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NEW CHARACTERIZATIONS OF p-SOLUBLE AND p-SUPERSOLUBLE FINITE GROUPS

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Abstract

Let G be a finite group and H a subgroup of G. H is said to be S-quasinormal in G if $HP = PH$ for all Sylow subgroups P of G. Let H_{sG} be the subgroup of H generated by all those subgroups of H which are S-quasinormal in G and H^{sG} the intersection of all S-quasinormal subgroups of G containing H. The symbol $|G|_p$ denotes the order of a Sylow p-subgroup of G. We prove the following

Theorem A. *Let G be a finite group and p a prime dividing $|G|$. Then G is p-supersoluble if and only if for every cyclic subgroup H of $\overline{G} = G/O_{p'}(G)$ of prime order or order 4 (if $p = 2$), $\overline{G}$ has a normal subgroup T such that $HT = H^{s\overline{G}}$ and $H \cap T = H_{s\overline{G}} \cap T$.*

Theorem B. *A soluble finite group G is p-supersoluble if and only if for every 2-maximal subgroup E of G such that $O_{p'}(G) \leqq E$ and $|G : E|$ is not a power of p, G has an S-quasinormal subgroup T with cyclic Sylow p-subgroups such that $E^{sG} = ET$ and $|E \cap T|_p = |E_{sG} \cap T|_p$.*

Theorem C. *A finite group G is p-soluble if for every 2-maximal subgroup E of G such that $O_{p'}(G) \leqq E$ and $|G : E|$ is not a power of p, G has an S-quasinormal subgroup T such that $E^{sG} = ET$ and $|E \cap T|_p = |E_{sG} \cap T|_p$.*

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1. Introduction

There are a large number of criteria for solubility, p-solubility, nilpotency, p-nilpotency and supersolubility of finite groups. Moreover people found a large number of various characterizations of such classes of groups. Nevertheless, the p-supersoluble groups remain little-studied subject of the group theory. The present paper adds some result to this line of research.

Throughout this paper, all groups are finite, G is a finite group, p denotes a prime divisor of $|G|$ and $|H|_p$ denotes the order of a Sylow p-subgroup of a group H.

Recall that a subgroup A of G is said to permute with a subgroup B if $AB = BA$. If A permutes with all Sylow subgroups of G, then A is called S-permutable, S-quasinormal or $\pi(G)$-permutable [12] in G. Let H be a subgroup of G, H^{sG} the intersection of all S-permutable subgroups of G containing H [9] and let H_{sG} be the subgroup of H generated by all those subgroups of H which are S-permutable in G [17]. Since S-quasinormal subgroups of G form a sublattice of the lattice of all subgroups of G (O. Kegel [12]), both subgroups H^{sG} and H_{sG} are S-quasinormal in G. We call H_{sG} the *S-quasinormal core* of H in G and call H^{sG} the *S-quasinormal closure* of H in G.

Our main goal here is to prove the following theorems.

Theorem A. *G is p-supersoluble if and only if for every cyclic subgroup H of $\overline{G} = G/O_{p'}(G)$ of prime order or order 4 (if $p = 2$), $\overline{G}$ has a normal subgroup T such that $HT = H^{s\overline{G}}$ and $H \cap T = H_{s\overline{G}} \cap T$.*

Corollary 1.1. *Suppose that for every cyclic subgroup H of G of prime order or order 4 (if $p = 2$), G has a normal subgroup T such that $HT = H^{sG}$ and $H \cap T = H_{sG} \cap T$. Then G is p-supersoluble.*

Corollary 1.2 (Buckley [4]). *Let G be a group of odd order. If every minimal subgroup of G is normal in G, then G is supersoluble.*

Corollary 1.2 (Gaschütz [11, IV, Theorem 5.7]). *If every minimal subgroup of a group G is normal in G, then the commutator subgroup G' of G is 2-closed.*

Proof. By Theorem A, G is p-supersoluble for all odd primes p. Hence $G/O_{p'}(G)$ is supersoluble (see below Lemma 2.5). Then, since $O_2(G)$ is the intersection of all such subgroups $O_{p'}(G)$, we see that G' is 2-closed.

Note that if a subgroup H is S-quasinormal in G, then $H^{sG} = H = H_{sG}$. Hence, by Theorem A, we obtain

Corollary 1.3 (Shaalan [15]). *If every cyclic subgroup of G of prime order or order 4 is S-quasinormal in G, then G is supersoluble.*

A subgroup H of a group G is said to be c-normal in G [19] if G has a normal subgroup T such that $HT = G$ (which implies $H(T \cap H^G) = H^G$) and $H \cap T = H_G \cap T$ (which implies $H \cap T = H_{sG} \cap T$). Hence, by Theorem A, we also have the following

Corollary 1.4 (Wang [19]). *If every cyclic subgroup of G of prime order or order 4 is c-normal in G, then G is supersoluble.*

Theorem B. *Suppose that G is soluble. Then G is p-supersoluble if and only if for every 2-maximal subgroup E of G such that $O_{p'}(G) \leqq E$ and $|G : E|$ is not a power of p, G has an S-quasinormal subgroup T with cyclic Sylow p-subgroups such that $E^{sG} = ET$ and $|E \cap T|_p = |E_{sG} \cap T|_p$.*

Theorem C. *G is p-soluble if for every 2-maximal subgroup E of G such that $O_{p'}(G) \leqq E$ and $|G : E|$ is not a power of p, G has an S-quasinormal subgroup T such that $E^{sG} = ET$ and $|E \cap T|_p = |E_{sG} \cap T|_p$.*

From Theorems B and C, we directly get

Corollary 1.5 (Guo, Skiba [9]). *Suppose that for every 2-maximal subgroup E of G such that $|G : E|$ is not a power prime, G has an S-quasinormal cyclic subgroup T satisfying $E^{sG} = ET$ and $E \cap T = E_{sG} \cap T$. Then G is supersoluble.*

Corollary 1.6 (Agrawal [1]). *If every 2-maximal subgroup of G is S-quasinormal in G, then G is supersoluble.*

Corollary 1.7 (Huppert [10]). *If every 2-maximal subgroup of G is normal in G, then G is supersoluble.*

All unexplained notations and terminologies in this paper are standard. The reader is refereed to [2], [8], [6] if necessary.

2. Preliminaries

The following known results about subnormal and S-quasinormal subgroups will be used in many places of our proofs.

Lemma 2.1. *Let $A \leqq K \leqq G$ and $B \leqq G$. Then*

(1) *If A is subnormal in G and A is a π-subgroup of G, then $A \leqq O_\pi(G)$* [21].

(2) *If A is subnormal in G, then $A \cap B$ is subnormal in B* [6, A, (14.1)].

(3) *If A is subnormal in G and B is a Hall π-subgroup of G, then $A \cap B$ is a Hall π-subgroup of A* [21].

(4) *If A is subnormal in G and A is soluble (nilpotent), then A is contained in some soluble normal (nilpotent) subgroup of G* [21].

(5) *If A is subnormal in G and B is a minimal normal subgroup of G, then $B \leqq N_G(A)$* [6, A, (14.3)].

Lemma 2.2. *Let $H \leqq K \leqq G$.*

(1) *If H is S-quasinormal in G, then H is S-quasinormal in K* [12].

(2) *Suppose that H is normal in G. Then K/H is S-quasinormal in G if and only if K is S-quasinormal in G* [12].

(3) *If H is S-quasinormal in G, then H is subnormal in G* [12].

(4) *If H and F are S-quasinormal subgroups of G, then $H \cap F$ and $\langle H, F\rangle$ are S-quasinormal in G* [12].

(5) *If H is S-quasinormal in G, then H/H_G is nilpotent* [5].

(6) *If H is S-quasinormal in G and $M \leqq G$, then $H \cap M$ is S-quasinormal in M* [5].

(7) *If H is S-quasinormal in G and H is a q-group for some prime q, then $O^q(G) \leqq N_G(H)$* [14, Lemma A].

Lemma 2.3 [17, Lemma 2.8]. *Let $H \leqq K \leqq G$. Then:*

(1) *H_{sG} is an S-quasinormal subgroup of G and $H_G \leqq H_{sG}$.*

(2) *$H_{sG} \leqq H_{sK}$.*

(3) *If H is normal in G, then $(K/H)_{s(G/H)} = K_{sG}/H$.*

(4) *If H is either a Hall subgroup of G or a maximal subgroup of G, then $H_{sG} = H_G$.*

Lemma 2.4 [9, Lemma 2.5]. *Let G be a group and $H \leqq K \leqq G$. Then:*

(1) *H^{sG} is an S-quasinormal subgroup of G and $H^{sG} \leqq H^G$.*

(2) *$H^{sK} \leqq H^{sG}$.*

(3) *If H is normal in G, then $(K/H)^{s(G/H)} = K^{sG}/H$.*

(4) *If H is either a Hall subgroup of G or a maximal subgroup of G, then $H^{sG} = H^G$.*

Lemma 2.5. *Let p be a prime and G a p-soluble group. Assume that $O_{p'}(G) = 1$. Then the following statements are equivalent.*

(i) *G is p-supersoluble;*

(ii) *G is supersoluble;*

(iii) *$G/O_p(G)$ is an abelian group of exponent dividing $p-1$.*

Proof. (i) $\Longrightarrow$ (ii). Since G is p-supersoluble, for every chief p-factor H/K of G, we have $|H/K| = p$ and so $G/C_G(H/K)$ is an abelian group of exponent dividing $p-1$ (see [20, Chapter 1, Theorem 1.4]. Since $O_{p'}(G) = 1$, the intersection of the centralizers of all chief factors H/K of $|H/K| = p$ is $O_{p',p}(G) = O_p(G)$. Hence G is supersoluble by [20, Chapter 1, Theorem 1.9]. By using the same arguments, we also see that (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i). $\square$

Let $\mathcal{F}$ be any non-empty class of groups. We use $G^{\mathcal{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathcal{F}$. $\mathcal{A}(p-1)$ denotes the formation of all abelian groups of exponent dividing $p-1$. The symbol $Z_{\mathcal{U}}(G)$ denotes the largest normal subgroup of a group G such that every chief factor of G below $Z_{\mathcal{U}}(G)$ is cyclic.

LEMMA 2.6 [18, Lemma 2.2]. *Let E be a normal p-subgroup of G. If $E \leqq Z_{\mathcal{U}}(G)$, then* $\big(G/C_G(E)\big)^{\mathcal{A}(p-1)} \leqq O_p\big(G/C_G(E)\big)$.

The following lemma may be proved based on some results in [13] on f-hypercentral action (see [16, Chapter II] or [6, Chapter IV, Section 6]). For reader's convenience, we give a direct proof.

LEMMA 2.7. *Let P be a normal p-subgroup of G and D a characteristic subgroup of P such that every non-trivial p'-automorphism of P induces a non-trivial automorphism of D. Suppose that $D \leqq Z_{\mathcal{U}}(G)$. Then $P \leqq Z_{\mathcal{U}}(G)$.*

PROOF. Let $C = C_G(P)$ and H/K be an arbitrary chief factor of G below P. Then $O_p\big(G/C_G(H/K)\big) = 1$ by [20, Appendix C, Corollary 6.4]. Since $D \leqq Z_{\mathcal{U}}(G)$, we have $\big(G/C_G(D)\big)^{\mathcal{A}(p-1)}$ is a p-group by Lemma 2.5. Hence $(G/C)^{\mathcal{A}(p-1)}$ is a p-group. It follows that $G/C_G(H/K) \in \mathcal{A}(p-1)$ and so $|H/K| = p$ by [20, Chapter 1, Theorem 1.4]. Therefore $P \leqq Z_{\mathcal{U}}(G)$. □

Let P be a p-group. If P is not a non-abelian 2-group we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

LEMMA 2.8 [3]. *Let P be a p-group of class at most 2. Suppose that* $\exp\big(P/Z(P)\big)$ *divides p.*

(1) *If $p > 2$, then* $\exp\big(\Omega(P)\big) = p$.

(2) *If P is a non-abelian 2-group, then* $\exp\big(\Omega(P)\big) = 4$.

PROOF. See page 3 in [3].

Let H be a subgroup of G and p a prime. Then we say that H is *S_p-embedded in G* if G has a subgroup T such that $HT = H^{sG}$ and $|H \cap T|_p = |H_{sG} \cap T|_p$. □

LEMMA 2.9. *Let H be a normal subgroup of G and $H \leqq K \leqq G$.*

(1) *If H is p-soluble and K/H is S_p-embedded in G/H, then K is S_p-embedded in G.*

(2) *If K is S_p-embedded in G, then K/H is S_p-embedded in G/H.*

(3) *If L is an S_p-embedded subgroup of G and $L \leqq K$, then L is S_p-embedded in K.*

(4) *The subgroup HE/H is S_p-embedded in G/H, for every S_p-embedded in G subgroup E satisfying* $\big(|H|, |E|\big) = 1$.

PROOF. (1) We prove that K is S_p-embedded in G by induction on $|G|$. Let L be a minimal normal subgroup of G such that $L \leqq H$. Then, obviously, $(K/L)/(H/L)$ is S_p-embedded in $(G/L)/(H/L)$. If $L \neq H$, then by induction, K/L is S_p-embedded in G/L. We may, therefore, assume that H is a minimal normal subgroup of G. Let T/H be an S-quasinormal subgroup of G/H such that $KT/H = (K/H)(T/H) = (K/H)^{s(G/H)}$ and $\big|(T/H) \cap (K/H)\big|_p = \big|(T/H) \cap (K/H)_{s(G/H)}\big|_p$. By Lemma 2.2(2), T is S-quasinormal in G. By Lemma 2.4(3), $(K/H)^{s(G/H)} = K^{sG}/H$. Hence $K^{sG} = KT$. Since H is p-soluble, H is either a p-group or a p'-group. Then, since $\big|(T/H) \cap (K/H)\big|_p = \big|(T/H) \cap (K/H)_{s(G/H)}\big|_p$, we obtain $|T \cap K|_p = |T \cap K_{sG}|_p$. Hence K is S_p-embedded in G.

(2) Assume that $KT = K^{sG}$ and $|T \cap K|_p = |T \cap K_{sG}|_p$, for some S-quasinormal subgroup T of G. Then HT/H is an S-quasinormal in G/H and $(HT/H)(K/H) = KT/H = K^{sG}/H = (K/H)^{sG}$ by Lemma 2.4(3). Besides, clearly, $H \leqq K_{sG}$. Hence $H \cap T \cap K = H \cap T = H \cap T \cap K_{sG}$. This implies that $\big|(TH/H) \cap (K/H)\big|_p = \big|H(T \cap K)/H\big|_p = \big|H(T \cap K_{sG})/H\big|_p = \big|(TH/H) \cap (K/H)_{s(G/H)}\big|_p$. Thus K/H is S_p-embedded in G/H.

(3) Let T be an S-quasinormal subgroup of G such that $LT = L^{sG}$ and $|T \cap L|_p = |T \cap L_{sG}|_p$. Let $T_0 = T \cap L^{sK}$. Then $T_0 = K \cap T \cap L^{sK}$ and $T_0 \cap L_{sK} = T \cap L_{sK}$. By Lemma 2.2(6), $K \cap T$ is S-quasinormal in K and so by Lemma 2.2(4), T_0 is S-quasinormal in K. Besides, by Lemma 2.4(2), $L^{sK} \leqq L^{sG}$ and so $L^{sK} = L^{sK} \cap L^{sG} = L^{sK} \cap LT = L\big(L^{sK} \cap T\big) = LT_0$. Finally, we show that $|T_0 \cap L|_p = |T_0 \cap L_{sK}|_p$. In fact, we only need to prove that $|P_1| \leqq |P_2|$, for some Sylow p-subgroups P_1 of $T_0 \cap L$ and some Sylow p-subgroup P_2 of $T_0 \cap L_{sK}$. Since $H \cap T_0 \leqq L \cap T$, we have $P_1 \leqq P_3$, for some Sylow p-subgroup P_3 of $L \cap T$. On the other hand, by Lemma 2.3(2), $L_{sG} \leqq L_{sK}$. Hence $T \cap L_{sG} \leqq T \cap L_{sK} = T_0 \cap L_{sK}$. It follows that $|P_1| \leqq |P_3| = |T \cap L|_p = |T \cap L_{sG}|_p \leqq |P_2|$. Hence L is S_p-embedded in K.

(4) By (2), we only need to prove that HE is S_p-embedded in G. Assume that E is S_p-embedded in G and let T be an S-quasinormal subgroup of G such that $ET = E^{sG}$ and $|T \cap E|_p = |T \cap E_{sG}|_p$. Let $T_0 = HT$. Then, obviously, T_o is an S-quasinormal subgroup of G and $HET_0 = HE^{sG} = (HE)^{sG}$. Next we show that $|T_0 \cap HE|_p = \big|T_0 \cap (HE)_{sG}\big|_p$.

Since $\big(|E|, |H|\big) = 1$, E is a Hall π-subgroup of EH and H is a Hall π'-subgroup of EH, for some set π of primes. If p divides $|H|$, then E is p'-group. Hence $|T_0 \cap HE|_p = |H| = |T_0 \cap HE_{sG}|_p = \big|T_0 \cap (HE)_{sG}\big|_p$. Now we assume that p divide $|E|$. In this case, H is a p'-group. Let $D = T \cap HE$. By Lemmas 2.1(2) and 2.2(3), D is subnormal in HE and so $D = (D \cap H)(D \cap E) \leqq H(T \cap E)$. It follows that $T_0 \cap HE = H(T \cap HE) = HD \leqq H(T \cap E)$ and so

$$|T_0 \cap HE|_p = |T \cap E|_p = |T \cap E_{sG}|_p \leqq \big|HT \cap (HE)_{sG}\big|_p \leqq |T_0 \cap HE|_p.$$

Therefore $|T_0 \cap HE|_p = \left|T_0 \cap (HE)_{sG}\right|_p$. This shows that HE is S_p-embedded in G. □

LEMMA 2.10. *Suppose that every maximal subgroup E of G with $\big(|G:E|, p\big) = 1$ is normal in G. Let P be a Sylow p-subgroup of G. Then G is p-closed and G/P is nilpotent.*

PROOF. Suppose that this lemma is false and let G be a counterexample of minimal order. Obviously, the hypothesis is true for any factor group of G. Hence G has a unique minimal normal subgroup, L say, and G/L is p-closed with nilpotent factor $(G/L)/(PL/L)$. If L is a p-group, then G is p-closed with nilpotent factor G/P, which contradicts the choice of G. Hence L is not a p-group. It is well known that the class of all p-closed groups G with nilpotent G/P is a saturated formation. Hence $L \nleqq \Phi(G)$. Let M be a maximal subgroup of G such that $ML = G$. Suppose that p divides $|G:M|$. Then p divides $|L|$ and by Fratinni argument, for some maximal subgroup E of G we have $EL = G$ and p does not divide $|G:E|$. Hence E is normal in G by hypothesis, which implies $|E| = 1$. Consequently, $G = L$. This contradiction completes the proof. □

LEMMA 2.11 [20, Chapter 4, Theorem 1.6]. *Let p be an odd prime number and $\mathbf{F}$ field of characteristic p. Let G be a completely reducible soluble linear group of degree n over $\mathbf{F}$. Suppose that a Sylow p-subgroup of G has order $p^{\lambda(n)}$. Then $\lambda(n) \leqq n-1$ with equality only if $n=1$ or $n=2$ and $p=3$.*

3. Proof of Theorem A

Let H be a subgroup of G. Then we say, following [9], that H is N-embedded in G if G has a normal subgroup T such that $HT = H^{sG}$ and $H \cap T = H_{sG} \cap T$.

PROOF OF THEOREM A. First suppose that every cyclic subgroup H of $G = G/O_{p'}(G)$ of prime order or order 4 is N-embedded in G. We shall show that G is p-supersoluble. Suppose that this is false and let G be a counterexample of minimal order. Let $Z = Z_{\mathcal{U}}(G)$.

(1) $O_{p'}(G) = 1$.

Since $O_{p'}\big(G/O_{p'}(G)\big) = 1$, the hypothesis is true for $G/O_{p'}(G)$. Hence, if $O_{p'}(G) \neq 1$, then $G/O_{p'}(G)$ is p-supersoluble by the choice of G. It follows that G is p-supersoluble, a contradiction. Hence (1) holds.

(2) $O_{p'}(L) = 1$ *for any subnormal subgroup L of G.*

Indeed, $O_{p'}(L) \leqq O_{p'}(G) = 1$ by Lemma 2.1(1).

(3) *Every proper normal subgroup L of G is supersoluble.*

The hypothesis holds for L by Lemma 2.7(2) in [9]. Hence L is p-supersoluble by the choice of G and therefore L is supersoluble by (2) and Lemma 2.5.

(4) *If N is a normal subgroup of G and $N \leqq O_p(G)$, then $N \leqq Z$.*

We will prove this assertion by induction on $|N|$. Suppose that $N \nleqq Z$. Then

(a) *G has a normal subgroup $R \leqq N$ such that N/R is a non-cyclic chief factor of G, $R \leqq Z$ and $V \leqq R$ for any normal subgroup $V \neq N$ of G contained in N.*

Let N/R be a chief factor of G. Then the hypothesis holds for (G, R). Therefore $R \leqq Z$ by induction and so N/R is not cyclic. Now let $V \neq N$ be any normal subgroup of G contained in N. Then $V \leqq Z$. If $V \nleqq R$, then from the G-isomorphism $N/R = VR/R \simeq V/V \cap R$, we obtain that $N \leqq Z$, a contradiction. Hence $V \leqq R$.

(b) *Let D be a Thompson critical subgroup of N (see* [7, p. 186]*). Then $\Omega(N) = N = D$.*

Indeed, suppose that $\Omega(N) < N$. Then, in view of (a), $\Omega(N) \leqq Z$. Hence $N \leqq Z$ by Lemmas 2.7 and Theorem 5.12 in [11, Chapter IV], a contradiction. Hence $\Omega(N) = N$. In view of Theorem 3.11 in [7, Chapter 5] we obtain similarly that $N = D$.

The final contradiction for (4).

Let H/R be any minimal subgroup of $N/R \cap Z(G_p/R)$, where G_p is a Sylow p-subgroup of G. Let $x \in H \setminus R$ and $L = \langle x \rangle$. Then $H/R = LR/R$ and $|L|$ is either a prime or 4 by Lemma 2.8. Hence L is N-embedded in G by the hypothesis. Hence G has a normal subgroup T such that $LT = L^{sG}$ and $L \cap T = L_{sG} \cap T$. It is clear that $L_{sG} \leqq N$. Thus $T \leqq N$. Suppose that $T \leqq R$. Then $H/R = LT/R = LR/R = L^{sG}R/R$ is S-quasinormal subgroup of G/R by Lemmas 2.2 and 2.4(1). Therefore H/R is normal in G/R by Lemma 2.2(7) and consequently $H/R = N/R$, which contradicts (a). Thus (4) holds.

(5) *G is p-soluble.*

Suppose that this is false. Then:

(a) *G is non-simple.*

Suppose that G is a simple non-abelian group. Let H be any subgroup of G of order p, T a normal subgroup of G such that $HT = H^{sG}$ and $T \cap H = T \cap H_{sG}$. By Lemmas 2.2(3) and 2.4(1), H^{sG} is subnormal in G. Hence

$H^{sG} = G$ and either $T = 1$ or $T = G$. In the both cases, we have $H = G$ and thereby $G = H = H_{sG}$ is cyclic. This contradiction shows that (a) holds.

(b) *G has a non-identity supersoluble normal subgroup R such that G/R is a simple non-abelian group, p divides $|G/R|$ and every proper normal subgroup of G is contained in R.*

Let R be a normal subgroup of G such that G/R is simple. Then in view of (a), $R \neq 1$. By (3), R is supersoluble. Hence G/R is a simple non-abelian group and p divides $|G/R|$. Now let L be any proper normal subgroup of G. Suppose that $L \nleqq R$. Then $G = RL$ is the product of two supersoluble groups. Consequently, G is soluble, a contradiction.

(c) $R = Z_\infty(G) \leqq O_p(G)$.

In view of (2), p divides $|R|$. Let P be a Sylow p-subgroup of R. By (2), $O_{p'}(R) = 1$. Hence, by (3) and Lemma 2.5, $P = F(R)$. Since $F(R)$ is a characteristic subgroup of R, P is normal in G. Hence $P \leqq Z_{\mathcal{U}}(R)$ by (4). Then, by (b), we see that $C_G(H/K) = G$ for any chief factor H/K of G below P. Hence $P \leqq Z_\infty(G)$ and so R is nilpotent since $R/P = R/F(R)$ is abelian. It follows from (2) that $R = P = Z_\infty(G) \leqq O_p(G)$.

The final contradiction for (5).

Since G/R is not p-nilpotent, it has a p-closed Schmidt subgroup H/R (see [11, Chapter IV, Theorem 5.4]). Since $R \leqq Z_\infty(G)$, we have $R \leqq Z_\infty(H)$. Hence $H = H_p \rtimes H_q$ is a Schmidt subgroup of G. Let $\Phi = \Phi(H_p)$. Then by [16, VI, Theorem 25.4], H_p/Φ is a non-central chief factor of H and H_p is a group of exponent p or exponent 4 (if $p = 2$ and H_p is non-abelian). Moreover, if H_p is abelian, then $\Phi = 1$. Hence $|H_p/\Phi| > p$ (otherwise $|H_p| = p$, which is impossible).

Let X/Φ be a minimal subgroup of H_p/Φ, $x \in X \setminus \Phi$ and $L = \langle x \rangle$. Then $|L| = p$ or $|L| = 4$. Hence L is N-embedded in G and so L is N-embedded in H by Lemma 2.7(2) in [9]. Let T be a normal subgroup of H such that $LT = L^{sG}$ and $L \cap T \leqq L_{sG}$. Note that if V is an S-quasinormal subgroup of H such that $V \leqq H_p$ and $V \nleqq \Phi$, then $V = H_p$ (otherwise, $H_q\Phi < VH_q\Phi < H$, a contradiction). Hence $L \neq L_{sG}$ and so $T \neq H_p$. But then $T \leqq \Phi$, which implies that $H_p = L$. This contradiction completes the proof of (5).

The final contradiction for the sufficiency. Let $P = O_p(G)$. Then $P \leqq Z_{\mathcal{U}}(G)$ by (4). On the other hand, in view of (1) and (4) we have $P = C_G(P)$ by [7, Chapter 6, Theorem 3.2]. But by Lemma 2.6, $G/C_G(P)^{\mathcal{A}(p-1)} \leqq O_p\big(G/C_G(P)\big) = 1$. Hence G is supersoluble by [20, Chapter 1, Theorem 1.9]. This contradiction completes the proof of the fact that G is p-supersoluble.

Conversely, suppose that G is p-supersoluble, we show that every cyclic subgroup of $\overline{G} = G/O_{p'}(G)$ of prime order or order 4 is N-embedded in $\overline{G}$.

Without loss of generality, we may assume that $O_{p'}(G) = 1$. Let P be a Sylow p-subgroup of G. Then by Lemma 2.5, P is normal in G and p is the largest prime dividing $|G|$. Hence, we only need to consider the case that $p > 2$. Let L be any subgroup of G of order p and $L \leqq N$ where N is a normal subgroup of G contained in P. We shall show by induction on $|N|$ that there are normal subgroups A and B of G such that $LA = B \leqq N$ and $L \cap A = 1$. Let V be a normal subgroup of G such that V is a maximal subgroup of N. If $VL = N$, then $V \cap L = 1$. We may, therefore, assume that $L \leqq V$. Then the required is true by induction. Thus, L is N-embedded in G. This completes the proof. □

4. Proof of Theorem B

We shall prove Theorem B in the following more general form.

THEOREM 4.1. *A soluble group G is p-supersoluble if and only if every 2-maximal subgroup E of G with $O_{p'}(G) \leqq E$ and $|G : E|$ is not a power of p, both has a supplement in E^{sG} with cyclic Sylow p-subgroups and is S_p-embedded in G.*

PROOF. First suppose that G is p-supersoluble and let E be any 2-maximal subgroup of G such that $O_{p'}(G) \leqq E$ and $|G : E|$ is not a power of p. We show that E has a supplement T in E^{sG} with cyclic Sylow p-subgroups and E is S_p-embedded in G. If $E = E^{sG}$, then it is evident. We may, therefore, assume that $E \neq E^{sG}$. Suppose that $O_{p'}(G) \neq 1$. Then, by induction, $E/O_{p'}(G)$ is S_p-embedded in $G/O_{p'}(G)$ and $E/O_{p'}(G)$ has a supplement $T/O_{p'}(G)$ in $\big(E/O_{p'}(G)\big)^{s(G/O_{p'}(G)}$ with cyclic Sylow p-subgroups. By Lemma 2.9(1), E is is S_p-embedded in G. On the other hand, since $\big(E/O_{p'}(G)\big)^{s(G/O_{p'}(G)} = E^{sG}/O_{p'}(G)$ by Lemma 2.4(3), T is a supplement of E in E^{sG} and clearly the Sylow p-subgroups of T is cyclic.

Now suppose that $O_{p'}(G) = 1$. Then by Lemma 2.5, G is supersoluble, $P = O_p(G) = F(G)$ is a Sylow p-subgroup of G and G/P is abelian. Let M be a maximal subgroup of G such that E is maximal in M. Since G is p-supersoluble and $|G : E|$ is not a power of p, one of $|M : E|$ and $|G : M|$ is a p'-number (see [8, Theorem 1.9.4]. Hence $|G : E| = pn$, where $(p, n) = 1$. It follows that $P \cap E$ is a maximal subgroup of P with $|P : P \cap E| = p$ and so $\Phi(P) \leqq E$. Since $\Phi(P)$ is a characteristic subgroup of P, it is normal in G. Hence $\Phi(P) \leqq E_{sG}$. If $\Phi(P) \neq 1$, then as above we can show that E is S_p-embedded in G. Besides, PE is normal in G by Lemma 2.5(iii) and $|PE : E| = |P : E \cap P| = p$. Since $E \subsetneqq E^{sG} \subsetneqq PE$, $E^{sG} = PE$. Hence E has a cyclic supplement $\langle x\rangle$ in E^{sG}, where $x \in PE$ and $x \notin E$.

Finally, assume that $\Phi(P) = 1$. Then P is an elementary abelian p-group and $P = P_1 \times P_2 \times \cdots \times P_t$, where P_i is a minimal normal subgroup of G, for all $i = 1, 2, \ldots, t$. It is clear that for some i, $P_i \nleqq E \cap P$. Hence $P_i(E \cap P) = P$. Since G is p-supersoluble, $|P_i| = p$. It follows that $P_i \cap E \cap P = 1$ and $EP_i = E$. Since $E \neq E^{sG}$, E is not S-quasinormal in G. But since PE is normal in G, by Lemma 2.5(iii), $PE = E^{sG} = P_i(E \cap P)E = P_iE$. Since $P_i \cap E \trianglelefteq E$, $P_i \cap E \subseteqq E_{sG}$. Hence $|P_i \cap E|_p = |P_i \cap E_{sG}|_p$. This implies that E has a supplement P_i in E^{sG}, which is a cyclic Sylow p-subgroup and E is S_p-embedded in G.

Conversely, assume that G is soluble and every 2-maximal subgroup E of G with $O_{p'}(G) \leqq E$ and $|G : E|$ is not a power of p has a supplement in E^{sG} with cyclic Sylow p-subgroups and E is S_p-embedded in G. We show that G is p-supersoluble. Assume that this is false and let G be a counterexample of minimal order. Then

(1) *G has a unique minimal normal subgroup L, G/L is p-supersoluble, p divides $|L|$ and $L \nleqq \Phi(G)$.*

Let L be a minimal normal subgroup of G and E/L a 2-maximal subgroup of G/L such that $O_{p'}(G/L) \leqq E/L$ and $|G/L : E/L|$ is not a power of p. Since $O_{p'}(G)L/L \leqq O_{p'}(G/L)$, $O_{p'}(G) \leqq E$. Besides, $|G : E|$ is not a power of p. Hence by hypothesis, E is S_p-embedded in G and there is a subgroup T such that $ET = E^{sG}$ and a Sylow p-subgroup of T is cyclic. Obviously, $(EL)^{sG} = E^{sG}L$. Hence $(E/L)(TL/L) = E^{sG}L/L = (E/L)^{s(G/L)}$ by Lemma 2.4(3) and clearly TL/L has a cyclic Sylow p-subgroup. Besides, E/L is S_p-embedded in G/L by Lemma 2.9(2). This shows that the hypothesis still holds for G/L. The minimal choice of G implies that G/L is p-supersoluble. It is well known that the class of all p-supersoluble groups is a saturated formation. Hence we see that (1) holds.

(2) *$G = [L]M$ for some maximal subgroup M of G, $L = C_G(L) = F(G) = O_p(G)$ and $p \neq |L|$.*

By (1), there exists a maximal subgroup M of G such that $G = LM$. Since G is soluble, L is either a p'-group or a p-group. In the former case, G is clearly p-supersoluble, a contradiction. Hence L is a abelian p-group. It follows that $L = C_G(L) = F(G) = O_p(G)$ and $|L| > p$ since G is not p-supersoluble.

(3) *L is not a Sylow p-subgroup of G.*

Assume that L is a Sylow p-subgroup of G and let E be a maximal subgroup of M. Then $|G : E| = |L|\,|M : E| \neq p^a$ and $O_p(G) = 1 \leqq E$. Hence by hypothesis E is S_p-embedded in G and E has a supplement X in E^{sG} with cyclic Sylow p-subgroups. Suppose that $E = 1$ and let V be a maximal subgroup of L. Then by hypothesis, V is S_p-embedded in G. Let T be a S-

quasinormal subgroup of G such that $V^{sG} = VT$ and $|T \cap V|_p = |T \cap V_{sG}|_p$. Since L is a Sylow p-subgroup of G, the subgroups V_{sG}, V^{sG} and T are normal in G by Lemma 2.2(7). This implies that $T = L$ and $V = 1$. But then $|L| = p$, which contradicts (2). Therefore $E \neq 1$. Let q be prime dividing $|M : E|$ and Q be a Sylow q-subgroup of M. Clearly, Q is a Sylow q-subgroup of G and $A = E^{sG}Q = QE^{sG}$ is a subgroup of G. Since E is maximal in M and $Q \nleqq E$, we have $\langle E, Q\rangle = M$. Hence $M \leqq A$ and so $LA = G$. Since L is a minimal normal subgroup of G, either $L \cap A = L$ or $L \cap A = 1$. In the former case, $G = A = E^{sG}Q$ and so L is a Sylow p-subgroup of any supplement of E in E^{sG}. Therefore L is cyclic and hence $|L| = p$, which contradicts (2). Thus $L \cap A = 1$. Obviously, $E^{sG} \leqq M$. Since $M_G = 1$, $(E^{sG})_G = 1$. Hence E^{sG} is nilpotent by Lemma 2.2(5). Then by Lemma 2.2(3) and Lemma 2.1(4), $E^{sG} \leqq L$, a contradiction. This shows that L is not a Sylow p-subgroup of G.

(4) *M has a non-normal maximal subgroup E such that* $\big(|M : E|, p\big) = 1$.

Suppose that every maximal subgroup E of M with $\big(|M : E|, p\big) = 1$ is normal in M. Then by Lemma 2.10, M is p-closed. Besides, by (3), p divides $|M|$. But by (2), we have $O_p(G/L) = O_p(G/C_G(L) = 1$ (see [8, Lemma 1.7.11]). Hence L is a Sylow p-subgroup of G, which contradicts (3). Hence, (6) holds.

(5) $E^{sG} = G$.

Indeed, suppose that $D = E^{sG} \neq G$. Since D is subnormal in G by Lemma 2.2(3) and Lemma 2.4(1), $M \nleqq D$. Hence $E = D \cap M$ is a subnormal subgroup of M by Lemmas 2.2(3) and 2.1(2). But since E is maximal in M, E is normal in M, which contradicts (4).

(6) *If T is a supplement of E in G, then* $O_{p'}(T) = 1$.

Since $G = ET = MT$, $M = M \cap ET = E(M \cap T)$ and $|L| = |G : M| = |T : M \cap T|$. It follows that $O_{p'}(T) \leqq M$. Hence $\big(O_{p'}(T)\big)^G = O_{p'}(T)^{TM} = O_{p'}(T)^M \leqq M_G = 1$. Consequently, $O_{p'}(T) = 1$.

(7) $p = 3$ *and* $|L| = 9$.

Let T be a supplement of E in $E^{sG} = G$ with cyclic Sylow p-subgroups. Then $G = ET = MT$. Let T_p be a Sylow p-subgroup of T. Suppose that $p = 2$. Then by [11, IV, Theorem 2.8], T is 2-nilpotent. But by (6), $T_{2'} = 1$ and so T is a 2-group. It follows that 2 divides $|M : E|$, which contradicts (4). Therefore p must be an odd number. Suppose that either $p \neq 3$ or $p = 3$ and $|L| > 3^2$. Let $|L| = p^a$ and p^b be the order of a Sylow p-subgroup of M. Then by Lemma 2.11, $b < a - 1$. Since T_p is cyclic, $|T_p \cap L| \leqq p$. It follows from $G = MT = [L]M$ that $|L| \leqq |T_p|$. Hence $p^{a+b} < p^{2a-1} \leqq |LT_p| \leqq p^{a+b}$. This contradiction shows that $p = 3$ and $a = 2$.

(8) *The order of a Sylow* 2*-subgroup* P *of* T *is* 2.

Let T be a supplement of E in G. Since T is 3-soluble and its Sylow 3-subgroups are cyclic, T is 3-supersoluble. But by (6), $O_{3'}(T) = 1$. Hence T is supersoluble by Lemma 2.5. Then, for a Sylow 3-subgroup P of T, we have $P = F(T)$. Thus $C_T(P) \leqq P$ by [8, Theorem 1.8.18]. It follows from [7, Chapter 5, Lemma 4.1(iii)] that $|T/P| = 2$. Therefore, (8) holds.

Final contradiction.

In view of (7), M is isomorphic with some subgroup of $GL(2,3)$. Hence $|M| \leqq 48$. It follows that $|M : E| = 2^m$ for some $m > 1$ (since E is not normal in M). But since $G = ET$, we see that $|M : E| \mid |P| = 2$, which is impossible. The final contradiction completes the proof. □

5. Proof of Theorem C

First we shall prove the following Theorem.

THEOREM 5.1. *Suppose that* G *is not a* p*-group. Then the following are equivalent.*

(1) G *is* p*-soluble.*

(2) *Every maximal subgroup of* G *is* S_p*-embedded in* G*.*

(3) G *has two maximal* p*-soluble* S_p*-embedded subgroups* M_1 *and* M_2*, whose indices* $|G : M_1|$ *and* $|G : M_2|$ *are coprime.*

(4) *For every maximal subgroup* M *of* G*, either* $|G : M|$ *is a power of* p *or* M *is* S_p*-embedded in* G*.*

PROOF. (1) $\Rightarrow$ (2). Let M be a maximal subgroup of G and H/K a chief factor of G such that $HM = G$ and $K \leqq M$. If $M_G \neq 1$, M/M_G is S_p-embedded in G/M_G by induction and consequently M is S_p-embedded in G by Lemma 2.9(1). Suppose that $M_G = 1$. Then $K = 1$. If H is a p'-group, then $|H \cap M|_p = 1 = |H \cap M_{sG}|_p$. Hence M is S_p-embedded in G. If H is an abelian p-group, then $H \cap M = H \cap M_{sG} = K = 1$. Then M is also S_p-embedded in G.

(3) $\Rightarrow$ (1). Assume that this is false and let G be a counterexample of minimal order. Let N be a minimal normal subgroup of G. If $N \leqq M_1 \cap M_2$, then M_1/N and M_2/N are p-soluble maximal subgroups of G/N whose indices $|G/N : M_1/N| = |G : M_1|$ and $|G/N : M_2/N| = |G : M_2|$ are coprime and M_1/N and M_2/N are S_p-embedded in G by Lemma 2.9(2). This shows that the hypothesis holds for G/N. Therefore G/N is p-soluble by the choice of G. On the other hand, if $N \nleqq M_1 \cap M_2$, for example $N \nleqq M_1$, then $G/N \simeq M_1/M_1 \cap N$ is p-soluble. Therefore, N is the only minimal normal subgroup N of G, $N \neq \Phi(G)$ and N is a non-abelian group with p divides $|N|$. Then, clearly, $G = NM_1 = NM_2$ and $(M_1)_G = 1 = (M_2)_G$.

Let N_p be a Sylow p-subgroup of N and P a Sylow p-subgroup of G contained N_p. Since $|G : M_1|$ and $|G : M_2|$ are coprime. Without loss of generality, we may assume that P is contained in at least one of the subgroups M_1 and M_2, for example, $P \leqq M_1$. Since M_1 is S_p-embedded in G, $M^{sG} = MT$ and $|M \cap T|_p = |T \cap M_{sG}|_p$, for some S-quasinormal subgroup T of G. By Lemma 2.2(3), M^{sG}, T and M_{sG} are subnormal in G. Then since $(M_1)_G = 1$ and by Lemma 2.3(4), we have that $(M_1)_{sG} = 1$ and $(M_1)^{sG} = G$. This shows that T is a complement of M_1 in G. Since $P \leqq M_1$, p does not divide $|T|$. Let $N = N_1 \times N_2 \times \cdots \times N_t$, where $N_1, N_2, \ldots, N_t$ are isomorphic simple non-abelian groups. Let L be a minimal subnormal subgroup of G contained in T. Since, obviously, $C_G(N) = 1$, $L \nleqq C_G(N)$ and so $L \leqq N$ by Lemma 2.1(5). Hence $L = N_i$, for some i. It follows that p divides $|L|$ and therefore p divides $|T|$, a contradiction.

(1) $\Rightarrow$ (3). Since G is p-soluble and G is not a p-group, then there are two maximal subgroups M_1 and M_2 of G such that $|G : M_1| = p^a$ for some $a \in \mathbb{N}$ and p does not divide $|G : M_2|$ by [8, Theorem 1.7.13]. Then $\big(|G : M_1|, |G : M_2|\big) = 1$. By (2), we see that M_1 and M_2 are S_p-embedded in G. Thus (3) holds.

(4) $\Rightarrow$ (1). Let L be a minimal normal subgroup of G. Clearly, the hypothesis is true for G/L. By induction, G/L is p-soluble. We may, therefore, assume that L is non-abelian, p divides $|L|$ and L is the only minimal normal subroup of G. Thus $C_G(L) = 1$. By the Frattini argument, for any Sylow p-subgroup P of L, there is a maximal subgroup M of G such that $LM = G$ and $N_G(P) \leqq M$. It is clear that $M_G = 1$ and p does not divides $|G : M|$. By hypothesis, G has an S-quasinormal subgroup T such that $M^{sG} = M^G = G = MT$ and $|T \cap M|_p = |T \cap M_{sG}|_p$. But by Lemma 2.3(4), $M_{sG} = M_G = 1$. This implies that $T \cap M$ is a p'-group. Let X be a minimal subnormal subgroup of G contained in T. Since $C_G(L) = 1$, $X \leqq L$ by Lemma 2.1(5) and so p divides $|X|$. It follows that p divides $|T|$. Since T is subnormal in G, by Lemma 2.1(3), $|T \cap M|_p \neq 1$. This contradiction shows that G is p-soluble. The theorem is proved. □

Corollary 5.2 (Wang [19]). *A group G is soluble if and only if every maximal subgroup of G is c-normal in G.*

Proof of Theorem C. Assume that this theorem is is false and let G be a counterexample of minimal order. Then p divides $|G|$. We proceed the proof via the following steps.

(1) *G is not simple.*

Suppose that G is a simple non-abelian group. Then $O_{p'}(G) = 1$. Let M be a maximal subgroup of G containing a Sylow p-subgroup P of G and E any maximal subgroup of M. Then $|G : E|$ is not a power of p. Hence by hypothesis, E is S_p-embedded in G. Let T be an S-quasinormal subgroup of

G such that $TE = E^{sG}$ and $|T \cap E|_p = |T \cap E_{sG}|_p$. By Lemma 2.2(3), E^{sG}, E_{sG} and T are all subnormal subgroups of G. Since G is a simple group, we have that $T = G$ and $E_{sG} = 1$. It follows from $|T \cap E|_p = |T \cap E_{sG}|_p = 1$ that $|E|_p = 1$. If $M = P$, then $|M| = p$ and so G is soluble by [11, IV, Theorem 7.4]. Otherwise, we may assume that $P \leqq E$, which implies $P = 1$. This contradiction shows that (1) holds.

(2) *G has a unique minimal normal subgroup L, G/L is p-soluble, p divides $|L|$ and $L \nleqq \Phi(G)$.*

Let L be a minimal normal subgroup of G. Then by Lemma 2.9(2), the hypothesis still holds for G/L. The minimal choice implies that G/L is p-soluble by the choice of G. Hence (2) holds.

(3) *G is p-soluble.*

By (1), $L \neq G$. Let M be any maximal subgroup of G containing L. Suppose that $|G : M| = p^a$. Then for every maximal subgroup E of M with $|M : E|$ is not a power of p, we have that $|G : E|$ is also not a power of p. Hence E is S_p-embedded in G by (2) and hypothesis. It follows from Lemma 2.9(3) that E is S_p-embedded in M. Hence M is p-soluble by Theorem 5.1(4). Consequently, L is p-soluble and thereby G is p-soluble. The contradiction completes the proof. □

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