix} + \begin{pmatrix} b_1 & b'_{12} & b'_{13} & \cdots & {}'_{1n} \\ 0 & b_1 & b'_{23} & \cdots & b'_{2n} \\ 0 & 0 & b_1 & \cdots & b'_{3n} \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & b_1 \end{pmatrix} x$$

be two nonzero polynomials in $R_n(x)$ with $F(x)G(x) = 0$. Let $f(x) = a_0 + a_1x$, $f_{ij}(x) = a_{ij} + a'_{ij}x$, $g(x) = b_0 + b_1x$, $g_{ij}(x) = b_{ij} + b'_{ij}x$ for $i = 1, 2, \ldots, n$, $j = 2, 3, \ldots, n$ and $i < j$. We proceed with the following three cases.

Case 1. Suppose $f(x) \neq 0, g(x) \neq 0$. In this case, we obtain that $f(x)g(x) = 0$ since $F(x)G(x) = 0$. Then there exists $r \in R\backslash\{0\}$ such that $f(x)r = 0$. Take $R = e_{1n}r$, then $F(x)R = 0$.

Case 2. Suppose $f(x) \neq 0$, $g(x) = 0$. Since $G(x) \neq 0$, there exists $g_{i,j}(x) \neq 0$ such that $g_{i+k,j}(x) = 0$, $1 \leqq k \leqq n - i$. Then $f(x)g_{i,j}(x) = 0$ and so there exists $r \in R\backslash\{0\}$ such that $f(x)r = 0$ since S is a right linearly McCoy ring. Put $R = e_{1n}r$, then $F(x)R = 0$. If $f(x) = 0$, $g(x) \neq 0$, we can also prove the result.

Case 3. Suppose $f(x) = 0$, $g(x) = 0$. For any $0 \neq r \in R$, let $R = e_{1n}r$, then $F(x)R = 0$.

Conversely, let $f(x) = a_0 + a_1(x)$, $g(x) = b_0 + b_1(x) \in S[x]\backslash\{0\}$ with $f(x)g(x) = 0$. Let $F(x) = f(x)I_n$, $G(x) = g(x)I_n$, then $F(x), G(x) \in R_n[x]\backslash\{0\}$ with $F(x)G(x) = 0$. Since R_n is a right linearly McCoy ring, there exists $A \in R_n\backslash\{0\}$ such that $F(x)A = 0$. Now it is easy to check that there exists $r \in R\backslash\{0\}$ such that $f(x)r = 0$. This shows that S is a right linearly McCoy ring. □

The next example shows that there exists a nil-McCoy ring which is not right linearly McCoy.

Example 2.17. Let R be a reduced ring and let $S = T_n(R)$. Let $f(x) = -e_{12} + e_{11}x$ and $g(x) = e_{12} + e_{22}x \in T_2(R)$, then $f(x), g(x) \in S[x]\backslash\{0\}$ with $f(x)g(x) = 0$. It is straightforward to check that S is not right linearly McCoy. But S is a nil-McCoy ring by Example 2.3.

Note 2.18. From Example 2.17, we know that if R is right linearly McCoy, then every n-by-n upper triangular matrix ring $T_n(R)$ need not be right linearly McCoy. It was proved in [3, Proposition 5.3] that all semicommutative rings are right linearly McCoy rings. Hence Example 2.17 also shows that there exists a nil-McCoy ring which is not semicommutative. This implies that the inverse of Proposition 2.8 need not hold.

3. α-McCoy rings

Recall that for a ring R with a ring endomorphism $\alpha : R \to R$, a skew polynomial ring (also called an Ore extension of endomorphism type) $R[x;\alpha]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$. By analogy with the case of a nil-McCoy ring, we give the following definition.

DEFINITION 3.1. A ring R is said to be right α-McCoy if for each pair of nonzero skew polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\alpha]$ with $f(x)g(x) = 0$, then there exists a nonzero element $s \in R$ such that $a_i s \in \mathrm{nil}\,(R)$ for all $0 \leqq i \leqq m$. Left α-McCoy rings are defined analogously. A ring is α-McCoy if it is both left and right α-McCoy.

Following [6], an endomorphism α of a ring R is said to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for all $a \in R$. A ring R is said to be α-rigid if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced.

According to [4], a ring R is said to be α-compatible if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$, where α is an endomorphism of R. The notion of the α-compatible ring is a generalization of α-rigid rings to the more general case where R is not assumed to be reduced.

In the following we give an example of a right α-reversible ring which is α-compatible.

EXAMPLE 3.2. Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{Z}_4 \right\}.$$

Suppose that $\alpha : R \to R$ is an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

It is straightforward to verify that R is right α-reversible and α-compatible. Note that α is an automorphism of R and R is not α-rigid.

PROPOSITION 3.3. *Let R be right α-reversible and α-compatible, where α is a monomorphism. If $\{a_0, a_1, \dots, a_n\} \subseteqq \mathrm{nil}\,(R)$ for $f(x) = a_0 + a_1 x + \dots + a_n x^n \in R[x;\alpha]$, then $f(x)$ is a nilpotent element of $R[x;\alpha]$.*

PROOF. Note that R is semicommutative since R is right α-reversible and α is a monomorphism by [2]. Suppose that $a_i^{m_i} = 0$, $i = 0, 1, \dots, n$. Let

$k = m_0 + m_1 + \cdots + m_n + 1$. Then we have

$$\big(f(x)\big)^k = \sum_{s=0}^{kn} \Bigg(\sum_{i_1+i_2+\cdots+i_k=s} a_{i_1}\alpha^{i_1}(a_{i_2})\alpha^{i_1+i_2}(a_{i_3})\cdots\alpha^{i_1+i_2+\cdots+i_{k-1}}(a_{i_k}) \Bigg) x^s,$$

where $a_{i_1}, a_{i_2}, \ldots, a_{i_k} \in \{a_0, a_1, \ldots, a_n\}$. Consider

$$(*) \qquad a_{i_1}\alpha^{i_1}(a_{i_2})\alpha^{i_1+i_2}(a_{i_3})\cdots\alpha^{i_1+i_2+\cdots+i_{k-1}}(a_{i_k}).$$

There exists $a_t \in \{a_0, a_1, \ldots, a_n\}$ such that the numbers of a_t are more than m_t in $(*)$. We may assume that a_t appears $s > m_t$ times in $(*)$. Rewrite $(*)$ as

$$(\star) \qquad b_0\alpha^{j_1}(a_t)b_1\alpha^{j_1+j_2}(a_t)\ldots b_{s-1}\alpha^{j_1+j_2+\cdots+j_s}(a_t)b_s,$$

where $b_i \in R, j_1, j_2, \ldots, j_s \in N$. Since $a_t^s = 0$, R is semicommutative and α-compatible, we can obtain $(\star) = 0$, hence Equation $(*) = 0$. It follows that

$$\sum_{i_1+i_2+\cdots+i_k=s} a_{i_1}\alpha^{i_1}(a_{i_2})\alpha^{i_1+i_2}(a_{i_3})\ldots\alpha^{i_1+i_2+\cdots+i_{k-1}}(a_{i_k}) = 0,$$

which implies that $f(x)$ is a nilpotent element of $R[x;\alpha]$. □

COROLLARY 3.4. *Let R be α-rigid,*

$$f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha].$$

If $\{a_0, a_1, \ldots, a_n\} \subseteqq \mathrm{nil}\,(R)$, then $f(x)$ is a nilpotent element of $R[x;\alpha]$.

The next example shows that there exists an α-compatible ring R such that R is not α-reversible. This shows that the assumption in Proposition 3.3 is not superfluous.

EXAMPLE 3.5. Let R be an α-rigid ring. Let

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d \in R \right\}.$$

The endomorphism α of R is extended to the endomorphism $\overline{\alpha} : S \to S$ defined by $\overline{\alpha}\big((a_{ij})\big) = \big(\alpha(a_{ij})\big)$. Then S is not $\bar{\alpha}$-reversible by [2, Example 2.20], but S is $\bar{\alpha}$-compatible by [4, Example 1.2].

In [10], Ouyang introduced weak α-rigid rings as a generalization of α-rigid rings. A ring R is weak α-rigid if $a\alpha(a) \in \mathrm{nil}\,(R) \Leftrightarrow a \in \mathrm{nil}\,(R)$ for $a \in R$. Note that Example 3.5 gives a ring which is weak α-rigid but not α-reversible by [10, Example 2.1].

Proposition 3.6. *Let α be a monomorphism of a ring R. If R is weak α-rigid and α-reversible, then R is an α-McCoy ring.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$. Then $f(x)g(x) = \sum_{k=0}^{m+n} \big(\sum_{i+j=k} a_i \alpha^i(b_j) \big) x^k$, and so $\sum_{i+j=k} a_i \alpha^i(b_j) = 0$ for any $0 \leqq k \leqq m+n$. In the following we claim that $a_i b_j \in \mathrm{nil}\,(R)$ by induction on $i+j$. Then we obtain $a_0 b_0 = 0$. This is true for $i+j=0$. Now suppose that our claim holds for $i+j<k$, where $1 \leqq k \leqq m+n$. Note that

$$(*) \qquad \sum_{i+j=k} a_i \alpha^i(b_j) = 0$$

By the induction hypothesis, we have $a_i b_0 \in \mathrm{nil}\,(R)$ for all $0 \leqq i < k$, and so $b_0 a_i \in \mathrm{nil}\,(R)$ for all $0 \leqq i < k$. Note that R is semicommutative since R is α-reversible and α is a monomorphism by [2]. If we multiply equation $(*)$ on the left side by b_0, then we have $b_0 a_k \alpha^k(b_0) \in \mathrm{nil}\,(R)$ since $\mathrm{nil}\,(R)$ is an ideal of R by [7, Lemma 3.1], and hence $b_0 a_k b_0 \in \mathrm{nil}\,(R)$ by [10, Proposition 2.3]. This implies that $b_0 a_k \in \mathrm{nil}\,(R)$, $a_k b_0 \in \mathrm{nil}\,(R)$. Thus $a_k \alpha^k(b_0) \in \mathrm{nil}\,(R)$ again by [10, Proposition 2.3]. Continuing this process, we can prove $a_i b_j \in \mathrm{nil}\,(R)$ for $i+j=k$. Therefore, we obtain $a_i b_j \in \mathrm{nil}\,(R)$ for all i, j by induction. This shows that R is α-McCoy. □

Corollary 3.7. *Let α be an endomorphism of a ring R. If R is an α-rigid ring, then R is α-McCoy.*

Acknowledgments. The authors wish to express their sincere thanks to the referee for his/her careful reading of the paper and the comments that improved the present article, especially the construction of Example 2.2. This work was supported by the Program Sponsored for Scientific Innovation Research of College Graduate in Jiangsu Province (No. CX10B_001Z) and was partially supported by the National Natural Science Foundation of China (No. 10971090 and No. 11071097).

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