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CHARACTERIZATIONS OF THE AMOROSO DISTRIBUTION

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Abstract

Characterizations of the Amoroso distribution based on a simple relationship between two truncated moments are presented. A remark regarding the characterization of certain special cases of the Amoroso distribution based on hazard function is given. We will also point out that a sub-family of the Amoroso family is a member of the generalized Pearson system.

1. Introduction

As we have mentioned in our previous characterization works, the problem of characterizing a distribution is an important problem which has recently attracted the attention of many researchers. Thus various characterizations have been established in many different directions. The present work deals with the characterization of the Amoroso distribution, the natural unification of the gamma and extreme value distributions, based on a simple relationship between two truncated moments.

It is pointed out by Crooks [1] that the Amoroso distribution, a four parameter, continuous, univariate, unimodel *pdf* (probability density function), with semi infinite range, was originally developed to model lifetimes (see [1] for more details). Moreover, many well-known and important distributions are special cases or limiting forms of the Amoroso distribution.

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Table 1 below is taken (with permission from G. E. Crooks for which we are grateful to him) from [1], which shows 35 special and 4 limiting cases of the Amoroso distribution.

The *pdf* of the Amoroso distribution is given by

$$f(x;a,\alpha,\tau,k)=\frac{1}{\Gamma(k)}\left|\frac{\tau}{\alpha}\right|\left(\frac{x-a}{\alpha}\right)^{\tau k-1}\exp\left\{-\left(\frac{x-a}{\alpha}\right)^{\tau}\right\}, \tag{1.1}$$

for x, a, α, τ in $\mathbb{R}$, $k>0$, support $x \geqq a$ if $\alpha>0$, $x \leqq a$ if $\alpha<0$.

For further details about the distributions listed in Table 1 and their applications, we refer the reader to Crooks [1].

We give below a table (Table 2) displaying four cases based on the signs of α and τ for the random variable $X \sim \text{Amoroso}\,(a,\alpha,\tau,k)$. Without loss of generality we assume $a=0$ throughout this work.

For $\alpha>0$ and $\tau>0$, $\text{Amoroso}\,(0,\alpha,\tau,k)=GG(\alpha,\tau,k)$, generalized gamma distribution, which has been characterized based on a simple relationship between two truncated moments in Hamedani [11] (subsection 2.5). Some special cases of $X\sim GG(\alpha,\tau,k)$ based on hazard function have been characterized in Hamedani and Ahsanullah [13] (subsection 2.10). These characterizations are valid for the distributions of $-X$ (when $\alpha<0$, $\tau>0$), $\frac{1}{X}$ (when $\alpha>0$, $\tau<0$) and $-\frac{1}{X}$ (when $\alpha<0$, $\tau<0$). Table 2 shows that for $\alpha<0$ a simple change of parameters $\alpha'=-\alpha$ will produce the cases on the second row of the table. So, we investigate here the characterization of the distribution of X when $\alpha>0$ and $\tau<0$. Letting $\gamma=-\tau>0$, we shall now express the *pdf* of the Amoroso random variable X as

$$f(x;\alpha,\gamma,k)=\frac{\gamma}{\alpha\Gamma(k)}\left(\frac{x}{\alpha}\right)^{-(\gamma k+1)}\exp\left\{-\left(\frac{x}{\alpha}\right)^{-\gamma}\right\},\quad x\geqq 0, \tag{1.2}$$

where all three parameters α, γ and k are positive.

The *cdf* (cumulative distribution function), F, corresponding to (1.2) is

$$F(x)=1-\frac{1}{\Gamma(k)}\int_0^{\left(\frac{x}{\alpha}\right)^{-\gamma}} u^{k-1}e^{-u}\,du,\quad x\geqq 0.$$

Table 1. The Amoroso family of distributions
m, n positive integers

Amoroso	a	α	k	τ	
Stacy	0	$\cdot$	$\cdot$	$\cdot$	
gen. Fisher–Tippet	$\cdot$	$\cdot$	n	$\cdot$	
Fisher–Tippet	$\cdot$	$\cdot$	1	$\cdot$	
Fréchet	$\cdot$	$\cdot$	1	< 0	
generalized Fréchet	$\cdot$	$\cdot$	n	< 0	
scaled inverse chi	0	$\cdot$	$\frac{m}{2}$	-2	
inverse chi	0	$\frac{1}{\sqrt{2}}$	$\frac{m}{2}$	-2	
inverse Rayleigh	0	$\cdot$	1	-2	
Pearson type V	$\cdot$	$\cdot$	$\cdot$	-1	
inverse gamma	0	$\cdot$	$\cdot$	-1	
scaled inverse chi-square	0	$\cdot$	$\frac{m}{2}$	-1	
inverse chi-square	0	$\frac{1}{2}$	$\frac{m}{2}$	-1	
Lévy	$\cdot$	$\cdot$	$\frac{1}{2}$	-1	
inverse exponential	0	$\cdot$	1	-1	
Pearson type III	$\cdot$	$\cdot$	$\cdot$	1	
gamma	0	$\cdot$	$\cdot$	1	
Erlang	0	> 0	n	1	
standard gamma	0	1	$\cdot$	1	
scaled chi-square	0	$\cdot$	$\frac{m}{2}$	1	
chi-square	0	2	$\frac{m}{2}$	1	
shifted exponential	$\cdot$	$\cdot$	1	1	
exponential	0	$\cdot$	1	1	
standard exponential	0	1	1	1	
Wien	0	$\cdot$	4	1	
Nakagami	$\cdot$	$\cdot$	$\cdot$	2	
scaled chi	0	$\cdot$	$\frac{m}{2}$	2	
chi	0	$\sqrt{2}$	$\frac{m}{2}$	2	
half-normal	0	$\cdot$	$\frac{1}{2}$	2	
Rayleigh	0	$\cdot$	1	2	
Maxwell	0	$\cdot$	$\frac{3}{2}$	2	
Wilson–Hilferty	0	$\cdot$	$\cdot$	3	
generalized Weibull	$\cdot$	$\cdot$	n	> 0	
Weibull	$\cdot$	$\cdot$	1	> 0	
pseudo-Weibull	$\cdot$	$\cdot$	$1 + \frac{1}{\tau}$	> 0	
stretched exponential	0	$\cdot$	1	> 0	
log-gamma	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\lim_{\tau \to \infty}$
power law	$\cdot$	$\cdot$	$\frac{1-p}{\tau}$	$\cdot$	$\lim_{\tau \to 0}$
log-normal	$\cdot$	$\cdot$	$\frac{1}{(\tau\sigma)^2}$	$\cdot$	$\lim_{\tau \to 0}$
normal	$\cdot$	$\cdot$	$\cdot$	1	$\lim_{k \to \infty}$

Table 2

	$\tau > 0$	$\tau < 0$
$\alpha > 0$	$X \sim GG(\alpha, \tau, k)$	$\frac{1}{X} \sim GG\big(\frac{1}{\alpha}, -\tau, k\big)$
$\alpha < 0$	$-X \sim GG(-\alpha, \tau, k)$	$-\frac{1}{X} \sim GG\big(-\frac{1}{\alpha}, -\tau, k\big)$

2. Characterization results

2.1. Characterizations based on two truncated moments

In this subsection we present characterizations of the Amoroso distribution with *pdf* (1.2) in terms of a simple relationship between two truncated moments. We like to mention here the work of Galambos and Kotz [2], Kotz and Shanbhag [15], Glänzel [3]–[5], Glänzel et al. [7]–[8], Glänzel and Hamedani [6] and Hamedani [9]–[11] in this direction. Our characterization results presented here will employ an interesting result due to Glänzel [4], which is stated here (Theorem G below) for the sake of completeness.

THEOREM G. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $a < b$ ($a = -\infty$, $b = \infty$ might as well be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that*

$$E\big[g(X) \mid X \geqq x\big] = E\big[h(X) \mid X \geqq x\big]\,\lambda(x), \quad x \in H,$$

defined with some real function λ. Assume that $g, h \in C^1(H)$, $\lambda \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $h\lambda = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and λ, particularly

$$F(x) = \int_a^x C \left| \frac{\lambda'(u)}{\lambda(u)h(u) - g(u)} \right| \exp\big(-s(u)\big)\,du,$$

where the function s is a solution of the differential equation $s' = \frac{\lambda' h}{\lambda h - g}$ and C is a constant to make $\int_H dF = 1$.

REMARKS 2.1.1. (a) In Theorem G, the interval H need not be closed. (b) The goal is to have the function λ as simple as possible. For a detailed discussion on the choice of λ, we refer the reader to Glänzel and Hamedani [6] and Hamedani [9]–[11].

We shall consider two cases:

Case (i). $\gamma k+1=\gamma$. Then *pdf* of Amoroso distribution (1.2), in this case, is

$$f(x;\alpha,\gamma,k)=\frac{\gamma}{\alpha\Gamma(k)}\left(\frac{x}{\alpha}\right)^{-\gamma}e^{-\left(\frac{x}{\alpha}\right)^{-\gamma}},\quad x>0, \tag{2.1.1}$$

where all three parameters α, γ and k are positive subject to $\gamma k+1=\gamma$.

PROPOSITION 2.1.2. *Let $X:\Omega\to[0,\infty)$ be a continuous random variable and let $h(x)=\frac{\gamma}{\alpha}\left[\left(\frac{x}{\alpha}\right)^{-1}-\left(\frac{x}{\alpha}\right)^{-\gamma-1}\right]$ for $x\in(0,\infty)$. The pdf of X is* (2.1.1) *if and only if there exist functions g and λ defined in Theorem* G *satisfying the differential equation*

$$\frac{\lambda'(x)}{\lambda(x)h(x)-g(x)}=1,\quad x>0. \tag{2.1.2}$$

PROOF. Let X have *pdf* (2.1.1) and let

$$g(x)=\frac{1}{\gamma}\left(\frac{x}{\alpha}\right)^{\gamma-2}-\left(\frac{x}{\alpha}\right)^{-2},\quad x>0$$

and

$$\lambda(x)=\frac{\alpha}{\gamma}\left(\frac{x}{\alpha}\right)^{\gamma-1},\quad x>0.$$

Then, after some computations we arrive at

$$\big(1-F(x)\big)E\big[h(X)\mid X\geqq x\big]=\frac{\gamma}{\alpha\Gamma(k)}\left(\frac{x}{\alpha}\right)^{-\gamma}e^{-\left(\frac{x}{\alpha}\right)^{-\gamma}},\quad x>0,$$

$$\big(1-F(x)\big)E\big[g(X)\mid X\geqq x\big]=\frac{1}{\Gamma(k)}\left(\frac{x}{\alpha}\right)^{-1}e^{-\left(\frac{x}{\alpha}\right)^{-\gamma}},\quad x>0,$$

and

$$\lambda(x)h(x)-g(x)=\left(\frac{\gamma-1}{\gamma}\right)\left(\frac{x}{\alpha}\right)^{\gamma-2}>0\ \text{ for }\ x>0,\quad\text{since}\quad\gamma=\gamma k+1>1.$$

The differential equation (2.1.2) clearly holds.

Conversely, if g and λ satisfy the differential equation (2.1.2), then

$$s'(x)=\frac{\lambda'(x)h(x)}{\lambda(x)h(x)-g(x)}=\frac{\gamma}{\alpha}\left[\left(\frac{x}{\alpha}\right)^{-1}-\left(\frac{x}{\alpha}\right)^{-\gamma-1}\right],\quad x>0,$$

and hence

$$s(x)=\ln(x^{\gamma})+\left(\frac{x}{\alpha}\right)^{-\gamma},\quad x>0.$$

Now from Theorem G, X has *pdf* (2.1.1). $\square$

COROLLARY 2.1.3. *Let $X:\Omega\to[0,\infty)$ be a continuous random variable and let $h(x)=\frac{1}{\gamma}\left(\frac{x}{\alpha}\right)^{\gamma-2}-\left(\frac{x}{\alpha}\right)^{-2}$ and $g(x)=\frac{\gamma}{\alpha}\left[\left(\frac{x}{\alpha}\right)^{-1}-\left(\frac{x}{\alpha}\right)^{-\gamma-1}\right]$ for $x\in(0,\infty)$. The pdf of X is* (2.1.1) *if and only if the function λ has the form*

$$\lambda(x)=\frac{\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{1-\gamma},\quad x>0.$$

Case (*ii*). $\gamma k+1\neq\gamma$. Then *pdf* of Amoroso distribution is (1.2), which we renumber it here as

$$f(x;\alpha,\gamma,k)=\frac{\gamma}{\alpha\Gamma(k)}\left(\frac{x}{\alpha}\right)^{-\gamma k-1}e^{-\left(\frac{x}{\alpha}\right)^{-\gamma}},\quad x>0, \tag{2.1.3}$$

where all three parameters α, γ and k are positive.

PROPOSITION 2.1.4. *Let $X:\Omega\to[0,\infty)$ be a continuous random variable and let $h(x)=(1+\gamma k)x^{-1}-\frac{\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{-\gamma-1}$ for $x\in(0,\infty)$. The pdf of X is* (2.1.3) *if and only if there exist functions g and λ defined in Theorem* G *satisfying the differential equation*

$$\frac{\lambda'(x)}{\lambda(x)h(x)-g(x)}=1,\quad x>0. \tag{2.1.4}$$

PROOF. Let X have *pdf* (2.1.3) and let

$$g(x)=\left(\frac{x}{\alpha}\right)^{\gamma(k-1)}\left(1-\left(\frac{x}{\alpha}\right)^{-\gamma}\right),\quad x>0$$

and

$$\lambda(x)=\frac{\alpha}{\gamma}\left(\frac{x}{\alpha}\right)^{\gamma(k-1)+1},\quad x>0.$$

Then, after some computations we arrive at

$$\big(1-F(x)\big)E\big[h(X)\mid X\geqq x\big]=\frac{\gamma}{\alpha\Gamma(k)}\left(\frac{x}{\alpha}\right)^{-(1+\gamma k)}e^{-\left(\frac{x}{\alpha}\right)^{-\gamma}},\quad x>0,$$

$$\big(1-F(x)\big)E\big[g(X)\mid X\geqq x\big]=\frac{1}{\Gamma(k)}\left(\frac{x}{\alpha}\right)^{-\gamma}e^{-\left(\frac{x}{\alpha}\right)^{-\gamma}},\quad x>0,$$

and

$$\lambda(x)h(x) - g(x) = \left(\frac{1+\gamma k-\gamma}{\gamma}\right)\left(\frac{x}{\alpha}\right)^{\gamma(k-1)} \neq 0 \quad \text{for} \quad x > 0,$$

since $\gamma \neq \gamma k + 1$. The differential equation (2.1.4) clearly holds.

Conversely, if g and λ satisfy differential equation (2.1.4), then

$$s'(x) = \frac{\lambda'(x)h(x)}{\lambda(x)h(x) - g(x)} = \frac{1+\gamma k}{\alpha}\left(\frac{x}{\alpha}\right)^{-1} - \frac{\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{-\gamma-1}, \quad x > 0,$$

and hence

$$s(x) = (1+\gamma k)\ln\left(\frac{x}{\alpha}\right) + \left(\frac{x}{\alpha}\right)^{-\gamma}, \quad x > 0.$$

Now from Theorem G, X has *pdf* (2.1.3). □

Corollary 2.1.5. *Let* $X : \Omega \to [0,\infty)$ *be a continuous random variable and let* $h(x) = \left(\frac{x}{\alpha}\right)^{\gamma(k-1)}\left(1 - \left(\frac{x}{\alpha}\right)^{-\gamma}\right)$ *and* $g(x) = (1+\gamma k)x^{-1} - \frac{\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{-\gamma-1}$ *for* $x \in (0,\infty)$. *The pdf of* X *is* (2.1.3) *if and only if the function* λ *has the form*

$$\lambda(x) = \frac{\gamma}{\alpha}\left(\frac{x}{\alpha}\right)^{-\gamma(k-1)-1}, \quad x > 0.$$

Remarks 2.1.6. (a) It is clear that Corollaries 2.1.3 and 2.1.5 are straightforward. The referee suggested the following statement for which we are grateful: "In general, if g and h are interchanged (e.g., $g^* = h$ and $h^* = g$) the triplet g^*, h^* and $\lambda^* = 1/\lambda$ provides a Theorem G characterization of the distribution in question, provided $\lambda \neq 0$ on the int$\,(H)$. This is a direct consequence of $E\big[g(X) \mid X \geqq x\big] = E\big[h(X) \mid X \geqq x\big] \cdot \lambda(x)$."

(b) The general solutions of the differential equations (2.1.2) and (2.1.4) are, respectively

$$\lambda(x) = \left(\frac{x}{\alpha}\right)^{\gamma} e^{\left(\frac{x}{\alpha}\right)^{-\gamma}}\left[-\int\left(\frac{x}{\alpha}\right)^{-\gamma} e^{-\left(\frac{x}{\alpha}\right)^{-\gamma}} g(x)\,dx + D_1\right], \quad x > 0,$$

and

$$\lambda(x) = \left(\frac{x}{\alpha}\right)^{1+\gamma k} e^{\left(\frac{x}{\alpha}\right)^{-\gamma}}\left[-\int\left(\frac{x}{\alpha}\right)^{-1-\gamma k} e^{-\left(\frac{x}{\alpha}\right)^{-\gamma}} g(x)\,dx + D_2\right], \quad x > 0,$$

where D_1 and D_2 are constants. One set of appropriate functions is given in each of the Propositions 2.1.2 and 2.1.3 for $D_1 = D_2 = 0$.

Remark 2.1.7. The above Propositions characterize the distribution of the Amoroso random variable X for the case $\alpha > 0$ and $\tau < 0$ directly, which is not part of Table 2.

2.2. Remark on characterization of Amoroso distribution based on hazard function

For the sake of completeness, we state the following definition.

Definition 2.2.1. Let F be an absolutely continuous distribution with the corresponding *pdf* f. The hazard function corresponding to F is denoted by η_F and is defined by

$$\eta_F(x) = \frac{f(x)}{1 - F(x)}, \quad x \in \operatorname{Supp} F, \tag{2.2.1}$$

where $\operatorname{Supp} F$ is the support of F.

It is obvious that the hazard function of twice differentiable function satisfies the first order differential equation

$$\frac{\eta_F'(x)}{\eta_F(x)} - \eta_F(x) = q(x),$$

where $q(x)$ is an appropriate integrable function. Although this differential equation has an obvious form since

$$\frac{f'(x)}{f(x)} = \frac{\eta_F'(x)}{\eta_F(x)} - \eta_F(x), \tag{2.2.2}$$

for many univariate continuous distributions (2.2.2) seems to be the only differential equation in terms of the hazard function. The goal of the characterizations based on hazard function is to establish a differential equation in terms of hazard function, which has as simple form as possible and is not of the trivial form (2.2.2). For some general families of distributions this may not be possible. For the Amoroso distribution with *pdf* (1.1) a nontrivial differential equation in terms of the hazard function is not possible, but for the case of (1.1) with $\alpha > 0$ and $\tau > 0$, generalized gamma, it is possible only for certain special cases (see, Hamedani and Ahsanullah [13], subsection 2.10). For these special distributions, the distributions of $-X$, $\frac{1}{X}$ and $-\frac{1}{X}$ will also have characterizations based on the hazard function. These random variables, however, were not mentioned in [13], when the characterization results based on hazard function for the case $\alpha > 0$ and $\tau > 0$ were first reported.

3. The Amoroso sub-family and generalized Pearson system

Various systems of distributions have been constructed to provide approximations to wide variety of distributions (see, e.g., [14]). These systems are designed with the requirements of ease of computation and feasibility of algebraic manipulation. To meet the requirements, there must be as few parameters as possible in defining a member of the system. One of these systems is Pearson system. A continuous distribution belongs to this system if its *pdf* f satisfies a differential equation of the form

$$\frac{1}{f(x)}\frac{df(x)}{dx} = -\frac{x+a}{bx^2+cx+d}, \tag{3.1}$$

where a, b, c and d are real parameters such that f is a *pdf*. The shape of the *pdf* depends on the values of these parameters. Pearson, [16], classified the different shapes into a number of Types **I–VII**. Many well-known distributions are special cases of Pearson Type distributions. The Pearson family is characterized via Theorem G in [6] (subsection 3.21).

Recently, some researchers have considered a generalization of (3.1), given by

$$\frac{1}{f(x)}\frac{df(x)}{dx} = \frac{\sum_{j=0}^{m} a_j x^j}{\sum_{j=0}^{n} b_j x^j}, \tag{3.2}$$

where $m, n \in \mathbb{N} \setminus \{0\}$ and the coefficients a_j's, b_j's are real parameters. The system of continuous univariate *pdf*'s generated by (3.2) is called generalized Pearson system, which includes a vast majority of continuous *pdf*'s. We make the observation that the *pdf* f of a sub-family of the Amoroso family satisfies the generalized Pearson differential equation (3.2) with, of course, appropriate boundary condition.

For $a = 0$, $\alpha > 0$ (or $\alpha < 0$), $\tau = -\gamma$, $\gamma \in \mathbb{N} \setminus \{0\}$, $k > 0$, the *pdf* f given by (1.1) satisfies (3.2) with $a_0 = \gamma\alpha^\gamma$, $a_j = 0$, $j = 1, 2, \ldots, \gamma - 1$, $a_\gamma = -(\gamma k + 1)$; $b_j = 0$, $j = 0, 1, \ldots, \gamma$, $b_{\gamma+1} = 1$, i.e.

$$\frac{1}{f(x)}\frac{df(x)}{dx} = \frac{\gamma\alpha^\gamma - (\gamma k + 1)x^\gamma}{x^{\gamma+1}}.$$

For $a = 0$, $\alpha > 0$ (or $\alpha < 0$), $\tau = \gamma$, $\gamma \in \mathbb{N} \setminus \{0\}$, $k > 0$, the *pdf* f given by (1.1) satisfies (3.2) with $a_0 = \gamma k - 1$, $a_j = 0$, $j = 1, 2, \ldots, \gamma - 1$, $a_\gamma = -\gamma\alpha^{-\gamma}$; $b_0 = 0$, $b_1 = 1$, i.e.

$$\frac{1}{f(x)}\frac{df(x)}{dx} = \frac{(\gamma k - 1) - \gamma\alpha^{-\gamma}x^\gamma}{x}.$$

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