ALMOST SURE FUNCTIONAL LIMIT THEOREMS

Part I. The general case

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In this paper we formulate and prove the almost sure functional limit theorem in fairly general cases. This limit theorem is a result which states that if a stochastic process $X(t,\omega)$, $t\ge 0$, is given on a probability space with some nice properties, then an appropriate probability measure $\bar\lambda_T$ can be defined on the interval $[1,T]$ for all $T>1$ in such a way that for almost all ω the distributions of the appropriate normalizations of the trajectories $X_t(\cdot,\omega)=X(t\cdot,\omega)$, considered as random variables $\xi_T(t)$, $t\in[1,T]$, on the probability spaces $([1,T],\mathcal{A},\lambda_T)$ with values in a function space have a weak limit independent of ω as $T\to\infty$. We shall consider self-similar processes which appear in different limit theorems. The almost sure functional limit theorem will be formulated and proved for them and their appropriate discretization under weak conditions. We also formulate and prove a coupling argument which makes it possible to prove the almost sure functional limit theorem for certain processes which converge to a self-similar process. In the second part of this work we shall prove and generalize — with the help of the results of the first part — some known almost sure functional limit theorems for independent random variables.

1. Introduction

The following "almost sure central limit theorem" is a popular subject in recent research. Let $X_1(\omega),X_2(\omega),\dots$ be a sequence of iid. random variables, $EX_1=0$, $EX_1^2=1$, $S_n(\omega)=\sum\limits_{k=1}^n X_k(\omega)$ on a probability space $(\Omega,\mathcal{A},P)$. (In the sequel we denote by $(\Omega,\mathcal{A},P)$ the probability space where the random variables we are considering exist.) Then

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^n\frac1k I\left(\frac{S_k(\omega)}{\sqrt k}<u\right)=\Phi(u)\quad\text{for almost all }\omega\in\Omega \tag{1.1}$$

and all numbers u, where $I(\mathbf{A})$ denotes the indicator function of a set $\mathbf{A}$, and $\Phi(u)$ is the standard normal distribution function. This result was discovered by Brosamler [2] and Schatte [7]. It states that appropriately normalized partial sums of iid. random variables satisfy not only the central limit theorem, but for a typical $\omega\in\Omega$ the weighted averages of the functions $g_k(u,\omega)=I\left(S_k(\omega)<u\sqrt k\right)$ with appropriate weights converge to the normal law. Later this result was formulated in a more general form which states that not only the weighted averages of the functions $I\left(S_k(\omega)<u\sqrt k\right)$ converge to the

normal distribution function for a typical ω, but a similar result also holds for sequences of random broken lines or polygons $G_n(u) = G_n(u,\omega)$, $n = 1,2,\dots$, defined in an appropriate way on the interval $[0,1]$ by means of the partial sums $S_1(\omega),\dots,S_n(\omega)$.

Define a random measure $\mu_n = \mu_n(\omega)$ for all n by attaching an appropriate weight $a_k = a_{k,n}$ to the functions $G_k(u,\omega)$ for all $1 \le k \le n$. Then these measures converge weakly to the Wiener measure for almost all ω. Such a result is called an almost sure functional limit theorem. Later we formulate this notion in a more detailed form.

The almost sure central (and also the functional) limit theorem shows some similarity to the ergodic theorem which states — in physical terminology — that the space and time averages of ergodic sequences agree. In the case of the almost sure central limit theorem an analogous result holds for the normalized partial sums $\frac{S_k(\omega)}{\sqrt{k}}$, $k = 1,2,\dots$. Now the time average is replaced by a weighted time average, where the k-th term gets weight $a_k = a_{k,n} = \frac{1}{\log(n+1)} \log\frac{k+1}{k} \sim \frac{1}{k\log n}$, $1 \le k \le n$, in the n-th block instead of the weight $\frac{1}{n}$ given to the first n terms in the ergodic theorem. On the other hand, $\frac{S_n(\omega)}{\sqrt{n}}$ is asymptotically normally distributed, with expectation zero and variance one. Hence the right-hand side in formula (1.1) equals $\lim\limits_{n\to\infty} EI\left(\frac{S_n(\omega)}{\sqrt{n}} < u\right)$, and this expression resembles to a space average. This similarity of the almost sure central limit theorem to the ergodic theorem may be put even stronger by an appropriate time scaling to be explained later.

The relation between the ergodic theorem and almost sure central (and functional) limit theorem is deeper than the above mentioned formal analogy. It was pointed out, — by our knowledge it was discovered by Brosamler [1], Fisher [5] and Lacey and Philipps in [6] — that these theorems can be deduced from the ergodic theorem applied to the Ornstein–Uhlenbeck process.

In the present paper we discuss how the almost sure central and functional limit theorem can be generalized and proved by means of the ergodic theorem in a natural way. The proof has two main ingredients. The first one is to show that a result analogous to the almost sure functional limit theorem holds for the Wiener process. This can be deduced from the ergodic theorem for the Ornstein–Uhlenbeck process. This is an ergodic process which can be obtained from the Wiener process by means of a well-known transformation. The second ingredient is to show that, since the random polygons or broken lines constructed from the partial sums of independent random variables in a natural way behave similarly to the Wiener process, the almost sure central limit theorem for the Wiener process also implies this result for the random polygons (or broken lines) made from normalized partial sums of independent random variables.

First we show that the method of proving the almost sure functional limit theorem for the Wiener process by means of the ergodic theorem for the Ornstein–Uhlenbeck process can be generalized for a large class of other processes, for the so-called self-similar processes. The stationarity property of the Ornstein–Uhlenbeck process is equivalent to the self-similarity property of the Wiener process, a property which holds for all self-similar processes. Actually, self-similar process are those processes which appear as the limit in different limit theorems. Similarly to the construction of the Ornstein–Uhlenbeck process generalized Ornstein–Uhlenbeck processes can be constructed as the transforms of self-similar processes. These generalized Ornstein-Uhlenbeck processes are stationary processes, and the application of the ergodic theorem for them enables us to prove the almost sure functional limit theorem for general self-similar processes. Then with the help of some further work we can also prove the almost sure functional limit theorem for their appropriate discretized versions.

In the next step we want to find a good coupling argument which enables us to prove the almost sure invariance principle not only for (self-similar) limit processes but also for processes in the domain of their attraction. To carry out such a program a coupling argument has to be introduced which is adapted to the present problem. We shall do it by introducing a notion we call the Property A.

In Part II. of this work we shall prove the almost sure functional limit theorem for independent random variables whose partial sums converge to the normal or to a stable law. In the proofs we shall exploit that the Wiener process and the stable process are self-similar, hence the results of the present paper can be applied for them. Then we can prove, by applying the coupling argument of the present paper, the almost sure invariance principle for independent random variables which satisfy certain (weak) conditions.

There are other processes which are natural candidates for almost sure functional limit theorem type results, e.g. random processes in the domain of attraction of a self-similar process subordinated to a Gaussian process (see Dobrushin [3]). But such problems will not be discussed here.

Several results of the present paper can be traced down in earlier works. Our main goal is to explain the main ideas behind these results and to present a unified treatment of various problems in this subject. The first part of this work considers general results where no independence type condition is assumed. In the second part different arguments — the techniques worked out for the study of independent random variables — are applied, and we deal there with almost sure functional limit theorems for independent random variables. This paper consists of three sections. In Section 2 we formulate the main results, and Section 3 contains the proofs.

2. The main results of the paper

To formulate our results first we recall the definition of self-similar processes with self-similarity parameter α and define with their help a new process which we call a generalized Ornstein–Uhlenbeck process.

Definition of self-similar processes. *We call a stochastic process $X(u,\omega)$, $u \ge 0$, $X(0,\omega) \equiv 0$, self-similar with self-similarity parameter α, $\alpha > 0$, if*

$$X(u,\omega) \stackrel{\Delta}{=} \frac{X(Tu,\omega)}{T^{1/\alpha}}, \quad 0 \le u < \infty, \tag{2.1}$$

for all $T > 0$, where $\stackrel{\Delta}{=}$ means that the processes at the two sides of the equation have the same distribution. (Here we consider the distribution of the whole process $X(u,\omega)$, $u \ge 0$, and not only its one-dimensional distributions.)

The Wiener process is self-similar with self-similarity parameter $\alpha = 2$. Similarly, for all stable laws G with parameter α, $0 < \alpha < 2$, $\alpha \ne 1$, a so-called stable process $X(u,\omega)$ can be constructed which has independent and stationary increments, $X(0,\omega) \equiv 0$, which is self-similar with self-similarity parameter α, and the distribution function of $X(1,\omega)$ is G. The case $\alpha = 1$ is exceptional. In this case (except the special case when $X(1,\omega)$ has symmetric distribution) only a modified version of formula (2.1) holds, where a norming factor const. $\log T$ must be added with an appropriate non-zero constant to one side in formula (2.1). Another example for self-similar processes was given by Dobrushin in paper [3], who could construct new kind of self-similar processes subordinated to a Gaussian process. He constructed them by working with non-linear functionals of Gaussian processes.

Now we introduce the following notion:

Definition of generalized Ornstein–Uhlenbeck processes. *Let $X(u,\omega)$, $u \ge 0$, be a self-similar process with self-similarity parameter $\alpha > 0$. We call the process $Z(t,\omega)$, $-\infty < t < \infty$, defined by formula*

$$Z(t,\omega) = \frac{X(e^t,\omega)}{e^{t/\alpha}}, \quad -\infty < t < \infty, \tag{2.2}$$

the generalized Ornstein–Uhlenbeck process corresponding to the process $X(u,\omega)$.

Let us remark that the generalized Ornstein–Uhlenbeck process corresponding to the Wiener process is the usual Ornstein–Uhlenbeck process.

A Wiener process $W(t,\omega)$, $t \ge 0$, has continuous trajectories, the trajectories of a stable process $X(t,\omega)$ are so-called càdlàg (continue à droite, limite à gauche) functions, i.e. all trajectories $X(\cdot,\omega)$ are continuous from the right, and have a left-hand side limit in all points $t > 0$. Hence the Wiener process $W(t,\omega)$ and any of its scaled version $A_T W(Tt,\omega)$, $0 \le t \le 1$, where $T > 0$ and $A_T > 0$ are arbitrary constants, can be considered as random variables taking values in the space $C([0,1])$ of continuous functions on the interval $[0,1]$. The processes $X(t,\omega)$, $A_T X(tT,\omega)$, $0 \le t \le 1$, where $X(t,\omega)$, $0 \le t < \infty$, is a stable process, can be considered as random variables on the space $D([0,1])$ of càdlàg functions on the interval $[0,1]$.

We shall work not only in the space $C([0,1])$ but also in the space $D([0,1])$. To work in the space $D([0,1])$ one has to handle some unpleasant technical problems. But since we also want to investigate stable processes in Part II. of this work, we also have to work in this space. We shall apply the book of P. Billingsley [1] as the main reference for this subject.

We consider both spaces $C([0,1])$ and $D([0,1])$ with the usual topology, and the Borel σ-algebra generated by this topology. Both spaces can be endowed with a metric which induces this topology, and with which these spaces are separable, complete metric spaces. A detailed discussion and proof of these results and definitions can be found in the book of P. Billingsley [1]. Since we shall need the exact form of these metrics we recall these results. In the $C([0,1])$ space the supremum metric $\rho(x,y)=\sup\limits_{0\le t\le 1}|x(t)-x(s)|$ is considered. In the space $D([0,1])$ the following metric $d_0(\cdot,\cdot)$ satisfies these properties: For a pair of functions $x,y\in D([0,1])$ $d_0(x,y)\le\varepsilon$, if there exists such a homeomorphism $\lambda(t)\colon[0,1]\to[0,1]$ of the interval $[0,1]$ into itself for which $\lambda(0)=0$, $\sup\limits_{t\ne s}\log\left|\dfrac{\lambda(t)-\lambda(s)}{t-s}\right|\le\varepsilon$, and $|x(t)-y(\lambda(t))|\le\varepsilon$ for all $t\in[0,1]$. (See for instance Theorems 14.1 and 14.2 in Billingsley's book [1].) In the sequel we shall apply these metrics in the spaces $C([0,1])$ and $D([0,1])$, and denote them by $\rho(\cdot,\cdot)$.

Let us also recall that given some probability measures μ_T on a metric space $\mathbf{K}$ indexed by $T\in[1,\infty)$ or $T=\{A_1,A_2,\dots\}$, $\lim\limits_{n\to\infty}A_N=\infty$, the measures μ_T converge weakly to a measure μ on $\mathbf{K}$ as $T\to\infty$ if $\lim\limits_{T\to\infty}\int\limits_{\mathbf{K}}\mathcal{F}(x)\mu_T(\,dx)=\int\limits_{\mathbf{K}}\mathcal{F}(x)\mu(\,dx)$ for all continuous and bounded functionals $\mathcal{F}$ on the space $\mathbf{K}$. The next result states the almost sure functional limit theorem for a self-similar process which satisfies some additional conditions. The proof is based on the ergodic theorem applied for the generalized Ornstein–Uhlenbeck process corresponding to this self-similar process.

Theorem 1. *Let $X(u,\omega)$ be a self-similar process with continuous or càdlàg trajectories, and $Z(t,\omega)$ the generalized Ornstein–Uhlenbeck process corresponding to it. The process $Z(t,\omega)$, $-\infty<t<\infty$, is stationary. Let us assume that the process $Z(t,\omega)$ is not only stationary, but also ergodic. Then for all measurable and bounded functionals $\mathcal{F}$ on the space $C([0,1])$ or $D([0,1])$ (depending on whether the trajectories of $X(\cdot,\omega)$ are continuous or only càdlàg functions)*

$$\lim_{T\to\infty}\frac{1}{\log T}\int_1^T\frac{1}{t}\mathcal{F}(X_t(u,\omega))\,dt=E\mathcal{F}(X_1(u,\omega))\quad\textit{for almost all }\omega,\tag{2.3}$$

where

$$X_t(u,\omega)=\frac{X(ut,\omega)}{t^{1/\alpha}},\quad 0\le u\le 1,\ t>0.\tag{2.4}$$

Let us define for all $\omega\in\Omega$ and $T\ge1$ the (random) probability measure $\mu_T(\omega)$ in the space $C([0,1])$ or $D([0,1])$ which is concentrated on the trajectories $X_t(\omega)$, $1\le t\le T$, and takes the value $X_t(\omega)$, $1\le t\le T$, with probability $\dfrac{1}{\log T}\dfrac{dt}{t}$. More formally, for a measurable set $\mathbf{A}\subset C([0,1])$ or $\mathbf{A}\subset D([0,1])$ put $\mu_T(\omega)(\mathbf{A})=\bar\lambda_T\{t\colon X_t(\omega)\in\mathbf{A}\}$, where $\bar\lambda_T$ is a measure on $[1,T]$ defined by the formula $\lambda_T(\mathbf{C})=\dfrac{1}{\log T}\int\limits_{\mathbf{C}}\dfrac{dt}{t}$ for all measurable sets $\mathbf{C}\subset[0,T]$.

The following version of Formula (2.3) also holds: For almost all $\omega\in\Omega$ the probability measures $\mu_T(\omega)$ converge weakly to the distribution of the process $X_1(u,\omega)$ defined in (2.4) with $t=1$, or in other words, there is a set of probability one such that if

ω is in this set then relation (2.3) *holds for this ω and* all *bounded and continuous functionals $\mathcal{F}$.*

If $X(u,\omega)$ is a Wiener or stable process, then the generalized Ornstein–Uhlenbeck process corresponding to it is not only stationary, but also ergodic. Hence the results of Theorem 1 are applicable in this case.

We want to prove a discretized version of the above result, where the measures $\mu_T(\omega)$ concentrated in the set of trajectories $X_t(\omega)$, $1 \le t \le T$, are replaced by some measures $\mu_N(\omega)$ which are concentrated on a set of trajectories $X_{a(j,N)}(\omega)$ with appropriate weights, and the numbers $a(j,N)$ constitute a finite set. Then we want to make a further discretization, where the trajectories $X_{a(j,N)}$ are replaced by their discretized version. To prove these results in the case when the trajectories of the process $X(\cdot,\omega)$ are càdlàg functions we impose the following additional condition.

$$P\left(\lim_{t\to 1-0} X(t,\omega) = X(1,\omega)\right) = 1. \tag{2.5}$$

First we formulate a result which serves as the basis of the discretization results formulated later.

Theorem 2. *Let $X(u,\omega)$, $X_t(u,\omega)$, $\mu_T(\omega)$ and μ_0 be the same as in Theorem 1. Let us assume that the conditions of Theorem 1 are satisfied, and also the additional condition* (2.5) *holds in the case when the process $X(\cdot,\omega)$ has càdlàg trajectories. Let us define, similarly to the trajectories $X_t(\cdot,\omega)$ defined in* (2.4)*, the following transformed functions $x_t = x_t(\cdot)$ of a function $x \in C([0,1])$ or $x \in D([0,1])$ by the formula*

$$x_t(u) = x_{t,\alpha}(u) = t^{-1/\alpha} x(ut), \quad 0 \le u \le 1,\ 0 < t \le 1, \tag{2.4$'$}$$

where α is the self-similarity parameter of the underlying self-similar process $X(\cdot,\omega)$. Then for almost all $\omega \in \Omega$

$$\lim_{\varepsilon\to 0}\lim_{T\to\infty} \mu_T(\omega)\left(\sup_{1-\varepsilon\le s,t\le 1} \rho(x_s,x_t) > \delta\right) = 0 \quad \textit{for all } \delta > 0. \tag{2.6}$$

where $\rho(\cdot,\cdot)$ is the metric whose definition was recalled before Theorem 1, and with which $C(0,1])$ or $D([0,1])$ are separable, complete metric spaces. (Let us recall that the (random) measure $\mu_T(\omega)$ is concentrated on the trajectories $X_u(\cdot,\omega)$, $1 \le u \le T$, of the process $X(\cdot,\omega)$ defined by formula (2.4).)

Condition (2.5) had to be imposed to control the behaviour of the trajectories of the processes $X_t(u,\omega)$ in the end point $u = 1$. This is not a strict restriction. For instance the next simple Lemma 1 gives a sufficient condition for its validity. It implies in particular, that the stable processes with self-similarity parameter α, $0 < \alpha < 2$, $\alpha \ne 1$, satisfy relation (2.5).

Lemma 1. *Let $X(\cdot,\omega)$ be a self-similar process with self-similarity parameter $\alpha > 0$ which is also a process with stationary increments, and whose trajectories are càdlàg functions. Then it satisfies relation* (2.5).

Now we formulate the result about "possible discretization" of the measures μ_T in the result of Theorem 1. Before this we make some comments which can explain the content of this result.

For all $T>1$ let us consider the probability space $([1,T],\mathcal{A},\bar\lambda_T)$, where $\mathcal{A}$ is the Borel σ-algebra, and $\bar\lambda_T$ is the measure defined in the formulation of Lemma 1. Fix an $\omega\in\Omega$, and let us consider the random variable $\xi(t)$, $1\le t\le T$, as $\xi(t)=X_t(\cdot,\omega)$, defined in formula (2.4), in the probability space $([1,T],\mathcal{A},\bar\lambda_T)$. This is a random variable which takes its value in the space $C([0,1])$ or $D([0,1])$, and it has distribution $\mu_T(\omega)$. Let us consider the above construction with some $T=B_N$, together with a dense splitting $1=B_{N,1}<B_{N,2}<\cdots<B_{N,k_N}=B_N$ of the interval $[1,B_N]$. Let us define the random variable $\hat\xi(t)$ such that $\hat\xi(t)=\xi(B_{k,N})=X_{B_{k,N}}(\cdot,\omega)$ if $t\in[B_{k,N},B_{k+1,N}]$. This random variable is close to the previously defined random variable $\xi(t)$, hence it is natural to expect that if $\hat\mu_{B_n}(\omega)$ denotes its distribution, then the measures $\hat\mu_{B_N}(\omega)$ have the same weak limit as the measures $\mu_{B_N}(\omega)$ as $N\to\infty$. The first statement of Theorem 3 is a result of this type. Then we prove that an appropriate small modification of the functions $\xi(B_{k,N})=X_{B_k,N}(\cdot,\omega)$ does not change the limit behaviour of the measures $\hat\mu_{B_N}(\omega)$. The second statement of Theorem 3 is such a result.

Theorem 3. *Let us assume that the conditions of Theorem 1 and Theorem 2 are satisfied. For all $N=0,1,\dots$ let us consider a finite increasing sequence of real numbers $1=B_{1,N}<B_{2,N}<\cdots<B_{k_N,N}$, and for the sake of simpler notation let us denote $B_{k_N,N}$ by B_N. Let us assume that these sequences satisfy the following properties:*

$$\lim_{N\to\infty}B_N=\infty,\qquad \lim_{N\to\infty}\frac{\log B_{j,N}}{\log B_N}=0 \text{ for all fixed } j,$$
$$\text{and}\quad \lim_{j\to\infty}\sup_{(k,N)\colon\, j\le k<N}\frac{B_{k+1,N}}{B_{k,N}}=1. \tag{2.7}$$

Moreover, assume the following strengthened form of the relation $\lim\limits_{N\to\infty}B_N=\infty$:

$$\lim_{j\to\infty}\inf_{N\colon\, N\ge j}\frac{B_{l,n}}{B_{j,N}}=\infty\quad \text{for all fixed } l=1,2,\dots. \tag{2.8}$$

For all $\omega\in\Omega$ define the (random) measures $\hat\mu_N(\omega)$, $N=1,2,\dots$, with the help of the sequences $1=B_{1,N}<B_{2,N}<\cdots<B_{k_N,N}$ in the following way:

The measure $\hat\mu_N(\omega)$, $N=1,2,\dots$, is concentrated on the trajectories $X_{B_{j,N}}(\cdot,\omega)$, $1\le j<k_N$, where $X_t(\cdot,\omega)$ is defined in (2.4), and

$$\hat\mu_N(\omega)(X_{B_{j,N}}(\cdot,\omega))=\frac{1}{\log B_N}\int_{B_{j,N}}^{B_{j+1,N}}\frac1u\,du=\frac{1}{\log B_N}\log\frac{B_{j+1,N}}{B_{j,N}}, \tag{2.9}$$
$$1\le j<k_N.$$

Then for almost all ω the measures $\hat\mu_N(\omega)$ converge weakly to μ_0.

For all $\omega \in \Omega$ let us also define the following random broken lines $\bar{X}_{B_{j,N}}(\cdot,\omega)$ which are "discretizations" of the trajectories $X_{B_{j,N}}(\cdot,\omega)$.

$$\bar{X}_{B_{j,N}}(s,\omega) = X_{B_{j,N}}\left(\frac{B_{l-1,N}}{B_{j,N}},\omega\right) \quad \text{if} \quad \frac{B_{l-1,N}}{B_{j,N}} \le s < \frac{B_{l,N}}{B_{j,N}},$$
$$1 \le l \le j, \ \ 1 \le j < k_N, \quad \text{and } \bar{X}_{B_{j,N}}(1,\omega) = X_{B_{j,N}}(1,\omega),$$

where $B_{0,N} = 0$. (The definition $B_{0,N} = 0$ is needed to define $\bar{X}_{B_{j,N}}(s,\omega)$ also for $0 \le sB_{j,N} < B_{1,N}$.)

Define the measures $\bar{\mu}_N(\omega)$ (with the help of the already defined measures $\hat{\mu}_N(\omega)$) as

$$\bar{\mu}_N(\omega)(\bar{X}_{B_{j,N}}(\cdot,\omega)) = \hat{\mu}_N(\omega)(X_{B_{j,N}}(\cdot,\omega)) = \frac{1}{\log B_N}\log\frac{B_{j+1,N}}{B_{j,N}}, \quad 1 \le j < k_N. \tag{2.9'}$$

Then for almost all $\omega \in \Omega$ the probability measures $\bar{\mu}_N(\omega)$ converge weakly to the probability measure μ_0 as $N \to \infty$.

We have defined $\bar{X}_{B_{j,N}}(\cdot,\omega)$ as a broken line with discontinuities and not as a polygon where the values of $X_{B_{j,N}}$ in the points $\frac{B_{l,N}}{B_{j,N}}$ are connected by linear segments. The reason for working with broken lines is that we want to prove results which are valid also in the case when the processes $X_t(\cdot,\omega)$ take their values in $D([0,1])$ but not necessarily in the space $C([0,1])$. In the general case the results we want to prove are valid only when broken lines are considered. In the case of processes with continuous trajectories we also could have defined them as random polygons. Moreover, it follows from some results of the general theory (see e.g. Section 18 in Billingsley's book [1]) that if the distribution of the processes consisting of the above defined random broken lines converge to a measure in the $C([0,1])$ space, then the distributions of the naturally defined random polygon version of these processes have the same limit in the $C([0,1])$ space.

Let $\xi_n(\omega)$, $n = 1,2,\dots$, be a sequence of random variables, and let us define the partial sums $S_n(\omega) = \sum_{k=1}^{n} \xi_k(\omega)$, $n = 1,2,\dots$, $S_0(\omega) \equiv 0$. Let us also consider two appropriate monotone increasing numerical sequences A_n and B_n, $n = 0,1,\dots$, of positive numbers such that

$$B_0 = 0, \quad \lim_{n\to\infty} A_n = \infty, \quad \lim_{n\to\infty} B_n = \infty, \quad \text{and} \quad \lim_{n\to\infty}\frac{B_{n+1}}{B_n} = 1. \tag{2.10}$$

For all $k = 1,2,\dots$ let us consider the partition $0 = s_{0,k} \le s_{1,k} \le \dots \le s_{k,k}$ of the interval $[0,1]$, defined by the formula $s_{j,k} = \frac{B_j}{B_k}$, $0 \le j \le k$. Let us also define with the help of the quantities $\xi_n(\omega)$, A_n and B_n, $n = 1,2,\dots$ the following random broken lines $S_k(s,\omega)$, $0 \le s \le 1$, $k = 1,2,\dots$,

$$S_k(s,\omega) = \frac{S_{j-1}(\omega)}{A_k} \quad \text{if } s_{j-1,k} \le s < s_{j,k}, \ 1 \le j \le k, \quad S_k(1,\omega) = \frac{S_k(\omega)}{A_k} \tag{2.11}$$

Now we introduce the following definition.

Definition of the almost sure functional limit theorem. *Let $\xi_n(\omega)$, $n = 1, 2, \dots$, be a sequence of random variables, and let two monotone increasing sequences of non-negative real numbers A_n and B_n, $n = 1, 2, \dots$, be given which satisfy formula (2.10). Let us consider the random broken lines $S_k(s, \omega)$, $0 \le s \le 1$, defined with the help of their partial sums $S_k(\omega)$, $k = 1, 2, \dots$, by formula (2.11). For all $\omega \in \Omega$ and $N = 1, 2, \dots$, define the random measure $\mu_N(\omega)$ in the following way: The measure $\mu_N(\omega)$ is concentrated on the random broken lines $S_k(\cdot, \omega)$, $1 \le k < N$, and*

$$\mu_N(\omega)(S_k(\cdot, \omega)) = \frac{1}{\log \frac{B_N}{B_1}} \log \frac{B_{k+1}}{B_k}, \quad 1 \le k < N. \tag{2.12}$$

We say that the sequence of random variables $\xi_n(\omega)$, $n = 1, 2, \dots$, satisfies the almost sure functional limit theorem with weight functions A_n and B_n, $n = 1, 2, \dots$, and limit measure μ_0 on the space $D([0,1])$ if for almost all $\omega \in \Omega$ the probability measures $\mu_N(\omega)$ converge weakly to the measure μ_0 as $N \to \infty$. In the special case when the limit measure μ_0 is the Wiener measure we say that these random variables satisfy the almost sure functional central limit theorem.

If the limit measure μ_0 is concentrated in the space $C([0,1])$, then the broken lines $S_k(\cdot, \omega)$ can be replaced by a natural modification which is a random polygon. Then we can consider a version of the measures $\mu_N(\omega)$ which are defined in the same way as the original ones, only the random processes $S_k(\cdot, \omega)$ are replaced by their random polygon version. Then the convergence of the original measures $\mu_N(\omega)$ to μ_0 in the space $D([0,1])$ implies the convergence of their modified version in the $C([0,1])$ space with the same limit. Let us also remark that although we allowed fairly large freedom in the definition of the sequence A_n in the definition of the almost sure functional limit theorem, nevertheless we shall always choose it in a very special way. Namely, in all almost sure functional limit theorems we shall prove the limit measure is the distribution of a self-similar process with a self-similarity parameter $\alpha > 0$ restricted to the interval $[0,1]$, and A_n is chosen as $A_n = B_n^{1/\alpha}$.

Let us remark that if the random variables $\xi_k(\omega)$ satisfy the almost sure functional central limit theorem with weight functions $A_n = \sqrt{n}$ and $B_n = n$, — and in Part II. we shall prove that under the conditions imposed for the validity of formula (1.1) this is the case, — then they also satisfy relation (1.1). To see this, fix a real number u and define the functional $\mathcal{F} = \mathcal{F}_t$ in the space $C([0,1])$ by the formula $\mathcal{F}(x) = 1$ if $x(1) < u$, and $\mathcal{F}(x) = 0$ if $x(1) \ge u$, where $x \in C([0,1])$, i.e. it is a continuous function on the interval $[0,1]$. This functional $\mathcal{F}$ is continuous with probability one with respect to the Wiener measure μ_0. Hence $\int \mathcal{F}(x)\, d\mu_n(\omega)(x) \to \int \mathcal{F}(x)\, d\mu_0(x)$ for almost all ω. This relation is equivalent to formula (1.1). Indeed, the right-hand side of this relation equals the right-hand side of formula (1.1), while the left-hand side is a slight modification of the left-hand side of (1.1). The difference between these formulas is that the weights $\frac{1}{k}$ in (1.1) are replaced by $\log \frac{k+1}{k}$ in the other formula, and summation goes from 1 to $n-1$ instead of summation from 1 to n. Since $\log \frac{k+1}{k} = \frac{1}{k} + O\left(\frac{1}{k^2}\right)$ these two relations are equivalent.

We formulate the following statement because of its importance in later applications in form of a Corollary.

Corollary. *Let $X(\cdot,\omega)$ be a self-similar process with self-similarity parameter $\alpha>0$ such that its trajectories are in the $C([0,1])$ or $D([0,1])$ space, it satisfies relation (2.5), and the generalized Ornstein–Uhlenbeck process corresponding to it is ergodic. Let t_n, $n=0,1,\dots$, $t_0=0$, be an increasing sequence of real numbers such that $\lim\limits_{n\to\infty} t_n=\infty$, $\lim\limits_{n\to\infty}\frac{t_{n+1}}{t_n}=1$. Put $\eta_n(\omega)=X(t_n,\omega)-X(t_{n-1},\omega)$, $B_n=t_n$, $A_n=B_n^{1/\alpha}$, $n=1,2,\dots$. Then the sequence $\eta_n(\omega)$, $n=1,2,\dots$, satisfies the almost sure functional limit theorem with weight functions A_n and B_n and limit measure μ_0 which is the distribution of the process $X(u,\omega)$, restricted to $0\le u\le 1$.*

To prove this Corollary define the process $X'(u,\omega)=A_1^{-1}X(B_1u,\omega)$ and observe that it has the same distribution as the process $X(u,\omega)$. Define the real numbers $B_{k,N}=\frac{t_k}{t_1}$, $1\le k\le N$, consider the random broken lines $\bar X'_{B_{j,N}}(\cdot,\omega)$, $1\le j\le N$, and the random measure $\bar\mu_N(\omega)$ defined in the formulation of Theorem 3 with this process $X'(\cdot,\omega)$ and these numbers $B_{k,N}$, (with the choice $k_N=N$), and apply Theorem 3, — whose conditions are satisfied, — for these random measures $\bar\mu_N(\omega)$.

On the other hand, define the random broken lines $S_k(s,\omega)$ by formula (2.11) with $B_N=t_N$, $A_N=B_n^{1/\alpha}$ and the partial sums $S_k(\omega)=\sum\limits_{l=1}^{k}(X(t_l,\omega)-X(t_{l-1},\omega))$, and let us also define the measure $\mu_N(\omega)$ by formula (2.12) with these random broken lines. Then a comparison shows that the above defined broken lines $\bar X'_{B_{j,N}}(\cdot,\omega)$ and $S_j(\cdot,\omega)$ and also their distributions, the random measures $\bar\mu_N(\omega)$ and $\mu_N(\omega)$ agree. Hence the second statement of Theorem 3 implies the almost sure functional limit theorem in this case.

If a sequence of random variables $\xi_n(\omega)$, $n=1,2,\dots$, is close to this sequence $\eta_n(\omega)$, then it is natural to except that this new sequence satisfies the same almost sure functional limit theorem. We want to give a good coupling argument that enables us to prove this for a large class of processes $\xi_n(\omega)$. For this aim we define a Property A. We prove that if Property A holds for a pair of sequences or random variables $(\xi_n(\omega),\eta_n(\omega))$, $n=1,2,\dots$, and the sequence $\eta_n(\omega)$, $n=1,2,\dots$, satisfies the almost sure functional limit theorem, then the sequence $\xi_n(\omega)$, $n=1,2,\dots$ also satisfies the almost sure functional limit theorem with the same norming constants and limit law.

Definition of Property A. *Let $\eta_n(\omega)$, $n=1,2,\dots$, be a sequence of random variables which satisfies the almost sure functional limit theorem with a limit measure μ_0 in the space $C([0,1])$ or $D([0,1])$ and some weight functions A_n and B_n satisfying relation (2.10). Let us also assume that the limit measure μ_0 is the distribution of the restriction of a self-similar process $X(u,\omega)$ with self-similarity parameter $\alpha>0$ to the interval*

$0 \le u \le 1$, *and the weight functions* A_n *and* B_n *are such that* $A_n = B_n^{1/\alpha}$.

Define the indices $N(n)$ *as* $N(n) = \inf\{k\colon B_k \ge 2^n\}$, $n = 0, 1, \dots$. *The pairs of sequences of random variables* $(\xi_n(\omega), \eta_n(\omega))$, $n = 1, 2, \dots$, *satisfy Property A if for all* $\varepsilon > 0$ *and* $\delta > 0$ *there exists a sequence of random variables* $\tilde\xi_n(\omega) = \tilde\xi_n(\varepsilon, \delta, \omega)$, $n = 1, 2, \dots$, *whose (joint) distribution agrees with the (joint) distribution of the sequence* $\xi_n(\omega)$, $n = 1, 2, \dots$, *and the partial sums* $\tilde S_n(\omega) = \sum\limits_{k=1}^{n} \tilde\xi_k(\omega)$ *and* $T_n(\omega) = \sum\limits_{k=1}^{n} \eta_k(\omega)$ *satisfy the following relation:*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{N(n)} \log \frac{B_{k+1}}{B_k} I\left(\left\{\frac{\sup\limits_{0\le j\le k} |\tilde S_j(\omega) - T_j(\omega)|}{A_k} > \varepsilon\right\}\right) < \delta \tag{2.13}$$

for almost all $\omega \in \Omega$, *where* $I(A)$ *denotes the indicator function of the set* A.

Remark: Let us remark that the joint distribution of the random variables $\xi_n(\omega)$, $n = 1, 2, \dots$, determines whether it satisfies the almost sure invariance principle. It is not important how and on which probability space these random variables are constructed. This can be seen for instance by applying the following "canonical representation" of the sequence $\xi_n(\omega)$, $n = 1, 2, \dots$, on the probability space $(\Omega, \mathcal{A}, P)$. Define the space $(R^\infty, \mathcal{B}^\infty, \bar\mu)$, where $R^\infty = \{(x_1, x_2, \dots)\colon x_j \in R,\ j = 1, 2, \dots\}$, $\mathcal{B}^\infty$ is the Borel σ-algebra on R^∞, $\bar\mu(\mathbf{B}) = P((\xi_1, \xi_2, \dots) \in \mathbf{B})$ for $\mathbf{B} \in \mathcal{B}^\infty$, and define the random variables $\bar\xi_n(x_1, x_2, \dots) = x_n$, $n = 1, 2, \dots$, on this space. Then the random variables $\bar\xi_n$ on the space $(R^\infty, \mathcal{B}^\infty, \bar\mu)$ have the same joint distribution as the random variables $\xi_n(\omega)$, and these two sequences satisfy the almost sure invariance principle simultaneously.

Theorem 4. *Let* $\eta_n(\omega)$, $n = 1, 2, \dots$, *be a sequence of random variables which satisfies the almost sure functional limit theorem, and let a pair of sequences of random variables* $(\xi_n(\omega), \eta_n(\omega))$, $n = 1, 2, \dots$, *satisfy Property A. Then the sequence of random variables* $\xi_n(\omega)$, $n = 1, 2, \dots$, *also satisfies the almost sure functional limit theorem with the same weight functions* A_n *and* B_n *and limit measure* μ_0 *as the sequence of random variables* $\eta_n(\omega)$.

We shall prove in Part II. of this work that Theorem 4 is applicable in several interesting cases. We shall prove with the help of a Basic Lemma formulated there that when partial sums of independent random variables are considered, then an appropriate construction satisfies the conditions of Theorem 4 under general conditions. In such a way it will turn out that the necessary and sufficient conditions of limit theorems for normalized partial sums of independent random variables are also sufficient conditions for the almost sure functional limit theorem.

We shall prove still another result which states that a small perturbation of the weight functions B_n does not affect the validity of the almost sure functional limit theorem. The reason to prove such a result is the following. We have certain freedom in the choice of the weight-functions B_n, and there are cases when no "most natural choice" of the weight functions exists. We want to show that different natural choices yield equivalent results. Let us remark that a modification of the weight-functions B_n also implies a modification of the random broken lines $S_n(t,\omega)$ appearing in the definition of the almost sure functional limit theorem.

Theorem 5. *Let a sequence of random variables $\xi_n(\omega)$, $n=1,2,\dots$, satisfy the almost sure functional limit theorem with some limit measure μ_0 and weight functions B_n, $A_n = B_n^{1/\alpha}$ with some $\alpha>0$, $n=0,1,\dots$, which satisfies relation (2.11). Let us also assume that a process $X(\cdot,\omega)$ in the space $D([0,1])$ whose distribution is the limit measure μ_0 satisfies condition (2.5). Let $\bar B_n$, $n=0,1,\dots$, $\bar B_0=1$, be another monotone increasing sequence such that $\lim\limits_{n\to\infty}\dfrac{\bar B_n}{B_n}=1$. Put $\bar A_n=\bar B_n^{1/\alpha}$. Then the sequence of random variables $\xi_n(\omega)$ also satisfies the almost sure functional limit theorem with the limit measure μ_0 and weight functions $\bar B_n$ and $\bar A_n$.*

We shall prove Theorem 5 with the help of the following Theorem 5A.*

Theorem 5A. *Let the conditions of Theorem 5 be satisfied. Define the partial sums $S_n(\omega)=\sum\limits_{k=1}^{n}\xi_k(\omega)$, $n=1,2,\dots$, and the random broken lines $S_k(s,\omega)$ and $\bar S_k(s,\omega)$, $0\le s\le 1$, $k=1,2,\dots$, by formula (2.11) with the help of the constants B_n, $A_n=B_n^{1/\alpha}$ and $\bar B_n$, $\bar A_n=\bar B_n^{1/\alpha}$ respectively. Let us also define the random measures $\hat\mu_N(\omega)$, $N=1,2,\dots$, on the product space $D([0,1])\times D([0,1])$ for all $\omega\in\Omega$ by the formula $\hat\mu_N(\omega)(S_k(\cdot,\omega),\bar S_k(\omega))=\dfrac{1}{\log\dfrac{B_N}{B_1}}\log\dfrac{B_{k+1}}{B_k}$, $1\le k\le N$. For almost all $\omega\in\Omega$ and all $\delta>0$ the relation $\lim\limits_{N\to\infty}\hat\mu_N(\omega)\{(x,y)\colon x,y\in D([0,1]),\ d(x,y)\ge\delta\}=0$ holds, where $d(\cdot,\cdot)$ is the (complete) metric introduced to define the topology in the space $D([0,1])$.*

Remark: Actually the proof of Theorem 5 yields a little bit more than the result formulated there. It shows that under the conditions of Theorem 5 the sequence of probability measures defined by formulas (2.11) and (2.12) have the same weak limit for almost all $\omega\in\Omega$ as the original one if the random broken lines $S_k(s,\omega)$ are replaced by $\bar S_k(s,\omega)$ or the weight functions $\dfrac{1}{\log\dfrac{B_N}{B_1}}\log\dfrac{B_{k+1}}{B_k}$ are replaced by $\dfrac{1}{\log\dfrac{\bar B_N}{\bar B_1}}\log\dfrac{\bar B_{k+1}}{\bar B_k}$ in formula (2.12) or if both replacements are made. Moreover, these statements hold if the condition $\lim\limits_{n\to\infty}\dfrac{\bar B_n}{B_n}=1$ in Theorem 5 is replaced by the weaker condition $\bar B_n=B_nL(B_n)$, where $L(\cdot)$ is a slowly varying function at infinity.

* In the first version of this paper the proof of Theorem 5 was incomplete. Unfortunately, I have observed this only after the appearance of the paper. This problem is settled in this version by the insertion and proof of Theorem 5A.

3. Proof of the results

Proof of Theorem 1. We can write

$$Z(t+T,\omega)=\frac{X(e^{t+T},\omega)}{e^{(t+T)/\alpha}}\overset{\Delta}{=}\frac{X(e^{t},\omega)}{e^{(t+T)/\alpha}e^{-T/\alpha}}=\frac{X(e^{t},\omega)}{e^{t/\alpha}}=Z(t,\omega)$$

for all $-\infty<T<\infty$. Hence the process $Z(t,\omega)$, $-\infty<t<\infty$, is stationary. If it is not only stationary, but also ergodic, then the ergodic theorem can be applied for the process $Z(\cdot,\omega)$ and all bounded and measurable functionals $\mathcal{G}$ on the space $(R^{(-\infty,\infty)},\mathcal{B},\mu)$, where $R^{(-\infty,\infty)}$ is the space of functions on the interval $(-\infty,\infty)$, $\mathcal{B}_0$ is the σ-algebra induced by the usual Borel (product) topology on $R^{(-\infty,\infty)}$, μ is the distribution of the process $Z(\cdot,\omega)$ on the space $(R^{(-\infty,\infty)},\mathcal{B}_0)$, and $\mathcal{B}$ is the closure of the σ-algebra $\mathcal{B}_0$ with respect to the measure μ. This means that $\mathbf{B}\in\mathcal{B}$ if and only if there exists some $\mathbf{B}_0\in\mathcal{B}_0$ such that $\mu(\mathbf{B}_0\Delta\mathbf{B})=0$ for the symmetric difference $\mathbf{B}_0\Delta\mathbf{B}$, or more precisely there is a $\mathcal{B}_0$ measurable set $\mathbf{C}$ such that $\mu(\mathbf{C})=0$ and $\mathbf{B}_0\Delta\mathbf{B}\subset\mathbf{C}$. Furthermore, we introduce the shift operators $\mathbf{T}_s$ defined by the formula $\mathbf{T}_s(z(\cdot))=z(s+\cdot)$ for all $z(\cdot)\in R^{(-\infty,\infty)}$ and put $Z_s(v,\omega)=Z(s+v,\omega)$, $-\infty<v<\infty$. Then the ergodic theorem implies that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{\log T}\int_0^{\log T}\mathcal{G}(\mathbf{T}_s(Z(u,\omega))\,ds&=\lim_{T\to\infty}\frac{1}{\log T}\int_0^{\log T}\mathcal{G}(Z_s(u,\omega))\,ds\\&=E\mathcal{G}(Z(u,\omega))\quad\text{for almost all }\omega\in\Omega.\end{aligned}\tag{3.1}$$

Given a bounded measurable functional* $\mathcal{F}$ on the space $C([0,1])$ or $D([0,1])$ let us extend it to the space of all measurable functions on the space $R^{[0,1]}$ of all functions on the interval $[0,1]$ by defining $\mathcal{F}(x)=0$ if the function $x=x(\cdot)$ is not in the space $C([0,1])$ or $D([0,1])$. Then we define the functional $\mathcal{G}=\mathcal{G}(\mathcal{F})$ on the space $R^{(-\infty,\infty)}$ by the formula $\mathcal{G}(z)=\mathcal{F}(x_z)$ with $x_z(u)=u^{1/\alpha}z(\log u)$, $0<u\le1$, $z(0)=0$. We can write

$$\frac{1}{\log T}\int_1^T\frac{1}{t}\mathcal{F}(X_t(\cdot,\omega))\,dt=\frac{1}{\log T}\int_0^{\log T}\mathcal{F}(X_{e^s}(\cdot,\omega))\,ds=\frac{1}{\log T}\int_0^{\log T}\mathcal{G}(Z_s(\cdot,\omega))\,ds,$$

* Yurij Davydov suggested to show that the procedure we follow is legitime also from measure theoretical point of view, i.e. no problem arises because $\mathcal{F}$ is defined in the space $C([0,1])$ or $D([0,1])$, and we work in the space of all functions in the interval [0,1] with the usual product topology and σ-algebra. This procedure can be justified for instance by means of the results proved in Billingsley's book [1], in the discussion after Theorem 8.3 and in Theorem 14.5. These results state that the σ-algebra defined by the usual topology in the spaces $C([0,1])$ and $D([0,1])$ agrees with the restriction of the usual σ-algebra in the space of all functions on the interval [0,1] to these spaces. We give a more detailed explanation in Appendix 2.

since $\mathcal{G}(Z_s(\cdot,\omega))=\mathcal{F}(X_{e^s}(\cdot,\omega))$. Indeed,

$$\begin{aligned}x_{Z_s(\cdot,\omega)}(u)&=u^{1/\alpha}Z_s(\log u,\omega)=u^{1/\alpha}Z(s+\log u,\omega)=u^{1/\alpha}\frac{X(e^{s+\log u},\omega)}{e^{(s+\log u)/\alpha}}\\&=\frac{X(ue^s,\omega)}{e^{s/\alpha}},\quad \text{for all } 0\le u\le 1,\end{aligned}$$

hence $x_{Z_s(\cdot,\omega)}=X_{e^s}(\cdot,\omega)$, where $X_s(\cdot,\omega)$ was defined in (2.4). This relation (with the choice $s=0$) implies in particular that

$$E\mathcal{G}(Z(\cdot,\omega))=E\mathcal{G}(Z_0(\cdot,\omega))=E\mathcal{F}(X_1(\cdot,\omega)).$$

These identities together with relation (3.1) and the definition of the measures $\mu_T(\omega)$ introduced in the formulation of Theorem 1 imply that

$$\begin{aligned}\lim_{T\to\infty}\int\mathcal{F}(x)\,d\mu_T(\omega)(x)&=\lim_{T\to\infty}\frac{1}{\log T}\int_1^T\frac{1}{t}\mathcal{F}(X_t(\cdot,\omega))\,dt\\&=\lim_{T\to\infty}\frac{1}{\log T}\int_0^{\log T}\mathcal{G}(Z_s(\cdot,\omega))\,ds=E\mathcal{G}(Z(\cdot,\omega))\\&=E\mathcal{F}(X_1(\cdot,\omega))=\int\mathcal{F}(x)\,d\mu_0(x)\quad\text{for almost all }\omega\in\Omega.\end{aligned}\tag{3.2}$$

To prove Theorem 1 we have to show that relation (3.2) holds simultaneously for all bounded and continuous functionals $\mathcal{F}$ for almost all $\omega\in\Omega$, and the exceptional set of $\omega\in\Omega$ of measure zero should not depend on the functional $\mathcal{F}$. We prove this* with the help of the following

Lemma A. *Under the conditions of Theorem 1 the closure of the set of (random) measures $\mu_T(\omega)$, $T\ge 1$, are compact in the topology defining weak convergence of probability measures in the space $C([0,1])$ or $D([0,1])$ (depending on where the distribution of the process $X(\cdot,\omega)$ is defined) for almost all $\omega\in\Omega$.*

Proof of Lemma A: We apply the result that a set of probability measures μ_T on a separable complete metric space (endowed with the topology inducing weak convergence) is compact if and only if for all $\varepsilon>0$ there is a compact set $\mathbf{K}=\mathbf{K}(\varepsilon)$ on the metric space such that $\mu_T(\mathbf{K})\ge 1-\varepsilon$ for all measures μ_T. Both spaces $C([0,1])$ and $D(0,1])$ can be endowed with a metric which turns them to a separable complete metric space.

* After having published this work I learned about the still unpublished paper "On almost sure type limit theorems" (in Russian) of I. A. Ibragimov and M. A. Lifshitz, where a much simpler proof of this statement is presented. I added an Appendix to this paper which supplies their proof. This can replace the remaining part of the proof of Theorem 1 except the last part, where the ergodicity of certain processes is proved.

(See e.g. Theorems 6.1 and 6.2, 14.1) in Billingsley's book [1].) Because of these results the following statement has to be proved. For almost all $\omega \in \Omega$ and all $\varepsilon > 0$ there exists a compact set $\mathbf{K} = \mathbf{K}(\varepsilon, \omega)$ in the space $C([0,1])$ or $D([0,1])$ such that $\mu_T(\omega)(\mathbf{K}) \ge 1-\varepsilon$ for all $T \ge 1$. In the proof we shall apply formula (3.2) which is valid for all bounded and measurable functionals $\mathcal{F}$ and some classical results which describe the compact sets in $C([0,1])$ and $D([0,1])$. These results can be found for instance in the book of Billingsley [1]. (Theorem 8.2 gives a description of compact sets in $C([0,1])$ and Theorem 14.4 a description of compact sets in $D([0,1])$.)

Let us first consider the case when the distribution of the processes $X_T(\cdot, \omega)$ defined in formula (2.4) are in the $C([0,1])$ space. We shall prove that for almost all $\omega \in \Omega$ and all $\varepsilon > 0$ and $\eta > 0$ there exist some numbers $K = K(\varepsilon, \omega)$ and $\delta = \delta(\varepsilon, \eta, \omega) > 0$ such that

$$\mu_T(\omega)\left(x \in C([0,1])\colon \sup_{0\le u\le 1} |x(u)| \ge K\right) \le \varepsilon,$$

and (3.3)

$$\mu_T(\omega)\left(x \in C([0,1])\colon |w_x(\delta)| \ge \eta\right) \le \varepsilon,$$

for all $T \ge 1$, where $w_x(\delta) = \sup\limits_{|t-s|\le\delta} |x(t)-x(s)|$ for a function $x \in C([0,1])$. First we show that relation (3.3) implies that for almost all $\omega \in \Omega$ and all $T \ge 1$ and $\varepsilon > 0$ there exists a compact set $\mathbf{K}(\varepsilon) = \mathbf{K}(\varepsilon, \omega) \subset C([0,1])$ for which $\mu_T(\omega)(\mathbf{K}(\varepsilon)) \ge 1-\varepsilon$. Indeed, let us fix some $\varepsilon > 0$, and consider the sets

$$\mathbf{J}_0 = \left(x \in C([0,1])\colon \sup_{0\le u\le 1} |x(u)| > K\right)$$

and

$$\mathbf{J}_n = \left(x \in C([0,1])\colon |w_x(\delta_n)| > 2^{-n}\varepsilon\right), \quad n = 1, 2, \dots$$

with such constants $K = K(\varepsilon, \omega)$ and $\delta_n = \delta_n(\varepsilon, \omega)$ for which $\mu_T(\omega)(\mathbf{J}_n) \le \varepsilon 2^{-n-1}$, $n = 0, 1, \dots$, $T \ge 1$. Such sets $\mathbf{J}_n$ really exist because of relation (3.3). (The numbers K and δ_n in the definition of the sets $\mathbf{J}_n$ and thus the sets $\mathbf{J}_n$ may depend on ω.) Define the set $\mathbf{K}(\varepsilon) = \bigcap\limits_{n=0}^{\infty} \bar{\mathbf{J}}_n$, where $\bar{\mathbf{J}}$ is the complement of the set $\mathbf{J}$. Then $\mathbf{K}(\varepsilon)$ is a compact set in $C([0,1])$, and for almost all ω and $T \ge 1$ $\mu_T(\omega)(\mathbf{K}(\varepsilon)) \ge 1-\varepsilon$. Applying this result for all $\varepsilon_n = 2^{-n}$, $n = 1, 2, \dots$, we get a set of $\bar{\Omega}$ of probability one, such that for all $\omega \in \bar{\Omega}$, $T \ge 1$ and $\varepsilon > 0$ there exists a compact set $\mathbf{K}(\varepsilon) = \mathbf{K}(\varepsilon, \omega)$ such that $\mu_T(\omega)(\mathbf{K}(\varepsilon)) \ge 1-\varepsilon$. In such a way we reduced the proof of Lemma A in the case of continuous trajectories $X(\cdot, \omega)$ to the proof of relation (3.3).

To prove formula (3.3) we shall apply relation (3.2) with appropriate functionals $\mathcal{F}_1$ and $\mathcal{F}_2$ on the space $C([0,1])$. Put

$$\mathcal{F}_1(x) = \mathcal{F}_{1,K}(x) = I\left(\sup_{0\le u\le 1} |x(u)| \ge K\right)$$

and

$$\mathcal{F}_2(x)=\mathcal{F}_{2,\delta,\eta}(x)=I\left(\sup_{s,t\in[0,1]\colon\,|t-s|\le\delta}|x(s)-x(t)|\ge\eta\right)$$

with appropriate constants $K>0$, $\eta>0$ and $\delta>0$. For fixed $\varepsilon>0$ and $\eta>0$ the constants $K=K(\varepsilon)>0$ and $\delta=\delta(\varepsilon,\eta)>0$ can be chosen in such a way that $E\mathcal{F}_1(X_1(\cdot,\omega))<\varepsilon^2$ and $E\mathcal{F}_2(X_1(\cdot,\omega))<\varepsilon^2$. Then, because of formula (3.2) for almost all $\omega\in\Omega$ there exists such a threshold $T_0=T_0(\omega)$ for which $\int\mathcal{F}_i(x)\,d\mu_T(\omega)(x)\le\varepsilon$ for all $T\ge T_0(\omega)$ and $i=1,2$. Since $\mathcal{F}_i(x)=0$ or $\mathcal{F}_i(x)=1$, $i=1,2$, this relation implies that $\mu_T(\omega)(x\colon\mathcal{F}_i(x)\ne0)\le\varepsilon$, for $T\ge T_0(\omega)$, $i=1,2$. This means that relation (3.3) holds for $T\ge T_0(\omega)$. Furthermore, since $X_{aT}(u,\omega)=a^{-1/\alpha}X_T\,(au,\omega)$ for all $0<a\le1$,

$$\mu_t(\omega)\left(x\colon\sup_{0\le u\le1}|x(u)|\ge K\right)\le\mu_{T_0(\omega)}(\omega)\left(x\colon\sup_{0\le u\le1}|x(u)|\ge KT_0(\omega)^{-1/\alpha}\right),$$

and

$$\begin{aligned}&\mu_t(\omega)\left(x\in C([0,1])\colon|w_x(\delta)|\ge\eta\right)\\&\qquad\le\mu_{T_0(\omega)}(\omega)\left(x\in C([0,1])\colon|w_x(\delta T_0(\omega))|\ge\eta T_0(\omega)^{-1/\alpha}\right).\end{aligned}$$

if $1\le t\le T_0(\omega)$. These probabilities can be taken small by choosing a sufficiently large $K>0$ and sufficiently small $\delta>0$ which depend only on $T_0(\omega)$. Hence relation (3.3) holds not only for $T\ge T_0(\omega)$ but also for all $T\ge1$ with a possible modification of the constants $\delta(\varepsilon,\eta,\omega)$ and $K(\omega)$ in it.

The proof in the case when the processes $X_T(\cdot,\omega)$ defined in (2.4) take their values in the space $D([0,1])$ is similar, hence we only indicate the necessary modifications. Because of the description of compact sets in the space $D([0,1])$ found for instance in Theorem (14.4) in Billingsley's book [1]) we can reduce the proof of Lemma A in this case, by a natural modification of the argument presented after the formulation of formula (3.3), to the following modified version of relation (3.3): For all $\varepsilon>0$ and $\eta>0$ there exist some $K>0$ and $\delta>0$ such that

$$\begin{aligned}&\mu_T(\omega)\left(x\in D([0,1])\colon\sup_{0\le u\le1}|x(u)|\ge K\right)\le\varepsilon,\\&\mu_T(\omega)\left(x\in D([0,1])\colon|w''_x(\delta)|\ge\eta\right)\le\varepsilon,\\&\mu_T(\omega)(x\in D([0,1])\colon w_x[0,\delta)\ge\eta)\le\varepsilon\\&\mu_T(\omega)(x\in D([0,1])\colon w_x[1-\delta,1)\ge\eta)\le\varepsilon\end{aligned}\tag{3.3'}$$

for all $T\ge1$, where

$$w''_x(\delta)=\sup_{0\le t_1\le t\le t_2,|t_2-t_1|\le\delta}\min\{|x(t)-x(t_1)|,|x(t_2)-x(t)|\},$$

and $w_x[a,b)=\sup\limits_{a\le s,t<b}|x(t)-x(s)|$ for all numbers $0\le a<b\le 1$.

The proof of formula (3.3′) is similar to that of formula (3.3). Let us introduce the functionals

$$\mathcal{F}_1(x)=I\left(\sup_{0\le t\le 1}|x(t)|\ge K\right),\quad \mathcal{F}_2(x)=I\left(w''_x(\delta)\ge\eta\right),$$
$$\mathcal{F}_3(x)=I\left(w_x[0,\delta)\ge\eta\right)\quad\text{and } \mathcal{F}_4(x)=I\left(w_x[1-\delta,1)\ge\eta\right)$$

on the space $D([0,1])$, where the constants $K=K(\varepsilon)$ and $\delta=\delta(\varepsilon,\eta)$ will be appropriately chosen. Let us observe that with their appropriate choice we can achieve that $E\mathcal{F}_i(X(\cdot,\omega)\le\varepsilon^2$ for $i=1,2,3,4$. To see this it is enough to observe that for all $x\in D([0,1])$ $\sup\limits_{0\le t\le 1}|x(t)|<\infty$, $\lim\limits_{\delta\to0}w''_x(\delta)=0$ (see e.g. formulas (14.8) and (14.46) in Billingsley's book [1]), $\lim\limits_{\delta\to0}w_x[0,\delta)=0$ and $\lim\limits_{\delta\to0}w_x[1-\delta,1)=0$. These functionals $\mathcal{F}_i$ take values 0 and 1, and formula $(3,3')$ can be proved similarly to (3.3) with the help of relation (3.2). In such a way Lemma A is proved.

Now we turn back to the proof of Theorem 1. We prove with the help of Lemma A, formula (3.2) and a compactness argument that for almost all $\omega\in\Omega$ the sequence of measures $\mu_T(\omega)$ converges weakly to μ_0 as $T\to\infty$. First we show that for all $\varepsilon_0>0$ and $\varepsilon>0$ there exists a set $\Omega_0=\Omega_0(\varepsilon_0,\varepsilon)\subset\Omega$ and a compact set $\mathbf{K}=\mathbf{K}(\varepsilon_0,\varepsilon)$ in $C([0,1])$ or $D([0,1])$ such that $P(\Omega_0)\ge1-\varepsilon_0$ and $\mu_T(\omega)(\mathbf{K})\ge1-\varepsilon$ for all $\omega\in\Omega_0$ and $T\ge1$. This can be deduced from formulas (3.3) in the space $C([0,1])$ and from formula (3.3′) in the space $D([0,1])$ by an argument similar to the proof of the compactness of the measures $\mu_T(\omega)$ by means of these relations. Thus for instance in the space $C([0,1])$ we define the sets $\mathbf{J}_n$, $n=1,2,\dots$, and $\mathbf{K}=\mathbf{K}(\varepsilon)$ similarly to the definition given after formula (3.3) with the only difference that in this case the numbers K and δ_n appearing in the definition of the sets $\mathbf{J}_n$ are chosen independently of ω in such a way that $P(\{\omega\colon \mu_T(\omega)(\mathbf{J}_n)\le\varepsilon2^{-n-1}\text{ for all }T\ge1\})\ge1-\varepsilon_02^{-n-1}$. The argument in the case of the $D([0,1])$ space with the help of relation (3.3′) is similar.

For a large number $L>0$ let $\mathbf{F}(L)$ denote the class of continuous and bounded functionals $\mathcal{F}$ on the space $C([0,1])$ or $D([0,1])$ such that $|\mathcal{F}(x)|\le L$ for all $x\in C([0,1])$ or $x\in D([0,1])$. Fix an $\varepsilon_0>0$ and $\varepsilon>0$, and choose a set $\Omega_0\subset\Omega$ and a compact set $\mathbf{K}=\mathbf{K}(\varepsilon_0,\varepsilon,L)$ in such a way that $P(\Omega_0)\ge1-\varepsilon_0$ and $\mu_T(\omega)(\mathbf{K})\ge1-\dfrac{\varepsilon}{L}$ for all $\omega\in\Omega_0$ and $T\ge1$. Fix two small numbers $\eta>0$ and $\delta>0$, and let the set $\mathbf{F}(L,\varepsilon_0,\varepsilon,\eta,\delta)\subset\mathbf{F}(L)$ consist of those functionals $\mathcal{F}\in\mathbf{F}(L)$ for which $\sup\limits_{x,y\in\mathbf{K},\,\rho(x,y)\le\delta}|\mathcal{F}(x)-\mathcal{F}(y)|\le\eta$. For all $\delta>0$ fix a finite δ-net in the compact set $\mathbf{K}$ corresponding to it, i.e. a finite set $\mathbf{J}_\delta=\{x_1,\dots,x_r\}\subset\mathbf{K}$ such that for all $x\in\mathbf{K}$ $\min\limits_{1\le s\le r}\rho(x,x_s)\le\delta$. Such a δ-net really exists because of the compactness of the set $\mathbf{K}$.

Consider the above fixed numbers $\varepsilon_0>0$, $\varepsilon>0$ and $L>0$, together with the sets Ω_0 and $\mathbf{K}$ corresponding to them. First we show that there exists an $\Omega_0'\subset\Omega_0$ such that

$P(\Omega_0 \setminus \Omega_0') = 0$, and

$$\limsup_{T\to\infty} \left| \int \mathcal{F}(x)\mu_T(\omega)(\,dx) - \int \mathcal{F}(x)\mu_0(\,dx) \right| < \varepsilon \qquad \text{for all } \mathcal{F} \in \mathbf{F}(L) \quad \text{and } \omega \in \Omega_0'. \tag{3.4}$$

To prove relation (3.4) let us first observe that because of the uniform continuity of the functionals $\mathcal{F} \in \mathbf{F}_L$ on the compact set $\mathbf{K}$ the relation

$$\bigcup_{n=1}^{\infty} \mathbf{F}\left(L, \varepsilon_0, \varepsilon, \eta, \frac{1}{n}\right) = \mathbf{F}(L) \tag{3.5}$$

holds for all fixed $\varepsilon_0 > 0$, $\varepsilon > 0$, $\eta > 0$ and $L > 0$.

Put $\delta = \frac{1}{n}$, consider the $\frac{1}{n}$-net $\mathbf{J}_{1/n} = \{x_1, \dots, x_r\}$ corresponding to it, and make a partition of the set $\mathbf{F}\left(L, \varepsilon_0, \varepsilon, \eta, \frac{1}{n}\right)$ into subclasses $\mathbf{F}\left(L, \varepsilon_0, \varepsilon, \eta, \frac{1}{n}, j(1), \dots, j(r)\right)$ with integers $|j(s)| \le (L+1)\eta^{-1}$, $s = 1, \dots, r$, which consist of those functionals $\mathcal{F} \in \mathbf{F}\left(L, \varepsilon_0, \varepsilon, \eta, \frac{1}{n}\right)$ for which $\mathcal{F}(x_s) \in [j_s\eta, (j_s+1)\eta)$, $s = 1, \dots, r$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ belong to the same subclass $\mathbf{F}(L, \varepsilon_0, \varepsilon, \eta, \frac{1}{n}, j(1), \dots, j(r))$, then $|\mathcal{F}_1(x) - \mathcal{F}_2(x)| < 2\eta$ for all $x \in \mathbf{K}$ because of the module of continuity of these functionals on the set $\mathbf{K}$, and because of the relation $\mu_T(\omega)(\mathbf{K}) \ge 1 - \dfrac{\varepsilon}{L}$ for all $\omega \in \Omega_0$, $\left|\int \mathcal{F}_1(x)\mu_T(\omega)(\,dx) - \mathcal{F}_2(x)\mu_T(\omega)(\,dx)\right| < \varepsilon + 2\eta$.

Let us choose an arbitrary functional $\mathcal{F}$ from all non-empty sets

$$\mathbf{F}\left(L, \varepsilon_0, \varepsilon, \eta, \frac{1}{n}, j(1), \dots, j(r)\right).$$

We get by applying formula (3.2) for these functionals $\mathcal{F}$ and the previous estimation a weakened version of relation (3.4) on a set $\omega \in \Omega_0''(n) \subset \Omega_0$ such that $P(\Omega_0 \setminus \Omega_0''(n)) = 0$, where $\mathbf{F}(L)$ is replaced by $\mathbf{F}\left(L, \varepsilon_0, \varepsilon.\eta, \frac{1}{n}\right)$, and the upper bound ε by $\varepsilon + 2\eta$. Then we get, by applying this relation for all $n = 1, 2, \dots$ together with relation (3.5) the weakened version of (3.4) for all $\omega \in \bigcap\limits_{n=1}^{\infty} \Omega_0''(n)$ and $\mathcal{F} \in \mathbf{F}(L)$ with upper bound $\varepsilon + 2\eta$ instead of ε. Finally, we get formula (3.4) in its original form by letting $\eta \to 0$.

It is not difficult to see that relation (3.4) implies the weak convergence $\mu_T(\omega)$ to μ_0 for almost all $\omega \in \Omega$. Indeed, let us fix a number $L > 0$ and $\varepsilon > 0$. Then we get, by applying relation (3.4) for all $\varepsilon_0(n) = n^{-1}$, $n = 1, 2, \dots$ that there exists a set $\Omega_0(n)$, $P(\Omega_0(n)) = 1 - \dfrac{1}{n}$, such that relation (3.4) holds for all $\omega \in \Omega_0(n)$. This implies that relation (3.4) holds for all $\omega \in \bar\Omega = \bigcup\limits_{n=1}^{\infty} \Omega_0(n)$, i.e. on a set of probability 1. Then, since relation (3.4) holds for all $L > 0$ and $\varepsilon > 0$ with probability 1 we get by letting $L \to \infty$ and $\varepsilon \to 0$ in this relation that the sequences of measures $\mu_T(\omega)$ converge weakly to the measure μ_0 for almost all $\omega \in \Omega$.

To complete the proof of Theorem 1 still we have to show that in the case of a Wiener or a stable process the generalized Ornstein–Uhlenbeck process corresponding to it is ergodic. This follows from a natural modification of the zero–one law for sums of independent identically distributed random variables to processes with independent and stationary increments which can be found for instance in Feller's book [4], Chapter 4, Section 7, Theorem 3. The continuous time version of this result which can be proved similarly, also holds. It states that if $X(t)$, $t \ge 0$, is a stable process with some parameter α, $0 < \alpha \le 2$, and a set $\mathbf{A}$ is measurable with respect to the (tail) σ-algebra $\mathcal{F}$ which is the intersection $\mathcal{F} = \bigcap\limits_{T>0} \mathcal{F}_T$, where $\mathcal{F}_T = \sigma\{X(t,\cdot)\colon t \ge T\}$, then $\mathbf{A}$ has probability zero or one. The same result holds if the set $\mathbf{A}$ is measurable with respect to the σ-algebra $\bigcap\limits_{T>0} \mathcal{F}'_T$, where $\mathcal{F}'_T = \sigma\{X(t,\cdot)\colon t \le T\}$. (This result follows for instance from the observation that $t^{-2/\alpha} X\left(\frac{1}{t},\omega\right)$ is also a stable process. These relations are equivalent to the statement that the generalized Ornstein–Uhlenbeck process $Z(t)$ corresponding to this stable process has trivial σ-algebra at infinity and minus infinity, i.e. all sets which are measurable with respect to the σ-algebra generated by the random variables $t \ge T$ (or $t \le T$) for all $-\infty < T < \infty$ have probability zero or one. This is a property which is actually stronger than the ergodicity of the process.

Proof of Theorem 2: Theorem 2 will be proved by means of formula (3.2) with an appropriately defined functional $\mathcal{F}$ in the space $C([0,1])$ or $D([0,1])$. Let us define the functional $\mathcal{F} = \mathcal{F}_{\varepsilon,\delta}$ with some $\varepsilon > 0$ and $\delta > 0$ as

$$\mathcal{F}_{\varepsilon,\delta}(x) = I\left(\sup_{1-\varepsilon\le s,t\le 1} \rho(x_s(\cdot), x_t(\cdot)) \ge \delta\right),$$

where the function x_t is defined in (2.4′), and $\rho(\cdot,\cdot)$ is the metric introduced in Section 2. We claim that under the conditions of Theorem 2

$$\lim_{\varepsilon\to 0} E\mathcal{F}_{\varepsilon,\delta}(X_1(\cdot,\omega)) = 0 \tag{3.6}$$

for all $\delta > 0$.

Let us also observe that by relation (3.2)

$$\lim_{T\to\infty} \mu_T(\omega)\left(\sup_{1-\varepsilon\le s,t\le 1} \rho(x_s, x_t) > \delta\right) = \lim_{T\to\infty} \int \mathcal{F}_{\varepsilon,\delta}(x)\, d\mu_T(\omega)(x) = E\mathcal{F}_{\varepsilon,\delta}(X_1(\cdot,\omega))$$

for all $\varepsilon > 0$ and $\delta > 0$ and almost all ω, where the function x_t was defined in formula (2.4′). Then we get relation (2.6) with the help of formula (3.6), by letting $\varepsilon \to 0$ in the last formula. Hence to prove relation (2.6) it is enough to prove formula (3.6).

If $X_1(\cdot,\omega) \in C([0,1])$, then this relation follows from the observation that for all $\eta > 0$ there is a compact set $\mathbf{K}_\eta$ in $C([0,1])$ such that $P(X_1(\cdot,\omega) \in \mathbf{K}_\eta) \ge 1-\eta$, and for all $\delta > 0$ there exists an $\varepsilon = \varepsilon(\eta) > 0$ such that $|x(u)-x(v)| < \delta$ if $x \in \mathbf{K}_\eta$, and $|u-v| \le \varepsilon$. There is also a constant $L > 0$ such that $\sup\limits_{x\in\mathbf{K}_\eta} |x(u)| \le L$. Since these relations hold for all $\delta > 0$ and appropriate $L > 0$ they imply that $\lim\limits_{\varepsilon\to 0} \sup\limits_{x\in\mathbf{K}_\eta, 1-\varepsilon\le t\le 1} \rho(x_t, x) = 0$. This means that for sufficiently small $\varepsilon > 0$ $\mathcal{F}_{\varepsilon,\delta}(X_1(\cdot,\omega)) = 0$ if $X_1(\cdot,\omega) \in \mathbf{K}_\eta$, i.e. in the case when an event of probability greater than $1-\eta$ occurs. Hence relation (3.6) holds in this case. The situation in the space $D([0,1])$ is more sophisticated. In this case formula (2.5) also has to be applied.

Since all functions $x(t)$ in the space $D([0,1])$ have a limit as $t \to 1-0$ it follows from relation (2.5) that for all $\delta > 0$

$$P\left(\lim_{\varepsilon\to 0} \sup_{1-\varepsilon\le t\le 1} |X(t,\omega) - X(1,\omega)| \ge \frac{\delta}{2}\right) = 0.$$

Hence there is a set $\mathbf{K} = \mathbf{K}_\eta$ in the space $D([0,1])$ such that $P(X_1(\cdot,\omega) \in \mathbf{K}) \ge 1-\eta$, the closure of the set $\mathbf{K}$ is compact, and for all $x \in \mathbf{K}$ $\lim\limits_{\varepsilon\to 0} \sup\limits_{1-\varepsilon\le t\le 1} |x_t - x| < \dfrac{\delta}{2}$, where the function x_t was defined in (2.4′). There is a finite $\dfrac{\delta}{5}$–net in $\mathbf{K}$, i.e. a finite set $\mathbf{J} = \{x^{(1)}, \dots, x^{(s)}\}$, $x^{(r)} \in \mathbf{K}$, $r = 1,\dots,s$, in such a way that for all $x \in \mathbf{K}$ there is some $x^{(r)} \in \mathbf{J}$ such that $\rho(x, x^{(r)}) \le \dfrac{\delta}{5}$. Then to prove formula (2.6) it is enough to show that for all $x^{(r)} \in \mathbf{J}$ there is some $\bar\varepsilon > 0$ such that $\rho(x_t^{(r)}, x^{(r)}) \le \dfrac{\delta}{4}$ for all $1-\bar\varepsilon \le t \le 1$. Indeed, if this statement holds, then for arbitrary $x \in \mathbf{K}$ there is some $x^{(r)} \in \mathbf{J}$ such that $\rho(x, x^{(r)}) \le \dfrac{\delta}{5}$. Then $\rho(x_s, x_t) \le \rho(x_s, x_s^{(r)}) + \rho(x_t, x_t^{(r)}) + \rho(x_s^{(r)}, x_t^{(r)})$. Let us also observe that because of the definition of the functions x_t for sufficiently small $\bar\varepsilon > 0$ for all $x \in D([0,1])$, $1-\bar\varepsilon \le t \le 1$ and $x^{(r)} \in \mathbf{J}$ the inequality $\rho(x_t, x_t^{(r)}) \le \dfrac{5}{4}\rho(x, x^{(r)})$ holds, and $\rho(x_s^{(r)}, x_t^{(r)}) \le \rho(x_s^{(r)}, x^{(r)}) + \rho(x_t^{(r)}, x^{(r)})$. The above inequalities imply that $\rho(x_s, x_t) \le \delta$ for $1-\bar\varepsilon \le s,t \le 1$ if $x \in \mathbf{K}$. Hence $\mathcal{F}_{\varepsilon,\delta}(X_1(\cdot,\omega)) = 0$ with $\varepsilon = \bar\varepsilon$ if $X_1(\cdot,\omega) \in \mathbf{K}$. Then formula (3.6) follows from the relation $P(X_1(\cdot,\omega) \in \mathbf{K}) \ge 1-\eta$.

Thus to complete the proof of formula (2.6) it is enough to show that for an arbitrary function $x \in D([0,1])$ such that $\lim\limits_{u\to 1-0} |x(u) - x(1)| < \dfrac{\delta}{2}$ the relation $\lim\limits_{\varepsilon\to 0} \rho(x_t, x) < \dfrac{\delta}{2}$ holds. (This relation means in particular that the limit exists.) To prove this relation let us define for all $\dfrac{1}{2} \le t < 1$ the mapping $\lambda_t(u)$ of the interval $[0,1]$ into itself as $\lambda_t(u) = tu$ for $0 \le u \le t^*(t)$ with $t^*(t) = 1 - \sqrt{1-t}$, and define $\lambda_t(u)$ in the remaining interval $(t^*(t), 1]$ also linearly, i.e. let $\lambda_t(u) = (\sqrt{1-t} + t)u + 1 - t - \sqrt{1-t}$ for $t^*(t) \le u \le 1$. Then $\lim\limits_{t\to 1} \sup\limits_{u\ne v} \log \left| \dfrac{\lambda_t(u) - \lambda_t(v)}{u-v} \right| = 0$. Because of the definition of the metric $\rho = d_0$ it is enough to show that

$$\lim_{t\to 1} \sup_{0\le u\le 1} |x_t(u) - x(\lambda_t(u))| = \lim_{u\to 1} |x(u) - x(1)| < \frac{\delta}{2}.$$

It is known that for an $x \in D([0,1])$ function $\sup\limits_{0\le u\le 1} |x(u)| < \infty$ (see e.g. Billingsley's book [1]). Hence

$$\sup_{0\le u\le t^*(t)} |x_t(u) - x(\lambda_t(u))| \le (t^{-1/\alpha} - 1) \sup_{0\le u\le 1} |x(u)|$$
$$\le \text{const.}\,(t^{-1/\alpha} - 1) \to 0 \quad \text{if } t \to 1-0.$$

Similarly, since a function $x \in D([0,1])$ has a right-hand side limit in the point 1, $\sup_{t^*(t)\le u<1} |x_t(u) - x(\lambda_t(u))| \to 0$ as $t \to 1-0$. Finally in the point $u = 1$ $\lambda_t(1) = 1$, and $\lim_{t\to 1-0} |x_t(1) - x(\lambda_t(1))| = \left| x(1) - \lim_{t\to 1-0} x(t) \right| < \dfrac{\delta}{2}$. These relations imply that $\lim_{t\to 1-0} \rho(x_t, x) = \lim_{t\to 1-0} |x(t) - x(1)| < \frac{\delta}{2}$. Theorem 2 is proved.

Proof of Lemma 1. We have to prove that for arbitrary $\delta > 0$

$$\lim_{\varepsilon\to 0} P\left(\sup_{1-\varepsilon\le t\le 1} |X(t,\omega) - X(1,\omega)| > \delta \right) = 0.$$

Because of the stationary increment and self-similarity property of the process $X(t,\omega)$ with parameter $\alpha > 0$ yields that

$$P\left(\sup_{1-\varepsilon\le t\le 1} |X(t,\omega) - X(1,\omega)| > \delta \right) = P\left(\sup_{0\le t\le \varepsilon} |X(t,\omega) - X(\varepsilon,\omega)| > \delta \right)$$
$$= P\left(\sup_{0\le t\le 1} |X(t,\omega) - X(1,\omega)| > \delta\varepsilon^{-1/\alpha} \right).$$

Then tending with $\varepsilon \to 0$ we get that $\delta\varepsilon^{-1/\alpha} \to \infty$, and the required property holds.

To prove Theorem 3 first we formulate and prove the following technical Lemma:

Lemma B. *Let $(M, \mathcal{M}, \rho)$ be a separable, complete metric space such that $\mathcal{M}$ is the σ-algebra generated by the open sets of this space. Let two sequences of probability measures μ_N and $\bar\mu_N$, $N = 1, 2, \dots$, be given on the space $(M, \mathcal{M}, \rho)$ such that the measures μ_N weakly converge to a probability measure μ_0 on $(M, \mathcal{M}, \rho)$ as $N \to \infty$, and*

$$\liminf_{N\to\infty} (\bar\mu_N(\mathbf{F}^\varepsilon) - \mu_N(\mathbf{F})) \ge 0 \quad \textit{for all closed sets } \mathbf{F} \in \mathcal{M} \textit{ and } \varepsilon > 0, \tag{3.7}$$

where $\mathbf{A}^\varepsilon = \{x\colon \rho(x, \mathbf{A}) < \varepsilon\}$ denotes the ε-neighborhood of a set $\mathbf{A} \in \mathcal{M}$. Then the measures $\bar\mu_N$ converge weakly to the same limit measure μ_0 as $N \to \infty$. Moreover, condition (3.7) can be slightly weakened. It is enough to assume that it holds for all compact sets $\mathbf{K} \in \mathcal{M}$ and $\varepsilon > 0$.

Proof of Lemma B. The weak convergence of the measures $\bar\mu_N$ to μ_0 as $N \to \infty$ is equivalent to the relation $\liminf_{N\to\infty} \bar\mu_N(\mathbf{G}) \ge \mu_0(\mathbf{G})$ for all open sets $\mathbf{G} \in \mathcal{M}$. For all open sets $\mathbf{G} \in \mathcal{M}$ and $\varepsilon > 0$ there exists a compact set $\mathbf{K} = \mathbf{K}_\varepsilon \in \mathcal{M}$ such that $\mathbf{K} \subset \mathbf{G}$ and $\mu_0(\mathbf{K}) \ge \mu_0(\mathbf{G}) - \varepsilon$. Then there exists some $\eta > 0$ such that also the η-neighborhood of $\mathbf{K}$ satisfies the relation $\mathbf{K}^\eta \subset \mathbf{G}$. Consider the $\eta/2$ neighborhood $\mathbf{K}^{\eta/2}$ of the set $\mathbf{K}$. Since $\mathbf{G}$ contains the $\eta/2$ neighborhood of the closure of $\mathbf{K}^{\eta/2}$, and the measures μ_N converge weakly to the measure μ_0 as $N \to \infty$ we can write with the help of relation (3.7) that $\liminf\limits_{N\to\infty} \bar\mu_N(\mathbf{G}) \ge \liminf\limits_{N\to\infty} \mu_N(\mathbf{K}^{\eta/2}) \ge \mu_0(\mathbf{K}^{\eta/2}) \ge \mu_0(\mathbf{G}) - \varepsilon$. Since the last relation holds for all $\varepsilon > 0$ and open sets $\mathbf{G}$, it implies the convergence of the measures $\bar\mu_N$ to μ_0 as $N \to \infty$.

To complete the proof of Lemma B let us observe that because of the compactness (convergence) of the measures μ_N in the weak convergence topology for all $\varepsilon > 0$ there is a compact set $\mathbf{K} \in \mathcal{M}$ such that $\mu_N(\mathbf{K}) > 1 - \varepsilon$ for all $N = 1, 2, \dots$. Then for a closed set $\mathbf{F} \in \mathcal{M}$ the set $\mathbf{F} \cap \mathbf{K}$ is also compact, and $\liminf\limits_{N\to\infty} (\bar\mu_N(\mathbf{F}^\varepsilon) - \mu_N(\mathbf{F})) \ge \liminf\limits_{N\to\infty} (\bar\mu_N((\mathbf{F} \cap \mathbf{K})^\varepsilon) - \mu_N(\mathbf{F} \cap \mathbf{K})) - \varepsilon \ge -\varepsilon$. Since this relation holds for all $\varepsilon > 0$, it is enough to assume relation (3.7) for compact sets $\mathbf{K}$.

Now we introduce the notion of good coupling we shall use later and formulate a simple consequence of Lemma B.

Definition of good coupling: *Let two sequences of probability measures μ_N and $\bar\mu_N$, $N = 1, 2, \dots$, be given on a separable complete metric space $(M, \mathcal{M}, \rho)$, where $\mathcal{M}$ denotes the σ-algebra generated by the topology induced by the metric ρ. These two sequences of measures have a good coupling if for all $\varepsilon > 0$ and $\delta > 0$ there is a sequence of probability measures $P_N^{\varepsilon,\delta}$, $N = 1, 2, \dots$, on the product space $(M \times M, \mathcal{M} \times \mathcal{M}, \bar\rho)$, $\bar\rho((x_1, y_1), (x_2, y_2)) = \rho(x_1, x_2) + \rho(y_1, y_2)$ which satisfies the following properties.*

- *i.) The marginal distributions of $P_N^{\varepsilon,\delta}$ are μ_N and $\bar\mu_N$, i.e. $P_N^{\varepsilon,\delta}(\mathbf{A} \times M) = \mu_N(\mathbf{A})$ and $P_N^{\varepsilon,\delta}(M \times \mathbf{A}) = \bar\mu_N(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{M}$, and $n = 1, 2, \dots$.*
- *ii.)* $\limsup\limits_{N\to\infty} P_N^{\varepsilon,\delta}(\{(x, y) \colon \rho(x, y) > \varepsilon\}) \le \delta$.

Corollary of Lemma B. *If two sequences of probability measures μ_N and $\bar\mu_N$, $N = 1, 2, \dots$, on a complete separable metric space $(M, \mathcal{M}, \rho)$ have a good coupling, and the sequence of measures μ_N converge weakly to a probability measure μ_0, then the measures $\bar\mu_N$ converge weakly to the same measure μ_0.*

Proof of the Corollary. Fix an $\varepsilon > 0$. For all $\delta > 0$ we can write

$$\liminf_{N\to\infty} (\bar\mu_N(\mathbf{F}^\varepsilon) - \mu_N(\mathbf{F})) \ge -\limsup_{N\to\infty} P_N^{\varepsilon,\delta}(\{(x, y) \colon \rho(x, y) > \varepsilon\}) \ge -\delta.$$

We get the statement of the Corollary by letting $\delta \to 0$.

Proof of Theorem 3. We shall prove the weak convergence of the measures $\hat\mu_N(\omega)$ for almost all ω with the help of Lemma B with the choice of $\mu_{B_N}(\omega)$ as μ_N and $\hat\mu_N(\omega)$ as $\bar\mu_N$. Then (for almost all ω) the measures μ_N converge weakly to μ_0, and it is enough to show that for almost all $\omega \in \Omega$

$$\begin{gathered} \liminf_{N\to\infty} (\hat\mu_N(\omega)(\mathbf{F}^\varepsilon) - \mu_{B_N}(\omega)(\mathbf{F})) \ge 0 \quad \text{for all closed sets} \\ \mathbf{F} \subset D([0,1]) \text{ or } \mathbf{F} \subset C([0,1]) \text{ and } \varepsilon \ge 0. \end{gathered} \tag{3.8}$$

Let us recall that for arbitrary measurable set $\mathbf{B}\subset D([0,1])$ (or $\mathbf{B}\subset C([0,1])$)

$$\mu_{B_N}(\omega)(\mathbf{B})=\bar\lambda_{B_N}\{s\colon s\in[1,B_N],\ X_s(\cdot,\omega)\in\mathbf{B}\}$$

and

$$\begin{aligned}\bar\mu_N(\omega)(\mathbf{B})=\bar\lambda_{B_N}\{s\colon &\text{there is some } 1\le j<k_N \text{ such that}\\ &B_{j,N}\le s<B_{j+1,N},\quad \text{and } X_{B_{j,N}}(\cdot,\omega)\in\mathbf{B}\},\end{aligned}$$

where the measure $\bar\lambda_T$ was defined in the formulation of Lemma 1.

For a pair of numbers $\varepsilon>0$ and $\eta>0$ define the set

$$\mathbf{A}(\varepsilon,\eta)=\left\{x\in D([0,1])\colon \sup_{1-\eta<s\le t\le 1}\rho(x_s,x_t)\le\varepsilon\right\}.$$

Given some $\varepsilon>0$ and $\delta>0$ fix some $\eta=\eta(\omega,\varepsilon,\delta)>0$ and $N_0=N_0(\omega,\varepsilon,\delta)$ in such a way that $\mu_{B_N}(\omega)(\mathbf{A}(\varepsilon,\eta))>1-\delta$ for $N\ge N_0$. By Theorem 2 such a choice of η and N_0 is possible for almost all $\omega\in\Omega$. Then we can choose, since the numbers $B_{k,j}$ satisfy condition (2.7), some number $j_0=j_0(\eta)$ and $N_1\ge N_0$ in such a way that $\dfrac{B_{k+1,N}}{B_{k,N}}\le 1+\frac{\eta}{2}$, if $N\ge N_1$ and $j_0\le k<N$, and $\dfrac{\log B_{j_0,N}}{\log B_N}<\delta$ if $N\ge N_1$. Then for all $N\ge N_1$

$$\begin{aligned}\hat\mu_N(\omega)(\mathbf{F}^\varepsilon)&\ge\hat\mu_N(\omega)(X_{B_{k,N}}(\cdot,\omega)\in\mathbf{F}^\varepsilon,\ \text{for some } k\ge j_0)\\ &=\bar\lambda_{B_N}(\{s\colon \text{there is some } j_0\le j<k_N \text{ such that}\\ &\qquad\qquad B_{j,N}\le s<B_{j+1,N} \text{ and } X_{B_j,N}(\cdot,\omega)\in\mathbf{F}^\varepsilon\})\\ &\ge\bar\lambda_{B_N}(\{s\colon B_{j_0,N}\le s<B_N \text{ and } X_s(\cdot,\omega)\in\mathbf{F}\cap\mathbf{A}(\varepsilon,\eta)\})\end{aligned}$$

The last inequality in this relation holds, because, in the case when $X_s(\cdot,\omega)\in\mathbf{F}\cap\mathbf{A}_N$ and $s\in[B_{j,N},B_{j+1,N})$ with some $j_0\le j<k_N$ (observe that the relation $[B_{j_0,N},B_N)=\bigcup\limits_{j=j_0}^{k_N-1}[B_{j,N},B_{j+1,N})$ holds), then $X_{B_{j,N}}(\cdot,\omega)\in\mathbf{F}^\varepsilon$, and this implies that all points $s\in(B_{j,N},B_{j+1,N}]$ are contained in the set whose $\bar\lambda_T$ measure is considered in the previous expression. To see the validity of this statement observe that with the notation $x=X_s(\cdot,\omega)$, $x\in D([0,1])$ $X_{B_{j,N}}(\cdot,\omega)=x_u$ with $u=\dfrac{B_{j,N}}{s}$, which satisfies the inequality $1-\eta\le\dfrac{1}{1+\frac{\eta}{2}}\le u\le 1$, where the function x_u is defined in formula (2.4′). Hence $x\in\mathbf{A}(\varepsilon,\eta)\cap\mathbf{F}$ implies that $x_u\in\mathbf{F}^\varepsilon$, as we claimed. Then we get that

$$\begin{aligned}\hat\mu_N(\omega)(\mathbf{F}^\varepsilon)&\ge\bar\lambda_{B_N}(s\colon s\in[1,B_N),\ \text{and } X_s(\cdot,\omega)\in\mathbf{F})\\ &\qquad-\bar\lambda_{B_N}([1,B_{j_0,N}))-\mu_{B_N}(D([0,1])\setminus\mathbf{A}(\varepsilon,\eta))\\ &\ge\bar\lambda_{B_N}(s\colon s\in[1,B_N),\ \text{and } X_s(\cdot,\omega)\in\mathbf{F})-2\delta=\mu_{B_N}(\mathbf{F})-2\delta,\end{aligned}\tag{3.9}$$

because $\mu_{B_N}(D([0,1])\setminus \mathbf{A}(\varepsilon,\eta))\le\delta$ and

$$\bar\lambda_{B_N}([1,B_{j_0,N}))=\frac{1}{\log B_N}\int\limits_1^{B_{j_0,N}}\frac1t\,dt=\frac{\log B_{j_0,N}}{\log B_N}\le\delta.$$

Letting $\delta\to 0$ in formula (3.9) we get formula (3.8). This implies the first part of Theorem 3.

We prove the second statement of Theorem 3 with the help of the Corollary of Lemma B, where $\hat\mu_N(\omega)$ plays the role of μ_N and $\bar\mu_N(\omega)$ the role of $\bar\mu_N$. We define the measure $P_N^\varepsilon=P_N(\omega)$ on the space $D([0,1])\times D([0,1])$ independently of the parameter ε in the following way: The measure $P_N(\omega)$ is concentrated on the trajectories $(X_{B_j,N}(\cdot,\omega),\bar X_{B_j,N}(\cdot,\omega))$, and

$$P_N(\omega)((X_{B_j,N}(\cdot,\omega),\bar X_{B_j,N}(\cdot,\omega))=\frac{1}{\log B_N}\log\frac{B_{j+1,N}}{B_{j,N}}.$$

Such a coupling can be constructed e.g. in the following way: For all $N=1,2,\dots$ let A_N denote the set $\mathbf{A}_N=\{1,\dots,k_N\}$, $\mathcal{A}_N$ the σ-algebra consisting of all subsets of $\mathbf{A}_N$, and define the probability measure ν_N, $\nu_N(j)=\dfrac{1}{\log B_N}\log\dfrac{B_{j+1,N}}{B_{j,N}}$, $1\le j<k_N$ on $(\mathbf{A}_N,\mathcal{A}_N)$. Then for all $\omega\in\Omega$ define the random variable $\xi_\omega(j)=(X_{B_j,N}(\cdot,\omega),\bar X_{B_j,N}(\cdot,\omega))$, $1\le j\le k_N$, on the probability space $(\mathbf{A}_N,\mathcal{A}_N,\nu_N)$, and let $P_N(\omega)$ be the distribution of the random variables ξ_ω in the space $D([0,1])\times D([0,1])$.

The marginal distributions of the measures $P_N(\omega)$ are $\hat\mu_N(\omega)$ and $\bar\mu_N(\omega)$. Hence by Corollary of Lemma B it is enough to prove that for almost all ω the relation

$$\lim_{N\to\infty}P_N(\omega)(\mathbf{A}_N(\varepsilon,\omega))=0 \tag{3.10}$$

holds with

$$\mathbf{A}_N(\varepsilon,\omega)=\left\{(X_{B_j,N}(\cdot,\omega),\bar X_{B_j,N}(\cdot,\omega))\colon \rho(X_{B_j,N}(\cdot,\omega),\bar X_{B_j,N}(\cdot,\omega))>\varepsilon\right\}$$

for all $\varepsilon>0$. Since the measures $\hat\mu_N$ are compact for all $\eta>0$ there is a compact set $\mathbf{K}=\mathbf{K}(\eta)\subset D([0,1])$ such that $\hat\mu_N(\mathbf{K})>1-\eta$ for all $N=1,2,\dots$, and formula (3.10) can be reduced to the statement

$$\lim_{N\to\infty}P_N(\omega)(\mathbf{A}_N(\varepsilon,\omega)\cap(\mathbf{K}\times D([0,1])))=0 \tag{3.11}$$

for arbitrary compact set $\mathbf{K}\subset D([0,1])$. Moreover, this statement can be reduced to a slightly weaker statement. To formulate it let us define for all $\eta>0$ and $N=1,2,\dots$ the number $\hat\jmath(N)=\hat\jmath(N,\eta)$ as $\hat\jmath(N)=\max\{j\colon \log B_{j,N}\le\eta\log B_N\}$. Because of condition (2.7) imposed on the numbers $B_{j,k}$ in Theorem 3 $\hat\jmath(N)\to\infty$ as $N\to\infty$.

Because of the definition of the measures $\hat\mu_N(\omega)$ and the number $\hat\jmath(N)$ the inequality $\hat\mu_N(\omega)\left\{\bigcup_{j\colon\, j\le \hat\jmath(N)} X_{B_{j,N}}(\cdot,\omega)\right\}\le\eta$ holds. Define the set

$$\begin{aligned}\mathbf{A}_N^\eta(\varepsilon,\omega)=\bigl\{&(X_{B_j,N}(\cdot,\omega),\bar X_{B_j,N}(\cdot,\omega))\colon \hat\jmath(N,\eta)\le j\le k_N,\\ &\rho(X_{B_j,N}(\cdot,\omega),\bar X_{B_j,N}(\cdot,\omega))>\varepsilon\bigr\}.\end{aligned}$$

Then $\hat\mu_N(\omega)(\mathbf{A}_N(\varepsilon,\omega)\backslash\mathbf{A}_N^\eta(\varepsilon,\omega))\le\eta$, and relation (3.11) can be reduced to the relation

$$\lim_{N\to\infty} P_N(\omega)(\mathbf{A}_N^\eta(\varepsilon,\omega)\cap(\mathbf{K}\times D([0,1])))=0 \tag{3.11$'$}$$

by letting $\eta\to 0$.

We claim that for an arbitrary compact set $\mathbf{K}\subset D([0,1])$, $\varepsilon>0$ and $\eta>0$ there is some $N_0=N_0(\mathbf{K},\varepsilon,\eta,\omega)$ such that for all $N\ge N_0$ and $j\ge\hat\jmath(N)$ the relation $X_{B_{j,N}}(\cdot,\omega)\in\mathbf{K}$ implies that $\rho(X_{B_{j,N}}(\cdot,\omega),\bar X_{B_{j,N}}(\cdot,\omega))<\varepsilon$, hence the set $\mathbf{A}_N^\eta(\varepsilon,\omega)\cap(\mathbf{K}\times D([0,1]))$ is empty for large enough N. This statement clearly implies relation (3.11$'$).

To prove this statement let us observe that the trajectory $\bar X_{B_{j,N}}(\cdot,\omega)$ is obtained as a discretization of the trajectory $X_{B_{j,N}}(\cdot,\omega)$ of the following type: There is a partition $0=t_{j,0,N}<t_{j,1,N}<\cdots<t_{j,j,N}=1$ of the interval $[0,1]$ such that $\bar X_{B_{j,N}}(t,\omega)=X_{B_{j,N}}(t_{j,l-1,N},\omega)$ if $t_{j,l-1,N}\le t<t_{j,l,N}$, $1\le l\le j$, and $\bar X_{B_{j,N}}(1,\omega)=X_{B_{j,N}}(1,\omega)$. The numbers $t_{j,l,N}$ could be given explicitly as $t_{j,l,N}=\dfrac{B_{l-1,N}}{B_{j,N}}$, but we do not need their explicit form. What we need is the fact that conditions (2.7) and (2.8) imposed on the numbers $B_{j,N}$ imply that $\lim\limits_{\hat\jmath\to\infty}\limsup\limits_{N\to\infty}\sup\limits_{N\ge j\ge\hat\jmath}\sup\limits_{1\le l\le j}(t_{j,l,N}-t_{j,l-1,N})=0$. This relation holds since for all $\eta>0$ there exist some $\hat\jmath_1=\hat\jmath_1(\eta)$, $\hat\jmath_2=\hat\jmath_2(\eta)$ and $N_0=N_0(\eta)$ in such a way that $\dfrac{B_{l,N}}{B_{l-1,N}}\le 1+\frac{\eta}{2}$ if $\hat\jmath_1\le l\le N$ and $N\ge N_0$, and $\eta B_{\hat\jmath_2,N}\ge B_{\hat\jmath_1,N}$ if $N\ge N_0$. Then for all $N\ge j\ge\hat\jmath_2$ and $N\ge N_0$ $t_{j,l,N}-t_{j,l-1,N}\le\dfrac{B_{l,N}-B_{l-1,N}}{B_{l,N}}\le\eta$ for $j\ge l\ge\hat\jmath_1$, and $t_{j,l,N}-t_{j,l-1,N}\le\dfrac{B_{\hat\jmath_1,N}}{B_{\hat\jmath_2,N}}\le\eta$ if $1\le l\le\hat\jmath_1$. The width of the partitions considered above tends to zero if $\hat\jmath=\hat\jmath(N)\to\infty$, as we claimed. Indeed, the previous calculations imply that it is less than η for $\hat\jmath\ge\hat\jmath_2(\eta)$.

We claim that this relation implies that

$$\lim_{N\to\infty}\sup_{j\colon\, j\ge\hat\jmath(N),\, X_{j,N}(\cdot,\omega)\in\mathbf{K}}\rho(X_{j,N}(\cdot,\omega),\bar X_{j,N}(\cdot,\omega))=0$$

for all compact sets $\mathbf{K}\subset D([0,1])$, and this relation implies formula (3.11$'$) and hence the second part of Theorem 3.

Let us define the following function $g(x,\delta)$ for $x\in D([0,1])$ and $\delta>0$:

$$g(x,\delta)=\sup_{\substack{0=t_0<t_1<\cdots<t_s=1\\ t_j-t_{j-1}\le\delta,\ j=1,\dots,s}}\rho(x,\bar x_{t_0,\cdots,t_s}), \tag{3.12}$$

where

$$\bar{x}_{t_0,\cdots,t_s}(t) = x(t_{j-1}) \text{ if } t_{j-1} \le t < t_j,\ j = 1,\dots,s \quad\text{and}\quad \bar{x}_{t_0,\cdots,t_s}(1) = x(1).$$

We shall prove the following Lemma C which is probably well-known among experts, but whose explicit formulation we did not find in the literature.

Lemma C. *For all functions $x \in D([0,1])$ $\lim\limits_{\delta\to 0} g(x,\delta) = 0$. Moreover, for all compact sets $\mathbf{K} \subset D([0,1])$*

$$\lim_{\delta\to 0} \sup_{x\in\mathbf{K}} g(x,\delta) = 0.$$

Then to finish the proof of Theorem 3 it is enough to show that $\lim\limits_{\delta\to 0} \sup\limits_{x\in\mathbf{K}} g(x,\delta) = 0$ for all compact sets $\mathbf{K} \subset D([0,1])$, where the function $g(x,\delta)$ is defined in formula (3.12), and this is the content of Lemma C.

Proof of Lemma C. It is known (see e.g. Billingsley's book [1] formulas (14.6) and (14.7)) that for all $\eta > 0$ there is some $\alpha = \alpha(\eta) > 0$ and a partition $0 = u_0 < u_1 < \cdots < u_r = 1$ of the interval $[0,1]$ such that for $u_j - u_{j-1} > \alpha$, and $\sup\limits_{1\le j\le r} \sup\limits_{u_{j-1}\le s,t<u_j} |x(s) - x(t)| < \eta$. Let us consider an arbitrary partition $0 = t_0 < t_1 < \cdots < t_s = 1$ of the interval $[0,1]$ such that $\sup\limits_{1\le j\le s} |t_j - t_{j-1}| < \alpha\eta$. We claim that in this case $\rho(x, \bar{x}_{t_1,\dots,t_s}) \le \eta$. Since this relation holds for all $\eta > 0$, it implies the first statement of Lemma C.

To prove this statement let us consider the partition $0 = T_0 < T_1 < \cdots < T_r$, such that the interval $[T_j, T_{j+1})$ is the union of those intervals $[t_l, t_{l+1})$ for which $t_l \in [u_j, u_{j+1})$. Let $\lambda(\cdot)$ be that mapping of the interval $[0,1]$ into itself which maps the interval $[u_j, u_{j+1})$ linearly to the interval $[T_j, T_{j+1})$. Then $\sup\limits_{0\le u\le 1} |x(\lambda(u)) - \bar{x}_{t_1,\dots,t_s}(u)| \le \eta$, and also $\sup\limits_{t\ne s} \log \left| \dfrac{\lambda(t)-\lambda(s)}{t-s} \right| \le \eta$. Hence $\rho(x, \bar{x}_{t_1,\dots,t_s}) \le \eta$, as we claimed. This implies the first statement of Lemma C.

The second, more general statement follows in the same way. We only have to observe that the number $\alpha = \alpha(\eta)$ can be chosen as the same number for all $x \in \mathbf{K}$ in a compact set $\mathbf{K} \in D([0,1])$. This follows from the characterization of compact sets in the space $D([0,1])$. (See relation (14.33) in Theorem 14.3 in the book of Billingsley [1].)

Proof of Theorem 4. Let us construct the following coupling of the random broken lines $\tilde{S}_N(\cdot,\omega)$ and $T_N(\cdot,\omega)$ which are defined with the help of the random variables $\tilde{S}_k(\omega)$ and $T_k(\omega)$, $k = 1,2,\dots$, in formula (2.13). Let $P_N^{\varepsilon,\delta}(\omega)$, $\omega \in \Omega$, be a measure on $D([0,1]) \times D([0,1])$ concentrated on the pairs, $(\tilde{S}_k(\cdot,\omega), T_k(\cdot,\omega))$ in such a way that

$$P_N^{\varepsilon,\delta}(\omega)(\tilde{S}_k(\cdot,\omega), T_k(\cdot,\omega)) = \mu_N(\omega)(T_k(\cdot,\omega)) = \frac{1}{\log \dfrac{B_N}{B_1}} \log \frac{B_{k+1}}{B_k}, \quad 1 \le k < N.$$

(The parameters $\varepsilon > 0$ and $\delta > 0$ in the definition are the same ε and δ which appear in formula (2.13).)

Then the marginal distributions of $P_N^{\varepsilon,\delta}(\omega)$ are the distributions $\mu_N(\omega)$ and $\bar\mu_N(\omega)$ appearing in the definition of the almost sure functional limit theorem. By the Corollary of Lemma B it is enough to prove that

$$\limsup_{N\to\infty} P_N^{\varepsilon,\delta}(\omega)\{(x,y)\colon \rho(x,y) > \varepsilon\} < \delta$$

for almost all $\omega \in \Omega$. Since $\rho(x,y) \le d(x,y)$ with $d(x,y) = \sup\limits_{0\le u\le 1} |x(u) - y(u)|$,

$$P_N^{\varepsilon,\delta}(\omega)\{(x,y)\colon \rho(x,y) > \varepsilon\} \le \frac{2}{\log B_N} \sum_{k=1}^{N-1} \log \frac{B_{k+1}}{B_k} I(d(\tilde S_k(\cdot,\omega), T_k(\cdot,\omega)) > \varepsilon)$$

for sufficiently large N. For a number N choose the number $\bar n = \bar n(N)$ such that $2^{\bar n - 1} \le B_N < 2^{\bar n}$. Then $N \le N(\bar n)$, and $\log B_N \ge \bar n - 1$. Hence

$$P_N^{\varepsilon,\delta}(\omega)\{(x,y)\colon \rho(x,y) > \varepsilon\} \le \frac{1}{\bar n - 1} \sum_{k=1}^{N(\bar n)} \log \frac{B_{k+1}}{B_k} I\left(\left\{ \frac{\sup\limits_{1\le j\le k} |\tilde S_j(\omega) - T_j(\omega)|}{A_k} > \varepsilon \right\}\right)$$

with this $\bar n = \bar n(N)$. As $\bar n(N)$ tends to infinity as $N \to \infty$ relation (2.13) implies that the $\limsup$ of the right-hand side of the last expression is less than δ for almost all ω as $N \to \infty$. Theorem 4 is proved.

Proof of Theorem 5A. Let us consider the partial sums $S_k(\omega) = \sum\limits_{j=1}^{k} \xi_j(\omega)$, $k = 1, 2, \dots$, and the random polygons $S_n(s,\omega)$ and $\bar S_n(s,\omega)$, $n = 1, 2, \dots$, defined by formula (2.11) with weight functions B_n, $A_n = B_n^{1/\alpha}$ and $\bar B_n$, $\bar A_n = \bar B_n^{1/\alpha}$ respectively. Let us also introduce the random polygons $\bar S'_n(\omega)$ defined with the help of the partial sums $S_k(\omega)$ with the new weight functions $\bar B_n$ and the original sequence $A_n = B_n^{1/\alpha}$ by formulas (2.11). We have to compare the distance $\rho(S_N(\cdot,\omega), \bar S'_N(\cdot,\omega)) \le \varepsilon$.

It is not difficult to show that $\lim\limits_{N\to\infty} d(S_N(\cdot,\omega), \bar S'_N(\cdot,\omega)) = 0$ under the conditions of Theorem 5, if the metric $\rho = d_0$ applied in this paper is replaced by the following metric $d(\cdot,\cdot)$ in the space $D([0,1])$: The relation $d(x,y) \le \varepsilon$, $x, y \in D([0,1])$, holds, if there is a strictly monotone increasing continuous function $\lambda(t)$ which is a homeomorphism of the interval $[0,1]$ into itself, and $\sup\limits_{0\le t\le 1} |\lambda(t) - t| \le \varepsilon$, $\sup\limits_{0\le t\le 1} |y(t) - x(\lambda(t))| \le \varepsilon$. The metric d induces the same topology as the metric $\rho = d_0$ in the space $D([0,1])$, but it has the unpleasant property that the space $D([0,1])$ is not a complete metric space with this metric. A detailed discussion about the relation between the metrics $d(\cdot,\cdot)$ and $d_0(\cdot,\cdot)$ is contained in the book of Billingsley [1].

In the proof we have to overcome the following difficulty. The natural transformation $\lambda(\cdot)$ for which $\bar S_N(\lambda(\cdot,\omega))$ is close to $\bar S'_N(\cdot,\omega)$ is the map which transforms the point $\frac{\bar{B}_k}{B_N}$ to the point $\frac{B_k}{B_N}$, and is linear between these points. This transformation shows that for large N the corresponding trajectories are close in the $d(\cdot,\cdot)$ metric, but it supplies no good estimate for the distance in the $d_0(\cdot,\cdot)$ metric.

We recall the following result from Billingsley's book [1] (see Lemma 2 in Section 14): If $d(x,y)\le\delta^2$, $0<\delta\le 1/4$, then $\rho(x,y)=d_0(x,y)\le 4\delta+w'_x(\delta)$, where the inequality $w'_x(\delta)\le\varepsilon$ for a function $x\in D([0,1])$ means that there exist some numbers $0=t_0<t_1<\cdots<t_s=1$ such that $t_j-t_{j-1}\ge\varepsilon$, and $\sup\limits_{t_{j-1}\le u,v<t_j}|x(u)-x(v)|\le\varepsilon$ for all $j=1,2,\dots$.

For all $\omega\in\Omega$ for which the sequence of probability measures $\mu_N(\omega)$, $N=1,2,\dots$, defined by relations (2.11) and (2.12) are compact fix a compact set $\mathbf{K}=\mathbf{K}(\varepsilon,\omega)\in D([0,1])$ in such a way that $\mu_N(\omega)(\mathbf{K})\ge 1-\varepsilon$. We have $\lim\limits_{\delta\to0}w'_x(\delta)=0$ for arbitrary $x\in D([0,1])$. Moreover, $\lim\limits_{\delta\to0}\sup\limits_{x\in\mathbf{K}}w'_x(\delta)=0$ for an arbitrary compact set $\mathbf{K}\subset D([0,1])$. (See, Theorem 14.3 in Billingsley book [1].) Given some $\delta/2>0$ and the compact set $\mathbf{K}=\mathbf{K}(\varepsilon,\omega)\subset D([0,1])$ we have fixed choose a number $0<\eta<1/4$ such that $5\eta<\delta/2$ and a number $\bar\eta>0$ such that $w'_x(\bar\eta)<\eta$ if $x\in\mathbf{K}$. Then there is an index $N_0=N_0(\eta,\bar\eta)$ such that $d(S_N(\cdot,\omega),\bar S'_N(\cdot,\omega))\le\min(\eta^2,\bar\eta^2)$, if $N\ge N_0$. The above relations imply that $\rho(S_N(\cdot,\omega),\bar S_N(\cdot,\omega))\le 4\min(\eta,\bar\eta)+w'_{S_N(\cdot,\omega)}(\bar\eta)\le\delta/2$, if $N\ge N_0$ and $S_N(\cdot,\omega)\in\mathbf{K}$.

To complete the proof of Theorem 5A we compare the random broken lines $\bar S'_n(\omega)$ and $\bar S_n(\omega)$. Observe that $\bar S'_k(\cdot,\omega)=\dfrac{\bar A_k}{A_k}\bar S_k(\cdot,\omega)$, and $\lim\limits_{k\to\infty}\dfrac{\bar A_k}{A_k}=1$. On the other hand, given the compact set $\mathbf{K}=\mathbf{K}(\varepsilon,\omega)$, there is a number $K=K(\varepsilon,\omega)>0$ such that $\sup\limits_{x\in\mathbf{K}}\sup\limits_{0\le s\le1}|x(s)|\le K$. These facts imply that there exists some threshold index $N_1=N_1(\omega,\varepsilon)$ such that $\rho(\bar S_N(\cdot,\omega),\bar S'_N(\omega))\le\delta/2$ if $N\ge N_1$.

The previous arguments imply that there is some index $\bar N=\max(N_0,N_1)$ and a compact set $\mathbf{K}\in D([0,1])$ such that $\mu_N(\omega)(\mathbf{K})\ge1-\varepsilon$, and $\rho(S_N(\cdot,\omega),\bar S_N(\cdot,\omega))\le\delta$ if $N\ge N_1$ and $S_N(\cdot,\omega)\in\mathbf{K}$. Since $\lim\limits_{N\to\infty}\mu_N(S_k(\cdot,\omega))=0$ for all fixed $k>0$, the $\mu_N(\omega)$ probability of the random broken lines $S_n(\cdot,\omega)$ for which $\rho(S_n(\cdot,\omega),\bar S_n(\omega))\le\delta$ is less than 2ε. Since this relation holds for all $\varepsilon>0$, it implies Theorem 5A.

Proof of Theorem 5. First we prove the following statement. Let us fix some $\delta>0$ and let $\bar{\mathbf{K}}$ be a compact set in the space $D([0,1])$ which also satisfies the following property: There is some $\eta_0>0$ such that

$$\sup_{1-\eta_0\le u\le1}|x(u)-x(1)|\le\frac{\delta}{4}\quad\text{for all } x\in\mathbf{K}. \tag{3.13}$$

We claim that there exists a number $\eta=\eta(\delta,\eta_0,\bar{\mathbf{K}})>0$ such that for all functions $x\in\bar{\mathbf{K}}$ and numbers $1-\eta\le t\le1$ the inequality $d(x,x_t)<\delta$ holds, where the function x_t is defined in formula (2.4′), and $d(\cdot,\cdot)$ is the complete metric we introduced to define the topology in the space $D([0,1])$.

To prove this statement let us first observe that because of the compactness of the set $\bar{\mathbf{K}}$ there exists a number $K>0$ such that $\sup\limits_{x\in\bar{\mathbf{K}}}\sup\limits_{0\le u\le 1}|x(u)|\le K$. Given a function $x(\cdot)\in D([0,1])$ and a number $0<t\le 1$ define the function $\bar x\in D([0,1])$ as $\bar x_t(u)=x(tu)$, $0\le u\le 1$. Then there exists an $\eta_1>0$ such that $d(x_t,\bar x_t)<K(t^{-\alpha/2}-1)\le\frac{\delta}{2}$ if $x\in\bar{\mathbf{K}}$ and $1-\eta_1\le t\le 1$. Hence it is enough to show that there is some $\eta'>0$ in such a way that $d(x,\bar x_t)\le\frac{\delta}{2}$ if $x\in\bar{\mathbf{K}}$ and $1-\eta'\le t\le 1$.

To prove this statement let us define for all $\frac12\le t<1$ the mapping $\lambda_t(u)$ of the interval $[0,1]$ into itself as $\lambda_t(u)=tu$ for $0\le u\le t^*(t)$ with $t^*(t)=1-\sqrt{1-t}$, and define $\lambda_t(u)$ in the remaining interval $(t^*(t),1]$ also linearly, i.e. let $\lambda_t(u)=(\sqrt{1-t}+t)u+1-t-\sqrt{1-t}$ for $t^*(t)\le u\le 1$. There is some $\eta_2>0$ such that $\sup\limits_{u\ne v}\log\left|\dfrac{\lambda_t(u)-\lambda_t(v)}{u-v}\right|\le\dfrac{\delta}{2}$ if $1-\eta_2\le t\le 1$. By recalling the definition of the metric $d(\cdot,\cdot)$ we see that to complete the proof of the statement we claimed to hold it is enough to show that there is some $\eta_3>0$ such that for all $x\in\bar{\mathbf{K}}$ and $1-\eta_3\le t\le 1$ $\sup\limits_{0\le u\le 1}|x(\lambda_t(u))-\bar x_t(u)|\le\frac{\delta}{2}$. Then the relation formulated at the start of the proof of Theorem 5 holds with $\eta=\min(\eta_1,\eta_2,\eta_3)$. But $x(\lambda_t(u))-\bar x_t(u)=0$ if $0\le u\le t^*(t)$, and $|x(\lambda_t(u))-\bar x_t(u)|\le\frac{\delta}{2}$ for $t^*(t)\le t\le 1$ if $\eta_3>0$ is chosen so small that $t^*(t)>1-\eta_0$ for $1-\eta_3<t\le 1$ with the number η_0 appearing in relation (3.13).

By Lemma B to prove Theorem 5 it is enough to show that for an arbitrary compact set $\mathbf{K}\subset D([0,1])$ and $\alpha>0$

$$\liminf_{N\to\infty}(\bar\mu_N(\omega)(\mathbf{K}^\alpha)-\mu_N(\omega)(\mathbf{K}))\ge 0\quad\text{for almost all }\omega\in\Omega,\tag{3.14}$$

where $\mathbf{K}^\alpha=\{x\colon\rho(x,\mathbf{K})\le\alpha\}$ is the α-neighborhood of the set $\mathbf{K}$.

To prove relation (3.14) we define some quantities. Let us observe that because of Theorem 5A and Lemma B the sequence of probability measures $\mu'_N(\omega)$, $N=1,2,\dots$, $\mu'_N(\omega)(\bar S_k(\cdot,\omega))=\dfrac{1}{\log\dfrac{B_N}{B_1}}\log\dfrac{B_{k+1}}{B_k}$, $1\le k<N$, are convergent hence compact for almost all $\omega\in\Omega$. Let us fix some $\varepsilon>0$. There is some compact set $\bar{\mathbf{K}}_0\in D([0,1])$ $\eta=\eta(\varepsilon,\alpha,\omega)$ in such a way that $\mu'_N(\omega)(\bar{\mathbf{K}}_0)\ge 1-\frac{\varepsilon}{2}$ for all $N=1,2,\dots$. Because of the conditions of Theorem 5 (The condition that relation (2.14) holds) there exists some $\eta_0>0$ such that the set

$$\bar{\mathbf{K}}_1=\{x\colon x\in D([0,1])\sup_{1-\eta_0\le t\le 1}|x(t)-x(1)|\le\frac{\alpha}{8}$$

satisfies the inequality $\mu_0\left(\bar{\mathbf{K}}_1\right)\ge 1-\frac{\varepsilon}{3}$. The above defined set $\bar{\mathbf{K}}_1$ is closed, hence the compactness of the sequence of measures $\mu'_N(\omega)$ implies that there is some threshold $N_0=N_0(\omega)$ such that $\mu_N(\omega)\left(\bar{\mathbf{K}}_1\right)\ge 1-\frac{\varepsilon}{2}$ for all $N\ge N_0$. Define the set $\bar{\mathbf{K}}=\bar{\mathbf{K}}_0\cap\bar{\mathbf{K}}_1$. Then for almost all $\omega\in\Omega$ there is some threshold $N_0=N_0(\omega)$ $\mu_N(\omega)(\bar{\mathbf{K}})\ge 1-\varepsilon$ for all $N\ge N_0$.

There exists some $\eta>0$ such that $d(x,x_t)\le\frac{\alpha}{2}$ if $x\in\bar{\mathbf{K}}$ and $1-\eta\le t\le 1$. For all positive integers n define the number $\tilde n=\tilde n(\eta)$ as

$$\tilde n=\min\left\{k\colon B_k>\left(1-\frac{\eta}{2}\right)B_n \text{ and } \bar B_k>\left(1-\frac{\eta}{2}\right)\bar B_n\right\}. \tag{3.15}$$

If the index n is such that the relations $S_n(\cdot,\omega)\in\mathbf{K}$, $d(S_n(\cdot,\omega),\bar S_n(\cdot,\omega))<\frac{\alpha}{2}$, $\bar S_n(\cdot,\omega)\in\bar{\mathbf{K}}$ hold and $\tilde n(n)\le m\le n$, then $\bar S_m(\cdot,\omega)\in\mathbf{K}^\alpha$. Indeed,

$$d(\bar S_m(\cdot,\omega),\bar S_n(\cdot,\omega))<\frac{\alpha}{2}, \tag{3.16}$$

since with the notation $x(\cdot)=\bar S_n(\cdot,\omega)$ we have $\bar S_m(\cdot,\omega)=x_t(\cdot)$ with $t=\frac{B_m}{B_n}$ which satisfies the inequality $1-\frac{\eta}{2}\le t\le 1$, and this implies (3.16). Relation (3.16) and the other conditions we have imposed imply that $\bar S_m(\cdot,\omega))\in\mathbf{K}^\alpha$.

Let us fix some sufficiently large integer $M>0$ to be chosen later which may depend on α, ε, $\omega\in\Omega$ and the sequences B_n and $\bar B_n$, but does not depend on the index N for which the measures $\mu_N(\omega)$ and $\bar\mu_N(\omega)$ are considered. Define the set of indices

$$\mathcal{C}=\mathcal{C}(\alpha,\varepsilon,\omega,\mathbf{K})=\left\{k\colon k\ge M,\ S_k(\cdot,\omega)\in\mathbf{K},\ \rho(S_n(\cdot,\omega),\bar S_n(\cdot,\omega))<\frac{\alpha}{2},\ \bar S_n(\cdot,\omega)\in\bar{\mathbf{K}}\right\}$$

and the sets $\mathbf{K}_0(N)\in\mathbf{D}([0,1])$

$$\mathbf{K}_0(N)=\{S_k(\cdot,\omega)\colon k\in\mathcal{C},\ 1\le k<N\},\quad N=1,2,\dots.$$

Then $\mathbf{K}_0(N)\subset\mathbf{K}$, and

$$\limsup_{N\to\infty}(\mu_N(\omega)(\mathbf{K})-\mu_N(\omega)(\mathbf{K}_0(N)))<2\varepsilon \tag{3.17}$$

for almost all $\omega\in\Omega$. To see the last relation observe that $\lim\limits_{N\to\infty}\mu_N(\omega)(S_k(\omega))=0$ for all fixed k, and because of Theorem 5A

$$\lim_{N\to\infty}\mu_N(\omega)\left\{\text{the union of random broken lines } S_n(\omega)\colon \rho(S_n(\cdot,\omega),\bar S_n(\cdot,\omega))\ge\frac{\alpha}{2}\right\}=0.$$

Let us define the following enlargements of the set $\mathcal{C}\cap\{k\colon 1\le k\le N\}$:

$$\bar{\mathcal{C}}_N=\bar{\mathcal{C}}_N(\alpha,\varepsilon,\omega,\mathbf{K})=\{m\colon \text{there is some } n\in\mathcal{C}, 1\le n\le N, \text{ such that } \tilde n(n)\le m\le n\},$$

$N=1,2,\dots$, and the sets of trajectories

$$\mathbf{K}^*(N)=\left\{\bar S_m(\cdot,\omega)\colon m\in\bar{\mathcal{C}}_N\right\}.$$

Then $\bar{\mathbf{K}}^*(N)\subset\mathbf{K}^\alpha$ because of the properties of the sets $\mathcal{C}$ and $\bar{\mathcal{C}}_N$. On the other hand, the set $\bar{\mathcal{C}}_N$ consists of disjoint intervals of integers $[L_j,R_j)$ such that $\dfrac{L_j}{R_j}\ge 1-\dfrac{\eta}{3}$, and

$L_j \ge (1-\eta)M$. (Here we use that if the number M is chose sufficiently large, then for all $n \ge M$ the two sides in the defining inequalities in relation (3.15) are almost equal.)

We claim that if we choose the constant M sufficiently large, then

$$\limsup_{N\to\infty} \frac{\bar\mu_N(\omega)(\mathbf{K}^*(N))}{\mu_N(\omega)(\mathbf{K}^*(N))} \ge 1-\varepsilon \tag{3.18}$$

Indeed,

$$\frac{\bar\mu_N(\mathbf{K}^*(N))}{\mu_N(\mathbf{K}^*(N))} \ge \inf_j \frac{\bar\mu_N\left(\bigcup\limits_{L_j\le m<R_j} S_m(\cdot,\omega)\right)}{\mu_N\left(\bigcup\limits_{L_j\le m<R_j} S_m(\cdot,\omega)\right)} = \inf_j \frac{\log\frac{B_N}{B_1}}{\log\frac{\bar B_N}{\bar B_1}} \frac{\log\frac{\bar B_{R_j}}{\bar B_{L_j}}}{\log\frac{B_{R_j}}{B_{L_j}}}, \tag{3.19}$$

where $[L_j,R_j)$ are the (disjoint) intervals whose union is the set $\bar{\mathcal{C}}_N$. To prove (3.18) make the following observations: Since $\lim\limits_{N\to\infty} B_N=\infty$ and $\lim\limits_{N\to\infty}\frac{\bar B_N}{B_N}=1$, the relation $\lim\limits_{N\to\infty}\frac{\log\frac{B_N}{B_1}}{\log\frac{\bar B_N}{\bar B_1}}=1$ holds. On the other hand $\log\frac{\bar B_{R_j}}{\bar B_{L_j}}\ge\frac{\eta}{4}$ for sufficiently large M, since the interval $[L_j,R_j)$ contains an interval $[\tilde n(n),n]$. This inequality together with the relation $\lim\limits_{N\to\infty}\frac{\bar B_N}{B_N}=1$ imply that the inequality $\frac{\bar B_{R_j}}{\bar B_{L_j}}\ge\left(1-\frac{\varepsilon\eta}{100}\right)\frac{B_{R_j}}{\bar B_{L_j}}$, hence $\log\frac{\bar B_{R_j}}{\bar B_{L_j}}\ge\left(1-\frac{\varepsilon}{4}\right)\log\frac{B_{R_j}}{B_{L_j}}$. The above estimates together with relation (3.18) imply inequalities (3.19). Relations (3.17), (3.18) and the relation $\mathbf{K}^\alpha\supset\mathbf{K}^*(N)\supset\mathbf{K}_0(N)$ imply that for sufficiently large N

$$\begin{aligned}\bar\mu_N(\omega)(\mathbf{K}^\alpha) &\ge \bar\mu_N(\omega)(\mathbf{K}^*(N)) \ge (1-\varepsilon)\mu_N(\omega)(\mathbf{K}^*(N))\\ &\ge (1-\varepsilon)\mu_N(\omega)(\mathbf{K}_0(N))) \ge (1-\varepsilon)\mu_N(\omega)(\mathbf{K}))-2\varepsilon.\end{aligned}$$

Since this relation holds for all $\varepsilon>0$ it implies relation (3.14) hence Theorem 5.

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Appendix

A simpler proof of the almost sure functional limit theorem part of Theorem 1 by means of formula (3.2).

This proof is a simple adaptation of an argument of I. A. Ibragimov and M. A. Lifshitz made in their paper *On almost sure type limit theorems*.

The weak convergence of the measures $\mu_T(\omega)$ to the measure μ_0 as $T \to \infty$ is equivalent to the statement

$$\liminf_{T\to\infty} \mu_T(\omega)(\mathbf{G}) \ge \mu_0(\mathbf{G}) \quad \text{for all open sets } \mathbf{G}.$$

On the other hand, the following simple lemma holds:

Lemma A. *Let $(M, \mathcal{M}, \rho)$ be a separable complete metric space with the σ-algebra $\mathcal{M}$ generated by the topology induced by the metric ρ in M. Let μ be a probability measure on $(M, \mathcal{M})$. There is a sequence of (countably many) open subsets $\mathbf{G}_n$, $n = 1, 2, \dots$, of the space M in such a way that*

$$\mu(\mathbf{G}) = \sup_{\mathbf{G}_n : \mathbf{G}_n \subset \mathbf{G}} \mu(\mathbf{G}_n) \quad \textit{for all open sets } \mathbf{G} \subset M.$$

Applying this lemma for the space $C([0,1])$ or $D([0,1])$ and the measure μ_0 the proof of the weak convergence of the probability measures $\mu_T(\omega)$ to μ_0 for almost all $\omega \in \Omega$, as $T \to \infty$ can be reduced to the statement

$$\liminf_{T\to\infty} \mu_T(\omega)(\mathbf{G}_n) \ge \mu_0(\mathbf{G}_n) \quad \text{for almost all } \omega \in \Omega \text{ and } n = 1, 2, \dots, \tag{A1}$$

where $\mathbf{G}_n$, $n = 1, 2, \dots$, are the open sets appearing in Lemma A. Indeed, this statement implies that for all open sets $\mathbf{G}$ and $\varepsilon > 0$ there exists a set $\mathbf{G}_n \subset \mathbf{G}$ such that $\mu_0(\mathbf{G}) \le \mu_0(\mathbf{G}_n) + \varepsilon$, and

$$\mu_0(\mathbf{G}) \le \mu_0(\mathbf{G}_n) + \varepsilon \le \liminf_{T\to\infty} \mu_T(\omega)(\mathbf{G}_n) + \varepsilon \le \liminf_{T\to\infty} \mu_T(\omega)(\mathbf{G}) + \varepsilon$$

for almost all $\omega \in \Omega$. Then, by letting ε tend to zero we get the almost sure functional limit theorem.

On the other hand, defining the functionals $\mathcal{F}_n$ in the space $C([0,1])$ or $D([0,1])$ as $\mathcal{F}_n(x) = 1$ if $x \in \mathbf{G}_n$, and $\mathcal{F}_n(x) = 0$ if $x \notin \mathbf{G}_n$ we get the following relation by means of formula (3.2).

$$\lim_{T\to\infty} \mu_T(\omega)(\mathbf{G}_n) = \lim_{T\to\infty} \int \mathcal{F}_n(x)\mu_T(\omega)(\,dx) = \int \mathcal{F}_n(x)\mu_0(\,dx) = \mu_0(\mathbf{G}_n)$$

for almost all $\omega \in \Omega$, i.e. even a stronger version of formula (A1) holds. This implies the weak convergence formulated in Theorem 1.

Proof of Lemma A. Let x_k, $k = 1, 2, \dots$, be an everywhere dense sequence in the space M. Let $\mathbf{H}_{k,m}$ denote the open ball with center x_k and radius $\frac{1}{m}$ in the space $(M, \mathcal{M})$. Let us consider all sets $\mathbf{H}_{k,m}$, $k = 1, 2, \dots$, $m = 1, 2, \dots$, and all finite union of the sets $\mathbf{H}_{k,m}$. This a countable collection $\mathbf{G}_n$ of open sets, and we claim that such a choice of the open sets $\mathbf{G}_n$ satisfies Lemma A.

Indeed, for all open sets $\mathbf{G} \subset M$ and $\varepsilon > 0$ there exists a compact set $\mathbf{K} \subset \mathbf{G}$ such that $\mu(\mathbf{G}) \le \mu(\mathbf{K}) + \varepsilon$. Since all points $x \in \mathbf{K}$ have a positive distance from the complement of the set $\mathbf{G}$, for all $x \in \mathbf{K}$ there is a set $\mathbf{H}_{k,m}$ such that $x \in \mathbf{H}_{k,m} \subset \mathbf{G}$. Hence the union of those sets $\mathbf{H}_{k,m}$ which are contained in $\mathbf{G}$ supply a cover of the set $\mathbf{K}$. The compact set $\mathbf{K}$ also has a finite cover consisting of such sets $\mathbf{H}_{k,m}$. This means that there exists a set $\mathbf{G}_n$ such that $\mathbf{K} \subset \mathbf{G}_n \subset \mathbf{G}$. This relation also implies that $\mu(\mathbf{G}_n) \ge \mu(\mathbf{K}) \ge \mu\mathbf{G}) - \varepsilon$. Since such a construction can be made for all $\varepsilon > 0$ these relations imply Lemma A.

Appendix 2.

The argument below gives a possible measure theoretical justification of the procedure leading to the proof of formula (3.2).

We need the following results in Billingsley book [1] (the discussion after Theorem 8.3 for the space $C([0,1])$ and Theorem 14.5 for the space $D([0,1])$). Put $(X, \mathcal{A}) = \left(\mathbf{R}^{[0,1]}, \mathcal{C}^{[0,1]}\right)$, where $\mathbf{R}^{[0,1]}$ is the direct product of the real line with indices $0 \le t \le 1$ and $\mathcal{C}^{[0,1]}$ is the direct product of the usual topology on the real line with indices $0 \le t \le 1$. Beside this, let us denote by $(Y, \mathcal{B})$ the space $C([0,1])$ or $D([0,1])$ with the usual topology. The results quoted from Billingsley's book state that if we denote by $\mathcal{M}$ the σ-algebra generated by the open sets in $(X, \mathcal{A})$ and by $\mathcal{N}$ the σ-algebra generated by the open sets in $(Y, \mathcal{B})$, then all $B \in \mathcal{N}$ can be written in the form $B = A \cap Y$ with some $A \in \mathcal{M}$. Billingsley's book also proves that $A \cap Y \in \mathcal{N}$ for all $A \in \mathcal{M}$.

Given any probability measure μ on the space $(Y, \mathcal{B})$ we can define its extension $\bar{\mu}$ by defining $\bar{\mu}(C) = \mu(C \cap Y)$ for all $C \subset X$ such that $C \cap Y \in \mathcal{N}$. The class of sets C with the property $C \subset X$ and $C \cap Y \in \mathcal{M}$ is a σ-algebra $\mathcal{G}$ such that $\mathcal{M} \subset \mathcal{G}$, $\mathcal{N} \subset \mathcal{G}$, and $\bar{\mu}$ is a probability measure in $\mathcal{G}$. Let us remark that since all $B \in \mathcal{N}$ can be written in the form $B = A \cap Y$ with $A \in \mathcal{M}$, the restriction of the measure $\bar{\mu}$ to $\mathcal{M}$, determines the measure $\bar{\mu}$. This implies that the finite dimensional distributions of the $C([0,1])$ or $D([0,1])$ valued stochastic process determine the distribution μ of the process and its extension $\bar{\mu}$. Given a measurable function $\mathcal{F}$ on the space $(Y, \mathcal{B})$ we call its extension any measurable function $\bar{\mathcal{F}}$ on the space $(X, \mathcal{G})$ such that $\bar{\mathcal{F}}(y) = \mathcal{F}(y)$ for all $y \in Y$. For instance we can define the extension of $\mathcal{F}$ by the formula $\bar{\mathcal{F}}(y) = \mathcal{F}(y)$ if $y \in Y$, and $\bar{\mathcal{F}}(y) = 0$ if $y \notin Y$.

We can prove formula (3.2) if the functional $\mathcal{F}$ and measures μ_0 and $\mu_T(\omega)$ are replaced by their extensions defined in the way described above. Observe that since all trajectories $X_t(\cdot, \omega)$ defined in (2.4) are in the space $C([0,1])$ or $D([0,1])$ the measures $\mu_T(\omega)$ are concentrated on the set Y. Then both the left and right-hand side of (3.2) remain the same if we rewrite the original functional $\mathcal{F}$ and measures μ_0 and $\mu_T(\omega)$ on the space $(Y, \mathcal{B})$ in this formula.

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