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VARIETIES GENERATED BY 2-TESTABLE MONOIDS

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Abstract

The smallest monoid containing a 2-testable semigroup is defined to be a 2-*testable monoid.* The well-known Brandt monoid B_2^1 of order six is an example of a 2-testable monoid. The finite basis problem for 2-testable monoids was recently addressed and solved. The present article continues with the investigation by describing all monoid varieties generated by 2-testable monoids. It is shown that there are 28 such varieties, all of which are finitely generated and precisely 19 of which are finitely based. As a comparison, the subvariety lattice of the monoid variety generated by the monoid B_2^1 is examined. This lattice has infinite width, satisfies neither the ascending chain condition nor the descending chain condition, and contains non-finitely generated varieties.

1. Introduction

A class of algebras is a *variety* if it is closed under the formation of homomorphic images, subalgebras, and arbitrary direct products. An algebra is *finitely based* if the identities it satisfies are finitely axiomatizable. The algebras considered in the present article are semigroups and monoids. For any class $\mathfrak{C}$ of semigroups or monoids, let $\mathbf{V_S}\,\mathfrak{C}$ denote the semigroup variety generated by $\mathfrak{C}$. For any class $\mathfrak{C}$ of monoids, let $\mathbb{V}_{\mathbb{M}}\,\mathfrak{C}$ denote the monoid variety generated by $\mathfrak{C}$. Refer to the surveys of Shevrin and Volkov [20] and Volkov [28] for a wealth of information on varieties, identities, and the finite basis problem for semigroups and monoids.

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A semigroup is 2-*testable* if it satisfies any identity formed by a pair of words that begin with the same letter, end with the same letter, and share the same set of factors of length two. Trahtman proved that the class of all 2-testable semigroups coincides with the variety $\mathbf{A_2} = \mathbf{V_S}\{A_2\}$ generated by the 0-simple semigroup

$$A_2 = \langle a, b \mid a^2 = aba = a,\ b^2 = 0,\ bab = b \rangle$$

of order five [26], and that the identities

$$x^3 \approx x^2, \quad xyxyx \approx xyx, \quad xyxzx \approx xzxyx \tag{1.1}$$

constitute a finite basis for the variety $\mathbf{A_2}$ [25]. It follows that any semigroup that satisfies the identities (1.1) is 2-testable. In particular, the Brandt semigroup

$$B_2 = \langle a, b \mid a^2 = b^2 = 0,\ aba = a,\ bab = b \rangle$$

of order five is 2-testable; this semigroup was also shown by Trahtman to be finitely based [22].

For any semigroup S, let S^1 denote the smallest monoid containing S. Since the variety $\mathbf{A_2}$ coincides with the class of 2-testable semigroups [26], it is convenient to refer to a monoid of the form S^1, where S is any semigroup from $\mathbf{A_2}$, as a 2-*testable monoid.* The 2-testable monoids A_2^1 [23] and B_2^1 [17] are non-finitely based. In fact, a well-known result of M. V. Sapir [19] implies that the monoids A_2^1 and B_2^1 are *inherently non-finitely based,* that is, they are not contained in any finitely based locally finite variety. It follows that any 2-testable monoid S^1 is non-finitely based whenever $B_2 \in \mathbf{V_S}\{S\}$. Recently, the finite basis property of all 2-testable monoids S^1 for which $B_2 \notin \mathbf{V_S}\{S\}$ was established [14]. These results led to a solution of the finite basis problem for all 2-testable monoids.

THEOREM 1.1. *For any semigroup $S \in \mathbf{A_2}$, the monoid S^1 is finitely based if and only if $B_2 \notin \mathbf{V_S}\{S\}$.*

The present article continues with the investigation of 2-testable monoids by describing the monoid varieties they generate. Since the variety $\mathbf{A_2}$ is the largest variety generated by 2-testable semigroups [26], it follows that the lattice $\mathcal{L}(\mathbf{A_2})$ of subvarieties of $\mathbf{A_2}$ coincides with the lattice of all varieties of 2-testable semigroups. This lattice is countably infinite [10] and contains an isomorphic copy of any finite lattice [27]. In contrast, there are only 28 monoid varieties generated by 2-testable monoids; these varieties constitute the join-semilattice in Figure 1.

In Section 3, finite 2-testable semigroups are presented to show that the varieties in Figure 1 are all finitely generated by 2-testable monoids. Other information on these varieties that are required in later sections are also given. In particular, it is noted that the monoid variety $\mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$ coincides with the largest finitely based variety generated by 2-testable monoids.

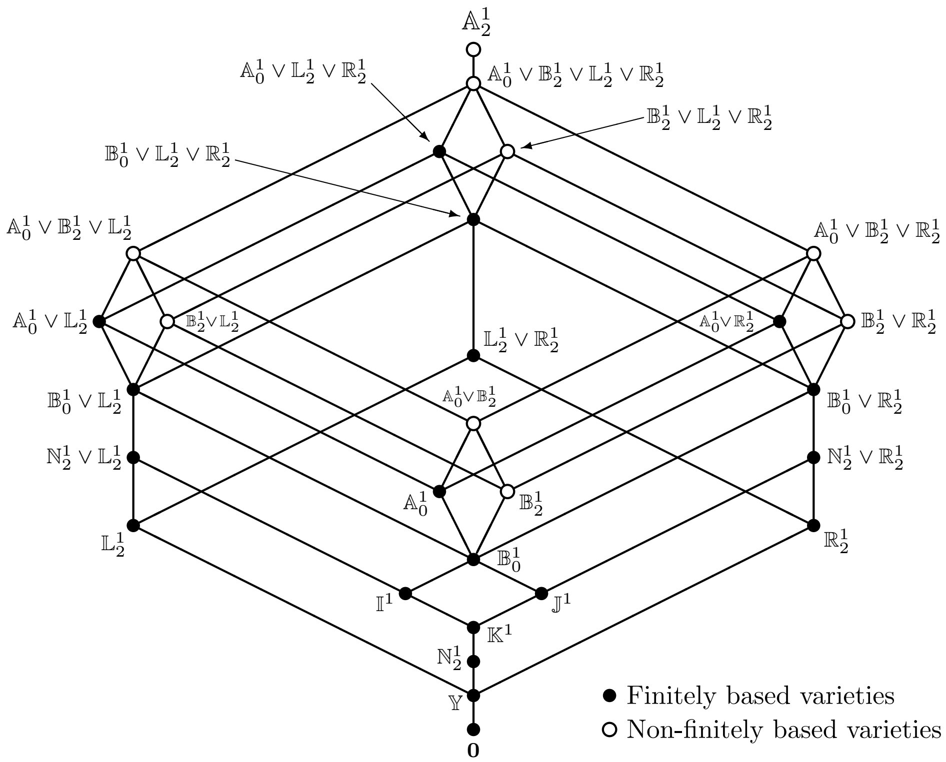

Fig. 1. The join-semilattice of varieties generated by 2-testable monoids

All subvarieties of $\mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$ are then identified in Section 4. These subvarieties coincide with the 19 finitely based varieties in Figure 1 and constitute a sublattice of the lattice of monoid varieties.

In Section 5, non-finitely based varieties generated by 2-testable monoids are examined. It is easily seen from Theorem 1.1 that these varieties belong to the interval $\left[\mathbb{B}_2^1, \mathbb{A}_2^1\right]$, where $\mathbb{A}_2^1 = \mathbb{V}_{\mathbb{M}}\left\{A_2^1\right\}$ and $\mathbb{B}_2^1 = \mathbb{V}_{\mathbb{M}}\left\{B_2^1\right\}$. Since all varieties in the interval $\left[\mathbb{B}_2^1, \mathbb{A}_2^1\right]$ are non-finitely based, none of them has a sufficiently well-described identity basis. It is thus extremely difficult, if not impossible, to identify all varieties in the interval $\left[\mathbb{B}_2^1, \mathbb{A}_2^1\right]$. Fortunately, based on recent results regarding subvarieties of $\mathbf{A_2}$ [13], the varieties in the interval $\left[\mathbb{B}_2^1, \mathbb{A}_2^1\right]$ generated by 2-testable monoids are shown to coincide with the nine non-finitely based varieties in Figure 1. These nine varieties constitute a join-semilattice, but unlike the 19 finitely based varieties in Section 4, it is unknown if this join-semilattice is a lattice. Nevertheless, the join-semilattice in Figure 1 is verified by results established in Sections 3–5.

Now regardless of whether or not the join-semilattice in Figure 1 is a lattice, the varieties it contains are far from all subvarieties of $\mathbb{A}_2^1$. This is demonstrated in Section 6, where results of Jackson [5], Jackson and O. Sapir [7], and M. V. Sapir [18, 19] are used to establish extreme properties of the lattice of subvarieties of the smaller variety $\mathbb{B}_2^1$. Specifically, this lattice has infinite width, satisfies neither the ascending chain condition nor the descending chain condition, and contains non-finitely generated varieties.

The article ends with Section 7.0, where several open questions regarding the join-semilattice in Figure 1 and subvarieties of $\mathbb{A}_2^1$ are posed.

2. Preliminaries

Most of the notation and background material of this article are given in this section. Refer to the monograph of Burris and Sankappanavar [2] for more information on universal algebra.

2.1. Letters and words

Let $\mathcal{X}$ be a fixed countably infinite alphabet throughout. Denote by $\mathcal{X}^+$ and $\mathcal{X}^*$ the free semigroup and the free monoid over $\mathcal{X}$ respectively. Elements of $\mathcal{X}$ and $\mathcal{X}^*$ are referred to as *letters* and *words* respectively.

Let x be any letter and $\mathbf{w}$ be any word. Then

- the *content* of $\mathbf{w}$, denoted by $\mathsf{con}\,(\mathbf{w})$, is the set of letters occurring in $\mathbf{w}$;
- the *head* of $\mathbf{w}$, denoted by $\mathsf{h}(\mathbf{w})$, is the first letter occurring in $\mathbf{w}$;
- the *tail* of $\mathbf{w}$, denoted by $\mathsf{t}(\mathbf{w})$, is the last letter occurring in $\mathbf{w}$;
- the *initial part* of $\mathbf{w}$, denoted by $\mathsf{ini}\,(\mathbf{w})$, is the word obtained from $\mathbf{w}$ by retaining the first occurrence of each letter;
- the *final part* of $\mathbf{w}$, denoted by $\mathsf{fin}\,(\mathbf{w})$, is the word obtained from $\mathbf{w}$ by retaining the last occurrence of each letter;
- the number of times x occurs in $\mathbf{w}$ is denoted by $\mathsf{occ}\,(x, \mathbf{w})$;
- x is *simple* in $\mathbf{w}$ if $\mathsf{occ}\,(x, \mathbf{w}) = 1$;
- $\mathbf{w}$ is *simple* if $\mathsf{occ}\,(y, \mathbf{w}) \leqq 1$ for any $y \in \mathcal{X}$;
- $\mathbf{w}$ is *quadratic* if $\mathsf{occ}\,(y, \mathbf{w}) \leqq 2$ for any $y \in \mathcal{X}$.

Let $\mathbf{w}$ be any quadratic word. If $\mathbf{w} = \mathbf{a}x\mathbf{b}x\mathbf{c}$ for some $x \in \mathcal{X}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^*$ such that $x \notin \mathsf{con}\,(\mathbf{abc})$, then the *distance* between the two occurrences of x in $\mathbf{w}$ is the length of $\mathbf{b}$. If $x_1, \dots, x_r$ are all the distinct non-simple letters of $\mathbf{w}$, then the *separation degree* of $\mathbf{w}$ is the sum $d_1 + \cdots + d_r$ where d_i is the distance between the two occurrences of x_i in $\mathbf{w}$.

2.2. Identities and varieties

An identity is written in the form $\mathbf{w} \approx \mathbf{w}'$ where $\mathbf{w}, \mathbf{w}' \in \mathcal{X}^+$. An identity $\mathbf{w} \approx \mathbf{w}'$ is *nontrivial* if $\mathbf{w} \neq \mathbf{w}'$. A semigroup S *satisfies* an identity $\mathbf{w} \approx \mathbf{w}'$ if, for any substitution φ from $\mathcal{X}$ into S, the elements $\mathbf{w}\varphi$ and $\mathbf{w}'\varphi$ coincide in S. A class $\mathfrak{C}$ of semigroups *satisfies* an identity $\mathbf{w} \approx \mathbf{w}'$ if every semigroup in $\mathfrak{C}$ satisfies $\mathbf{w} \approx \mathbf{w}'$; this is indicated by $\mathfrak{C} \models \mathbf{w} \approx \mathbf{w}'$.

Let Σ be any set of identities. The deducibility of an identity $\mathbf{w} \approx \mathbf{w}'$ from Σ is indicated by $\mathbf{w} \overset{\Sigma}{\approx} \mathbf{w}'$. The monoid variety *defined* by Σ, denoted by $[\Sigma]$, is the class of all monoids that satisfy all identities in Σ; in this case, Σ is said to be a *basis* for the variety. A variety is *finitely based* if it possesses a finite basis.

For any variety $\mathfrak{V}$ and any subvariety $\mathfrak{V}'$ of $\mathfrak{V}$, the *interval* $[\mathfrak{V}', \mathfrak{V}]$ is the set of all subvarieties of $\mathfrak{V}$ containing $\mathfrak{V}'$. Let $\mathcal{L}(\mathfrak{V})$ denote the lattice of subvarieties of $\mathfrak{V}$. Equivalently, $\mathcal{L}(\mathfrak{V}) = [\mathbf{0}, \mathfrak{V}]$ where $\mathbf{0}$ is the trivial variety.

LEMMA 2.1 (Almeida [1, Lemma 7.1.1]). *Let S be any semigroup and $\mathbb{V}$ be any monoid variety such that $S \in \mathbf{V_S}\,\mathbb{V}$. Then $S^1 \in \mathbb{V}$.*

3. Some 2-testable monoids

The present section introduces 2-testable monoids that generate the varieties in Figure 1. Some identities and identity bases that are required in later sections are also given.

3.1. Monoids generating varieties in Figure 1

It is routinely checked that the following semigroups satisfy the identities (1.1) and so are 2-testable:

$$A_0 = \langle a, b \mid a^2 = a,\ b^2 = b,\ ba = 0 \rangle,$$

$$B_0 = \langle a, b, c \mid a^2 = a,\ b^2 = b,\ ab = ba = 0,\ ac = cb = c \rangle,$$

$$I = \langle a, b \mid ab = a,\ ba = 0,\ b^2 = b \rangle,$$

$$J = \langle a, b \mid ba = a,\ ab = 0,\ b^2 = b \rangle,$$

$$K = \langle a, b \mid a^2 = b^2 = ba = 0 \rangle,$$

$$L_2 = \langle a, b \mid a^2 = ab = a,\ b^2 = ba = b \rangle,$$

$$N_2 = \langle a \mid a^2 = 0 \rangle,$$

$$R_2 = \langle a, b \mid a^2 = ba = a,\ b^2 = ab = b \rangle.$$

In fact, the semigroups A_0, I, J, L_2, N_2, and R_2 are isomorphic to subsemigroups of A_2, and the semigroup B_0 is isomorphic to a subsemigroup of B_2. Note that L_2 is a left-zero semigroup, N_2 is a null semigroup, and R_2 is a right-zero semigroup. Let $\mathbb{Y}$ be the variety of semilattice monoids and let

$$\mathbb{A}_0^1 = \mathbb{V}_{\mathbb{M}}\{A_0^1\}, \quad \mathbb{B}_0^1 = \mathbb{V}_{\mathbb{M}}\{B_0^1\}, \quad \mathbb{I}^1 = \mathbb{V}_{\mathbb{M}}\{I^1\}, \quad \mathbb{J}^1 = \mathbb{V}_{\mathbb{M}}\{J^1\},$$

$$\mathbb{K}^1 = \mathbb{V}_{\mathbb{M}}\{K^1\}, \quad \mathbb{L}_2^1 = \mathbb{V}_{\mathbb{M}}\{L_2^1\}, \quad \mathbb{N}_2^1 = \mathbb{V}_{\mathbb{M}}\{N_2^1\}, \quad \mathbb{R}_2^1 = \mathbb{V}_{\mathbb{M}}\{R_2^1\}.$$

The *join* of two varieties $\mathfrak{V}$ and $\mathfrak{V}'$, denoted by $\mathfrak{V} \vee \mathfrak{V}'$, is the smallest variety containing $\mathfrak{V}$ and $\mathfrak{V}'$. If S and T are 2-testable semigroups, then the direct product $S \times T$ is a 2-testable semigroup such that $\mathbb{V}_{\mathbb{M}}\{S^1\} \vee \mathbb{V}_{\mathbb{M}}\{T^1\} = \mathbb{V}_{\mathbb{M}}\{(S \times T)^1\}$. It follows that the varieties in Figure 1 are finitely generated by 2-testable monoids.

LEMMA 3.1. *The variety $\mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$ is the largest finitely based variety generated by 2-testable monoids.*

PROOF. The variety $\mathbf{A_2}$ contains a subvariety that is largest with respect to not containing the semigroup B_2 [8, Theorem 3.6]; this subvariety of $\mathbf{A_2}$ is generated by a certain 2-testable semigroup C_0 of order six [15, Theorem 4.2(iii)]. By Theorem 1.1, the variety $\mathbb{V}_{\mathbb{M}}\{C_0^1\}$ coincides with the largest finitely based variety generated by 2-testable monoids. The present lemma then follows since the varieties $\mathbb{V}_{\mathbb{M}}\{C_0^1\}$ and $\mathbb{V}_{\mathbb{M}}\{A_0^1 \times L_2^1 \times R_2^1\}$ coincide [14, Lemma 3.2]. □

3.2. Bases and identities

LEMMA 3.2. *Let $\mathbf{w} \approx \mathbf{w}'$ be any identity. Then*

(i) $L_2^1 \models \mathbf{w} \approx \mathbf{w}'$ *if and only if* $\mathsf{ini}(\mathbf{w}) = \mathsf{ini}(\mathbf{w}')$;

(ii) $R_2^1 \models \mathbf{w} \approx \mathbf{w}'$ *if and only if* $\mathsf{fin}(\mathbf{w}) = \mathsf{fin}(\mathbf{w}')$.

Further, if the words $\mathbf{w}$ and $\mathbf{w}'$ are quadratic, then

(iii) $N_2^1 \models \mathbf{w} \approx \mathbf{w}'$ *if and only if* $\mathsf{occ}(x, \mathbf{w}) = \mathsf{occ}(x, \mathbf{w}')$ *for all* $x \in \mathcal{X}$.

PROOF. These results are well known and easily verified. □

LEMMA 3.3.

(i) $\mathbb{L}_2^1 = [xyx \approx xy]$.

(ii) $\mathbb{R}_2^1 = [xyx \approx yx]$.

(iii) $\mathbb{L}_2^1 \vee \mathbb{R}_2^1 = \left[x^2 \approx x, xyxzx \approx xyzx\right]$.

(iv) $\mathbb{N}_2^1 \vee \mathbb{L}_2^1 = \left[x^3 \approx x^2, xyx \approx x^2y\right]$.

(v) $\mathbb{N}_2^1 \vee \mathbb{R}_2^1 = \left[x^3 \approx x^2, xyx \approx yx^2\right]$.

PROOF. Parts (i)–(iii) are well known and can be found in Almeida [1, Section 5.5]. The arguments of Edmunds [3, proof of Proposition 3.1(c)] can be repeated to establish part (iv). Part (v) is symmetrical to part (iv). □

The following identities are required in the bases for some varieties containing the monoids A_0^1 and B_0^1:

$$(\bigstar) \qquad xyxzx \approx xyzx,$$

$$(\blacklozenge) \qquad xyxy \approx x^2y^2,$$

$$(\blacktriangleright) \qquad xyxy \approx xy^2x,$$

$$(\blacktriangleleft) \qquad xyxy \approx yx^2y.$$

LEMMA 3.4.

(i) $\mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1 = \left[(\bigstar)\right]$.

(ii) $\mathbb{A}_0^1 \vee \mathbb{L}_2^1 = \left[(\bigstar), (\blacktriangleright)\right]$.

(iii) $\mathbb{A}_0^1 \vee \mathbb{R}_2^1 = \left[(\bigstar), (\blacktriangleleft)\right]$.

(iv) $\mathbb{A}_0^1 = \left[(\bigstar), (\blacktriangleright), (\blacktriangleleft)\right]$.

(v) $\mathbb{B}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1 = \left[(\bigstar), (\blacklozenge)\right]$.

(vi) $\mathbb{B}_0^1 \vee \mathbb{L}_2^1 = \left[(\bigstar), (\blacklozenge), (\blacktriangleright)\right]$.

(vii) $\mathbb{B}_0^1 \vee \mathbb{R}_2^1 = \left[(\bigstar), (\blacklozenge), (\blacktriangleleft)\right]$.

(viii) $\mathbb{B}_0^1 = \left[(\bigstar), (\blacklozenge), (\blacktriangleright), (\blacktriangleleft)\right]$.

PROOF. Parts (i)–(iv) follow from Lee [14, Proposition 2.3], part (viii) follows from Edmunds [3, Proposition 3.1(i)], and part (v) is established at the end of this subsection.

It follows from Edmunds [3, Proposition 3.1(i)] that the identities $\{(\bigstar), (\blacklozenge), (\blacktriangleright)\}$ constitute a basis for a certain monoid of order five. The proof of this result can easily be repeated to show the same identities also constitute a basis for the variety $\mathbb{B}_0^1 \vee \mathbb{L}_2^1$. Hence part (vi) holds. By symmetry, part (vii) also holds. □

REMARK 3.5. Note that if a letter x occurs three or more times in a word $\mathbf{w}$, then all except the first and last occurrences of x in $\mathbf{w}$ can be eliminated by the identity ($\bigstar$) and its consequences $x^2yx \approx xyx \approx xyx^2$. Therefore any word can be converted by the identity ($\bigstar$) into a unique quadratic word.

Let x be any letter in a quadratic word $\mathbf{w}$ such that $\mathbf{w} = \mathbf{a}x\mathbf{b}x\mathbf{c}$ for some $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathscr{X}^*$. Then x is said to be *tight* in $\mathbf{w}$ if $\mathsf{h}(\mathbf{b}) \notin \mathsf{con}\,(\mathbf{a})$ and $\mathsf{t}(\mathbf{b}) \notin \mathsf{con}\,(\mathbf{c})$. Note that x is vacuously tight in $\mathbf{w}$ if $\mathbf{b} = \emptyset$. A quadratic word is *tight* if all its non-simple letters are tight in it.

LEMMA 3.6. *Let* $\mathbf{w}$ *be any word. Then there exists a tight quadratic word* $\widehat{\mathbf{w}}$ *such that the identity* $\mathbf{w} \approx \widehat{\mathbf{w}}$ *is a consequence of the identities* $\{(\bigstar), (\blacklozenge)\}$.

PROOF. By Remark 3.5, the word $\mathbf{w}$ can be chosen to be quadratic. Suppose that $\mathbf{w} = \mathbf{a}x\mathbf{b}x\mathbf{c}$ for some $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathscr{X}^*$ and x is not tight in $\mathbf{w}$. If $\mathsf{h}(\mathbf{b}) = h \in \mathsf{con}\,(\mathbf{a})$, say $\mathbf{a} = \mathbf{a}'h\mathbf{a}''$ and $\mathbf{b} = h\mathbf{b}'$ for some $\mathbf{a}', \mathbf{a}'', \mathbf{b}' \in \mathscr{X}^*$, then the identities $\{(\bigstar), (\blacklozenge)\}$ can be applied to interchange the first x with the first letter of $\mathbf{b}$:

$$\mathbf{w} = \mathbf{a}'h\mathbf{a}''xh\mathbf{b}'x\mathbf{c} \overset{(\bigstar)}{\approx} \mathbf{a}'h\mathbf{a}''hxhx\mathbf{b}'x\mathbf{c} \overset{(\blacklozenge)}{\approx} \mathbf{a}'h\mathbf{a}''h^2x^2\mathbf{b}'x\mathbf{c} \overset{(\bigstar)}{\approx} \mathbf{a}hx\mathbf{b}'x\mathbf{c}.$$

Similarly, if $\mathsf{t}(\mathbf{b}) \in \mathsf{con}\,(\mathbf{c})$, then the identities $\{(\bigstar), (\blacklozenge)\}$ can be applied to interchange the second x with the last letter of $\mathbf{b}$. These interchanges can be repeated until the letter x is tight. In other words, the identities $\{(\bigstar), (\blacklozenge)\}$ *tightened* the letter x. Observe that in the process of tightening a non-simple letter x, each time an identity from $\{(\bigstar), (\blacklozenge)\}$ is applied to interchange x with another non-simple letter y,

- the distance between the two occurrences of x decreases,
- the distance between the two occurrences of y decreases, and
- the distance between the two occurrences of any other non-simple letter remains unchanged.

It follows that whenever the identities $\{(\bigstar), (\blacklozenge)\}$ tighten a non-tight letter x, the separation degree of the word decreases. Since the separation degree of any word is nonnegative, only finitely many applications of the identities $\{(\bigstar), (\blacklozenge)\}$ are required to tighten every non-simple letter of $\mathbf{w}$. □

PROOF OF LEMMA 3.4(V). Luo and Zhang have shown that the identities

$$x^8y \approx x^2y, \quad xy^8 \approx xy^2, \quad x^7yx \approx xyx, \quad x^2yx \approx xyx^2, \quad xyxzx \approx x^2yzx,$$

and ($\blacklozenge$) constitute a basis for the variety $\mathbf{S}_3$ generated by all semigroups of order three [16, Corollary 4.6]. The monoid B_0^1 can then be shown to belong to the variety $\mathbf{S}_3$. It follows that results of Luo and Zhang [16] will be useful in the present proof.

It is routinely verified that the monoids B_0^1, L_2^1, and R_2^1 satisfy the identities $\{(\bigstar), (\blacklozenge)\}$. Therefore, to complete the proof, it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by the variety $\mathbb{B}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$ is deducible from the identities $\{(\bigstar), (\blacklozenge)\}$. By Remark 3.5 and Lemma 3.6, the words $\mathbf{w}$ and $\mathbf{w}'$ can be chosen to be quadratic and tight. Then the words $\mathbf{w}$ and $\mathbf{w}'$ satisfy conditions (CF1)–(CF4) in Luo and Zhang [16, Section 4] and so are said to be in *canonical form.* Now the monoid N_2^1 is isomorphic to a submonoid of B_0^1 and so satisfies the identity $\mathbf{w} \approx \mathbf{w}'$; since the monoids L_2^1 and R_2^1 also satisfy the identity $\mathbf{w} \approx \mathbf{w}'$, the conditions $\mathsf{ini}(\mathbf{w}) = \mathsf{ini}(\mathbf{w}')$, $\mathsf{fin}(\mathbf{w}) = \mathsf{fin}(\mathbf{w}')$, and $\mathsf{occ}(x, \mathbf{w}) = \mathsf{occ}(x, \mathbf{w}')$ for all $x \in \mathscr{X}$ hold by Lemma 3.2. It then follows from Luo and Zhang [16, Lemma 4.5] that the words $\mathbf{w}$ and $\mathbf{w}'$ are identical. Consequently, the identity $\mathbf{w} \approx \mathbf{w}'$ is deducible from the identities $\{(\bigstar), (\blacklozenge)\}$. □

3.3. Other required identities

LEMMA 3.7. *Let $\mathbb{V}$ be any subvariety of $\mathbb{A}_2^1$.*

(i) *If $A_0^1 \notin \mathbb{V}$, then $\mathbb{V}$ satisfies the identity*

$$(x^2y^2)^2 \approx x^2y^2. \tag{3.1}$$

(ii) *If $A_2^1 \notin \mathbb{V}$, then $\mathbb{V}$ satisfies the identity*

$$\big((x^2y)^2(yx^2)^2\big)^2 \approx (x^2yx^2)^2. \tag{3.2}$$

(iii) *If $R_2^1 \notin \mathbb{V}$, then $\mathbb{V}$ satisfies the identity*

$$(x^2y)^2x^2 \approx (x^2y)^2. \tag{3.3}$$

(iv) *If $L_2^1 \notin \mathbb{V}$, then $\mathbb{V}$ satisfies the identity*

$$x^2(yx^2)^2 \approx (yx^2)^2. \tag{3.4}$$

(v) *If $B_0^1 \notin \mathbb{V}$, then $\mathbb{V}$ satisfies one of the identities*

$$x^2 \approx x, \tag{3.5}$$

$$x^2yx^2 \approx x^2y, \tag{3.6}$$

$$x^2yx^2 \approx yx^2. \tag{3.7}$$

Proof. Part (v) follows from Almeida [1, Proposition 11.10.2]. Since parts (iii) and (iv) are dual results, it suffices to verify parts (i)–(iii). Let $S \in \{A_0, A_2, R_2\}$. Suppose that $S^1 \notin \mathbb{V}$. Then $S \notin \mathbf{V_S}\,\mathbb{V}$ by Lemma 2.1.

(i) If $S = A_0$, then $\mathbf{V_S}\,\mathbb{V} \models$ (3.1) by Torlopova [21].

(ii) If $S = A_2$, then $\mathbf{V_S}\,\mathbb{V} \models$ (3.2) by Lee [9].

(iii) If $S = R_2$, then $\mathbf{V_S}\,\mathbb{V} \models$ (3.3) by Almeida [1, Proposition 10.10.2(c)]. □

4. Finitely based varieties generated by 2-testable monoids

Proposition 4.1.

(i) *The lattice in Figure 2 coincides with* $\mathcal{L}\big(\mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1\big)$.

(ii) *The varieties in Figure 2 are precisely all finitely based varieties generated by 2-testable monoids.*

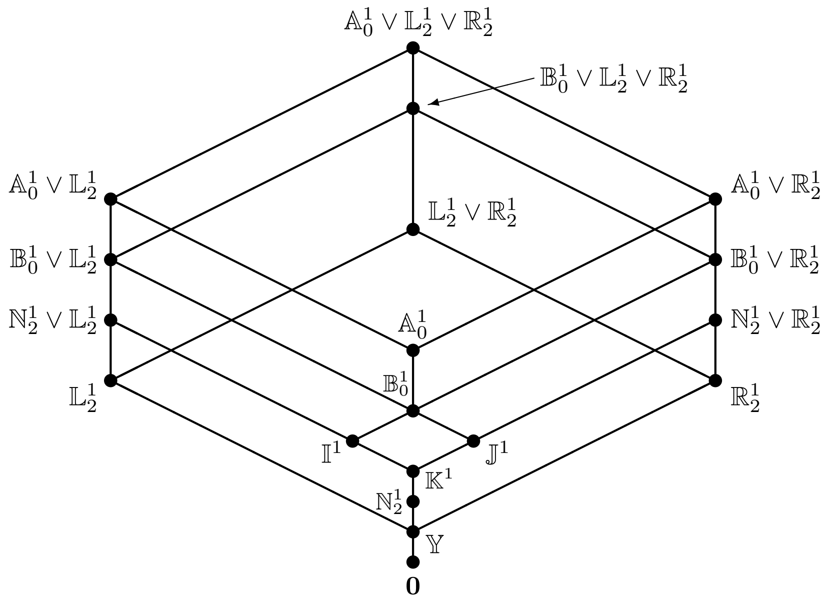

Fig. 2. The lattice of finitely based varieties generated by 2-testable monoids

In Lemma 4.2, the subvarieties of $\mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$ are partitioned into four disjoint intervals. The varieties in these intervals are then described in Lemma 4.3. Based on these results, the proof of Proposition 4.1 is given at the end of the section.

LEMMA 4.2. *The lattice* $\mathcal{L}\big(\mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1\big)$ *is the disjoint union of the intervals*

$$\mathcal{I}_1 = \big[\mathbb{L}_2^1 \vee \mathbb{R}_2^1, \mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1\big],$$
$$\mathcal{I}_2 = \big[\mathbb{L}_2^1, \mathbb{A}_0^1 \vee \mathbb{L}_2^1\big],$$
$$\mathcal{I}_3 = \big[\mathbb{R}_2^1, \mathbb{A}_0^1 \vee \mathbb{R}_2^1\big],$$
$$\mathcal{I}_4 = \mathcal{L}\big(\mathbb{A}_0^1\big).$$

PROOF. Let $\mathbb{V} \in \mathcal{L}\big(\mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1\big)$. Then by Lemma 3.4(i), the variety $\mathbb{V}$ satisfies the identity ($\bigstar$). There are four cases.
CASE 1. $L_2^1, R_2^1 \in \mathbb{V}$. Then $\mathbb{V} \in \mathcal{I}_1$.
CASE 2. $L_2^1 \in \mathbb{V}$ and $R_2^1 \notin \mathbb{V}$. Then by Lemma 3.7(iii), the variety $\mathbb{V}$ satisfies the identity (3.3). Since

$$xyxy \overset{(\bigstar)}{\approx} xy\big(x^2y\big)^2 \overset{(3.3)}{\approx} xy\big(x^2y\big)^2x^2 \overset{(\bigstar)}{\approx} xy^2x,$$

the variety $\mathbb{V}$ also satisfies the identity ($\blacktriangleright$). Hence $\mathbb{V} \in \mathcal{I}_2$ by Lemma 3.4(ii).
CASE 3. $L_2^1 \notin \mathbb{V}$ and $R_2^1 \in \mathbb{V}$. By an argument that is symmetrical to Case 2, the variety $\mathbb{V}$ satisfies the identity ($\blacktriangleleft$) so that $\mathbb{V} \in \mathcal{I}_3$ by Lemma 3.4(iii).
CASE 4. $L_2^1, R_2^1 \notin \mathbb{V}$. By Cases 2 and 3, the variety $\mathbb{V}$ satisfies the identities ($\blacktriangleright$) and ($\blacktriangleleft$). Therefore $\mathbb{V} \in \mathcal{I}_4$ by Lemma 3.4(iv). □

LEMMA 4.3.

(i) *The varieties in the interval* $\mathcal{I}_1$ *constitute the chain*

$$\mathbb{L}_2^1 \vee \mathbb{R}_2^1 \subset \mathbb{B}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1 \subset \mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1.$$

(ii) *The varieties in the interval* $\mathcal{I}_2$ *constitute the chain*

$$\mathbb{L}_2^1 \subset \mathbb{N}_2^1 \vee \mathbb{L}_2^1 \subset \mathbb{B}_0^1 \vee \mathbb{L}_2^1 \subset \mathbb{A}_0^1 \vee \mathbb{L}_2^1.$$

(iii) *The varieties in the interval* $\mathcal{I}_3$ *constitute the chain*

$$\mathbb{R}_2^1 \subset \mathbb{N}_2^1 \vee \mathbb{R}_2^1 \subset \mathbb{B}_0^1 \vee \mathbb{R}_2^1 \subset \mathbb{A}_0^1 \vee \mathbb{R}_2^1.$$

(iv) *The eight subvarieties of* $\mathbb{A}_0^1$ *in Figure* 2 *constitute the interval* $\mathcal{I}_4$.

PROOF. (i) Let $\mathbb{V} \in \mathcal{I}_1$ so that $\mathbb{V}$ satisfies the identity ($\bigstar$) by Lemma 3.4(i). Suppose that $\mathbb{V} \neq \mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$. Then $A_0^1 \notin \mathbb{V}$ because $L_2^1, R_2^1 \in \mathbb{V}$ by assumption. By Lemma 3.7(i), the variety $\mathbb{V}$ satisfies the identity (3.1). Since

$$xyxy \overset{(\bigstar)}{\approx} x^2y^2x^2y^2 \overset{(3.1)}{\approx} x^2y^2,$$

the variety $\mathbb{V}$ satisfies the identity ($\blacklozenge$) so that $\mathbb{V} \subseteq \mathbb{B}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$ by Lemma 3.4(v).

Suppose that $\mathbb{V} \neq \mathbb{B}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$. Then $B_0^1 \notin \mathbb{V}$ because $L_2^1, R_2^1 \in \mathbb{V}$ by assumption. By Lemmas 3.2 and 3.7(v), the variety $\mathbb{V}$ satisfies the identity (3.5). It then follows from Lemma 3.3(iii) that $\mathbb{V} = \mathbb{L}_2^1 \vee \mathbb{R}_2^1$.

(ii) Let $\mathbb{V} \in \mathcal{I}_2$ so that $\mathbb{V}$ satisfies the identities $\{(\bigstar), (\blacktriangleright)\}$ by Lemma 3.4(ii). Suppose that $\mathbb{V} \neq \mathbb{A}_0^1 \vee \mathbb{L}_2^1$. Then $A_0^1 \notin \mathbb{V}$ since $L_2^1 \in \mathbb{V}$ by assumption. By the same argument in part (i), the variety $\mathbb{V}$ satisfies the identity ($\blacklozenge$) so that $\mathbb{V} \subseteq \mathbb{B}_0^1 \vee \mathbb{L}_2^1$ by Lemma 3.4(vi).

Suppose that $\mathbb{V} \neq \mathbb{B}_0^1 \vee \mathbb{L}_2^1$. Then $B_0^1 \notin \mathbb{V}$ because $L_2^1 \in \mathbb{V}$ by assumption. By Lemmas 3.2(i) and 3.7(v), the variety $\mathbb{V}$ satisfies either the identity (3.5) or the identity (3.6). But since

$$xyx \overset{(3.5)}{\approx} x^2y^2x^2 \overset{(\blacktriangleright)}{\approx} x^2yx^2y \overset{(3.5)}{\approx} x^2y \quad \text{and} \quad xyx \overset{(\bigstar)}{\approx} x^2yx^2 \overset{(3.6)}{\approx} x^2y,$$

the variety $\mathbb{V}$ satisfies the identity $xyx \approx x^2y$ in either case so that $\mathbb{V} \subseteq \mathbb{N}_2^1 \vee \mathbb{L}_2^1$ by Lemma 3.3(iv).

Suppose that $\mathbb{V} \neq \mathbb{N}_2^1 \vee \mathbb{L}_2^1$. Then $N_2^1 \notin \mathbb{V}$ because $L_2^1 \in \mathbb{V}$ by assumption. By Lemma 3.2(iii), the variety satisfies the identity $x^2 \approx x$ so that $\mathbb{V} = \mathbb{L}_2^1$ by Lemma 3.3(i).

(iii) This is symmetrical to part (ii).

(iv) This was established by Lee [11, Section 5]. □

PROOF OF PROPOSITION 4.1. (i) This follows from Lemmas 4.2 and 4.3.

(ii) Let $\mathbb{V}$ be any finitely based variety generated by 2-testable monoids. Then $\mathbb{V} \subseteq \mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$ by Lemma 3.1. By part (i), the variety $\mathbb{V}$ is one of the varieties in Figure 2. □

5. Non-finitely based varieties generated by 2-testable monoids

PROPOSITION 5.1. *The non-finitely based varieties generated by* 2*-testable monoids constitute the join-semilattice in Figure* 3.

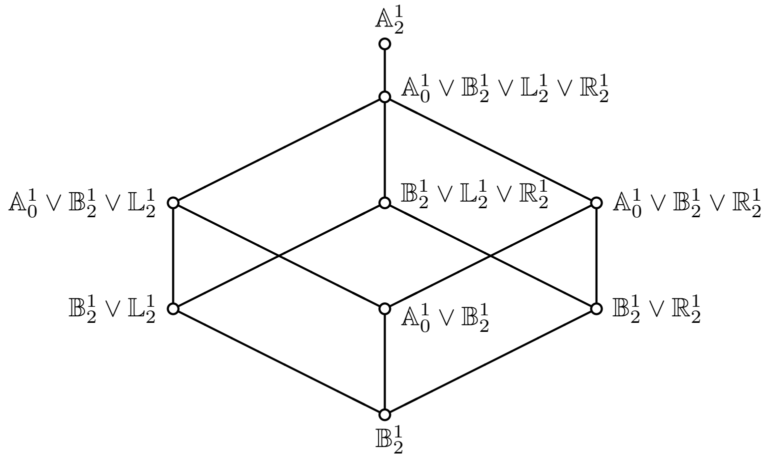

Fig. 3. The join-semilattice of non-finitely based varieties generated by 2-testable monoids

Let P and Q be semigroups with the following multiplication tables:

P	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	0	0	b
b	0	0	0	0	b	b
c	0	a	b	c	0	0
d	0	0	0	0	d	d
e	0	0	0	0	e	e

Q	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	a	a
b	0	a	b	a	0
c	0	0	0	c	c
d	0	0	0	d	d

The present section requires the semigroup varieties

$$\mathbf{A_0} = \mathbf{V_S}\{A_0\}, \qquad \mathbf{B_2} = \mathbf{V_S}\{B_2\}, \qquad \mathbf{L_2} = \mathbf{V_S}\{L_2\},$$
$$\mathbf{P} = \mathbf{V_S}\{P\}, \qquad \mathbf{Q} = \mathbf{V_S}\{Q\}, \qquad \mathbf{R_2} = \mathbf{V_S}\{R_2\},$$

and the monoid varieties

$$\mathbb{P}^1 = \mathbb{V}_{\mathbb{M}}\left\{P^1\right\}, \quad \mathbb{Q}^1 = \mathbb{V}_{\mathbb{M}}\left\{Q^1\right\}.$$

For any variety $\mathfrak{V}$, the *dual* variety of $\mathfrak{V}$ is

$$\mathfrak{V}_\delta = \{V \mid V \text{ is anti-isomorphic to some member of } \mathfrak{V}\}.$$

For example, $\left(\mathbb{L}_2^1\right)_\delta = \mathbb{R}_2^1$ and $\mathbb{I}_\delta^1 = \mathbb{J}^1$. The varieties $\mathbb{A}_0^1$, $\mathbb{A}_2^1$, $\mathbb{B}_0^1$, $\mathbb{B}_2^1$, $\mathbb{K}^1$, $\mathbb{N}^1$, and $\mathbb{Y}$ are *self-dual* in the sense that they satisfy the equation $\mathbb{V}_\delta = \mathbb{V}$.

Some results regarding the varieties $\mathbb{P}^1$ and $\mathbb{Q}^1$ are established in Subsection 5.1. The proof of Proposition 5.1 is then given in Subsection 5.2.

5.1. The varieties $\mathbb{P}^1$ and $\mathbb{Q}^1$

LEMMA 5.2 (Lee [12, Proposition 3.3]).
(i) $\mathbb{A}_0^1 \vee \mathbb{L}_2^1 = \mathbb{Q}^1$.
(ii) $\mathbb{A}_0^1 \vee \mathbb{R}_2^1 = \mathbb{Q}_\delta^1$.

Any words $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are said to be *disjoint* if the sets $\mathsf{con}(\mathbf{w}_1), \ldots, \mathsf{con}(\mathbf{w}_n)$ are pairwise disjoint. A word of length at least two is *connected* if it cannot be written as a product of two disjoint nonempty words. Any word $\mathbf{w}$ can be uniquely written in *natural form,* that is,

$$\mathbf{w} = \prod_{i=1}^{n} (\mathbf{s}_i \mathbf{w}_i)$$

where each $\mathbf{s}_i$ is a simple word with $\mathbf{s}_1$ possibly being empty, each $\mathbf{w}_i$ is a product of disjoint connected words with $\mathbf{w}_n$ possibly being empty, and the words $\mathbf{s}_1, \mathbf{w}_1, \ldots, \mathbf{s}_n, \mathbf{w}_n$ are disjoint.

LEMMA 5.3 (Lee and Volkov [15, Proposition 3.2(ii)]). *Let* $\mathbf{w} = \prod_{i=1}^{n} (\mathbf{s}_i \mathbf{w}_i)$ *and* $\mathbf{w}' = \prod_{i=1}^{n'} (\mathbf{s}'_i \mathbf{w}'_i)$ *be any words written in natural form. Then* $B_2 \models \mathbf{w} \approx \mathbf{w}'$ *if and only if* $n = n'$, $\mathbf{s}_i = \mathbf{s}'_i$, *and* $B_2 \models \mathbf{w}_i \approx \mathbf{w}'_i$ *for all* i.

LEMMA 5.4 (Lee [13, Corollary 5.6(iii) and Lemma 6.2(i)]). *Let* $\mathbf{w} \approx \mathbf{w}'$ *be any identity satisfied by the semigroup* B_2. *Suppose that the words* $\mathbf{w} = \prod_{i=1}^{n} (\mathbf{s}_i \mathbf{w}_i)$ *and* $\mathbf{w}' = \prod_{i=1}^{n} (\mathbf{s}_i \mathbf{w}'_i)$ *are in natural form. Then* $P \models \mathbf{w} \approx \mathbf{w}'$ *if and only if* $\mathsf{h}(\mathbf{w}_i) = \mathsf{h}(\mathbf{w}'_i)$ *for all* i.

LEMMA 5.5.
(i) $\mathbb{B}_2^1 \vee \mathbb{L}_2^1 = \mathbb{B}_2^1 \vee \mathbb{P}^1$.
(ii) $\mathbb{B}_2^1 \vee \mathbb{R}_2^1 = \mathbb{B}_2^1 \vee \mathbb{P}_\delta^1$.

PROOF. By symmetry, it suffices to verify part (i). Let $\mathbf{w} \approx \mathbf{w}'$ be any identity satisfied by the variety $\mathbb{B}_2^1 \vee \mathbb{L}_2^1$. By Lemma 5.3,

$$\mathbf{w} = \prod_{i=1}^{n} (\mathbf{s}_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{n} (\mathbf{s}_i \mathbf{w}'_i)$$

when $\mathbf{w}$ and $\mathbf{w}'$ are written in natural form. Since the words $\mathbf{s}_1, \mathbf{w}_1, \ldots, \mathbf{s}_n, \mathbf{w}_n$ are disjoint, the words $\mathbf{s}_1, \mathbf{w}'_1, \ldots, \mathbf{s}_n, \mathbf{w}'_n$ are disjoint, and that $\mathsf{ini}(\mathbf{w}) = \mathsf{ini}(\mathbf{w}')$ by Lemma 3.2(i), it follows that $\mathsf{h}(\mathbf{w}_i) = \mathsf{h}(\mathbf{w}'_i)$ for all i. Hence by Lemma 5.4, the semigroup P satisfies the identity $\mathbf{w} \approx \mathbf{w}'$. Since the identity $\mathbf{w} \approx \mathbf{w}'$ was arbitrarily chosen, $P \in \mathbf{V_S}\left(\mathbb{B}_2^1 \vee \mathbb{L}_2^1\right)$. Therefore $P^1 \in \mathbb{B}_2^1 \vee \mathbb{L}_2^1$

by Lemma 2.1. Consequently, the inclusion $\mathbb{B}_2^1 \vee \mathbb{P}^1 \subseteq \mathbb{B}_2^1 \vee \mathbb{L}_2^1$ holds. The inclusion $\mathbb{B}_2^1 \vee \mathbb{L}_2^1 \subseteq \mathbb{B}_2^1 \vee \mathbb{P}^1$ holds since the monoid L_2^1 is isomorphic to the submonoid $\{d, e, 1\}$ of P^1. $\square$

5.2. Proof of Proposition 5.1

As observed in Subsection 3.1, the nine varieties in Figure 3 are all generated by 2-testable monoids, and it is evident from the generating monoids that the varieties form a join-semilattice. Further, these varieties contain the inherently non-finitely based monoid B_2^1 and so are non-finitely based. Lemma 3.7 can be used to distinguish these varieties. For instance, the identity (3.2) is satisfied by the monoids A_0^1, B_2^1, L_2^1, and R_2^1, but not by the monoid A_2^1. Hence $\mathbb{A}_2^1 \neq \mathbb{A}_0^1 \vee \mathbb{B}_2^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$.

It remains to show that if S^1 is any non-finitely based 2-testable monoid, then it generates one of the varieties in Figure 3. Let $\mathbf{S} = \mathbf{V_S}\{S\}$ and $\mathbb{S}^1 = \mathbb{V}_{\mathbb{M}}\left\{S^1\right\}$. By Theorem 1.1, the variety $\mathbf{S}$ belongs to the interval $[\mathbf{B_2}, \mathbf{A_2}]$. The structure of the interval $[\mathbf{B_2}, \mathbf{A_2}]$, shown in Figure 4, follows from results of Lee [13, Section 5].

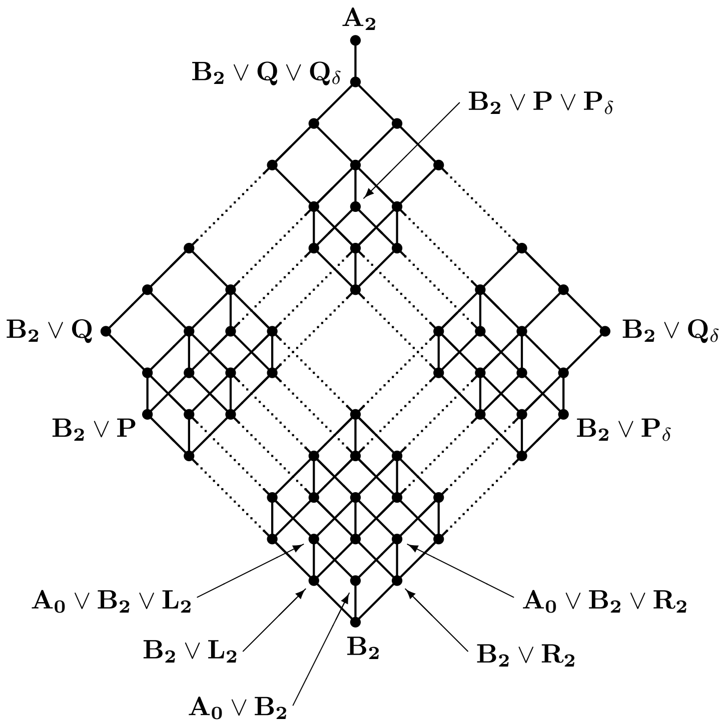

Fig. 4. The interval $[\mathbf{B_2}, \mathbf{A_2}]$

Remark 5.6.

(i) In Figure 4, only varieties in the interval $[\mathbf{B_2}, \mathbf{A_2}]$ that are required in the present section are labeled. Refer to Lee [13] for more information on the unlabeled varieties and other subvarieties of $\mathbf{A_2}$.

(ii) The interval $[\mathbf{A_0} \vee \mathbf{B_2}, \mathbf{B_2} \vee \mathbf{Q} \vee \mathbf{Q}_\delta]$ is isomorphic to the direct product of two $\omega + 3$ chains, while the interval $[\mathbf{B_2}, \mathbf{B_2} \vee \mathbf{P} \vee \mathbf{P}_\delta]$ is isomorphic to the direct product of two $\omega + 2$ chains.

It is clear that if $\mathbf{S} \in \{\mathbf{A_2}, \mathbf{B_2}, \mathbf{A_0} \vee \mathbf{B_2}\}$, then $\mathbb{S}^1 \in \left\{\mathbb{A}_2^1, \mathbb{B}_2^1, \mathbb{A}_0^1 \vee \mathbb{B}_2^1\right\}$. Therefore it suffices to consider the case when the variety $\mathbf{S}$ belongs to one of the following subintervals of $[\mathbf{B_2}, \mathbf{A_2}]$:

$$\mathscr{J}_1 = [\mathbf{A_0} \vee \mathbf{B_2} \vee \mathbf{L_2} \vee \mathbf{R_2}, \mathbf{B_2} \vee \mathbf{Q} \vee \mathbf{Q}_\delta],$$

$$\mathscr{J}_2 = [\mathbf{A_0} \vee \mathbf{B_2} \vee \mathbf{L_2}, \mathbf{B_2} \vee \mathbf{Q}],$$

$$\mathscr{J}_3 = [\mathbf{A_0} \vee \mathbf{B_2} \vee \mathbf{R_2}, \mathbf{B_2} \vee \mathbf{Q}_\delta],$$

$$\mathscr{J}_4 = [\mathbf{B_2} \vee \mathbf{L_2} \vee \mathbf{R_2}, \mathbf{B_2} \vee \mathbf{P} \vee \mathbf{P}_\delta],$$

$$\mathscr{J}_5 = [\mathbf{B_2} \vee \mathbf{L_2}, \mathbf{B_2} \vee \mathbf{P}],$$

$$\mathscr{J}_6 = [\mathbf{B_2} \vee \mathbf{R_2}, \mathbf{B_2} \vee \mathbf{P}_\delta].$$

The following result then verifies that the variety $\mathbb{S}^1$ coincides with one of the varieties in Figure 3.

Lemma 5.7.

(i) *If* $\mathbf{S} \in \mathscr{J}_1$, *then* $\mathbb{S}^1 = \mathbb{A}_0^1 \vee \mathbb{B}_2^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$.

(ii) *If* $\mathbf{S} \in \mathscr{J}_2$, *then* $\mathbb{S}^1 = \mathbb{A}_0^1 \vee \mathbb{B}_2^1 \vee \mathbb{L}_2^1$.

(iii) *If* $\mathbf{S} \in \mathscr{J}_3$, *then* $\mathbb{S}^1 = \mathbb{A}_0^1 \vee \mathbb{B}_2^1 \vee \mathbb{R}_2^1$.

(iv) *If* $\mathbf{S} \in \mathscr{J}_4$, *then* $\mathbb{S}^1 = \mathbb{B}_2^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$.

(v) *If* $\mathbf{S} \in \mathscr{J}_5$, *then* $\mathbb{S}^1 = \mathbb{B}_2^1 \vee \mathbb{L}_2^1$.

(vi) *If* $\mathbf{S} \in \mathscr{J}_6$, *then* $\mathbb{S}^1 = \mathbb{B}_2^1 \vee \mathbb{R}_2^1$.

Proof. Part (i) holds because if $\mathbf{S} \in \mathscr{J}_1$, then

$$\mathbb{A}_0^1 \vee \mathbb{B}_2^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1 \subseteq \mathbb{S}^1 \subseteq \mathbb{B}_2^1 \vee \mathbb{Q}^1 \vee \mathbb{Q}_\delta^1 = \mathbb{A}_0^1 \vee \mathbb{B}_2^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$$

by Lemma 5.2. Parts (ii) and (iii) hold similarly. Part (iv) holds because if $\mathbf{S} \in \mathscr{J}_4$, then

$$\mathbb{B}_2^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1 \subseteq \mathbb{S}^1 \subseteq \mathbb{B}_2^1 \vee \mathbb{P}^1 \vee \mathbb{P}_\delta^1 = \mathbb{B}_2^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1$$

by Lemma 5.5. Parts (v) and (vi) hold similarly. □

6. The lattice $\mathcal{L}(\mathbb{B}_2^1)$

Subsection 6.2 presents a chain in the lattice $\mathcal{L}(\mathbb{B}_2^1)$ that is isomorphic to the integers; this lattice thus violates both the ascending chain and descending chain conditions. Subsection 6.3 demonstrates that the lattice $\mathcal{L}(\mathbb{B}_2^1)$ contains finite anti-chains of arbitrary order and so has infinite width. The lattice $\mathcal{L}(\mathbb{B}_2^1)$ is also shown in Subsection 6.4 to contain non-finitely generated varieties.

For any word $\mathbf{w}$, let $\mathrm{S}(\mathbf{w})$ denote the Rees quotient monoid of $\mathcal{X}^*$ over the ideal of all words that are not factors of $\mathbf{w}$. Equivalently, $\mathrm{S}(\mathbf{w})$ can be treated as the monoid that consists of every factor of the word $\mathbf{w}$, together with a zero element 0, with binary operation $\cdot$ given by

$$\mathbf{a} \cdot \mathbf{b} = \begin{cases} \mathbf{ab}, & \text{if } \mathbf{ab} \text{ is a factor of } \mathbf{w}; \\ 0, & \text{otherwise.} \end{cases}$$

The empty factor, more conveniently written as 1, is the identity element of the monoid $\mathrm{S}(\mathbf{w})$. Note that 0 and 1 are the only idempotents of the monoid $\mathrm{S}(\mathbf{w})$.

6.1. Isoterms and Zimin words

A word $\mathbf{w}$ is an *isoterm* for a semigroup S if S does not satisfy any nontrivial identity of the form $\mathbf{w} \approx \mathbf{w}'$.

LEMMA 6.1 (Jackson [5, Lemma 3.3]). *Let $\mathbf{w}$ be any word and let M be any monoid. Then $\mathbf{w}$ is an isoterm for M if and only if $\mathrm{S}(\mathbf{w}) \in \mathbb{V}_{\mathbb{M}}\{M\}$.*

The *Zimin words* [29] are defined by $\mathbf{z}_1 = x_1$ and $\mathbf{z}_{n+1} = \mathbf{z}_n x_{n+1} \mathbf{z}_n$ for all $n \geqq 1$. For each $n \geqq 1$, define the monoid variety

$$\mathbb{Z}_n = \mathbb{V}_{\mathbb{M}} \left\{ \mathrm{S}(\mathbf{z}_n) \right\}.$$

LEMMA 6.2 (M. V. Sapir [19, Proposition 7 and Lemma 3.7]).

(i) *A finite semigroup S is inherently non-finitely based if and only if every Zimin word is an isoterm for S.*

(ii) *Every Zimin word is an isoterm for the monoid B_2^1. Consequently, the monoid B_2^1 is inherently non-finitely based and $\mathbb{Z}_n \subseteqq \mathbb{B}_2^1$ for all $n \geqq 1$.*

6.2. An infinite chain in $\mathcal{L}\big(\mathbb{B}_2^1\big)$

LEMMA 6.3. *For any $n \geqq 1$, the inclusions $\mathbb{Z}_n \subset \mathbb{Z}_{n+1} \subset \mathbb{B}_2^1$ are proper.*

PROOF. The word $\mathbf{z}_n$ is clearly an isoterm for the monoid $\mathrm{S}(\mathbf{z}_{n+1})$. Hence the inclusions $\mathbb{Z}_n \subseteqq \mathbb{Z}_{n+1} \subseteqq \mathbb{B}_2^1$ hold by Lemmas 6.1 and 6.2(ii). It is routinely shown that the identity $\mathbf{z}_{n+1} \approx x_1\mathbf{z}_{n+1}$ is satisfied by the monoid $\mathrm{S}(\mathbf{z}_n)$ but not by the monoid $\mathrm{S}(\mathbf{z}_{n+1})$. Therefore $\mathbb{Z}_n \neq \mathbb{Z}_{n+1}$. Since all Zimin words are square-free, the monoid $\mathrm{S}(\mathbf{z}_{n+1})$ satisfies the identity $x^2y \approx yx^2$. But the substitution $(x, y) \mapsto (ab, a)$ shows that the monoid B_2^1 does not satisfy the identity $x^2y \approx yx^2$. Hence $\mathbb{Z}_{n+1} \neq \mathbb{B}_2^1$. □

LEMMA 6.4. *The words $xyzxy$ and $xyzyx$ are isoterms for the monoid $\mathrm{S}(\mathbf{z}_3)$.*

PROOF. Suppose that some word from $\{xyzxy, xyzyx\}$ is not an isoterm for the monoid $\mathrm{S}(\mathbf{z}_3)$ so that this monoid satisfies some nontrivial identity $xyzxy \approx \mathbf{w}$ or some nontrivial identity $xyzyx \approx \mathbf{w}$. Performing the substitution $y \mapsto yx$ on the former identity and the substitution $z \mapsto xzx$ on the latter identity, a nontrivial identity of the form $xyxzxyx \approx \mathbf{w}'$ is obtained in either case. Hence the word $xyxzxyx$ is not an isoterm for the monoid $\mathrm{S}(\mathbf{z}_3)$. But this is impossible since the words $xyxzxyx$ and $\mathbf{z}_3$ are the same up to a permutation on the letters in $\mathcal{X}$. □

PROPOSITION 6.5. *The lattice $\mathcal{L}\big(\mathbb{B}_2^1\big)$ contains a chain that is isomorphic to the integers.*

PROOF. In the presence of Lemma 6.3, it suffices to show that the lattice $\mathcal{L}(\mathbb{Z}_3)$ contains an infinite decreasing chain. It follows from Lemmas 6.1 and 6.4 that the varieties $\mathbb{W} = \mathbb{V}_{\mathbb{M}}\big\{\mathrm{S}(xyzxy)\big\}$ and $\mathbb{W}' = \mathbb{V}_{\mathbb{M}}\big\{\mathrm{S}(xyzyx)\big\}$ are contained in $\mathbb{Z}_3$. Jackson and O. Sapir [7, Section 5] proved that the varieties $\mathbb{W}$ and $\mathbb{W} \vee \mathbb{W}'$ are non-finitely based and finitely based respectively. Consequently, the subinterval $[\mathbb{W}, \mathbb{W} \vee \mathbb{W}']$ of $\mathcal{L}(\mathbb{Z}_3)$ contains the required infinite decreasing chain. □

6.3. Finite anti-chains in $\mathcal{L}\big(\mathbb{B}_2^1\big)$ of arbitrary order

LEMMA 6.6. *Let $\mathbf{w}_0, \ldots, \mathbf{w}_n \in \{y_1y_2, y_2y_1\}$ with $n \geqq 1$. Then the word*

$$\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^{n} (h_i \mathbf{w}_i)$$

is an isoterm for the monoid $\mathrm{S}(\mathbf{z}_{m+2})$ for any m such that $2^m > n$.

PROOF. Without loss of generality, assume that $\mathbf{w}_0 = y_1y_2$. By definition, the word $\mathbf{z}_2 = x_1x_2x_1$ is a factor of the word $\mathbf{z}_{m+2}$. More specifically, it is routinely shown by induction on m that

$$\mathbf{z}_{m+2} = \mathbf{z}_2 \prod_{i=1}^{2^m-1} (t_i\mathbf{z}_2)$$

for some $t_1, \ldots, t_{2^m-1} \in \{x_3, \ldots, x_{m+2}\}$. Let φ denote the substitution given by $y_1\varphi = x_1$, $y_2\varphi = x_2$, and for each $i \in \{1, \ldots, n\}$,

$$h_i\varphi = \begin{cases} t_i, & \text{if } \mathbf{w}_{i-1} = y_2y_1 \text{ and } \mathbf{w}_i = y_1y_2; \\ x_1t_i, & \text{if } \mathbf{w}_{i-1} = y_1y_2 \text{ and } \mathbf{w}_i = y_1y_2; \\ t_ix_1, & \text{if } \mathbf{w}_{i-1} = y_2y_1 \text{ and } \mathbf{w}_i = y_2y_1; \\ x_1t_ix_1, & \text{if } \mathbf{w}_{i-1} = y_1y_2 \text{ and } \mathbf{w}_i = y_2y_1. \end{cases}$$

Then $\mathbf{w}\varphi$ is a prefix of the word $\mathbf{z}_{m+2}$. (For example, consider the word

$$\mathbf{w} = y_1y_2\, h_1\, y_1y_2\, h_2\, y_2y_1\, h_3\, y_2y_1\, h_4\, y_1y_2\, h_5\, y_2y_1$$

with $n = 5$. Since $2^3 > 5$, it suffice to choose $m = 3$. Then

$$\begin{aligned} \mathbf{w}\varphi &= x_1x_2 \cdot h_1\varphi \cdot x_1x_2 \cdot h_2\varphi \cdot x_2x_1 \cdot h_3\varphi \cdot x_2x_1 \cdot h_4\varphi \cdot x_1x_2 \cdot h_5\varphi \cdot x_2x_1 \\ &= x_1x_2 \cdot x_1t_1 \cdot x_1x_2 \cdot x_1t_2x_1 \cdot x_2x_1 \cdot t_3x_1 \cdot x_2x_1 \cdot t_4 \cdot x_1x_2 \cdot x_1t_5x_1 \cdot x_2x_1 \end{aligned}$$

is a prefix of the word $\mathbf{z}_5$.) Let $\mathbf{s} \in \mathcal{X}^*$ be such that $\mathbf{z}_{m+2} = (\mathbf{w}\varphi)\mathbf{s}$.

Working toward a contradiction, suppose that the word $\mathbf{w}$ is not an isoterm for the monoid $\mathrm{S}(\mathbf{z}_{m+2})$ so that this monoid satisfies a nontrivial identity of the form $\mathbf{w} \approx \mathbf{w}'$. Since every simple word is an isoterm for the monoid $\mathrm{S}(\mathbf{z}_{m+2})$, it follows that $\mathbf{w}' = \mathbf{w}_0' \prod_{i=1}^n (h_i\mathbf{w}_i')$ for some $\mathbf{w}_0', \ldots, \mathbf{w}_n' \in \{y_1, y_2\}^*$. It is then easily shown that $\mathbf{w}\varphi \neq \mathbf{w}'\varphi$. Now the monoid $\mathrm{S}(\mathbf{z}_{m+2})$ satisfies the nontrivial identity $(\mathbf{w}\varphi)\mathbf{s} \approx (\mathbf{w}'\varphi)\mathbf{s}$ where $(\mathbf{w}\varphi)\mathbf{s} = \mathbf{z}_{m+2}$, and this is impossible. □

PROPOSITION 6.7. *For each $m \geqq 0$, the lattice $\mathcal{L}(\mathbb{Z}_{m+2})$ has width at least 2^m. Consequently, the lattice $\mathcal{L}(\mathbb{B}_2^1)$ contains an anti-chain of each finite order.*

PROOF. The result clearly holds if $m = 0$. Since the subvarieties $\mathbb{W}$ and $\mathbb{W}'$ of $\mathbb{Z}_3$ in the proof of Proposition 6.5 are incomparable, the lattice $\mathcal{L}(\mathbb{Z}_3)$ has width at least two. Therefore it suffices to assume that $m \geqq 2$. Let $2^m = n + 1$ (so that $n \geqq 3$) and let $\mathbf{y} = y_1y_2 \prod_{i=1}^n (h_iy_1y_2)$. For each

$j \in \{1, \ldots, n+1\}$, replace the jth factor y_1y_2 in the word $\mathbf{y}$ by y_2y_1, and denote the resulting word by $\mathbf{y}_j$, that is,

$$\begin{aligned} \mathbf{y}_1 &= y_2y_1 \cdot h_1y_1y_2 \cdot h_2y_1y_2 \cdots h_{n-1}y_1y_2 \cdot h_ny_1y_2, \\ \mathbf{y}_2 &= y_1y_2 \cdot h_1y_2y_1 \cdot h_2y_1y_2 \cdots h_{n-1}y_1y_2 \cdot h_ny_1y_2, \\ &\vdots \\ \mathbf{y}_{n+1} &= y_1y_2 \cdot h_1y_1y_2 \cdot h_2y_1y_2 \cdots h_{n-1}y_1y_2 \cdot h_ny_2y_1. \end{aligned}$$

By Lemmas 6.1 and 6.6, the varieties $\mathbb{V}_{\mathbb{M}}\{\mathrm{S}(\mathbf{y}_1)\}, \ldots, \mathbb{V}_{\mathbb{M}}\{\mathrm{S}(\mathbf{y}_{n+1})\}$ are contained in the variety $\mathbb{Z}_{m+2}$; these $n+1$ subvarieties of $\mathbb{Z}_{m+2}$ are pairwise incomparable since it is routinely checked that $\mathrm{S}(\mathbf{y}_j) \models \mathbf{y}_k \approx \mathbf{y}$ if and only if $j \neq k$. □

6.4. Non-finitely generated varieties in $\mathcal{L}(\mathbb{B}_2^1)$

Lemma 6.8 (M. V. Sapir [18, Theorem 2]). *A finite aperiodic monoid M is inherently non-finitely based if and only if $B_2^1 \in \mathbb{V}_{\mathbb{M}}\{M\}$.*

Recall from Lemma 6.3 that $\mathbb{Z}_1 \subset \mathbb{Z}_2 \subset \cdots \subset \mathbb{B}_2^1$. Since the identity $x^2y \approx yx^2$ is satisfied by any monoid $\mathrm{S}(\mathbf{z}_n)$ but not by the monoid B_2^1, the complete join

$$\mathbb{Z}_\infty = \mathbb{Z}_1 \vee \mathbb{Z}_2 \vee \cdots$$

is a proper subvariety of the variety $\mathbb{B}_2^1$.

Proposition 6.9. *Let $\mathbb{V}$ be any aperiodic monoid variety such that $\mathbb{Z}_\infty \subseteqq \mathbb{V}$ and $B_2^1 \notin \mathbb{V}$. Then the variety $\mathbb{V}$ is non-finitely generated. Consequently, every proper subvariety of $\mathbb{B}_2^1$ that contains $\mathbb{Z}_\infty$ is non-finitely generated.*

Proof. Suppose that $\mathbb{V} = \mathbb{V}_{\mathbb{M}}\{M\}$ for some finite monoid M. Since $\mathbb{Z}_\infty \subseteqq \mathbb{V}$, it follows from Lemma 6.1 that every Zimin word is an isoterm for the monoid M. By Lemma 6.2(i), the monoid M is inherently non-finitely based. The contradiction $B_2^1 \in \mathbb{V}$ then follows from Lemma 6.8. □

7. Open questions

7.1. Varieties in Figure 1

Let $\mathfrak{X}$ denote the join-semilattice in Figure 1. Let $\mathfrak{X} = \mathfrak{F} \cup \mathfrak{N}$ where $\mathfrak{F}$ consists of finitely based varieties in $\mathfrak{X}$, and $\mathfrak{N}$ consists of non-finitely based varieties in $\mathfrak{X}$. Recall from Propositions 4.1 and 5.1 that $\mathfrak{F}$ coincides with the lattice $\mathcal{L}\big(\mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1\big)$ and that $\mathfrak{N}$ is a join-semilattice.

Question 7.1. Is the join-semilattice $\mathfrak{N}$ a lattice?

Let $\mathbb{V}, \mathbb{V}' \in \mathfrak{X}$. By Proposition 4.1 and the 2-testable monoids given in Subsection 3.1 that generate the varieties in $\mathfrak{X}$, it is easily verified that $\mathbb{V} \vee \mathbb{V}' \in \mathfrak{X}$. Since $\mathfrak{F} = \mathcal{L}\big(\mathbb{A}_0^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1\big)$, either $\mathbb{V} \in \mathfrak{F}$ or $\mathbb{V}' \in \mathfrak{F}$ implies that $\mathbb{V} \cap \mathbb{V}' \in \mathfrak{F} \subset \mathfrak{X}$. It follows that an affirmative answer to Question 7.1 implies that the join-semilattice $\mathfrak{X}$ is a lattice.

7.2. The interval $\big[\mathbb{B}_2^1, \mathbb{A}_2^1\big]$

As commented in Section 1, the task of identifying all varieties in the interval $\big[\mathbb{B}_2^1, \mathbb{A}_2^1\big]$ is hindered by the presence of non-finitely based varieties within it. But the complete description of the interval $[\mathbf{B_2}, \mathbf{A_2}]$ (see Figure 4) inspires the conjecture of bases of some varieties within the variety $\mathbb{A}_2^1$.

Question 7.2. Which of the following equations hold?

$$\mathbb{B}_2^1 = \mathbb{A}_2^1 \cap \big[x^2y^2 \approx y^2x^2\big], \tag{7.1}$$

$$\mathbb{B}_2^1 \vee \mathbb{L}_2^1 = \mathbb{A}_2^1 \cap \big[x^2y^2x^2 \approx x^2y^2\big], \tag{7.2}$$

$$\mathbb{B}_2^1 \vee \mathbb{R}_2^1 = \mathbb{A}_2^1 \cap \big[x^2y^2x^2 \approx y^2x^2\big], \tag{7.3}$$

$$\mathbb{B}_2^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1 = \mathbb{A}_2^1 \cap \big[x^2y^2x^2y^2 \approx x^2y^2\big], \tag{7.4}$$

$$\mathbb{A}_0^1 \vee \mathbb{B}_2^1 = \mathbb{A}_2^1 \cap \big[x^2y^2x^2 \approx y^2x^2y^2\big], \tag{7.5}$$

$$\mathbb{A}_0^1 \vee \mathbb{B}_2^1 \vee \mathbb{L}_2^1 = \mathbb{A}_2^1 \cap \big[x^2y^2x^2y^2 \approx x^2y^2x^2\big], \tag{7.6}$$

$$\mathbb{A}_0^1 \vee \mathbb{B}_2^1 \vee \mathbb{R}_2^1 = \mathbb{A}_2^1 \cap \big[x^2y^2x^2y^2 \approx y^2x^2y^2\big], \tag{7.7}$$

$$\mathbb{A}_0^1 \vee \mathbb{B}_2^1 \vee \mathbb{L}_2^1 \vee \mathbb{R}_2^1 = \mathbb{A}_2^1 \cap \big[x^2y^2x^2z^2x^2 \approx x^2y^2z^2x^2\big]. \tag{7.8}$$

It is routinely shown that if (7.1)–(7.7) hold, then the answer to Question 7.1 is affirmative.

QUESTION 7.3. Is every variety in the interval $\left[\mathbb{B}_2^1, \mathbb{A}_2^1\right]$ of the form $\mathbb{A}_2^1 \cap [\Sigma]$ for some finite set Σ of identities?

7.3. Number of subvarieties

Trahtman [24] proved that the semigroup variety $\mathbf{A_2^1} = \mathbf{V_S}\left\{A_2^1\right\}$ contains continuum many subvarieties while Jackson [4] later proved that the smaller variety $\mathbf{B_2^1} = \mathbf{V_S}\left\{B_2^1\right\}$ also has the same property. However, the only monoids that belong to these subvarieties are semilattices. Therefore no conclusion on the number of subvarieties of the monoid varieties $\mathbb{A}_2^1$ and $\mathbb{B}_2^1$ can be drawn from the aforementioned results of Jackson and Trahtman.

QUESTION 7.4. Does any of the monoid varieties $\mathbb{A}_2^1$ and $\mathbb{B}_2^1$ contain continuum many subvarieties?

Jackson and McKenzie [6] presented a monoid of order 56 that generates a monoid variety with continuum many subvarieties. An affirmative answer to Question 7.4 thus provides a significantly smaller example.

7.4. Non-finitely generated varieties

It follows from Proposition 6.9 that every variety in the interval $\left[\mathbb{Z}_\infty, \mathbb{A}_2^1 \cap \left[x^2y \approx yx^2\right]\right]$ is non-finitely generated.

QUESTION 7.5. Which of the inclusions $\mathbb{Z}_\infty \subseteqq \mathbb{B}_2^1 \cap \left[x^2y \approx yx^2\right] \subseteqq \mathbb{A}_2^1 \cap \left[x^2y \approx yx^2\right]$ is proper?

Note that if (7.1) holds, then $\mathbb{B}_2^1 \cap \left[x^2y \approx yx^2\right] = \mathbb{A}_2^1 \cap \left[x^2y \approx yx^2\right]$.

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