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FUNDAMENTAL GROUP OF DESARGUES CONFIGURATION SPACES

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Abstract

We compute the fundamental group of various spaces of Desargues configurations in complex projective spaces: planar and non-planar configurations, with a fixed center and also with an arbitrary center.

1. Introduction

Let M be a manifold and $\mathcal{F}_k(M)$ be its *ordered configuration space* of k-tuples $\left\{(x_1,\dots,x_k)\in M^k \mid x_i\neq x_j,\, i\neq j\right\}$. The k^{th} *pure braid group* of M is the fundamental group of $\mathcal{F}_k(M)$. The pure braid group of the plane, denoted by $\mathcal{PB}_n$, has the presentation [4]

$$\pi_1\big(\mathcal{F}_n(\mathbb{C})\big)=\mathcal{PB}_n\cong\left\langle \alpha_{ij},\ 1\leqq i<j\leqq n \mid (YB\,3)_n,(YB\,4)_n\right\rangle$$

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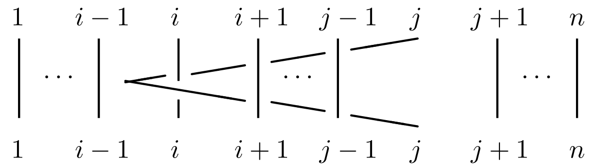
where generators α_{ij} are represented in the figure and the Yang–Baxter relations

α_{ij}



$(YB\,3)_n$ and $(YB\,4)_n$ are, for any $1 \leqq i < j < k \leqq n$,

$$(YB\,3)_n : \alpha_{ij}\alpha_{ik}\alpha_{jk} = \alpha_{ik}\alpha_{jk}\alpha_{ij} = \alpha_{jk}\alpha_{ij}\alpha_{ik}$$

and, for any $1 \leqq i < j < k < l \leqq n$,

$$(YB\,4)_n : [\alpha_{kl}, \alpha_{ij}] = \left[\alpha_{jl}, \alpha_{jk}^{-1}\alpha_{ik}\alpha_{jk}\right] = [\alpha_{il}, \alpha_{jk}] = \left[\alpha_{jl}, \alpha_{kl}\alpha_{ik}\alpha_{kl}^{-1}\right] = 1.$$

The pure braid group of $S^2 \approx \mathbb{C}\mathrm{P}^1$ has the presentation (see [5] and [4]):

$$\pi_1\big(\mathcal{F}_{k+1}\big(S^2\big)\big) \cong \big\langle \alpha_{ij}, 1 \leqq i < j \leqq k \mid (YB\,3)_k, (YB\,4)_k, D_k^2 = 1 \big\rangle,$$

where $D_k = \alpha_{12}(\alpha_{13}\alpha_{23})\dots(\alpha_{1k}\dots\alpha_{k-1,k})$ (in $\mathcal{B}_k$, the Artin braid group, D_k is the square of the Garside element Δ_k, see [6] and [2]). In [2] we started to study the topology of configuration spaces under simple geometrical restrictions. Using the geometry of the projective space we can stratify the configuration space $\mathcal{F}_k(\mathbb{C}\mathrm{P}^n)$ with complex submanifolds:

$$\mathcal{F}_k(\mathbb{C}\mathrm{P}^n) = \coprod_{i=1}^{n} \mathcal{F}_k^{i,n},$$

where $\mathcal{F}_k^{i,n}$ is the ordered configuration space of all k-tuples in $\mathbb{C}\mathrm{P}^n$ generating a subspace of dimension i. Their fundamental groups are given by (see [2]):

Theorem 1.1. *The spaces $\mathcal{F}_k^{i,n}$ are simply connected with the following exceptions*

(1) *for $k \geqq 2$,*

$$\pi_1\big(\mathcal{F}_{k+1}^{1,1}\big) \cong \big\langle \alpha_{ij},\ 1 \leqq i < j \leqq k \mid (YB\,3)_k, (YB\,4)_k,\ D_k^2 = 1 \big\rangle;$$

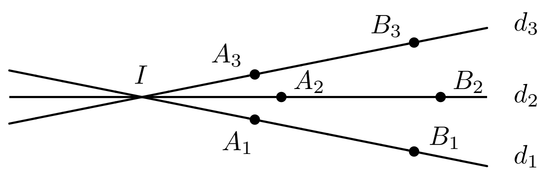
(2) *for $k \geqq 3$ and $n \geqq 2$,*

$$\pi_1\big(\mathcal{F}_{k+1}^{1,n}\big) \cong \big\langle \alpha_{ij},\ 1 \leqq i < j \leqq k \mid (YB\,3)_k, (YB\,4)_k,\ D_k = 1 \big\rangle.$$

In this paper we compute the fundamental groups of various configuration spaces related to projective Desargues configurations. We do not use special notations for the dual projective space: if P_1, P_2, P_3 are three points and d_1, d_2, d_3 are three lines in $\mathbb{C}\mathrm{P}^2$, $(P_1, P_2, P_3) \in \mathcal{F}_3^{1,2}$ is equivalent with the collinearity of these points and $(d_1, d_2, d_3) \in \mathcal{F}_3^{1,2}$ is equivalent with the concurrency of these lines. We define $\mathcal{D}^{2,n}$, the *space of planar Desargues configurations in* $\mathbb{C}\mathrm{P}^n$ $(n \geqq 2)$, by

$$\mathcal{D}^{2,n} = \big\{ (A_1, B_1, A_2, B_2, A_3, B_3) \in \mathcal{F}_6^{2,n} \mid (d_1, d_2, d_3) \in \mathcal{F}_3^{1,2}, \\ A_i, B_i \in d_i \setminus \{I\} \big\}$$

(here $I = d_1 \cap d_2 \cap d_3$).



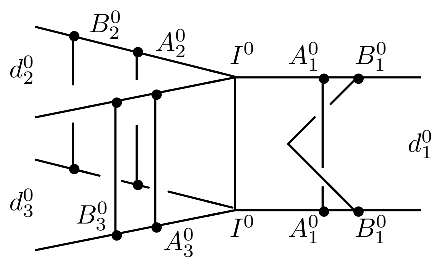
We consider also $\mathcal{D}_I^{2,n}$, *the space of planar Desargues configuration with a fixed intersection point* $I \in \mathbb{C}\mathrm{P}^n$, defined by

$$\mathcal{D}_I^{2,n} = \big\{ (A_1, B_1, A_2, B_2, A_3, B_3) \in \mathcal{D}^{2,n} \mid d_1 \cap d_2 \cap d_3 = I \big\}.$$

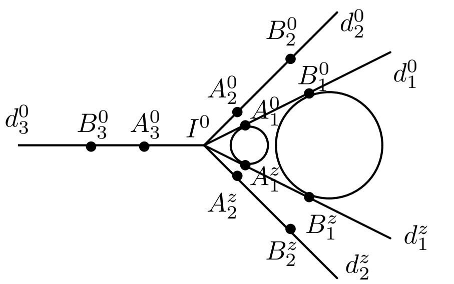
Theorem 1.2. *The fundamental group of* $\mathcal{D}_I^{2,n}$ *is given by*

$$\pi_1\big(\mathcal{D}_I^{2,n}\big) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n \geqq 3. \end{cases}$$

The first group is generated by $[\alpha]$, $[\beta]$, and $[\sigma]$, and the second group is generated by $[\alpha]$ and $[\beta]$. Precise formulae for α, β and σ are given in Section 2; here is a diagram representing these generators (there is a similar picture for β):



$\alpha:$ B_1 is moving on the line $d_1^0 \setminus \{I^0, A_1^0\}$



$\sigma:$ the lines d_1 and d_2 are moving

THEOREM 1.3. *The fundamental group of $\mathcal{D}^{2,n}$ is given by:*

$$\pi_1\big(\mathcal{D}^{2,n}\big) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \textit{if } n = 2, \\ \mathbb{Z} & \textit{if } n \geqq 3. \end{cases}$$

The first group is generated by $[\alpha]$ and $[\beta]$ and the second group is generated by $[\alpha]$ (or by $[\beta]$); we will use the same notations for $[\alpha]$, $[\beta]$, $[\sigma]$ and their images through different natural maps: $\mathcal{D}_I^{*,*} \to \mathcal{D}^{*,*}$, $\mathcal{D}_I^{*,*} \to \mathcal{D}_I^{*,*+1}$, $\mathcal{D}^{*,*} \to \mathcal{D}^{*,*+1}$.

We define $\mathcal{D}^{3,n}$, the *space of non-planar Desargues configurations in* $\mathbb{C}\mathrm{P}^n$ ($n \geqq 3$):

$$\mathcal{D}^{3,n} = \big\{ (A_1, B_1, A_2, B_2, A_3, B_3) \in \mathcal{F}_6^{3,n} \mid d_1 \cap d_2 \cap d_3 = I,\ A_i, B_i \in d_i \setminus \{I\} \big\}$$

and $\mathcal{D}_I^{3,n}$, the associated *space of non-planar Desargues configurations with a fixed intersection point* $I \in \mathbb{C}\mathrm{P}^n$.

THEOREM 1.4. *The fundamental group of $\mathcal{D}_I^{3,n}$ is given by:*

$$\pi_1\big(\mathcal{D}_I^{3,n}\big) \cong \begin{cases} \mathbb{Z} & \textit{if } n = 3, \\ 1 & \textit{if } n \geqq 4. \end{cases}$$

THEOREM 1.5. *The fundamental group of $\mathcal{D}^{3,n}$ is given by:*

$$\pi_1\big(\mathcal{D}^{3,n}\big) \cong \begin{cases} \mathbb{Z}_4 & \textit{if } n = 3, \\ 1 & \textit{if } n \geqq 4. \end{cases}$$

In the last two theorems, in the non-simply connected cases, the fundamental groups are generated by $[\alpha]$.

2. Desargues configurations in the projective plane

In order to find the fundamental groups of the spaces $\mathcal{D}=\mathcal{D}^{2,2}$ and $\mathcal{D}_I=\mathcal{D}_I^{2,2}$ we use two fibrations and their homotopy exact sequences.

LEMMA 2.1. *The projection*

$$\mu:\,\mathcal{D}\to\mathbb{C}\mathrm{P}^2,\quad (A_1,B_1,A_2,B_2,A_3,B_3)\mapsto I=d_1\cap d_2\cap d_3$$

is a locally trivial fibration with fiber $\mathcal{D}_I$.

PROOF. Fix a point $I^0\in\mathbb{C}\mathrm{P}^2$ and choose a line $l\subset\mathbb{C}\mathrm{P}^2\setminus\{I^0\}$ and the neighborhood $\mathcal{U}_l=\mathbb{C}\mathrm{P}^2\setminus l$ of I^0. For a point I in this neighborhood and a Desargues configuration $\left(A_1^0,B_1^0,A_2^0,B_2^0,A_3^0,B_3^0\right)$ on three lines d_1^0, d_2^0, d_3^0 containing I^0 construct lines d_1, d_2, d_3 containing I and the configuration $(A_1,B_1,\dots,A_3,B_3)$ as follows: consider the points $D_i=l\cap d_i^0$ and $Q=l\cap I^0I$ and define $d_i=ID_i,\ A_i=d_i\cap QA_i^0$ and in the same way B_i $(i=1,2,3)$. We describe this construction using coordinates to show that the map

$$\left(I,\left(A_1^0,B_1^0,A_2^0,B_2^0,A_3^0,B_3^0\right)\right)\mapsto(A_1,B_1,A_2,B_2,A_3,B_3)$$

has a continuous extension on the singular locus $(d_1^0\cup d_2^0\cup d_3^0\setminus l)$. Choose a projective frame such that $I^0=[0:0:1]$, $l:X_2=0$. If $I=[s:t:1]$ and $A_i^0=[n_i:-m_i:a_i]$, $B_i^0=[n_i:-m_i:b_i]$ (a_i, b_i are distinct and non zero and also $n_im_j\neq m_in_j$ for distinct $i,j=1,2,3$), then we define $A_i=[n_i+sa_i:-m_i+ta_i:a_i]$ and $B_i=[n_i+sb_i:-m_i+tb_i:b_i]$, $(i=1,2,3)$, and these formulae agree with the geometrical construction given for nondegenerate positions of $I\in\mathbb{C}\mathrm{P}^2\setminus\left(d_1^0\cup d_2^0\cup d_3^0\cup l\right)$. The trivialization over $\mathcal{U}_l$ is given by

$$\varphi:\mathcal{U}_l\times\mathcal{D}_{I^0}\to\gamma^{-1}(\mathcal{U}_l),$$

$$\varphi\left(I,\left(A_1^0,B_1^0,A_2^0,B_2^0,A_3^0,B_3^0\right)\right)=(A_1,B_1,A_2,B_2,A_3,B_3).\qquad\square$$

LEMMA 2.2. *The projection*

$$\lambda:\,\mathcal{D}_I\to\mathcal{F}_3\left(\mathbb{C}\mathrm{P}^1\right),\quad (A_1,B_1,A_2,B_2,A_3,B_3)\mapsto(d_1,d_2,d_3)$$

is a locally trivial fibration with fiber $\mathcal{F}_2(\mathbb{C})\times\mathcal{F}_2(\mathbb{C})\times\mathcal{F}_2(\mathbb{C})$.

Proof. Fix a point $d^0_* = \left(d^0_1, d^0_2, d^0_3\right)$ in $\mathcal{F}_3\left(\mathbb{C}\mathrm{P}^1\right)$ and choose a point Q in $\mathbb{C}\mathrm{P}^2 \setminus \left(d^0_1 \cup d^0_2 \cup d^0_3\right)$ and the neighborhood $\mathcal{U}_Q = \left\{(d_1, d_2, d_3) \in \mathcal{F}_3\left(\mathbb{C}\mathrm{P}^1\right) \mid Q \notin d_1 \cup d_2 \cup d_3\right\}$. The trivialization over $\mathcal{U}_Q$ is given by

$$\psi : \mathcal{U}_Q \times \mathcal{F}_2\left(d^0_1 \setminus \{I\}\right) \times \mathcal{F}_2\left(d^0_2 \setminus \{I\}\right) \times \mathcal{F}_2\left(d^0_3 \setminus \{I\}\right) \to \lambda^{-1}(\mathcal{U}_Q)$$

$$\psi\Big((d_1, d_2, d_3), \left(A^0_1, B^0_1\right), \left(A^0_2, B^0_2\right), \left(A^0_3, B^0_3\right)\Big) = (A_1, B_1, A_2, B_2, A_3, B_3),$$

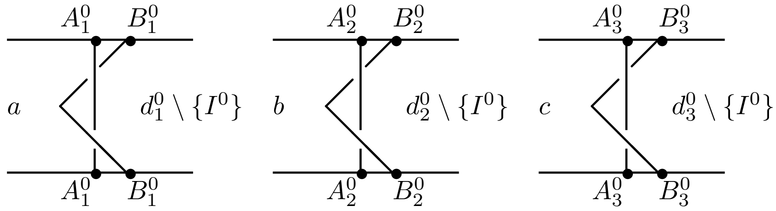
where $A_i = d_i \cap QA^0_i$ and similarly for B_i $(i = 1, 2, 3)$. Obviously, A_i, B_i and I are three distinct points on d_i. □

In $\mathcal{D}_{I^0=[0:0:1]}$ we choose the base point $D^0 = \left(A^0_1, B^0_1, A^0_2, B^0_2, A^0_3, B^0_3\right)$ where, for $k = 1, 2$, $A^0_k = [-1 : k : 1]$, $B^0_k = [-1 : k : 2]$, $A^0_3 = [0 : 1 : 1]$, $B^0_3 = [0 : 1 : 2]$. The corresponding lines are given by the equations $d^0_k : kX_0 + X_1 = 0$, $d^0_3 : X_0 = 0$ and we identify the affine line $\mathbb{C}$ with d^0_k as follows: for $k = 1, 2$, $z \mapsto [-1 : k : z]$, and for $k = 3$, $z \mapsto [0 : 1 : z]$ (therefore the intersection point $I^0 = [0 : 0 : 1]$ is the point at infinity of these lines). We identify the set of three distinct lines through I^0 with the configuration space $\mathcal{F}_3\left(\mathbb{C}\mathrm{P}^1\right)$; in this space the base point is $d^0_* = \left(d^0_1, d^0_2, d^0_3\right)$. In the configuration spaces $\mathcal{F}_2\left(d^0_i \setminus \{I^0\}\right)$ we choose the base points $\left(A^0_i, B^0_i\right)$, $i = 1, 2, 3$. The homotopy exact sequence from Lemma 2.2 and the triviality of $\pi_2\left(\mathcal{F}_3\left(\mathbb{C}\mathrm{P}^1\right)\right)$ (see [3]) give the short exact sequence

$$1 \longrightarrow \pi_1\left(\mathcal{F}_2(\mathbb{C})\right) \times \pi_1\left(\mathcal{F}_2(\mathbb{C})\right) \times \pi_1\left(\mathcal{F}_2(\mathbb{C})\right)$$

$$\xrightarrow{j_*} \pi_1\left(\mathcal{D}_{I^0}\right) \xrightarrow{\lambda_*} \pi_1\left(\mathcal{F}_3\left(\mathbb{C}\mathrm{P}^1\right)\right) \longrightarrow 1.$$

Proof of Theorem 1.2 (the case $n = 2$). The first group, isomorphic to $\mathbb{Z}^3$, is generated by the pure braids a, b, c, hence their images in $\pi_1\left(\mathcal{D}_{I^0}\right)$ are given by the



homotopy classes of the maps $\alpha, \beta, \gamma : \big(S^1, 1\big) \to \big(\mathcal{D}_{I^0}, D^0\big)$

$$\alpha(z) = \big(A_1^0, B_1^{\alpha(z)}, A_2^0, B_2^0, A_3^0, B_3^0\big), \qquad B_1^{\alpha(z)} = [-1 : 1 : 1 + z],$$

$$\beta(z) = \big(A_1^0, B_1^0, A_2^0, B_2^{\beta(z)}, A_3^0, B_3^0\big), \qquad B_2^{\beta(z)} = [-1 : 2 : 1 + z],$$

$$\gamma(z) = \big(A_1^0, B_1^0, A_2^0, B_2^0, A_3^0, B_3^{\gamma(z)}\big), \qquad B_3^{\gamma(z)} = [0 : 1 : 1 + z].$$

The third group, $\pi_1(\mathcal{F}_3\big(\mathbb{C}\mathrm{P}^1\big) \cong \mathbb{Z}_2$, is generated by the homotopy class of the map

$$s : \big(S^1, 1\big) \to \big(\mathcal{F}_3\big(\mathbb{C}\mathrm{P}^1\big), d_*^0\big),$$

$$z \mapsto \big(d_1^{s(z)} : zX_0 + X_1 = 0, d_2^{s(z)} : 2zX_0 + X_1 = 0, d_3^0\big),$$

because this corresponds to the braid α_{12} in $\mathbb{C}\mathrm{P}^1$. We lift the map s to the map

$$\sigma : (S^1, 1) \to \big(\mathcal{D}_I^0, D^0\big), \quad z \mapsto \big(A_1^{\sigma(z)}, B_1^{\sigma(z)}, A_2^{\sigma(z)}, B_2^{\sigma(z)}, A_3^0, B_3^0\big),$$

where $A_k^{\sigma(z)} = [-1 : kz : 1]$, $B_k^{\sigma(z)} = [-1 : kz : 2]$, $k = 1, 2$.

The group $\pi_1\big(\mathcal{D}_{I^0}, D^0\big)$ is generated by the homotopy classes of α, β, γ and σ; the defining relations are commutation relations between $[\alpha]$, $[\beta]$ and $[\gamma]$ from $\pi_1\big(\mathcal{F}_2(\mathbb{C})^3\big)$ and the four relations, to be proved in the next two lemmas:

α) $$[\sigma][\alpha][\sigma]^{-1} = [\alpha],$$

β) $$[\sigma][\beta][\sigma]^{-1} = [\beta],$$

γ) $$[\sigma][\gamma][\sigma]^{-1} = [\gamma],$$

σ) $$[\sigma]^2 = [\alpha]^{-1}[\beta]^{-1}[\gamma].$$

The generator $[\gamma]$ can be eliminated, $[\sigma]$ commutes with $[\alpha]$ and $[\beta]$, and the third relation, γ), is a consequence of the previous commutation relations.

Lemma 2.3. *In $\pi_1\big(\mathcal{D}_{I^0}, D^0\big)$ the next relation holds:*

σ) $$[\sigma]^2 = [\alpha]^{-1}[\beta]^{-1}[\gamma].$$

PROOF. The map

$$\Lambda:\ \big(D^2,S^1\big)\to\big(\mathcal{F}_3\big(\mathbb{C}\mathrm{P}^1\big),d_*^0=\big(d_1^0,d_2^0,d_3^0\big)\big),\quad z\mapsto\big(d_1^{\Lambda(z)},d_2^{\Lambda(z)},d_3^{\Lambda(z)}\big),$$

where $d_k^{\Lambda(z)}:\,(kz-r)X_0+(\overline{z}+kr)X_1=0,\ (k=1,2)$, and $d_3^{\Lambda(z)}:\,zX_0+rX_1=0$ (the notation $r=1-|z|$ will be used in this proof and the next ones), shows that $s^2\simeq\text{constant}_{d_*^0}$. We lift this homotopy to

$$\widetilde{\Lambda}:\ D^2\to\mathcal{D}_{I^0},\quad \widetilde{\Lambda}(z)=\big(A_1^{\widetilde{\Lambda}(z)},B_1^{\widetilde{\Lambda}(z)},A_2^{\widetilde{\Lambda}(z)},B_2^{\widetilde{\Lambda}(z)},A_3^{\widetilde{\Lambda}(z)},B_3^{\widetilde{\Lambda}(z)}\big),$$

where $A_k^{\widetilde{\Lambda}(z)}=[-\overline{z}-kr:kz-r:\overline{z}]$, $B_k^{\widetilde{\Lambda}(z)}=[-\overline{z}-kr:kz-r:\overline{z}+1]$, $(k=1,2)$, and $A_3^{\widetilde{\Lambda}(z)}=[-r:z:z]$, $B_3^{\widetilde{\Lambda}(z)}=[-r:z:z+1]$; the map

$$\widetilde{\Lambda}|_{S^1}:\ S^1\to\mathcal{D}_{I^0},\quad z\mapsto(A_1^z,B_1^z,A_2^z,B_2^z,A_3^0,B_3^z)$$

(with $A_k^z=\big[-1:kz^2:1\big]$, $B_k^z=\big[-1:kz^2:1+z\big]$, $k=1,2$, and $B_3^z=[0:1:1+\overline{z}]$) has a trivial homotopy class, therefore we have the relation $[\sigma]^2=\big[\sigma*\sigma*\big(\widetilde{\Lambda}|_{S^1}\big)^{-1}\big]$.

Now we construct a homotopy between $\sigma*\sigma*\big(\widetilde{\Lambda}|_{S^1}\big)^{-1}$ and $\alpha^{-1}*\beta^{-1}*\gamma$:

$$L:\ S^1\times I\to\mathcal{D}_{I^0},\quad (z,t)\mapsto\big(A_1^{L(z,t)},B_1^{L(z,t)},A_2^{L(z,t)},B_2^{L(z,t)},A_3^0,B_3^{L(z,t)}\big),$$

where $(k=1,2)$:

$$A_k^{L(z,t)}=\big[-1:kL^1(z,t):1\big],\quad B_k^{L(z,t)}=\big[-1:kL^1(z,t):L_k^2(z,t)\big]$$

$$B_3^{L(z,t)}=\begin{cases}[0:1:2] & 0\leqq\arg z\leqq\pi\\ \big[0:1:1+z^2\big] & \pi\leqq\arg z\leqq 2\pi,\end{cases}$$

and

$$L^1(z,t)=\begin{cases}z^4 & 0\leqq\arg z\leqq t\pi\\ \exp(4t\pi i) & t\pi\leqq\arg z\leqq(2-t)\pi\\ \overline{z}^4 & (2-t)\pi\leqq\arg z\leqq 2\pi,\end{cases}$$

$$L_k^2(z,t) = \begin{cases} 2 & 0 \leqq \arg z \leqq \frac{t+k-1}{k}\pi \\ 1+\exp\left(4\dfrac{(2-k)t\pi - \arg z}{1+t} i\right) & \\ & \dfrac{t+k-1}{k}\pi \leqq \arg z \leqq \dfrac{1+(5-2k)t}{3-k}\pi \\ 2 & \dfrac{1+(5-2k)t}{3-k}\pi \leqq \arg z \leqq 2\pi. \end{cases}$$

It is easy to check that $L(-,0) = \left(\alpha^{-1} * \beta^{-1}\right) * \gamma$ and $L(-,1) = (\sigma * \sigma) * \left(\widetilde{\Lambda}|_{S^1}\right)^{-1}$. □

LEMMA 2.4. *In* $\pi_1\left(\mathcal{D}_{I^0}, D^0\right)$ *the next relations hold:*

$\alpha)$ $$[\sigma][\alpha][\sigma]^{-1} = [\alpha];$$

$\beta)$ $$[\sigma][\beta][\sigma]^{-1} = [\beta];$$

$\gamma)$ $$[\sigma][\gamma][\sigma]^{-1} = [\gamma].$$

PROOF. The loop $\sigma * \alpha * \sigma^{-1}$ in $\mathcal{D}_{I^0}$ is given by $z \mapsto \left(A_1^{\widetilde{\alpha}(z)}, B_1^{\widetilde{\alpha}(z)}, A_2^{\widetilde{\alpha}(z)}, B_2^{\widetilde{\alpha}(z)}, A_3^0, B_3^0\right)$, where the points $A_k^{\widetilde{\alpha}(z)}$ $(k=1,2)$, $B_1^{\widetilde{\alpha}(z)}$ and $B_2^{\widetilde{\alpha}(z)}$ are given by:

$$\begin{array}{lll} A_k = \left[-1 : kz^3 : 1\right] & B_1 = \left[-1 : z^3 : 2\right] & \\ & B_2 = \left[-1 : 2z^3 : 2\right] & \arg z \in \left[0, \dfrac{2\pi}{3}\right] \\ A_k = A_k^0 & B_1 = \left[-1 : 1 : 1+z^3\right] & \\ & B_2 = B_2^0 & \arg z \in \left[\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right] \\ A_k = \left[-1 : k\overline{z}^3 : 1\right] & B_1 = \left[-1 : \overline{z}^3 : 2\right] & \\ & B_2 = \left[-1 : 2\overline{z}^3 : 2\right] & \arg z \in \left[\dfrac{4\pi}{3}, 2\pi\right]. \end{array}$$

We define two maps

$$\varepsilon : S^1 \times I \to S^1, \qquad \varepsilon(z,t) = \begin{cases} z^3 & 0 \leqq \arg z \leqq \frac{2t}{3}\pi \\ \exp(2t\pi i) & \frac{2t}{3}\pi \leqq \arg z \leqq \frac{2(3-t)}{3}\pi \\ \overline{z}^3 & \frac{2(3-t)}{3}\pi \leqq \arg z \leqq 2\pi, \end{cases}$$

$$\eta : S^1 \to \mathbb{C} \setminus \{1\}, \qquad \eta(z) = \begin{cases} 2 & \arg z \in \left[0, \frac{2\pi}{3}\right] \cup \left[\frac{4\pi}{3}, 2\pi\right] \\ 1 + z^3 & \arg z \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]. \end{cases}$$

and a new homotopy

$$K_\alpha(z,t) : S^1 \times I \to \mathcal{D}_{I^0},$$
$$K_\alpha(z,t) = \big(A_1(z,t), \widetilde{B}_1(z,t), A_2(z,t), B_2(z,t), A_3^0, B_3^0\big),$$

where $A_k(z,t) = \big[-1 : k\varepsilon(z,t) : 1\big]$, $B_k(z,t) = \big[-1 : k\varepsilon(z,t) : 2\big]$, $(k = 1, 2)$, $\widetilde{B}_1(z,t) = \big[-1 : \varepsilon(z,t) : \eta(z)\big]$. One can check that $K_\alpha|_{t=0} \simeq \alpha$ and $K_\alpha|_{t=1} = \sigma * \alpha * \sigma^{-1}$. Similarly we have a homotopy K_β between β and $K_\beta|_{t=1} = \sigma * \beta * \sigma^{-1}$. Next homotopy (we also use the notation $B_3(z,t) = \big[0 : 1 : \eta(z)\big]$)

$$K_\gamma(z,t) : S^1 \times I \to \mathcal{D}_{I^0},$$
$$(z,t) \mapsto \big(A_1(z,t), B_1(z,t), A_2(z,t), B_2(z,t), A_3^0, B_3(z,t)\big),$$

gives the last relation: $K_{\gamma|t=0} \simeq \gamma$, $K_{\gamma|t=1} = \sigma * \gamma * \sigma^{-1}$. □

Proof of Theorem 1.3 (the case $n = 2$). Lemma 2.1 gives the exact sequence

$$\ldots \longrightarrow \pi_2(\mathbb{C}\mathrm{P}^2) \xrightarrow{\delta_*} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \pi_1(\mathcal{D}) \longrightarrow 1$$

where the first group is cyclic generated by the homotopy class of the map

$$\Phi : \big(D^2, S^1\big) \to \big(\mathbb{C}\mathrm{P}^2, I^0\big), \quad z \mapsto [0 : r : z].$$

We choose the lift

$$\widetilde{\Phi} : \big(D^2, S^1\big) \to \big(\mathcal{D}, \mathcal{D}_{I^0}\big), \quad z \mapsto \big(A_1^{\widetilde{\Phi}(z)}, B_1^{\widetilde{\Phi}(z)}, A_2^{\widetilde{\Phi}(z)}, B_2^{\widetilde{\Phi}(z)}, A_3^{\widetilde{\Phi}(z)}, B_3^{\widetilde{\Phi}(z)}\big),$$

where $(k=1,2)$

$$A_k^{\widetilde{\Phi}(z)} = \big[-1 : (2k+1)r + k\overline{z} : (2k+1)z + k(r-2)\big],$$

$$B_k^{\widetilde{\Phi}(z)} = \big[-1 : (2k+2)r + k\overline{z} : (2k+2)z + k(r-2)\big],$$

$$A_3^{\widetilde{\Phi}(z)} = \big[-r : \overline{z} + 4r : 4z - 3(r+1)\big],$$

$$B_3^{\widetilde{\Phi}(z)} = \big[-r : \overline{z} + 5r : 5z - 3(r+1)\big],$$

hence $\operatorname{Im}\delta_*$ is generated by the homotopy class of the map

$$\widetilde{\Phi}\big|_{S^1} : S^1 \to \mathcal{D}_{I^0}, \quad z \mapsto \big(A_1^{\Phi(z)}, B_1^{\Phi(z)}, A_2^{\Phi(z)}, B_2^{\Phi(z)}, A_3^{\Phi(z)}, B_3^{\Phi(z)}\big),$$

with $(k=1,2)$

$$A_k^{\Phi(z)} = \big[-1 : k\overline{z} : (2k+1)z - 2k\big], \quad B_k^{\Phi(z)} = \big[-1 : k\overline{z} : (2k+2)z - 2k\big],$$

$$A_3^{\Phi(z)} = [0 : \overline{z} : 4z-3], \qquad B_3^{\Phi(z)} = [0 : \overline{z} : 5z-3].$$

The maps $\lambda \circ \widetilde{\Phi}\big|_{S^1}$ and s^{-1} coincide, therefore the product $\big[\widetilde{\Phi}\big|_{S^1}\big] \cdot [\sigma]$ belongs to $\ker\lambda_* = \operatorname{Im} j_*$. We show that $\big[\widetilde{\Phi}|_{S^1}\big] \cdot [\sigma] = [\alpha]\cdot[\beta]\cdot[\gamma]$ and this implies the claim of the theorem. We define the homotopy:

$$H : S^1 \times I \to \mathcal{D}_{I^0},$$

$$(z,t) \mapsto \big(A_1^{H(z,t)}, B_1^{H(z,t)}, A_2^{H(z,t)}, B_2^{H(z,t)}, A_3^{H(z,t)}, B_3^{H(z,t)}\big),$$

where $(k=1,2)$

$$A_k^{H(z,t)} = \big[-1 : H_k^1(z,t) : H_k^2(z,t)\big]$$

$$B_k^{H(z,t)} = \big[-1 : H_k^1(z,t) : H_k^2(z,t) + H_k^4(z,t)\big]$$

$$A_3^{H(z,t)} = \big[0 : 1 : H^3(z,t)\big] \quad B_3^{H(z,t)} = \big[0 : 1 : H^3(z,t) + H^5(z,t)\big]$$

and

$$H_k^1(z,t) = \begin{cases} k\overline{z}^2 & 0 \leqq \arg z \leqq t\pi \\ k\exp(-2t\pi i) & t\pi \leqq \arg z \leqq (2-t)\pi \\ kz^2 & (2-t)\pi \leqq \arg z \leqq 2\pi, \end{cases}$$

$$H_k^2(z,t) = \begin{cases} 1 + (2k+1)t(z^2-1) & 0 \leqq \arg z \leqq \pi \\ 1 & \pi \leqq \arg z \leqq 2\pi, \end{cases}$$

$$H^3(z,t) = \begin{cases} 1 + t(4z^4 - 3z^2 - 1) & 0 \leqq \arg z \leqq \pi \\ 1 & \pi \leqq \arg z \leqq 2\pi, \end{cases}$$

$$H_1^4(z,t) = \begin{cases} \exp\left(\dfrac{4\arg z}{1+t} i\right) & 0 \leqq \arg z \leqq \dfrac{1+t}{2}\pi \\ 1 & \dfrac{1+t}{2}\pi \leqq \arg z \leqq 2\pi, \end{cases}$$

$$H_2^4(z,t) = \begin{cases} 1 & 0 \leqq \arg z \leqq \dfrac{1-t}{2}\pi \\ \exp\left(2\dfrac{2\arg z - (1-t)\pi}{1+t} i\right) & \dfrac{1-t}{2}\pi \leqq \arg z \leqq \pi \\ 1 & \pi \leqq \arg z \leqq 2\pi. \end{cases}$$

$$H^5(z,t) = \begin{cases} 1 & 0 \leqq \arg z \leqq (1-t)\pi \\ \exp\left[4\big(\arg z - (1-t)\pi\big) i\right] & (1-t)\pi \leqq \arg z \leqq (2-t)\pi \\ 1 & (2-t)\pi \leqq \arg z \leqq 2\pi. \end{cases}$$

These computations give $\operatorname{Im}\delta_* = \mathbb{Z}\big\langle 2[\alpha] + 2[\beta] + [\sigma]\big\rangle$, therefore we can choose $[\alpha]$ and $[\beta]$ as generators of the fundamental group of $\mathcal{D}$. □

3. Planar Desargues configuration in $\mathbb{C}\mathrm{P}^n$

First we reduce the computations of $\pi_1\big(\mathcal{D}_I^{2,n}\big)$ and of $\pi_1\big(\mathcal{D}^{2,n}\big)$ to the case $n = 3$.

Lemma 3.1. *The following projections are locally trivial fibrations:*

a) $\mathcal{D}_I^{2,2} \hookrightarrow \mathcal{D}_I^{2,n} \to \mathrm{Gr}^1\big(\mathbb{C}\mathrm{P}^{n-1}\big)$,
$(A_1, B_1, A_2, B_2, A_3, B_3) \mapsto \text{line}\,(d_1, d_2, d_3)$;

b) $\mathcal{D}^{2,2} \hookrightarrow \mathcal{D}^{2,n} \to \mathrm{Gr}^2(\mathbb{C}\mathrm{P}^n)$,
$(A_1, B_1, A_2, B_2, A_3, B_3) \mapsto 2\text{-plane}\,(d_1, d_2, d_3)$.

Proof. *a*) Fix a 2-plane P_0 through I and choose a hyperplane $H \subset \mathbb{C}\mathrm{P}^n$ such that $I \notin H$ and an $(n-3)$ dimensional subspace $Q \subset H$ such that $Q \cap l_0 = \emptyset$, where $l_0 = P_0 \cap H$. Take as a neighborhood of P_0 the set $\{P$ a 2-plane in $\mathbb{C}\mathrm{P}^n \mid I \in P,\ P \cap Q = \emptyset\}$ and associate to a Desargues configuration in $\mathcal{D}_I(P_0)$ the projection from Q, an element in $\mathcal{D}_I(P)$: $C_i^0 = d_i^0 \cap l_0$, $l = P \cap H$, $C_i = \left(Q \vee C_i^0\right) \cap l$, $Q_i = Q \cap \left(C_i C_i^0\right)$, $d_i = IC_i$, $A_i = Q_i A_i^0 \cap d_i$, $B_i = Q_i B_i^0 \cap d_i$ (for $i = 1, 2, 3$). Using projective coordinates one can show that this trivialization is well defined on the singular locus $P = P_0$: if $I = [0 : \ldots : 0 : 1]$, $P_0 : X_0 = \ldots = X_{n-3} = 0$, $A_i^0 = \left[0 : \ldots : a^0_{n-2,i} : a^0_{n-1,i} : a^0_{n,i}\right]$, $B_i^0 = \left[0 : \ldots : b^0_{n-2,i} : b^0_{n-1,i} : b^0_{n,i}\right]$, and P is defined by the equations $X_k = p_{k,1}X_{n-2} + p_{k,2}X_{n-1} + p_{k,3}X_n$ $(k = 0, \ldots, n-3)$, then

$$\begin{aligned} A_i = \big[&p_{0,0}a_{n-2,i} + p_{0,1}a_{n-1,i} : \ldots : p_{n-3,0}a_{n-2,i} + p_{n-3,1}a_{n-1,i} : \\ &a^0_{n-2,i} : a^0_{n-1,i} : a^0_{n,i}\big], \end{aligned}$$

$$\begin{aligned} B_i = \big[&p_{0,0}b_{n-2,i} + p_{0,1}b_{n-1,i} : \ldots : p_{n-3,0}b_{n-2,i} + p_{n-3,1}b_{n-1,i} : \\ &b^0_{n-2,i} : b^0_{n-1,i} : b^0_{n,i}\big]. \end{aligned}$$

b) Fix a 2-plane P_0 and choose as center of projection a disjoint $n-3$ dimensional subspace Q. Take as a neighborhood of P_0 the set of 2-planes disjoint from Q. The projection from Q associate to a Desargues configuration in $\mathcal{D}^2(P_0)$ a Desargues configuration in $\mathcal{D}^2(P)$: $P \cap \left(Q \vee I^0\right) = I$, $P \cap \left(Q \vee d_i^0\right) = d_i$, $d_i \cap \left(Q \vee A_i^0\right) = A_i$, $d_i \cap \left(Q \vee B_i^0\right) = B_i$. □

Corollary 3.2. *For $n \geqq 3$ we have*

a) $$\pi_1\left(\mathcal{D}_I^{2,3}\right) \cong \pi_1\left(\mathcal{D}_I^{2,n}\right);$$

b) $$\pi_1\left(\mathcal{D}^{2,3}\right) \cong \pi_1\left(\mathcal{D}^{2,n}\right).$$

Proof. This is a consequence of the stability of the second homotopy group of the complex Grassmannians:

$$\begin{array}{ccccccc} \pi_2\left(\mathrm{Gr}^1\left(\mathbb{C}\mathrm{P}^2\right)\right) & \longrightarrow & \pi_1\left(\mathcal{D}_I^{2,2}\right) & \longrightarrow & \pi_1\left(\mathcal{D}_I^{2,3}\right) & \longrightarrow & 1 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ \pi_2\left(\mathrm{Gr}^1\left(\mathbb{C}\mathrm{P}^{n-1}\right)\right) & \longrightarrow & \pi_1\left(\mathcal{D}_I^{2,2}\right) & \longrightarrow & \pi_1\left(\mathcal{D}_I^{2,n}\right) & \longrightarrow & 1 \end{array}$$

and also

$$\begin{array}{ccccccc}
\pi_2\big(\mathrm{Gr}^2\big(\mathbb{C}\mathrm{P}^3\big)\big) & \longrightarrow & \pi_1\big(\mathcal{D}^{2,2}\big) & \longrightarrow & \pi_1\big(\mathcal{D}^{2,3}\big) & \longrightarrow & 1 \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \\
\pi_2\big(\mathrm{Gr}^2\big(\mathbb{C}\mathrm{P}^n\big)\big) & \longrightarrow & \pi_1\big(\mathcal{D}^{2,2}\big) & \longrightarrow & \pi_1\big(\mathcal{D}^{2,n}\big) & \longrightarrow & 1.
\end{array}$$ □

Using the fibration of Lemma 3.1*a*) for $n = 3$ we have the exact sequence

$$\ldots \to \pi_2\big(\mathbb{C}\mathrm{P}^2\big) \xrightarrow{\delta_*} \pi_1\big(\mathcal{D}_I^{2,2}\big) \to \pi_1\big(\mathcal{D}_I^{2,3}\big) \to 1.$$

We choose the base point in $\mathcal{D}_I^{2,3}$ the image of the base point in $\mathcal{D}_I$ through the embedding $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1 : x_2 : 0]$ and we denote the compositions $\alpha, \beta :\ S^1 \to \mathcal{D}_I^{2,2} \to \mathcal{D}_I^{2,3}$ with the same letters.

Proposition 3.3. *In the exact sequence of the fibration $\mathcal{D}_I^{2,3} \to \mathbb{C}\mathrm{P}^2$ we have:*
a) $\operatorname{Im}\delta_* = \mathbb{Z}\big([\alpha] + [\beta] + [\sigma]\big)$;
b) $\pi_1\big(\mathcal{D}_I^{2,3}\big) \cong \mathbb{Z} \oplus \mathbb{Z}$ *is generated by* $[\alpha]$ *and* $[\beta]$.

Proof. *a*) The base point in $\mathrm{Gr}^1\big(\mathbb{C}\mathrm{P}^2\big) \approx \mathbb{C}\mathrm{P}^2$ is the line $X_3 = 0$ (in the dual space of lines through $I^0 = [0 : 0 : 1 : 0]$) and we choose the generator of $\pi_2\big(\mathbb{C}\mathrm{P}^2\big)$ the homotopy class of the map

$$\Pi :\ \big(D^2, S^1\big) \to \mathrm{Gr}^1\big(\mathbb{C}\mathrm{P}^2\big), \quad z \mapsto \big(1 - |z|\big)\, X_1 + zX_3 = 0.$$

The lift $\widetilde{\Pi} :\ D^2 \to \mathcal{D}_{I^0}^{2,3},\ \ z \mapsto \big(A_1^{\widetilde{\Pi}(z)}, B_1^{\widetilde{\Pi}(z)}, A_2^{\widetilde{\Pi}(z)}, B_2^{\widetilde{\Pi}(z)}, A_3^{\widetilde{\Pi}(z)}, B_3^{\widetilde{\Pi}(z)}\big)$ is given by $(k = 1, 2)$

$$A_k^{\widetilde{\Pi}(z)} = \big[2r|z| - 1 : kz : 1 : -kr\big], \qquad A_3^{\widetilde{\Pi}(z)} = [0 : z : z : -r],$$

$$B_k^{\widetilde{\Pi}(z)} = \big[2r|z| - 1 : kz : 2 : -kr\big], \qquad B_3^{\widetilde{\Pi}(z)} = [0 : z : z + 1 : -r],$$

where the corresponding lines are

$$d_k^{\widetilde{\Pi}(z)} :\ kX_0 + \overline{z}X_1 - rX_3 = 0, \qquad rX_1 + zX_3 = 0,$$

$$d_3^{\widetilde{\Pi}(z)} :\ X_0 = 0, \qquad rX_1 + zX_3 = 0.$$

The homotopy

$$M:\, S^1\times I\to \mathcal{D}^{2,2}_{I^0},$$

$$(z,t)\mapsto \big(A_1^{M(z,t)},B_1^{M(z,t)},A_2^{M(z,t)},B_2^{M(z,t)},A_3^0,B_3^{M(z,t)}\big),$$

where $A_k^{M(z,t)}=\big[-1:km_1(z,t):1\big]$, $B_k^{M(z,t)}=\big[-1:km_1(z,t):2\big]$, and $B_3^{M(z,t)}=\big[0:1:1+m_2(z,t)\big]$ are defined by:

$$m_1(z,t)=\begin{cases}\exp\left(2\dfrac{\arg z}{2-t}i\right) & 0\leqq \arg z\leqq (2-t)\pi\\ 1 & (2-t)\pi\leqq \arg z\leqq 2\pi,\end{cases}$$

$$m_2(z,t)=\begin{cases}1 & 0\leqq \arg z\leqq t\pi\\ \exp\left(2\dfrac{t\pi-\arg z}{2-t}i\right) & t\pi\leqq \arg z\leqq 2\pi,\end{cases}$$

shows that the restriction $\widetilde{\Pi}|_{S^1}$ and the loop $\sigma*\gamma^{-1}$ are homotopic. Using this and the relation $[\gamma]=[\alpha]+[\beta]+2[\sigma]$ we find $\delta_*\big([\Pi]\big)=\big[\widetilde{\Pi}|_{S^1}\big]=-[\alpha]-[\beta]-[\sigma]$.

b) The second part is a consequence of part *a*). □

PROPOSITION 3.4. *The fundamental group of $\mathcal{D}^{2,3}$ is isomorphic to $\mathbb{Z}$ and it is generated by $[\alpha]$ (or by $[\beta]$).*

PROOF. This is a consequence of Proposition 3.3 and the computations in Section 2:

$$\begin{array}{ccccccc}
\pi_2\big(\mathbb{C}\mathrm{P}^2\big)=\mathbb{Z}\big\langle[\Phi]\big\rangle & \xrightarrow{\delta^2_*} & \pi_1\big(\mathcal{D}^{2,2}_I\big)=\mathbb{Z}\big\langle[\alpha],[\beta],[\sigma]\big\rangle & \longrightarrow & \pi_1\big(\mathcal{D}^{2,2}\big) & \longrightarrow & 1\\
\downarrow\cong & & \downarrow i_* & & \downarrow i_* & & \\
\pi_2\big(\mathbb{C}\mathrm{P}^3\big)=\mathbb{Z}\big\langle\big[\Phi^3\big]\big\rangle & \xrightarrow{\delta^3_*} & \pi_1\big(\mathcal{D}^{2,3}_I\big)=\mathbb{Z}\big\langle[\alpha],[\beta]\big\rangle & \longrightarrow & \pi_1\big(\mathcal{D}^{2,3}\big) & \longrightarrow & 1
\end{array}$$

hence $\delta^3_*\big([\Phi^3]\big)=i_*\delta^2_*\big([\Phi]\big)=i_*\big(\big[\widetilde{\Phi}|_{S^1}\big]\big)=i_*\big(2[\alpha]+2[\beta]+[\sigma]\big)=[\alpha]+[\beta]$.
□

4. Non planar Desargues Configurations

First we analyze the fundamental group of two three-dimensional configuration spaces $\mathcal{D}_I^3 = \mathcal{D}_I^{3,3}$ and $\mathcal{D}^3 = \mathcal{D}^{3,3}$.

LEMMA 4.1. *The following projections are locally trivial fibrations:*

a)
$$\mathcal{F}_2(\mathbb{C}) \times \mathcal{F}_2(\mathbb{C}) \times \mathcal{F}_2(\mathbb{C}) \hookrightarrow \mathcal{D}_I^3 \to \mathcal{F}_3^{2,2},$$
$$(A_1, B_1, A_2, B_2, A_3, B_3) \mapsto (d_1, d_2, d_3)$$

b)
$$\mathcal{D}_I^3 \hookrightarrow \mathcal{D}^3 \to \mathbb{C}\mathrm{P}^3,$$
$$(A_1, B_1, A_2, B_2, A_3, B_3) \mapsto I = d_1 \cap d_2 \cap d_3.$$

PROOF. The proofs are similar to those of Lemmas 2.1 and 2.2. □

PROOF OF THEOREM 1.4 (the case $n = 3$). We modify a little the previous notations: the base point in these solid Desargues configurations are related to the center $I^0 = [0:0:1:0]$ and to the points:

$$A_1^0 = [0:0:0:1], \quad B_1^0 = [0:0:1:1], \quad d_1^0 : X_0 = X_1 = 0,$$
$$A_2^0 = [0:1:0:0], \quad B_2^0 = [0:1:1:0], \quad d_2^0 : X_0 = X_3 = 0,$$
$$A_3^0 = [1:0:0:0], \quad B_3^0 = [1:0:1:0], \quad d_1^0 : X_1 = X_3 = 0.$$

Using the fibrations of Lemma 4.1 we find

$$\pi_2\big(\mathcal{F}_3^{2,2}\big) \xrightarrow{\delta_*} \pi_1\big(\mathcal{F}_2(\mathbb{C})^3\big) \cong \mathbb{Z}^3 \to \pi_1\big(\mathcal{D}_{I^0}^3\big) \to 1,$$

where the first group is isomorphic with $\pi_2\big(\mathcal{F}_2\big(\mathbb{C}\mathrm{P}^2\big)\big) \cong \mathbb{Z}^2 = \mathbb{Z}\big\langle [F], [B]\big\rangle$ (use the fibration $* \simeq \mathbb{C}\mathrm{P}^2 \setminus \mathbb{C}\mathrm{P}^1 \hookrightarrow \mathcal{F}_3^{2,2} \to \mathcal{F}_2\big(\mathbb{C}\mathrm{P}^2\big)$); the homotopy classes $[F]$ and $[B]$ correspond to the free generators of the second homotopy groups of the fiber and of the basis respectively, in the fibration (see [3]) $\mathbb{C}\mathrm{P}^1 \simeq \big(\mathbb{C}\mathrm{P}^2 \setminus \{*\}\big) \hookrightarrow \mathcal{F}_2\big(\mathbb{C}\mathrm{P}^2\big) \to \mathbb{C}\mathrm{P}^2$:

$$F : \big(D^2, S^1\big) \to \big(\mathcal{F}_3^{2,2}, d_*^0\big), \quad z \mapsto \big(d_1^0, d_2^{F(z)}, d_3^{F(z)}\big),$$

where $d_2^{F(z)} : zX_0 - rX_1 = 0 = X_3$ and $d_3^{F(z)} : rX_0 + \overline{z}X_1 = 0 = X_3$, and also

$$B : \big(D^2, S^1\big) \to \big(\mathcal{F}_3^{2,2}, *\big), \quad z \mapsto \big(d_1^{B(z)}, d_2^0, d_3^{B(z)}\big),$$

where $d_1^{B(z)} : zX_0 - rX_3 = 0 = X_1$, $d_3^{B(z)} : rX_0 + \overline{z}X_3 = 0 = X_1$. Choosing the lifts $\widetilde{F}, \widetilde{B} : \left(D^2, S^1\right) \to \left(\mathcal{D}^3_{I^0}, \mathcal{F}_2\left(d_1^0\right) \times \mathcal{F}_2\left(d_2^0\right) \times \mathcal{F}_2\left(d_3^0\right)\right)$:

$$\widetilde{F}(z) = \left(A_1^0, B_1^0, A_2^{\widetilde{F}(z)}, B_2^{\widetilde{F}(z)}, A_3^{\widetilde{F}(z)}, B_3^{\widetilde{F}(z)}\right)$$

with

$$A_2^{\widetilde{F}(z)} = [r : z : 0 : 0], \qquad B_2^{\widetilde{F}(z)} = [r : z : 1 : 0],$$
$$A_3^{\widetilde{F}(z)} = [\overline{z} : -r : 0 : 0], \qquad B_3^{\widetilde{F}(z)} = [\overline{z} : -r : 1 : 0],$$

respectively

$$\widetilde{B}(z) = \left(A_1^{\widetilde{B}(z)}, B_1^{\widetilde{B}(z)}, A_2^0, B_2^0, A_3^{\widetilde{B}(z)}, B_3^{\widetilde{B}(z)}\right)$$

with

$$A_1^{\widetilde{B}(z)} = [r : 0 : 0 : z], \qquad B_1^{\widetilde{B}(z)} = [r : 0 : 1 : z]$$
$$A_3^{\widetilde{B}(z)} = [\overline{z} : 0 : 0 : -r], \qquad B_3^{\widetilde{B}(z)} = [\overline{z} : 0 : 1 : -r],$$

we obtain the equalities $\delta_*\left([F]\right) = -[b] + [c]$, $\delta_*\left([B]\right) = -[a] + [c]$. Therefore we proved that

COROLLARY 4.2. *The fundamental group of the space $\mathcal{D}^3_I$ is infinite cyclic generated by $[\alpha]$.*

Using the second fibration of Lemma 4.1, we find the exact sequence

$$\to \pi_2\left(\mathbb{C}\mathrm{P}^3\right) \xrightarrow{\delta_*} \pi_1\left(\mathcal{D}^3_{I^0}\right) \to \pi_1\left(\mathcal{D}^3\right) \to 1$$

where the generator $\Psi : \left(D^2, S^1\right) \to \left(\mathbb{C}\mathrm{P}^3, I^0\right)$, $z \mapsto [r : 0 : z : 0]$ has the lift

$$\widetilde{\Psi} : \left(D^2, S^1\right) \to \mathcal{D}^3, \quad z \mapsto \left(A_1^0, B_1^{\widetilde{\Psi}(z)}, A_2^0, B_2^{\widetilde{\Psi}(z)}, A_3^{\widetilde{\Psi}(z)}, B_3^{\widetilde{\Psi}(z)}\right),$$

with

$$B_1^{\widetilde{\Psi}(z)} = [r : 0 : z : 1], \qquad B_2^{\widetilde{\Psi}(z)} = [r : 1 : z : 0],$$
$$A_3^{\widetilde{\Psi}(z)} = [\overline{z} : 0 : -r : 0], \qquad B_3^{\widetilde{\Psi}(z)} = [r + \overline{z} : 0 : z - r : 0].$$

Therefore $\delta_*\left([\Psi]\right) = \left[\widetilde{\Psi}|S^1\right] = [\alpha] + [\beta] + 2[\gamma] = 4[\alpha]$, and we proved:

COROLLARY 4.3. *The fundamental group of the space $\mathcal{D}^3$ is cyclic of order four and it is generated by $[\alpha]$.*

PROPOSITION 4.4.

$$\pi_1\big(\mathcal{D}_I^{3,4}\big) \cong \pi_1\big(\mathcal{D}_I^{3,n}\big) \quad (n \geqq 4);$$

$$\pi_1\big(\mathcal{D}^{3,4}\big) \cong \pi_1\big(\mathcal{D}^{3,n}\big) \quad (n \geqq 4).$$

PROOF. This is like in 3.2. □

PROOF OF THEOREM 1.4 AND OF THEOREM 1.5. We show that $\pi_1\big(\mathcal{D}_I^{3,4}\big) = 1$; this implies that $\pi_1\big(\mathcal{D}^{3,4}\big) = 1$. Choose as a generator for the fundamental group of the space of 3-planes in $\mathbb{C}\mathrm{P}^4$ containing the fixed point $I = [0:0:1:0:0]$ the class of the map

$$\Sigma : \big(D^2, S^1\big) \to \Big(\mathrm{Gr}^2\big(\mathbb{C}\mathrm{P}^3\big),\ X_4 = 0\Big), \quad z \mapsto rX_1 - zX_4 = 0.$$

The lift

$$\widetilde{\Sigma} : \big(D^2, S^1\big) \to \big(\mathcal{D}_{I^0}^{3,4}, \mathcal{D}_{I^0}^{3,3}\big), \quad z \mapsto \Big(A_1^{00}, B_1^{00}, A_2^{\widetilde{\Sigma}(z)}, B_2^{\widetilde{\Sigma}(z)}, A_3^{00}, B_3^{00}\Big),$$

where $A_1^{00} = [0:0:0:1:0], \ldots, B_3^{00} = [1:0:1:0:0]$ are fixed points and

$$A_2^{\widetilde{\Sigma}(z)} = [0:z:0:0:r], \quad B_2^{\widetilde{\Sigma}(z)} = [0:z:1:0:r],$$

shows that $\delta_* : \pi_2\Big(\mathrm{Gr}^2\big(\mathbb{C}\mathrm{P}^3\big)\Big) \to \pi_1\big(\mathcal{D}_{I^0}^{3,3}\big)$ is an isomorphism. □

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