ALMOST SURE FUNCTIONAL LIMIT THEOREMS

Part II. The case of independent random variables

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In the first part of this paper we formulated and proved the almost sure functional limit theorem under general conditions. In this paper we prove with its help that the usual conditions of limit theorems for the distribution of appropriately normalized sums of independent random variables are also sufficient for the almost sure functional limit theorem for these independent random variables.

1. Introduction

In this paper we study the almost sure functional limit theorem for independent random variables. To make the paper more accessible we recall some definitions given in Part I. of this work.

Let $\xi_n(\omega)$, $n=1,2,\dots$, be a sequence of random variables on a probability space $(\Omega,\mathcal{A},P)$, and let us define the partial sums $S_n(\omega)=\sum\limits_{k=1}^n \xi_k(\omega)$, $n=1,2,\dots$, $S_0(\omega)\equiv 0$. Let a monotone increasing sequence B_n, $n=0,1,\dots$, of real numbers be given such that

$$B_0=0, \quad \lim_{n\to\infty} B_n=\infty, \text{ and } \quad \lim_{n\to\infty}\frac{B_{n+1}}{B_n}=1 \tag{1.1}$$

together with a positive number $\alpha>0$, and define, with the help of the above partial sums $S_n(\omega)$, $n=1,2,\dots$, the broken lines

$$\begin{aligned} S(s,\omega)&=S_j(\omega) \quad \text{if } B_{j-1}\le s<B_j,\\ S_k(s,\omega)&=B_k^{-1/\alpha}S(B_ks,\omega), \quad 0\le s\le 1,\\ &=\frac{S_{j-1}(\omega)}{B_k^{1/\alpha}} \quad \text{if } s_{j-1,k}\le s<s_{j,k},\ 1\le j\le k, \quad S_k(1,\omega)=\frac{S_k(\omega)}{B_k^{1/\alpha}},\\ &\qquad\qquad k=1,2,\dots, \end{aligned} \tag{1.2}$$

where $s_{j,k}=\dfrac{B_j}{B_k}$, $0\le j\le k$. Now we introduce the following definition.

Definition of the almost sure functional limit theorem. *Let $\xi_n(\omega)$, $n=1,2,\dots$, be a sequence of random variables, and let a sequence of real numbers B_n, $n=1,2,\dots$, be given which satisfies formula (1.1) together with some $\alpha>0$. Let us consider the random broken lines $S_k(s,\omega)$, $0\le s\le 1$, defined with the help of their partial sums $S_k(\omega)=\sum\limits_{j=1}^k \xi_j(\omega)$, $k=1,2,\dots$, by formula (1.2). For all $\omega\in\Omega$ and $N=1,2,\dots$ define* *the random probability measures $\mu_N(\omega)$ on the space $D([0,1])$ of càdlàg (continuous from the right, limit from the left) functions on the interval $[0,1]$ in the following way: The measure $\mu_N(\omega)$ is concentrated on the random broken lines $S_k(\cdot,\omega)$, $1\le k\le N$, defined in formula (1.2), and*

$$\mu_N(\omega)(S_k(\cdot,\omega)) = \frac{1}{\log \dfrac{B_N}{B_1}} \log \frac{B_{k+1}}{B_k}, \quad 1\le k<N. \tag{1.3}$$

The sequence of random variables $\xi_n(\omega)$, $n=1,2,\dots$, satisfies the almost sure functional limit theorem with weight function B_n, $n=1,2,\dots$, parameter $\alpha>0$ and limit measure μ_0 on the space $D([0,1])$ if for almost all $\omega\in\Omega$ the probability measures $\mu_N(\omega)$ defined with the help of the above constants B_n and α (appearing in formula (1.2)) converge weakly to the measure μ_0 as $N\to\infty$.

This definition can be naturally modified to measures in the space $C([0,1])$ of continuous functions on the interval $[0,1]$. It follows from the general theory that if the almost sure functional limit theorem holds in the space $D([0,1])$ and the limit measure μ_0 is concentrated in the space $C([0,1])$, then the $C([0,1])$ version of the almost sure functional limit theorem also holds. We formulated the almost sure functional limit theorem in the $D([0,1])$ space, because we want to prove it also for random variables in the domain of attraction of a stable law. In this case we have to work in the space $D([0,1])$.

The definition of the almost sure functional limit theorem given here slightly differs from that given in Part I. of this paper (see [18].) In the definition given there we have considered a sequence of A_n instead of the number α. But since in all cases we prove the almost sure functional limit theorem a sequence of the form $A_n=B_n^{1/\alpha}$ is chosen, we made this modification. In Section 2 of Part I. of this paper we formulated and proved a Corollary which states the following: Let a self-similar process $X(t,\omega)$, $t\ge0$, be given with a self-similarity parameter $\alpha>0$, (see its definition in Part I.) which also satisfies some additional weak conditions which are not serious restriction in possible applications, together with a sequence B_n, $n=1,2,\dots$, of real numbers for which relation (1.1) holds. Then the random variables $\eta_n(\omega)=X(B_n,\omega)-X(B_{n-1},\omega)$, $n=1,2,\dots$, satisfy the almost sure functional limit theorem with weight function B_n, $n=1,2,\dots$, parameter α and limit measure μ_0 which is the distribution of the self-similar process restricted to the interval $[0,1]$.

In particular, by applying this result with the choice of the Wiener process $W(t,\omega)$ as the self-similar process together with a sequence B_n satisfying formula (1.1) we get that the almost sure functional limit theorem holds for a sequence of independent Gaussian random variables $\eta_n(\omega)$, $n=1,2,\dots$, $E\eta_n(\omega)=0$, $E\eta_n^2(\omega)=B_n-B_{n-1}$ with the weight functions B_n, parameter $\alpha=2$ and limit measure μ_0 which is the Wiener measure, i.e. the distribution of the process $W(t,\omega)$, $0\le t\le1$. Similarly, we get by considering a stable process $X(t,\omega)$ with self-similarity parameter α, $0<\alpha<2$, $\alpha\ne1$, as a self-similar process and a sequence of real numbers B_n satisfying relation (1.1), that a sequence of independent random variables $\eta_n(\omega)$, $n=1,2,\dots$, such that the distribution of $\eta_n(\omega)$ agrees with the distribution of $X(B_n - B_{n-1}, \omega)$ satisfies the almost sure functional limit theorem with weight function B_n, parameter α and limit measure μ_0 which is the distribution of the restriction of the process $X(t,\omega)$ to the interval $[0,1]$. We prove that if a sequence of independent random variables is given whose normalized partial sums converge in distribution either to the standard normal distribution or to a stable law, then this sequence also satisfies the almost sure functional limit theorem. Such statements are the main results of this paper. They are formulated in the following Theorems.

Theorem 1. *Let $\xi_n(\omega)$, $n = 1, 2, \dots$, be a sequence of independent random variables such that $E\xi_n(\omega) = 0$, $E\xi_n^2(\omega) = \sigma_n^2$, and it satisfies the Lindeberg condition, i.e.*

$$\lim_{n\to\infty} \frac{1}{D_n^2} \sum_{k=1}^{n} E\xi_k^2(\omega) I(|\xi_k(\omega)| \ge \varepsilon D_n) = 0 \quad \textit{for all } \varepsilon > 0, \tag{1.4}$$

where $D_n^2 = \sum\limits_{k=1}^{n} \sigma_k^2$. Then the sequence of random variables $\xi_n(\omega)$, $n = 1, 2, \dots$, satisfies the almost sure functional (central) limit theorem with the Wiener measure μ_0 as the limit measure, weight function $B_n = D_n^2$, $n = 1, 2, \dots$, and parameter $\alpha = 2$.

Theorem 2. *Let $\xi_n(\omega)$, $n = 1, 2, \dots$, be a sequence of independent, identically distributed random variables with a non-degenerated distribution such that $E\xi_1(\omega) = 0$ and $\mu(x) = E\xi_1^2(\omega) I(|\xi_1(\omega)| \le x)$ is a slowly varying function at infinity. Define the numbers a_n as $a_n = \sup\left\{u\colon n\dfrac{\mu(u)}{u^2} \ge 1\right\}$ for all $n \ge n_0$ with a sufficiently large integer n_0. (This definition is meaningful. To see this observe that $n\dfrac{\mu(u)}{u^2} > 1$ for an appropriate u and all sufficiently large n, and $\lim\limits_{u\to\infty} \dfrac{n\mu(u)}{u^2} = 0$ for all n if $\mu(u)$ is a slowly varying function.) Define the sequence a_n in the above way if $n \ge n_0$, and for the sake of a unique definition put $a_n = a_{n_0}$ for $n \le n_0$. Then the sequence $\xi_n(\omega)$, $n = 1, 2, \dots$, satisfies the almost sure (central) functional limit theorem with the Wiener measure μ_0 as the limit measure, weight function $B_n = \sum\limits_{k=1}^{n} \mu(a_k)$, $n = 1, 2, \dots$, and parameter $\alpha = 2$.*

To formulate the following Theorem 3 let us recall that the distribution function $G(x)$ of a stable law, $0 < \alpha < 2$, is determined by three parameters α, $C_1 \ge 0$ and $C_2 \ge 0$, $C_1 + C_2 > 0$, such that the relations

$$\begin{aligned} G(x) &\sim C_1 x^{-\alpha} \quad \text{if } x \to \infty, \\ 1 - G(-x) &\sim C_2 x^{-\alpha} \quad \text{if } x \to \infty \end{aligned} \tag{1.5}$$

hold. Let us also recall that for all stable laws $G(x)$ satisfying (1.5) with $0 < \alpha < 2$, $\alpha \ne 1$, there is a (stable) process $X(t,\omega)$, $0 \le t \le 1$, with trajectories in the space $D([0,1])$ such that it has independent and stationary increments, and $X(1,\omega)$ has the distribution function $G(x)$. The distribution of this process $X(t,\omega)$ in the space $D([0,1])$ is uniquely determined. It is a self-similar process with self-similarity parameter α.

In the next Theorem 3 we formulate the following result: If such conditons are imposed under which the normalized partial sums of a sequence of independent, identically distributed random converge in distribution to a stable law, then these random variables also satisfy the almost sure functional limit theorem. Before formulating the precise statement let us introduce some notations. Let us consider a distribution function $F(x)$ which satisfies the following condition:

$$\begin{aligned} 1-F(x) &\sim C_1 x^{-\alpha} L(x) \\ F(-x) &\sim C_2 x^{-\alpha} L(x) \end{aligned} \qquad \text{if } x \to \infty, \tag{1.6}$$

where $L(x)$ is a slowly varying function at infinity, $C_1 \ge 0$, $C_2 \ge 0$, $C_1 + C_2 > 0$, $0 < \alpha < 2$, $\alpha \ne 1$. Let us define the following functions $b(x)$ and $\bar{L}(x)$ which appear in the normalization of the almost sure functional theorem (and also in the usual limit theorem) for i.i.d. random variables with distribution function $F(x)$ which satisfies condition (1.6). For all $x > 0$ put $b(x) = \max\left\{u\colon \frac{L(u)x}{u^\alpha} \ge 1\right\}$, and let $\bar{L}(x) = b(x)^\alpha x^{-1}$, where the number α and function $L(\cdot)$ are the same as in formula (1.6). Let μ_0 denote the (uniquely determined) distribution of the stable process $X(t,\omega)$ with parameter $\alpha > 0$, $0 \le t \le 1$, in the space $D([0,1])$ for which $X(1,\omega)$ has that stable distribution $G(x)$ which satisfies relation (1.5). (The numbers α, C_1 and C_2 are the same in formulas (1.5) and (1.6).)

Theorem 3. *Let $\xi_n(\omega)$, $n = 1, 2, \dots$, be a sequence of independent and identically distributed random variables with a distribution function $F(x)$ which satisfies Condition (1.6). Then there is a sequence of real numbers a_n such that the sequence of random variables $\xi_n(\omega) - a_n$, $n = 1, 2, \dots$, satisfies the almost sure functional limit theorem with weight function $B_n = \sum_{k=1}^{n} \bar{L}(k)$, $n = 1, 2, \dots$ and limit measure μ_0, where the function $\bar{L}(x)$ and measure μ_0 were defined before the formulation of this Theorem 3.*

We also claim that the function $b(x)$ defined before the formulation of this Theorem 3 is a regularly varying function at infinity with parameter $1/\alpha$, hence $\bar{L}(x) = b(x)^\alpha x^{-1}$ is a slowly varying function.

The normalized partial sums $\frac{1}{B_n^{1/\alpha}} \sum_{k=1}^{n} (\xi_k(\omega) - a_n)$ converge in distribution to $G(x)$ as $n \to \infty$ if the constants a_n are the same as in the almost sure functional limit theorem formulated in this theorem.

Remark 1: With some modification in the proof the last statement of Theorem 3, the limit theorem for the distribution of the normalized partial sums, can be replaced by a stronger version of this result. The functional limit theorem also holds for the distributions of the random broken lines made for all $n = 1, 2, \dots$ from the normalized partial sum $S_k(\omega) = \frac{1}{B_n^{1/\alpha}} \sum_{j=1}^{k} (\xi_j(\omega) - a_n)$, $k = 1, \dots, n$, in the natural way. The limit is the same measure μ_0 which appears as the limit in the almost sure limit theorem.

We also explain without working out all details that Theorem 3 also holds for $\alpha = 1$ with some slight natural modifications.

Theorem 3′. *The results of Theorem 3 also hold in the case $\alpha = 1$ with the slight modification that in this case the "shift parameters" a_n, $n = 1, 2, \dots$ must be defined differently, the measure μ_0 is the distribution of the process $X(t) - \gamma t \log t$ with the constant $\gamma = \gamma(C_1, C_2) = C_1 - C_2$, where $X(t, \omega)$ is the stable process with parameter $\alpha = 1$ for which $X(1, \omega)$ has distribution function $G(x)$ which satisfies relation (1.5).*

Remark 2. The norming (shift) constants a_n in the limit theorem for the distribution of the normalized partial sums mentioned at the end of Theorem 3 are determined with a certain accuracy uniquely. Namely, the norming constants a_n can be replaced by another norming constant $\bar{a}_n$ if and only if $a_n - \bar{a}_n = o\left(\frac{B_n^{1/\alpha}}{n}\right)$. This implies that the norming sequence a_k, $k = 1, 2, \dots$, in the almost sure functional limit theorem can be replaced by any such sequence $\bar{a}_k$ for which the number $\bar{a}_n$ can be chosen as the norming (shift) constant in the partial sum of the first n term in the limit theorem for the distribution of the partial sums. To see this it is enough to observe that $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{a_k - \bar{a}_k}{B_n^{1/\alpha}} = 0$, since $\sum\limits_{k=1}^{n} \frac{a_k - \bar{a}_k}{B_n^{1/\alpha}} = o\left(\sum\limits_{k=1}^{n} \frac{B_k^{1/\alpha}}{k B_n^{1/\alpha}}\right) = o(1)$.

Let us remark that the conditions imposed in Theorems 1 and 2 are the natural conditions for the central limit theorem for sums of independent random variables. Theorem 1 contains the necessary and sufficient conditions of the central limit theorem for sums of independent random variables with the natural norming constants $D_n^{1/2}$ if these random variables satisfy the condition of uniform smallness. The conditions in Theorem 2 are the necessary and sufficient conditions of the central limit theorem with an appropriate normalization for sums of independent and identically distributed random variables. Similarly, in Theorem 3 the necessary and sufficient conditions of the limit theorem with a stable limit law for partial sums of independent and identically distributed random variables were imposed.

The above results will be proved by means of a coupling argument. In Part I. we introduced a notion we called the Property A. We showed that if Property A holds for a pair of sequences of random variables $(\xi_n(\omega), \eta_n(\omega))$, $n = 1, 2, \dots$, and the sequence $\eta_n(\omega)$, $n = 1, 2, \dots$, satisfies the almost sure functional limit theorem, then the sequence of random variables $\xi_n(\omega)$, $n = 1, 2, \dots$, also satisfies the almost sure functional limit theorem with the same weight function B_n, parameter α and limit measure μ_0 as the sequence $\eta_n(\omega)$, $n = 1, 2, \dots$. We shall prove Theorems 1, 2 and 3 with the help of this result which we recall in the next section.

A result we shall call the Basic Lemma will be formulated and proved. Then it will be proved with its help that the sequences of random variables considered in these theorems together with a sequence of appropriately defined independent random variables with normal or stable distributions satisfy Property A. The theorems follow from Property A for these pairs of sequences and the almost sure functional limit theorem for independent random variables with Gaussian or stable distribution mentioned in the beginning of this paper.

This paper consists of six sections. In Section 2 we formulate a result we call the Basic Lemma. This Basic Lemma will be proved in Section 3. In Section 4 we prove Theorems 1 and 2, some almost sure functional limit theorems with the Wiener measure as the limit measure. In Section 5 we prove Theorem 3, the almost sure functional limit theorem for independent, identically distributed random variables in the domain of attraction of a stable law. Finally, Section 6 contains the formulation of certain open problems and some comments. Here we also compare briefly our results with those of earlier papers.

2. Formulation of Property A and the Basic Lemma

First we recall the definition of Property A which enables us to prove the almost sure functional limit theorem in several interesting cases.

Definition of Property A. *Let $\eta_n(\omega)$, $n=1,2,\dots$, be a sequence of random variables which satisfies the almost sure functional limit theorem with a limit measure μ_0 in the space $D([0,1])$ with some weight function B_n, $n=0,1,\dots$, satisfying relation (1.1) and parameter $\alpha>0$. Let us also assume that the limit measure μ_0 is the distribution of the restriction of a self-similar process $X(u,\omega)$ with self-similarity parameter $\alpha>0$ to the interval $0\le u\le 1$.*

Define the indices $N(n)$ as $N(n)=\inf\{k\colon B_k\ge 2^n\}$, $n=0,1,\dots$. The pair of sequences of random variables $(\xi_n(\omega),\eta_n(\omega))$, $n=1,2,\dots$, satisfies Property A if for all $\varepsilon>0$ there exists a sequence of random variables $\tilde\xi_n(\omega)=\tilde\xi_n(\varepsilon,\omega)$, $n=1,2,\dots$, whose (joint) distribution agrees with the (joint) distribution of the sequence $\xi_n(\omega)$, $n=1,2,\dots$, and the partial sums $\tilde S_n(\omega)=\sum\limits_{k=1}^n\tilde\xi_k(\omega)$ and $T_n(\omega)=\sum\limits_{k=1}^n\eta_k(\omega)$, $n=1,2,\dots$, satisfy the relation

$$\limsup_{n\to\infty}\frac1n\sum_{k=1}^{N(n)}\log\frac{B_{k+1}}{B_k}I\left(\left\{\omega\colon \frac{\sup\limits_{0\le j\le k}|\tilde S_j(\omega)-T_j(\omega)|}{B_k^{1/\alpha}}>\varepsilon\right\}\right)\le\varepsilon \tag{2.1}$$

for almost all $\omega\in\Omega$, where $I(A)$ denotes the indicator function of the set A.

In Theorem 4 of Part I. we have proved that if the pair of sequences of random variables $(\xi_n(\omega),\eta_n(\omega))$, $n=1,2,\dots$, satisfies Property A, then the sequence of random variables $\xi_n(\omega)$, $n=1,2,\dots$, satisfies the almost sure functional limit theorem with the same weight function B_n, parameter $\alpha>0$ and limit measure μ_0 as the sequence of random variables $\eta_n(\omega)$, $n=1,2,\dots$. In this section we formulate a lemma which enables us to check Property A in several interesting cases.

To prove Property A we need a good construction of the pairs $(\tilde\xi_n(\omega),\eta_n(\omega))$, $n=1,2,\dots$. Let us first briefly describe a standard method which produces a construction for partial sums of independent random variables in such a way that the differences of the partial sums $\tilde S_n(\omega)-T_n(\omega)$, $n=1,2,\dots$, made from these random variables are relatively small for almost all ω.

Let us choose an appropriate subsequence n_j of the integers and apply a so-called quantile transform, to be described later, which makes the differences $(S_{n_j}(\omega) - S_{n_{j-1}}(\omega)) - (T_{n_j}(\omega) - T_{n_{j-1}}(\omega))$ relatively small, and for which the expressions $S_{n_j}(\omega) - S_{n_{j-1}}(\omega)$ and $T_{n_j}(\omega) - T_{n_{j-1}}(\omega)$ have the right distribution. We make such constructions independently for all $j = 1, 2, \dots$, and define the random variables $S_{n_j}(\omega)$ and $T_{n_j}(\omega)$ as the partial sums of these terms. Then these subsequences can be extended to two sequences $S_n(\omega)$ and $T_n(\omega)$ which have the same joint distribution as the partial sums of the independent random variables $\xi_k(\omega)$ and $\eta_k(\omega)$, $k = 1, 2, \dots$. The differences between the random variables $S_{n_j}(\omega) - S_{n_{j-1}}(\omega)$ and $T_{n_j}(\omega) - T_{n_{j-1}}(\omega)$ constructed in the above way can be well estimated if we have a good control on the distance of the distribution functions of these partial sums. This enables us to bound the differences $S_{n_j}(\omega) - T_{n_j}(\omega)$, and then by bounding the fluctuation of the sequences $S_n(\omega)$ and $T_n(\omega)$ between these points we get an estimate about the goodness of this approximation.

Such a construction, with the appropriate choice of the numbers n_j is made in certain papers to get an almost sure approximation of a sequence $T_n(\omega)$ of partial sums of independent random variables with a sequence $S_n(\omega)$, $n = 1, 2, \dots$, of partial sums of different independent random variables. The construction we make to satisfy Property A in case of independent random variables is similar. The only difference is that we want to get a good bound on the expression in formula (2.1) instead of an almost sure approximation, hence we choose the sequence n_j according to this requirement. Now we only demand that the differences $S_n(\omega) - T_n(\omega)$ be small for most indices n. The existence of some exceptional indices n (depending on ω) where this difference is large is allowed if they do not enlarge considerably the expression in formula (2.1). The role of the following Basic Lemma is that it enables us to show that a construction satisfying Property A can be made under relatively weak conditions. Before its formulation we give an informal explanation about the technical details n it, and also make some indication about its role in the proof of Theorems 1—3.

The Basic Lemma actually states that the above sketched construction with an appropriate choice of the points n_j satisfies inequality (2.1). In this lemma we give a bound for the partial sums of some random variables $\zeta_k(\omega)$, $k = 1, 2, \dots$, if they satisfy certain conditions. The bound (2.6) proved in the Basic Lemma will be applied with the choice $\zeta_k(\omega) = \tilde{\xi}_k(\omega) - \eta_k(\omega)$ where $\tilde{\xi}_k(\omega)$ and $\eta_k(\omega)$, $k = 1, 2, \dots$, are random variables constructed in the above way. Let us remark that when the sequences $S_n(\omega)$ and $T_n(\omega)$ are compared, then the natural time scale is measured by the sequence B_n. (To understand this let us look at the definition of the process $S(t, \omega)$ in formula (1.2).) Let us consider an exponentially rare sequence of the time parameter. This is the content of the definition of the numbers $N(n)$ in the formulation of the Basic Lemma which guarantees that $B_{N(n)} \sim 2^n$. We shall also define a refinement $N(n, k)$ of this sequence, and these points $N(n, k)$ will play the role of the points n_j where the quantile transform will be applied in the above sketched construction.

We make a decomposition of the random variables $\zeta_k(\omega)$ in formula (2.3) of the Basic Lemma which will be satisfied in our applications with the natural choice $\zeta_k^{(1)}(\omega) = \tilde{\xi}_k(\omega)$ and $\zeta_k^{(2)}(\omega) = \eta_k(\omega)$. The Basic Lemma also contains a condition about the independence of the random variables $U_{N(n,k)}(\omega) - U_{N(n,k-1)}(\omega)$, $n = 1, 2, \dots$, $k =$

$1, \dots, l_n$, defined there, but this is a condition which is automatically satisfied in the constructions we apply in the proof of Theorems 1—3. In formula (2.5) we formulate an estimate which can be satisfied in our applications if we define the points $N(n,k)$ in an appropriate way and give a good estimate for the difference of the partial sums of the random variables $\tilde\xi_j(\omega)$ and $\eta_j(\omega)$ with indices j between these points by means of an estimate on the quantile transform. In our applications formula (2.4) states a good bound for the fluctuation of the partial sums of the random variables $\xi_k(\omega)$ and $\eta_k(\omega)$, $k = 1, 2, \dots$. Let us also remark we need the bounds in formulas (2.4) and (2.5) only for large indices n, and the threshold index from which they must hold may depend on the parameter $\varepsilon > 0$ appearing in formula (2.6). These are the conditions imposed in the Basic Lemma to satisfy formula (2.6).

We shall prove Theorems 1 — 3 by means of the Basic Lemma. An important step of the proof is a good choice of the points $N(n,k)$ between which the quantile transform will be applied. This choice is essentially different in the proof of Theorem 3 and Theorem 1. In Theorem 3 the points $N(n,k)$ contain all integer points between $N(n-1)$ and $N(n)$. In this case formula (2.4) is an empty condition. In the proof of Theorem 1 the numbers $N(n,k)$ will be chosen in such a way that $B_{N(n,k+1)} - B_{N(n,k)} \sim \bar\varepsilon 2^n$, where the coefficient $\bar\varepsilon > 0$ is a very small number, but it does not depend on the number n. This means that the numbers $N(n,k)$ (with fixed n) are relatively uniformly distributed in the interval $[N(n-1), N(n))$, and the difference between them is relatively large. The cause of the different choice of $N(n,k)$ in the proof of Theorem 1 and Theorem 3 is that under the conditions of Theorem 3 the single terms $\xi_n(\omega)$ and $\eta_n(\omega)$ have a similar distribution, while under the conditions of Theorem 1 one can guarantee the similar distribution of the partial sums of the random variables $\xi_n(\omega)$ and $\eta_n(\omega)$, by means of the central limit theorem, only if these partial sums have sufficiently many terms. In the proof of Theorem 2 the single terms $\xi_n(\omega)$ will be written up as sums of random variables by means of an appropriate truncation. Then in the proof of Theorem 2 different partial sums have to be handled. All of them will be investigated by means of the Basic Lemma, but some of them will be estimated with a choice of the numbers $N(n,k)$ similar to that given in the proof of Theorem 1 and some of them with a choice of the numbers $N(n,k)$ similar to that given in the proof of Theorem 3.

Now we turn to the formulation of the Basic Lemma. First we introduce the following definition.

Definition of refinement of a sequence of integers. *Given a sequence* $0 = N(0) < N(1) < N(2) < \cdots$ *of integers we call the refinement of this sequence* $N(n)$, $n = 0, 1, 2, \dots$ *a set of non-negative integers* $N(n,k)$ *indexed by two parameters* $n = 1, 2, \dots$, *and* $0 \le k \le l_n$ *with some positive integer* l_n *such that*

$$N(n-1) = N(n,0) < N(n,1) < \cdots < N(n,l_n) = N(n), \quad n = 1, 2, \dots.$$

Basic Lemma. *Let* B_n, $n = 0, 1, \dots$, $B_0 = 1$, *be a sequence of real numbers which satisfies relation (1.1). Let a sequence of random variables* $\zeta_n(\omega)$, $n = 1, 2, \dots$ *and a number* $\alpha > 0$ *also be given. Define the sequence* $N(n) = \inf\{k\colon B_k \ge 2^n\}$, $n = 1, 2, \dots$,

$N(0)=0$. We give an estimate on the maximum of the partial sums of the random variables $\zeta_n(\omega)$ under appropriate conditions.

Fix a number $\varepsilon>0$ and a refinement $N(n,k)$, $n=0,1,\dots$, $0\le k\le l_n$, of the sequence $N(n)$, $n=1,2,\dots$, which may depend on ε. Put

$$U_n(\omega)=\sum_{j=1}^n \zeta_j(\omega), \quad n=1,2,\dots,$$

and

$$V_n(\omega)=\sup_{0<j\le l_n}\left|U_{N(n,j)}(\omega)-U_{N(n,0)}(\omega)\right|, \quad n=1,2,\dots. \tag{2.2}$$

Let us assume that there is a decomposition

$$\zeta_k(\omega)=\zeta_k^{(1)}(\omega)-\zeta_k^{(2)}(\omega), \quad k=1,2,\dots \tag{2.3}$$

of the random variables $\zeta_k(\omega)$ in such a way that both sequences $\zeta_k^{(i)}(\omega)$, $k=1,2,\dots$, $i=1,2$, consist of independent random variables which satisfy certain inequalities formulated below. To formulate them let us introduce the notation

$$\begin{gathered}\zeta_{n,k,m}^{(i)}(\omega)=\zeta_{N(n,k-1)+m}^{(i)}(\omega), \quad n=1,2,\dots,\ k=1,\dots,l_n,\\ 1\le m\le N(n,k)-N(n,k-1), \quad i=1,2\end{gathered}$$

with the help of the refinement $N(n,k)$ of the sequence $N(n)$, $n=1,2,\dots$, fixed in this lemma. Let us assume that the following inequalities hold:

$$\begin{gathered}P\left(\sup_{1\le p<N(n,k)-N(n,k-1)}\left|\sum_{m=1}^p \zeta_{n,k,m}^{(i)}(\omega)\right|>\varepsilon x 2^{n/\alpha}\right)\le\frac{C_1\varepsilon x^{-\gamma}}{l_n}\\ i=1,2, \quad \text{for all } x\ge 1,\ \ n\ge n_0,\ \text{ and } 1\le k\le l_n\end{gathered} \tag{2.4}$$

with some constants $n_0=n_0(\varepsilon)>0$, $\gamma>0$ and $C_1>0$. (In the case $N(n,k)=N(n,k-1)+1$ this sum is empty. In this case we assume that relation (2.4) is satisfied.) Let us also assume that the random variables $U_{N(n,k)}(\omega)-U_{N(n,k-1)}(\omega)$, $n=1,2,\dots$, $k=1,\dots,l_n$, are independent, and the inequality

$$P\left(V_n(\omega)\ge\varepsilon x 2^{n/\alpha}\right)\le C_2\varepsilon x^{-\gamma} \quad \text{for all } x\ge 1 \text{ and } n\ge n_0 \tag{2.5}$$

holds with some $n_0=n_0(\varepsilon)$, $\gamma>0$ and $C_2>0$. Then

$$\limsup_{n\to\infty}\frac1n\sum_{k=1}^{N(n)}\log\frac{B_{k+1}}{B_k} I\left(\frac{\sup\limits_{0\le s\le k}|U_s(\omega)|}{B_k^{1/\alpha}}>K\varepsilon\right)\le K\varepsilon \tag{2.6}$$

for almost all $\omega\in\Omega$.

with an appropriate constant $K = K(C_1, C_2, \gamma, \alpha) > 0$. (Such a constant K could be given explicitly, but we do not need such a formula. It is enough to know that this constant K does not depend on ε.)

We shall be able to prove Property A under weak conditions by applying the Basic Lemma for arbitrary small $\varepsilon > 0$. In such a way we show that the conditions sufficient for a limit theorem for partial sums of independent random variables also imply Property A with an appropriate construction. On the other hand, a construction by means of an almost sure approximation only supplies a weaker result. Indeed, by applying such a construction we get bounds sufficient for our purposes only under some additional conditions. The reason for this difference is that the condition of Property A formulated in formula (2.1) only demands that the differences $S_n(\omega) - T_n(\omega)$ be small in some average. Let us remark that there are even results, (see Berkes and Csáki [4]) which state that there are cases when the almost sure functional limit theorem holds for a sequence of independent random variables, but their partial sums do not satisfy a limit theorem.

Finally we remark that in Conditions (2.4) and (2.5) an estimate on the tail behaviour of the partial sums was formulated. In a limit theorem for normalized partial sums of independent random variables we do not require such an estimate. But the conditions formulated in Theorems 1, 2 and 3, i.e. the necessary and sufficient conditions of certain limit theorems also imply an estimate on the tail behaviour appropriate for our purposes. Actually, conditions (2.4) and (2.5) can be considerably weakened. The power $|x|^{-\gamma}$ at the right-hand side of these formulas could be replaced by $(1+\log|x|)^{-\gamma'}$ with a sufficiently large $\gamma' > 0$. But such a condition does not seem to be better applicable in the problems we are interested in.

3. Proof of the Basic Lemma

Proof of the Basic Lemma. We shall prove relation (2.6) with the help of two inequalities. In these inequalities the supremum is taken for an appropriate subsequence. To formulate them we define the numbers $L_0 = 1$, $L_p = \sum_{k=1}^{p} l_k$, $p = 1, 2, \dots$, and the sequence $m(j)$, $j = 0, 1, \dots$, by the formula $m(0) = 0$ and $m(j) = N(p-1, j - L_{p-1})$, if $L_{p-1} < j \le L_p$. The numbers l_p and $N(p, j)$ in these formulas are the same as those considered in the formulation of the Basic Lemma. The number $m(j)$ counts the value of the j-th term among the numbers $N(n, k)$. Observe that in particular $m(L_n) = N(n-1, l_n) = N(n)$, $n = 1, 2, \dots$. Put $A_n = B_n^{1/\alpha}$. We shall prove that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{L_n} \log \frac{B_{m(j)+1}}{B_{m(j-1)+1}} I\left(\frac{\sup\limits_{1\le s\le j} |U_{m(s)}(\omega)|}{A_{m(j)}} > K_1\varepsilon \right) \le K_1\varepsilon \quad \text{for almost all } \omega \in \Omega \tag{3.1}$$

with an appropriate constant $K_1 > 0$, and

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{L_n} \log \frac{B_{m(j)+1}}{B_{m(j-1)+1}} I\left(\frac{\sup\limits_{1\le s\le j} \sup\limits_{m(s-1)<p<m(s)} |U_p(\omega) - U_{m(s-1)}(\omega)|}{A_{m(j)}} > K_2\varepsilon \right) \le K_2\varepsilon \quad \text{for almost all } \omega \in \Omega \tag{3.2}$$

with an appropriate constant $K_2 > 0$. First we show that relations (3.1) and (3.2) imply relation (2.6) with $K = 3^{1/\alpha}(K_1 + K_2)$.

Indeed, if for some $\omega \in \Omega$ there is an index k, $n_0 \le k \le N(n)$ with some $n_0 = n_0(\varepsilon)$, such that it gives a non-zero contribution to the sum in (2.6) with the choice $K = 3^{1/\alpha}(K_1 + K_2)$, i.e. $\sup\limits_{1\le s\le k} |U_s(\omega)| > 3^{1/\alpha}(K_1 + K_2)A_k\varepsilon$, then consider that interval $(m(j-1), m(j)]$, $1 \le j \le L_n$, which contains this number k. In this case one of the following relations holds. Either

$$\sup_{1\le s\le j} |U_{m(s)}(\omega)| > 3^{1/\alpha} K_1 A_k \varepsilon \ge K_1 A_{m(j)}\varepsilon$$

or

$$\sup_{1\le s\le j} \sup_{m(s-1)<p<m(s)} |U_p(\omega) - U_{m(s-1)}(\omega)| > 3^{1/\alpha} K_2 A_k\varepsilon \ge K_2 A_{m(j)}\varepsilon.$$

Then the contribution of the terms with indices in the interval $(m(j-1), m(j)]$ to the sum in the expression (2.6) is not greater than $\log \frac{B_{m(j)+1}}{B_{m(j-1)+1}}$, and such a contribution appears in the j-th term of one of the sum (3.1) or (3.2). Hence relations (3.1) and

(3.2), the identity $m(L_n)=N(n)$ together with a summation for $1\le j\le L_n$ imply formula (2.6).

To prove relation (3.1) introduce the random variables

$$T_s(\omega)=U_{m(s)}(\omega)-U_{m(s-1)}(\omega),\quad s=1,2,\dots.$$

The random variables $T_s(\omega)$, $s=1,2,\dots$ are independent. This statement is equivalent to the independence of the random variables $U_{N(n,k)}(\omega)-U_{N(n,k-1)}(\omega)$, $n=1,2,\dots$, $k=1,\dots,l_n$, and this is a condition imposed in the formulation of the Basic Lemma.

Since $\lim\limits_{n\to\infty}\dfrac{B_{N(n+1)}}{B_{N(n)}}=2$, $A_n=B_n^{1/\alpha}$, there is some $n_0>0$ such that

$$A_{N(n)}\ge 2^{(n-k)/2\alpha}A_{N(k)}\quad\text{for arbitrary } n\ge n_0 \text{ and } k\le n.$$

For all $s=1,2,\dots$ define the number $R(s)$ which satisfies the inequality $L_{R(s)-1}<s\le L_{R(s)}$. The number $R(s)$ counts the number of the form $N(l,0)=N(l-1)$ among the first s terms of the sequence $N(n,k)$. This fact and the content of the value of $m(j)$ imply that $N(R(s)-1))<m(s)\le N(R(s))$. (The sequence $R(s)$ is the "inverse" of the monotone sequence L_s. The relation $R(L_s)=s$ holds.) Hence

$$\frac{A_{m(j)}}{A_{m(s)}}\ge 2^{(R(j)-R(s)-1)/2\alpha}\quad\text{for } 1\le s\le j \text{ and } j\ge n_0.$$

Let us fix some $j\ge n_0$. We shall show by applying the above relation for $s\le j$ and by putting in one block those indices s for which $N(r-1)<m(s)\le N(r)$, or equivalently $L_{r-1}<s\le L_r$ that

$$\begin{aligned}\left\{\omega\colon \frac{\sup\limits_{1\le s\le j}\left|U_{m(s)}(\omega)\right|}{A_{m(j)}}\ge K_1\varepsilon\right\}&=\left\{\omega\colon \left|\sum_{p=1}^{s}T_p(\omega)\right|\ge K_1\varepsilon A_{m(j)}\ \text{ for some } 1\le s\le j\right\}\\ &\subset\bigcup_{r=1}^{R(j)}\left\{\omega\colon \sup_{L_{r-1}<u\le L_r}\left|\sum_{p=L_{r-1}+1}^{u}T_p(\omega)\right|\ge CK_1\varepsilon 2^{(R(j)-r)/4\alpha}A_{m(L_{r-1})}\right\}\end{aligned}\tag{3.3}$$

with $C=1-2^{-1/4\alpha}$. To prove relation (3.3) observe that if $L_{r-1}<j$, then

$$A_{m(j)}\ge 2^{(R(j)-R(L_{r-1})-1)/2\alpha}A_{m(L_{r-1})}=2^{(R(j)-r)/2\alpha}A_{m(L_{r-1})},$$

hence if some ω is not contained in the set at the right-hand side, i.e.

$$\sup_{L_{r-1}<u\le L_r}\left|\sum_{p=L_{r-1}+1}^{u}T_p(\omega)\right|<CK_1\varepsilon 2^{(R(j)-r)/4\alpha}A_{m(L_{r-1})}\quad\text{for all } 1\le r\le R(j)$$

then

$$\left|\sum_{p=1}^{s} T_p(\omega)\right| < \sum_{r=1}^{R(j)} \sup_{L_{r-1}<u\le L_r} \left|\sum_{p=L_{r-1}+1}^{u} T_p(\omega)\right|$$
$$\le CK_1\varepsilon \sum_{r=1}^{R(j)} 2^{-(R(j)-r)/4\alpha} 2^{(R(j)-r)/2\alpha} A_{m(L_{r-1})}$$
$$\le CK_1\varepsilon \sum_{r=1}^{R(j)} 2^{-(R(j)-r)/4\alpha} A_{m(j)} < K_1\varepsilon A_{m(j)}$$

for all $s \le L_{R(j)}$, hence for all $s \le j$, and this means that ω is not contained in the set at the left-hand side of formula (3.3).

We get from formula (3.3), the definition of the random variables $V_r(\omega)$ introduced in formula (2.2) and the relation $A_{m(L_{r-1})} = B^{1/\alpha}_{m(L_{r-1})} = B^{1/\alpha}_{N(r-1)} \ge 2^{(r-1)/\alpha}$ that

$$\left\{\omega\colon \frac{\sup\limits_{1\le s\le j} \left|U_{m(s)}(\omega)\right|}{A_{m(j)}} \ge K_1\varepsilon\right\} \subset \bigcup_{r=1}^{R(j)} \left\{\omega\colon V_r(\omega) \ge CK_1\varepsilon 2^{(R(j)-r)/4\alpha} \times 2^{(r-1)/\alpha}\right\}$$
$$= \bigcup_{r=1}^{R(j)} \left\{\omega\colon V_r(\omega) \ge CK_1 2^{(R(j)+3r-4)/4\alpha}\varepsilon\right\} \qquad (3.4)$$

We shall prove relation (3.1) with the help of (3.4). Let us first sum for $R(j) = p$ with a fixed $p \ge n_0$. (Observe that the right-hand side of (3.4) depends on j only through $R(j)$.) We get that

$$\sum_{j\colon R(j)=p} \log \frac{B_{m(j)+1}}{B_{m(j-1)+1}} I\left(\frac{\sup\limits_{1\le s\le j} |U_{m(s)}(\omega)|}{A_{m(j)}} > K_1\varepsilon\right)$$
$$\le \sum_{j\colon R(j)=p} \log \frac{B_{m(j)+1}}{B_{m(j-1)+1}} \sum_{r=1}^{p} I\left(V_r(\omega) \ge CK_1 2^{(p+3r-4)/4\alpha}\varepsilon\right)$$
$$= \sum_{r=1}^{p} I\left(V_r(\omega) \ge CK_1 2^{(p+3r-4)/4\alpha}\varepsilon\right) \sum_{j\colon R(j)=p} \log \frac{B_{m(j)+1}}{B_{m(j-1)+1}} \qquad (3.5)$$
$$\le \log \frac{B_{N(p+1)+1}}{B_{N(p)}} \sum_{r=1}^{p} I\left(V_r(\omega) \ge CK_1 2^{(p+3r-4)/4\alpha}\varepsilon\right)$$
$$\le 2\sum_{r=1}^{p} I\left(V_r(\omega) \ge CK_1 2^{(p+3r-4)/4\alpha}\varepsilon\right)$$

if $p \ge n_0$ with a sufficiently large threshold n_0.

We get a good bound on the expression in (3.1) by summing the estimates (3.5) for $p=L_{n_0},L_{n_0}+1,\dots$, exploiting that the terms of (3.1) not considered in such a way give only a bounded contribution, and the relation $R(j)\le n$ holds for $j\le L_n$. In such a way we obtain that

$$\begin{aligned}&\sum_{j=1}^{L_n}\log\frac{B_{m(j)+1}}{B_{m(j-1)+1}}I\left(\frac{\sup\limits_{1\le s\le j}|U_{m(s)}(\omega)|}{A_{m(j)}}>K_1\varepsilon\right)\\&\qquad\le 2\sum_{p=1}^{n}\sum_{r=1}^{p}I\left(V_r(\omega)\ge CK_12^{(p+3r-4)/4\alpha}\varepsilon\right)+\text{const.}\end{aligned}\tag{3.6}$$

Define the random variables

$$\chi_r(\omega)=\sum_{p=0}^{\infty}I\left(2^{-r/\alpha}V_r(\omega)\ge CK_12^{(p-4)/4\alpha}\varepsilon\right),\quad r=1,2,\dots.$$

We can write with the help of relation (3.6) by changing the order of summation at the right-hand side that

$$\begin{aligned}&\limsup_{n\to\infty}\frac1n\sum_{j=1}^{L_n}\log\frac{B_{m(j)+1}}{B_{m(j-1)+1}}I\left(\frac{\sup\limits_{1\le s\le j}|U_{m(s)}(\omega)|}{A_{m(j)}}>K_1\varepsilon\right)\\&\quad\le\limsup_{n\to\infty}\frac2n\sum_{r=1}^{n}\sum_{p=r}^{n}I\left(V_r(\omega)\ge CK_12^{r/\alpha}2^{(p-r-4)/4\alpha}\varepsilon\right)\\&\quad=\limsup_{n\to\infty}\frac2n\sum_{r=1}^{n}\sum_{p=0}^{n-r}I\left(2^{-r/\alpha}V_r(\omega)\ge CK_12^{(p-4)/4\alpha}\varepsilon\right)\le\limsup_{n\to\infty}\frac2n\sum_{r=1}^{n}\chi_r(\omega).\end{aligned}\tag{3.7}$$

The random variables $\chi_r(\omega)$, $r=1,2,\dots$, are independent, the relations $0\le E\chi_r(\omega)\le K\varepsilon$ and $E\chi_r^2(\omega)\le\text{const.}$ hold for $r\ge n_0(\varepsilon)$ because of relation (2.5), where an explicit bound can be given for $K=K(\alpha,\gamma)$. Indeed, the random variables $\chi_r(\omega)$ take non-negative integer values, the set $\{\omega\colon \chi_r(\omega)\ge k\}$ agrees with the set $\{\omega\colon 2^{-r/\alpha}V_r(\omega)\ge CK_12^{(k-4)/4\alpha}\varepsilon\}$ whose probability can be bounded by $C_1\varepsilon2^{-k\gamma/4\alpha}$ by formula (2.5). (We may assume that $K_1>0$ is chosen so large that $CK_12^{-1/\alpha}>1$.) This implies that $P(\chi_r(\omega)\ge x)\le C_1\varepsilon 2^{-\gamma x/4\alpha}$ for $r\ge n_0$. Hence the laws of large numbers can be applied for these random variables, and we get that

$$\lim_{n\to0}\frac1n\sum_{r=1}^{n}(\chi_r(\omega)-E\chi_r(\omega))=0,\ \text{for almost all }\omega\in\Omega$$

$$\text{and}\quad\limsup_{n\to\infty}\frac1n\sum_{r=1}^{n}E\chi_r(\omega)\le K\varepsilon.$$

These estimates together with relation (3.7) imply (3.1).

To prove relation (3.2) let us introduce the random variables

$$Z_n^{(i)}(\omega)=\frac{1}{2^{n/\alpha}}\sup_{1\le k\le l_n}\sup_{1\le p<N(n,k)-N(n,k-1)}\left|\sum_{m=1}^p \zeta_{n,k,m}^{(i)}(\omega)\right|,\qquad n=1,2,\dots,\quad i=1,2.$$

Condition (2.4) implies that

$$P\left(Z_n^{(i)}(\omega)\ge\varepsilon x\right)\le C_1\varepsilon x^{-\gamma}\quad\text{for all } x\ge A\quad n\ge n_0(\varepsilon)\quad\text{and } i=1,2. \tag{3.8}$$

We claim that

$$\sum_{j=L_{r-1}+1}^{L_r}\log\frac{B_{m(j)+1}}{B_{m(j-1)+1}}I\left(\frac{\sup\limits_{1\le s\le j}\sup\limits_{m(s-1)<p<m(s)}|U_p(\omega)-U_{m(s-1)}(\omega)|}{A_{m(j)}}>K_2\varepsilon\right)$$
$$\le 2\sum_{u=1}^r\left(I\left(Z_u^{(1)}(\omega)>\frac{K_2\varepsilon 2^{(r-u-1)/\alpha}}{2}\right)+I\left(Z_s^{(2)}(\omega)>\frac{K_2\varepsilon 2^{(r-u-1)/\alpha}}{2}\right)\right)$$

for all $r\ge n_0$. Indeed, the left-hand side of this inequality is non-zero only if one of the term at the right side is non-zero. In this case the left hand-side is bounded by $\sum\limits_{j=L_{r-1}+1}^{L_r}\log\frac{B_{m(j)+1}}{B_{m(j-1)+1}}\le 2$, and one of the summands at the right-hand side is non-zero, since $Z_u^{(1)}(\omega)+Z_u^{(2)}(\omega)>2^{-u/\alpha}K_2\varepsilon A_{m(j)}\ge K_2\varepsilon 2^{(r-u-1)/\alpha}$ for some $1\le u\le r$. Hence the inequality also holds in this case. By summing up this inequality for $r=1,\dots,n$ we get the following bound for the expression in (3.2):

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{L_n}\log\frac{B_{m(j)+1}}{B_{m(j-1)+1}}I\left(\frac{\sup\limits_{1\le s\le j}\sup\limits_{m(s-1)<p<m(s)}|U_p(\omega)-U_{m(s-1)}(\omega)|}{A_{m(j)}}>K_2\varepsilon\right)$$
$$\le\limsup_{n\to\infty}\frac{2}{n}\sum_{r=1}^n\sum_{u=1}^r\Bigg(I\left(Z_u^{(1)}(\omega)>\frac{K_2\varepsilon 2^{(r-u-1)/\alpha}}{2}\right) \tag{3.9}$$
$$+I\left(Z_u^{(2)}(\omega)>\frac{K_2\varepsilon 2^{(r-u-1)/\alpha}}{2}\right)\Bigg).$$

Let us define the random variables

$$X_u^{(i)}(\omega)=\sum_{p=0}^\infty I\left(Z_u^{(i)}(\omega)\ge\frac{K_2\varepsilon}{4}2^{(p-1)/\alpha}\right),\quad u=0,1,2,\dots,\quad i=1,2.$$

Then by changing the order of summation at the right-hand side of (3.9) we get that the left-hand side of formula (3.2) can be bounded by the expression

$$\limsup_{n\to\infty} \frac{2}{n} \sum_{u=1}^{n} \left(X_u^{(1)}(\omega) + X_u^{(2)}(\omega) \right).$$

The averages of the random variables $X_u^{(1)}(\omega) + X_u^{(2)}(\omega) - EX_u^{(1)}(\omega) - EX_u^{(2)}(\omega)$ tend to zero with probability one. Indeed, the random variables $X_u^{(i)}(\omega)$ satisfy the laws of large numbers both for $i=1$ and $i=2$, because they are independent, and by relation (3.8) the moments of these random variables are finite. (The estimates $P(X_u^{(i)} > x) \le C_2 2^{-\gamma x/\alpha}$, $i=1,2$, $u \ge n_0$ follows from relation (2.4) if $K_2 > 0$ is chosen sufficiently large. This can be proved similarly to the estimate on the probability of $P(\chi_r(\omega) > x)$ made after formula (3.7).) Moreover, $EX_u^{(i)}(\omega) \le K\varepsilon$ for all $u \ge n_0(\varepsilon)$ and $i=1,2$, with an appropriate constant $K>0$, and as a consequence

$$\limsup_{n\to\infty} \frac{2}{n} \sum_{u=1}^{n} \left(EX_u^{(1)}(\omega) + EX_u^{(2)}(\omega) \right) \le 4K\varepsilon.$$

These relations imply formula (3.2). The Basic Lemma is proved.

4. The proof of Theorems 1 and 2

Proof of Theorem 1. Let $\eta_n(\omega)$, $n=1,2,\dots$, be a sequence of independent Gaussian random variables such that $E\eta_n(\omega)=0$ and $E\eta_n^2(\omega)=\sigma_n^2$. Let us fix a number $\varepsilon>0$. We want to construct a sequence of independent random variables $\tilde\xi_n^{(\varepsilon)}(\omega)$, $n=1,2,\dots$, which has the same distribution as the sequence $\xi_n(\omega)$, $n=1,2,\dots$, and the sequences $\zeta_n(\omega) = \tilde\xi_n^{(\varepsilon)}(\omega) - \eta_n(\omega)$, and $U_n(\omega) = \sum\limits_{j=1}^{n} \zeta_n(\omega)$, $n=1,2,\dots$, satisfy relation (2.6) with $B_n = D_n^2 = \sum\limits_{k=1}^{n} \sigma_k^2$, $\alpha=2$ and the number ε we have fixed. This relation will be proved with an application of the Basic Lemma. If we can do this for arbitrary $\varepsilon>0$, then Theorem 4 of Part I., recalled at the beginning of Section 2 and the almost sure functional (central) limit theorem for the sequence $\eta_n(\omega)$, $n=1,2,\dots$, imply Theorem 1.

We shall omit the sign "$\tilde{}$" and "$^{(\varepsilon)}$" and write ξ_n instead of $\tilde\xi_n^{(\varepsilon)}$. To apply the Basic Lemma we have to define some quantities. We fix a sufficiently small $\bar\varepsilon = \bar\varepsilon(\varepsilon) > 0$ to be defined later and define the numbers $N(n)$, $n=1,2,\dots$, by means of the sequence $B_n = D_n^2$ as in the formulation of the Basic Lemma. Then we define an "$\bar\varepsilon$ regular refinement" $N(n,k)$, $n=1,2,\dots$, $0\le k\le l_n$, of the sequence $N(n)$. By this regularity property we mean that

$$\begin{aligned}\bar\varepsilon(B_{N(n)} - B_{N(n-1)}) \le{} & B_{N(n,k)} - B_{N(n,k-1)} \le 3\bar\varepsilon(B_{N(n)} - B_{N(n-1)}) \\ & \text{for } n \ge n_0(\bar\varepsilon) \text{ and all } 1 \le k \le l_n.\end{aligned} \tag{4.1}$$

The numbers $N(n,k)$ will be defined recursively in the variable k for fixed n in the following way. Put $N(n,0)=N(n-1)$, and if $N(n,k)$ is already defined and $B_{N(n)}-B_{N(n,k)}>3\bar\varepsilon(B_{N(n)}-B_{N(n-1)})$, then

$$N(n,k+1)=\min\{j\colon B_j-B_{N(n,k)}\ge\bar\varepsilon(B_{N(n)}-B_{N(n-1)})\}.$$

If $B_{N(n)}-B_{N(n,k)}\le 3\bar\varepsilon(B_{N(n)}-B_{N(n-1)})$, then put $N(n,k+1)=N(n)$. Let us remark that the Lindeberg condition (1.4) implies that the sequence B_n, $n=1,2,\dots$, satisfies relation (1.1), and $\lim\limits_{N\to\infty}\sup\limits_{N(n-1)\le k\le N(n)}\dfrac{\sigma_k^2}{B_{N(n)}}=0$. Hence $\lim\limits_{n\to\infty}2^{-n}B_{N(n)}=1$, and $B_{N(n,k)}-B_{N(n,k-1)}\sim\bar\varepsilon(B_{N(n)}-B_{N(n-1)})$, if $1\le k\le l_n-1$. It is not difficult to see that the sequence $N(n,k)$ is an $\bar\varepsilon$ regular refinement of the sequence $N(n)$.

Let $F_{n,k}(x)=P(S_{n,k}(\omega)<x)$ denote the distribution function of $S_{n,k}(\omega)=\dfrac{1}{\bar A_{n,k}}\sum\limits_{j=N(n,k-1)+1}^{N(n,k)}\xi_j(\omega)$ with $\bar A_{n,k}^2=\sum\limits_{j=N(n,k-1)+1}^{N(n,k)}\sigma_j^2=B_{N(n,k)}-B_{N(n,k-1)}$, and define the random variables $T_{n,k}(\omega)=\dfrac{1}{\bar A_{n,k}}\sum\limits_{j=N(n,k-1)+1}^{N(n,k)}\eta_j(\omega)$, $n=1,2,\dots$, $k=1,\dots,l_n$. Let $\Phi(x)$ denote the standard normal distribution function. Then $T_{n,k}(\omega)$, $n=1,2,\dots$, $k=1,\dots,l_n$, are independent standard normal random variables, and the variables $\chi_{n,k}(\omega)=\Phi(T_{n,k}(\omega))$ are independent random variables, uniformly distributed in the interval $[0,1]$.

We shall construct the random variables $S_{n,k}(\omega)=\dfrac{1}{\bar A_{n,k}}\sum\limits_{j=N(n,k-1)+1}^{N(n,k)}\xi_j(\omega)$, $n=1,2,\dots$, $1\le k\le l_n$, by means of the so-called quantile transform as $S_{n,k}(\omega)=F_{n,k}^{-1}(\chi_{n,k}(\omega))$, where $F_{n,k}^{-1}(x)$ denotes the inverse of the distribution function $F_{n,k}(x)$. More precisely, we define this inverse function as $F_{n,k}^{-1}(x)=G_{n,k}(x)=\sup\{u\colon F_{n,k}(u)<x\}$, and $S_{n,k}(\omega)=G_{n,k}(\chi_{n,k}(\omega))$, $n=1,2,\dots$, $k=1,\dots,l_n$. Such a definition is meaningful for all distribution functions. The random variables $S_{n,k}(\omega)$, $n=1,2,\dots$, $k=1,\dots,l_n$, defined in this way are independent, and they have distribution function $F_{n,k}(x)$.

To see that the distribution function of the above defined random variable $S_{n,k}(\omega)$ is really $F_{n,k}(x)$ let us first observe that

$$\begin{aligned}P(S_{n,k}(\omega)<x)&=\lim_{h\colon h>0,h\to0}P(S_{n,k}(\omega)<x-h)\\&=\lim_{h\colon h>0,h\to0}P(G_{n,k}(\chi_{n,k}(\omega))<x-h)\le P(\chi_{n,k}(\omega)\le F(x))=F(x),\end{aligned}$$

since $\{\omega\colon G(\chi_{n,k}(\omega))<x-h\}\subset\{\omega\colon \chi_{n,k}(\omega)\le F(x)\}$ for all $h>0$. To see the estimate from the opposite direction observe that $P(S_{n,k}(\omega)<x)=P(G_{n,k}(\chi_{n,k}(\omega))<x)\ge P(\chi_{n,k}(\omega)<F(x))=F(x)$, since $\{\omega\colon \chi_{n,k}(\omega)<F(x)\}\subset\{\omega\colon G_{n,k}(\chi_{n,k}(\omega))<x\}$. The last relation holds, since in the case $\chi_{n,k}(\omega)<F_{n,k}(x)$ we have $\chi_{n,k}(\omega)=F_{n,k}(x)-h$ with some $h>0$, and $G_{n,k}(\chi_{n,k}(\omega))=\sup\{v\colon F_{n,k}(v)<F_{n,k}(x)-h\}<x$. This relation holds, since the continuity of the function $F_{n,k}(x)$ from the left implies that all

numbers v for which $F_{n,k}(v) < F_{n,k}(x) - h$, the inequality $v \le x - \delta$ holds with some $\delta = \delta(h) > 0$.

Define the random variables $S_{N(n,k)}(\omega) = \sum\limits_{\substack{(m,j)\colon\, m<n-1,\, 1\le j\le l_m \\ \text{or } m=n-1, \text{ and } j\le k}} \bar{A}_{m,j} S_{m,j}(\omega)$, for all $n = 1, 2, \dots$, $1 \le k \le l_n$. If we consider the partial sums $S'_n(\omega) = \sum\limits_{k=1}^{n} \xi_k(\omega)$ with the random variables $\xi_k(\omega)$ given in the formulation of Theorem 1 then the joint distribution of the random vectors $S_{N(n,k)}(\omega)$ and $S'_{N(n,k)}(\omega)$, $n = 1, 2, \dots$, $1 \le k \le l_n$ agree. It follows from the results of general measure theory that in a sufficiently rich probability space the sequence of random variables of the form $S_{N(p,k)}(\omega)$, $p = 1, 2, \dots$, $0 \le k \le l_p$, can be extended to a sequence of random variables $S_n(\omega)$, $n = 1, 2, \dots$ whose elements for the indices of the form $N(p,k)$ are the already constructed random variables $S_{N(p,k)}(\omega)$, and the distribution of the sequences $S_n(\omega)$ and $S'_n(\omega)$, $n = 1, 2, \dots$, agree. Define the random variables $\xi_n(\omega) = S_n(\omega) - S_{n-1}(\omega)$, $n = 1, 2, \dots$. This sequence has the same joint distribution as the original sequence of independent random variables $\xi_n(\omega)$ considered in the formulation of Theorem 1. In such a way we constructed a sequence of random variables $\xi_n(\omega)$ in dependence of a small parameter $\bar{\varepsilon} > 0$.

Put $\zeta_n(\omega) = \xi_n(\omega) - \eta_n(\omega)$, $n = 1, 2, \dots$. We claim that if the parameter $\bar{\varepsilon} = \bar{\varepsilon}(\varepsilon)$ is chosen sufficiently small, then this sequence together with the already constructed "$\bar{\varepsilon}$ regular refinement" $N(n,k)$ of the sequence $N(n)$ and with the choice of the random variables $\zeta_n^{(1)}(\omega) = \xi_n(\omega)$, $\zeta_n^{(2)}(\omega) = \eta_n(\omega)$, $\zeta_{n,k,m}^{(1)}(\omega) = \xi_{N(n,k-1)+m}(\omega)$ and $\zeta_{n,k,m}^{(2)}(\omega) = \eta_{N(n,k-1)+m}(\omega)$, $n = 1, 2, \dots$, $k = 1, \dots, l_n$, $0 \le m < N(n,k) - N(n,k-1)$, satisfy the Basic Lemma with parameter ε. This statement implies Theorem 1.

The most important step in the proof of this statement is to show the following estimate. Because of the central limit theorem for all $\delta > 0$ there is a threshold $n_0 = n_0(\delta)$ such that

$$E(S_{n,k}(\omega) - T_{n,k}(\omega))^2 \le \delta, \quad \text{for all } n \ge n_0 \quad \text{and } 1 \le k \le l_n. \tag{4.2}$$

To prove relation (4.2) let us observe that the Lindeberg condition appearing in the formulation of Theorem 1 makes it possible to apply the central limit theorem for the normalized sums $S_{n,k}(\omega)$. This result yields that the distribution functions $F_{n,k}$ of the random variables $S_{n,k}(\omega)$ satisfy the relation $\lim\limits_{n\to\infty} \sup\limits_{1\le k\le l_n} \sup\limits_{|x|<\infty} |F_{n,k}(x) - \Phi(x)| = 0$. This implies that the random variables $T_{n,k}(\omega)$ and the $S_{n,k}(\omega)$ constructed by the above described quantile transform from the random variables $\chi_{n,k}(\omega) = \Phi(T_{n,k}(\omega))$ satisfy the following relation: For all $L > 1$ and $\eta > 0$ there exists some $n_0 = n_0(L, \eta)$ such that

$$|S_{n,k}(\omega) - T_{n,k}(\omega)| < \frac{\eta}{L} \quad \text{on the set } \{\omega\colon |T_{n,k}(\omega)| \le L\}$$

for all $n \ge n_0$ and $1 \le k \le l_n$. Let us choose the number $L > 1$ in such a way that the standard normal random variables $T_{n,k}(\omega)$ satisfy the inequality $ET^2_{n,k}(\omega)I(|T_{n,k}(\omega)| \ge$

$L) < \dfrac{\delta}{10}$, and let $\eta = \dfrac{\delta}{20}$. Then $E(T_{n,k}(\omega) - S_{n,k}(\omega))^2 I(|T_{n,k}(\omega)| \le L) < \dfrac{\delta^2}{400}$, and

$$\left|E(T_{n,k}^2(\omega) - S_{n,k}^2(\omega))I(|T_{n,k}(\omega)| \le L)\right| \le \left(E\left(T_{n,k}(\omega) + S_{n,k}(\omega)\right)^2\right)^{1/2} \left(E\left(T_{n,k}(\omega) - S_{n,k}(\omega)\right)^2 I(|T_{n,k}(\omega)| \le L)\right)^{1/2} < \frac{\delta}{10}.$$

Since $ES_{n,k}^2(\omega) = ET_{n,k}^2(\omega) = 1$, the inequalities $ES_{n,k}^2(\omega)I(|T_{n,k}(\omega)| \le L) \ge 1 - \dfrac{\delta}{5}$, and $ES_{n,k}^2(\omega)I(|T_{n,k}(\omega)| > L) \le \dfrac{\delta}{5}$ hold. Because of the simple inequality

$$\begin{aligned} E(S_{N(n,k)}(\omega) - T_{N(n,k)}(\omega))^2 &\le E(S_{N(n,k)}(\omega) - T_{N(n,k)}(\omega))^2 I(|T_{n,k}(\omega)| \le L) \\ &\quad + 2ES_{N(n,k)}^2(\omega)I(|T_{n,k}(\omega)| \ge L) + 2ET_{N(n,k)}^2(\omega))^2 I(|T_{n,k}(\omega)| \ge L) \end{aligned}$$

the above relations imply (4.2).

Let us remark that the estimate (4.2) does not depend on the parameter $\bar\varepsilon$ appearing in our construction. Only the threshold index $n_0 = n_0(\delta) = n_0(\delta, \bar\varepsilon)$ in (4.2) depends on this parameter.

Put

$$Z_{n,k}(\omega) = \sum_{j=N(n,k-1)+1}^{N(n,k)} \zeta_j(\omega) = \sum_{j=N(n,k-1)+1}^{N(n,k)} (\xi_j(\omega) - \eta_j(\omega))$$
$$n = 1, 2, \dots, \ 1 \le k \le l_n$$

Observe that $EZ_{n,k}(\omega) = 0$, $EZ_{n,k}^2(\omega) \le \delta \bar A_{n,k}^2$, if $n \ge n_0$, $\sum\limits_{k=1}^{l_n} \bar A_{n,k}^2 = \sum\limits_{j=N(n-1)+1}^{N(n)} \sigma_j^2 = B(N(n)) - B(N(n-1)) = 2^{n-1}(1 + o(1))$, and the expression to be estimated in formula (2.5) can be written in the form $V_n(\omega) = \sup\limits_{1 \le j \le l_n} \left|\sum\limits_{l=1}^{j} Z_j(\omega)\right|$. These relations together with the Kolmogorov inequality imply that

$$P(|V_n(\omega)| \ge \varepsilon x 2^{n/2}) = P\left(\sup_{1 \le j \le l_n} \left|\sum_{l=1}^{j} Z_j(\omega)\right| \ge \varepsilon x 2^{n/2}\right) \le \frac{\delta \sum\limits_{j=1}^{l_n} \bar A_{n,j}^2}{\varepsilon^2 x^2 2^n} \le \frac{2\delta}{\varepsilon^2 x^2}$$

Hence we get relation (2.5) with $\gamma = 2$ by choosing $\delta = \varepsilon^3$.

Now we turn to the proof of formula (2.4). We shall prove it only for the random variables $\zeta_{n,k,p}^{(1)}(\omega) = \xi_{N(n,k-1)+p}(\omega)$, $n = 1, 2, \dots$, $k = 1, \dots, l_n$, $0 \le p < N(n,k) - N(n,k-1)$. The same proof also applies to the sequence $\zeta_{n,k,p}^{(2)}(\omega) = \eta_{N(n,k-1)+p}(\omega)$, but this case can be checked simply, because the random variables $\zeta_{n,k,p}^{(2)}(\omega)$ are Gaussian. The appropriate choice of the small parameter $\bar\varepsilon = \bar\varepsilon(\varepsilon) > 0$ is important at this step.

Let us observe that by relation (4.1) the norming constants $\bar{A}_{n,k}$ satisfy the relation $2^{n-1}\bar{\varepsilon} \le \bar{A}_{n,k}^2 = B_{N(n,k)} - B_{N(n,k-1)} \le 2^{n+2}\bar{\varepsilon}$, and $l_n \le \dfrac{2}{\bar{\varepsilon}}$ for $n \ge n_0(\bar{\varepsilon})$. We have to bound the probability of the event in formula (2.4). Since the random variables $\zeta_{n,k,p}^{(1)}(\omega) = \xi_{N(n,k-1)+p}(\omega)$, are independent, we could apply the Kolmogorov inequality to get an estimate for this expression. But a direct application of this estimate is not good enough for our purposes, and we have to apply a more refined argument. It is the factor l_n^{-1} at the right-hand side of (2.4) which makes the problem harder.

First we show that for all $\delta > 0$ and $n > n_0(\delta)$ there exists some $K = K(\delta)$ such that

$$ES_{n,k}^2(\omega)I(|S_{n,k}(\omega))| > K) \le 4\delta. \tag{4.3}$$

Relation (4.3) follows from relation (4.2) and the observations that there exists some $L > 0$ such that $ET_{n,k}^2(\omega)I(T_{n,k}(\omega) \ge L) \le \delta$ and some $K = K(L) > 0$ such that $\{\omega\colon |S_{n,k}(\omega)| \ge K\} \subset \{\omega\colon |T_{n,k}(\omega)| \ge L\}$ for all sufficiently large n and $1 \le k \le l_n$. The latter statement follows from the special structure of the quantile transform. Then $ES_{n,k}^2(\omega)I(|S_{n,k}(\omega))| > K) \le 2ET_{n,k}^2(\omega)I(|S_{n,k}(\omega))| > K) + 2E(S_{n,k}(\omega) - T_{n,k}(\omega))^2 \le 4\delta$, as we claimed.

We can write by the Chebishev inequality and formula (4.3) that

$$\begin{aligned} P\left(\left|\sum_{p=N(n,k-1)+1}^{N(n,k)} \zeta_{n,k,p}^{(1)}(\omega)\right| > \varepsilon x 2^{n/2}\right) &= P(\bar{A}_{n,k}|S_{n,k}(\omega)| > \varepsilon x 2^{n/2}) \\ \le ES_{n,k}^2(\omega)I(|S_{n,k}(\omega)| \ge K)\frac{\bar{A}_{n,k}^2}{\varepsilon^2 x^2 2^n} &\le \frac{4\bar{A}_{n,k}^2\delta}{\varepsilon^2 x^2 2^n} \\ \text{for all } x \ge 1,\ \ n \ge n_0,\ \text{and } 1 \le k \le l_n& \end{aligned} \tag{4.4}$$

provided that $\bar{\varepsilon} > 0$ is chosen so small that $\bar{A}_{n,k}K \le \varepsilon 2^{n/2}$, which relation makes it possible to replace the second moment of $S_{n,k}(\omega)$ by the second moment of the random variable $S_{n,k}(\omega)I(|S_{n,k}(\omega)| \ge K)$ in the above estimate. Such a choice of $\bar{\varepsilon}$ is possible, since $\bar{A}_{n,k}^2 \le \bar{\varepsilon}2^{n+2}$ if $n \ge n_0$. (We remark that the constants in the estimations applied to get (4.4) do not depend on the parameter $\bar{\varepsilon}$. Only the threshold index n_0 depends on it.) Since $l_n\bar{A}_{n,k}^2 \le 2^{n+3}$ the number $\delta > 0$ can be chosen in such a way that $\delta\bar{A}_{n,k}^2 l_n \le \varepsilon^3 2^n$ for all sufficiently large n. For instance $\delta = \varepsilon^3/8$ is a good choice. With such a choice of δ we get a weakened form of formula (2.4) with $\gamma = 2$. Here the supremum is dropped, only the last term $p = N(n,k) - N(n,k-1)$ is taken in the supremum in expression (2.4).

Formula (2.4) in its original form can be proved for instance with the help of formula (4.4) and the following maximum inequality (see e.g. [8], Lemma 3.21 at p. 45). Let $\xi_1(\omega), \dots, \xi_n(\omega)$ be independent random variables, and put $S_k(\omega) = \sum\limits_{j=1}^{k} \xi_j(\omega)$. Then for all $y > 0$ such that $\max\limits_{1 \le k \le n} P(S_k(\omega) > y) \le \dfrac{1}{4}$

$$P\left(\sup_{1 \le k \le n} S_k(\omega) \ge 2y\right) \le \frac{4}{3} P(S_n(\omega) \ge y). \tag{4.5}$$

We apply this inequality for the partial sums

$$S_m(\omega) = \pm \sum_{p=1}^{m} \zeta^{(1)}_{n,k,p}(\omega), \quad 1 \le m \le N(n,k) - N(n,k-1).$$

Since $E\zeta^{(1)}_{n,k,p}(\omega) = 0$, and the variance $D_n^2 = \sum\limits_{p=N(n,k-1)+1}^{N(n,k)-N(n,k-1)} E(\zeta^{(1)}_{n,k,p})^2(\omega) = \bar{A}^2_{n,k}$, satisfies the inequality $D_n^2 = \bar{A}^2_{n,k} \le \bar{\varepsilon} 2^{n+2}$. Because of these estimates Chebishev's inequality yields that for $n \ge n_0(\bar{\varepsilon})$

$$\max_{1 \le m \le N(n,k)-N(n,k-1)} P\left(S_m(\omega) > \varepsilon \frac{x}{2} 2^{n/2}\right) \le \frac{1}{4} \quad \text{for } x \ge \frac{1}{2}$$

if $\bar{\varepsilon} = \bar{\varepsilon}(\varepsilon)$ is sufficiently small. Hence formula (4.5) is applicable with $y = \frac{x}{2} 2^{n/2}$ if $x \ge \frac{1}{2}$. Let us observe that relation (4.4) also holds for $x \ge 1/2$ and not only for $x \ge 1$. These relations imply that for sufficiently small $\bar{\varepsilon} = \bar{\varepsilon}(\varepsilon) > 0$

$$P\left(\sup_{1 \le m \le N(n,k)-N(n,k-1)} \left| \sum_{p=1}^{m} \zeta^{(1)}_{n,k,p}(\omega) \right| > \varepsilon x 2^{n/2} \right) \le \frac{50 \bar{A}^2_{n,k} \delta}{\varepsilon^2 x^2 2^n}$$
$$\text{for all } x \ge 1, \ \ n \ge 0, \ \text{and } 1 \le k \le l_n.$$

This relation implies formula (2.4) with $\gamma = 2$ in the same way as formula (4.4) implied its weakened form. Theorem 1 is proved.

Proof of Theorem 2. We may assume that $\lim\limits_{x \to \infty} \mu(x) = E\xi_1^2(\omega) = \infty$, because in the case $E\xi_1^2(\omega) < \infty$ Theorem 1 can be applied. More precisely Theorem 1 supplies a modified version of Theorem 2 in this case with the norming sequence $\bar{B}_n = nE\xi_1^2(\omega)$ instead of the original sequence B_n. But $\lim\limits_{n \to \infty} \frac{\bar{B}_n}{B_n} = 1$, in this case, and Theorem 5 of Part I. implies the result in this case.

Since the function $\mu(x) = E\xi_1^2(\omega) I(|\xi_1(\omega)| \le x)$ is a slowly varying function, it follows for instance from Theorem 2 of [13] in Chapter VIII, Section 9 (and its proof) that

$$\begin{aligned} P(|\xi_1(\omega)| > x) &= o\left(x^{-2}\mu(x)\right) \quad \text{if } x \to \infty, \\ E|\xi_1| I(|\xi_1(\omega)| > x) &= o\left(x^{-1}\mu(x)\right) \quad \text{if } x \to \infty. \end{aligned} \tag{4.6}$$

Define the random variables $\bar{\xi}_n(\omega) = \xi_n(\omega) I(|\xi_n(\omega)| \le a_n)$, $\chi_n^{(1)}(\omega) = \xi_n(\omega) I(\xi_n(\omega) \ge a_n)$, and $\chi_n^{(2)}(\omega) = \xi_n(\omega) I(\xi_n(\omega) \le -a_n)$, $n = 1, 2, \dots$, with the numbers a_n defined in the formulation of Theorem 2. We claim that

a.) The sequence of random variables $\bar{\xi}_n(\omega)$, $n = 1, 2, \dots$, satisfies the almost sure functional limit theorem with the weight function $B_n = \sum\limits_{k=1}^{n} a_k$, $n = 1, 2, \dots$, parameter $\alpha = 2$, and the Wiener measure μ_0 as the limit measure.

b.) Both sequences of random variables $\chi_n^{(i)}(\omega)$, $n=1,2,\dots$, $i=1,2$, satisfy the almost sure functional limit theorem with the same weight function B_n as in Statement a.), $n=1,2,\dots$, parameter $\alpha=2$, and (degenerated) limit measure μ_0 on the space $D([0,1])$ which is concentrated on the function $x(t)\equiv 0$.

Statement b.) can be reformulated in the following way: Put $S_n^{(i)}(\omega)=\sum\limits_{k=1}^n \chi_k^{(i)}(\omega)$, $i=1,2$, $n=1,2,\dots$, and define the broken lines $S_N^{(i)}(\cdot,\omega)$, $i=1,2$, $N=1,2,\dots$, with the above random variables $S_n^{(i)}(\omega)$, the numbers B_n and parameter $\alpha=2$ by formula (1.2). For all $\omega\in\Omega$, $i=1,2$, and $n=1,2,\dots$, introduce the (random) measure $P_{\omega,i,n}$ by the formula $P_{\omega,i,n}\left(S_k^{(i)}(\cdot,\omega)\right)=\dfrac{1}{\log\frac{B_n}{B_1}}\log\dfrac{B_{k+1}}{B_k}$, $1\le k<n$. Then for any open neighbourhhood $\mathbf{G}$ of the function $x(t)\equiv 0$ in the space $D([0,1])$ $\lim\limits_{n\to\infty}P_{\omega,i,n}(\mathbf{G})=0$ for almost all $\omega\in\Omega$, $i=1,2$.

Since $\xi_n(\omega)=\bar\xi_n(\omega)+\chi_n^{(1)}(\omega)+\chi_n^{(2)}(\omega)$, Statements a.) and b.) imply Theorem 2. Indeed, it can be seen for instance with the help of Lemma B of Part I.) and Statement b.) that for almost all $\omega\in\Omega$ the sequence $\xi_n(\omega)$, $n=1,2,\dots$, satisfies the same almost sure functional limit theorem as the sequence $\bar\xi_n(\omega)$, $n=1,2,\dots$.

We shall prove Statement a.) with the help of Theorem 1. Let us first observe that $\lim\limits_{n\to\infty}\dfrac{\operatorname{Var}\bar\xi_n(\omega)}{\mu(a_n)}=1$. Indeed, $\dfrac{\operatorname{Var}\bar\xi_n(\omega)}{\mu(a_n)}-1=\dfrac{(E\bar\xi_n(\omega))^2}{\mu(a_n)}$, and by the relations (4.6) and $E\xi_n(\omega)=0$, the definition of the sequence of a_n and $a_n\to\infty$ as $n\to\infty$ we have $(E\bar\xi_n(\omega))^2=(E\xi_n(\omega)I(|\xi_n(\omega)|>a_n)^2=o(a_n^{-2}\mu(a_n)^2)=o\left(\dfrac{\mu(a_n)}{n}\right)$. This means that for $\bar B_n=\sum\limits_{k=1}^n\operatorname{Var}\bar\xi_k(\omega)$, $\lim\limits_{n\to\infty}\dfrac{\bar B_n}{B_n}=1$.

To prove Statement a.) first we show that for all $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{B_n}\sum_{k=1}^n E\left[(\bar\xi_k(\omega)-E\bar\xi_k(\omega))^2 I\left(|\bar\xi_k(\omega)-E\bar\xi_k(\omega)|>\varepsilon B_n^{1/2}\right)\right]=0, \tag{4.7}$$

which implies that the Lindeberg condition holds for the sequence $\bar\xi_k(\omega)-E\bar\xi_k(\omega)$, $k=1,2,\dots$.

First we show that

$$\sqrt{c}\le\liminf_{n\to\infty}\sup_{cn\le k\le n}\frac{a_k}{a_n}\le\limsup_{n\to\infty}\sup_{cn\le k\le n}\frac{a_k}{a_n}\le 1\quad\text{for all } 0<c<1. \tag{4.8}$$

Indeed, it follows from the definition of the sequence a_n that $a_k\le a_n$ for all $k\le n$, and this implies the right-hand side of (4.8). On the other hand, since $\mu(\cdot)$ is a slowly varying function the numbers a_n satisfy the relation $\lim\limits_{n\to\infty}n\dfrac{\mu(a_n)}{a_n^2}=1$, and for any $\varepsilon>0$, $k\ge cn$ and $n\ge n_0(c,\varepsilon)$, $k\dfrac{\mu((\sqrt{c}-\varepsilon)a_n)}{((\sqrt{c}-\varepsilon)a_n)^2}>cn\dfrac{\mu(a_n)}{(c-\sqrt{c}\varepsilon)a_n^2}\ge 1$, hence $a_k\ge(\sqrt{c}-\varepsilon)a_n$. This relation implies the left-hand side of (4.8). Since $\mu(\cdot)$ is a slowly varying function, it follows from relation (4.8) that

$$\lim_{n\to\infty} \frac{B_n}{n\mu(a_n)} = 1. \tag{4.9}$$

By relation (4.9) the expression B_n can be replaced by $n\mu(a_n)$ in (4.7). Let us also observe that $B_n^{1/2} \sim \sqrt{n\mu(a_n)} = \sqrt{\frac{n\mu(a_n)}{a_n^2}}a_n \sim a_n$, and $|E\bar\xi_k(\omega)| \le E|\bar\xi_1(\omega)| \le \text{const.}$ for large n. Hence the terms in the sum (4.7) can be estimated as

$$\begin{aligned}
&E\left[(\bar\xi_k(\omega)-E\bar\xi_k(\omega))^2 I\left(|\bar\xi_k(\omega)-E\bar\xi_k(\omega)| \ge \varepsilon B_n^{1/2}\right)\right] \\
&\qquad \le 2\left(E|\xi_1(\omega)|\right)^2 + 2E\xi_k(\omega)^2 I\left(\frac{\varepsilon}{2}a_n \le |\xi_k| \le a_k\right) \\
&\qquad \le \text{const.} + \max\left(\mu(a_k) - \mu\left(\frac{\varepsilon}{2}a_n\right), 0\right).
\end{aligned}$$

Since $\mu(x)$ is a slowly varying function tending to infinity as $x \to \infty$, $\lim\limits_{n\to\infty} a_n = \infty$, and $a_k \ge \text{const.}\sqrt{c}a_n$ if $k \ge cn$, with some $0 < c \le 1$ the above estimate implies that for all $\varepsilon > 0$ and $\delta > 0$ there is some threshold $n_0 = n_0(\varepsilon,\delta)$ such that for all $n \ge n_0$ and $1 \le k \le n$

$$E\left[(\bar\xi_k(\omega)-E\bar\xi_k(\omega))^2 I\left(|\bar\xi_k(\omega)-E\bar\xi_k(\omega)| \ge \varepsilon B_n^{1/2}\right)\right] \le \delta\mu(a_n).$$

Summing these inequalities for all $1 \le k \le n$ and exploiting that they hold (for sufficiently large n) for all $\delta > 0$ we get relation (4.7).

The above relations together with Theorem 5 of Part I. of this paper (which states that the weight function B_n in the almost sure functional limit theorem can be replaced by a weight function $\bar B_n$ such that $\lim\limits_{n\to\infty} \frac{\bar B_n}{B_n} = 1$) imply that the sequence of random variables $\bar\xi_n(\omega) - E\bar\xi_n(\omega)$, $n = 1, 2, \dots$, satisfy the almost sure functional limit theorem with parameter $\alpha = 2$ and the Wiener measure μ_0 as limit measure. Hence to finish the proof Statement a.) it is enough to show that

$$\lim_{n\to\infty} \frac{1}{B_n^{1/2}} \sum_{k=1}^n E\bar\xi_k(\omega) = 0. \tag{4.10}$$

To prove relation (4.10) let us first observe that because of the identity $E\xi_1(\omega) = 0$, relations (4.6) and (4.9) we can write, fixing an $\varepsilon > 0$ for all $r > r_0(\varepsilon)$ that

$$\left|\sum_{j=2^r+1}^{2^{r+1}} E\bar\xi_j(\omega)\right| \le \varepsilon \sum_{j=2^r+1}^{2^{r+1}} \frac{\mu(a_j)}{a_j} \le \text{const.}\, \varepsilon 2^r \frac{\mu(a_{2^r})}{a_{2^r}} \le \text{const.}\, \varepsilon a_{2^r}.$$

Given an integer n, choose the integer R such that $2^R < n \le 2^{R+1}$, and apply the above estimate for all $r_0(\varepsilon) \le r \le R$. Then we get by using the relations $B_n^{1/2} \sim \sqrt{n\mu(a_n)} \sim a_n$ and $a_n \to \infty$ if $n \to \infty$ that $\lim\limits_{r\to\infty} \dfrac{a_{2^{r+1}}}{a_{2^r}} = \sqrt{2}$ and

$$\left|\sum_{j=1}^n E\bar{\xi}_j(\omega)\right| \le \text{const.}\,(\varepsilon) + \text{const.}\,\varepsilon \sum_{r=1}^R a_{2^r}$$
$$\le \text{const.}\,(\varepsilon) + \text{const.}\,'\varepsilon a_n \le \text{const.}\,(\varepsilon) + \text{const.}\,'\varepsilon B_n^{1/2}.$$

Since the above relation holds for all $\varepsilon > 0$, it implies relation (4.10).

We shall prove Statement b.) for the sequence $\chi_n^{(1)}(\omega)$, $n = 1, 2, \dots$, the proof for the sequence $\chi_n^{(2)}(\omega)$ is the same. Put $S_n(\omega) = \sum\limits_{j=1}^n \chi_j^{(1)}(\omega)$. First we show that to verify Statement b.) it is enough to prove the inequality

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{N(n)} \log \frac{B_k}{B_{k-1}} I\left(\left\{\frac{\sup\limits_{0\le j\le k} S_j(\omega)}{B_k^{1/2}} > \varepsilon\right\}\right) \le \varepsilon, \quad \text{for all } \varepsilon > 0 \tag{4.11}$$

with $N(n) = \inf\{k\colon B_k \ge 2^n\}$. One can argue for instance in the following way. Let us remark that the sequence of (degenerated) random variables $\eta_n(\omega) \equiv 0$ satisfies the almost sure functional limit theorem with the weight function B_n, parameter $\alpha = 2$ and limit measure μ_0 which is concentrated to the function $x(t) \equiv 0$, $0 \le t \le 1$. Then to prove Statement b.) it is enough to check that the pair of sequences $(\chi_n^{(1)}(\omega), \eta_n(\omega))$, $n = 1, 2, \dots$, satisfy Property A. Formula (4.11) agrees with Property A in the present case.

We shall prove formula (4.11) by means of the Basic Lemma with the same sequence of numbers B_n which appears in Theorem 2, $\zeta_n(\omega) = \chi_n^{(1)}(\omega)$, $n = 1, 2, \dots$, and the (trivial) refinement of the sequence $N(n)$ for which the numbers $N(n, k)$ contain all integers in the interval $[N(n-1), N(n)]$. (This refinement of the sequence $N(n)$ does not depend on ε.) With this choice of the refinement of the sequence $N(n)$ condition (2.4) is an empty statement in the application of the Basic Lemma, and it is enough to check relation (2.5). In the present case

$$V_n(\omega) = \sup_{N(n-1)<k\le N(n)} \left|\sum_{j=N(n-1)+1}^{k} \zeta_j(\omega)\right| = \sum_{j=N(n-1)}^{N(n)} \zeta_j(\omega), \tag{4.12}$$

and we have to prove relation (2.5) with this random variable, $\alpha = 2$ and appropriate $\gamma > 0$. Then the validity of relation (2.6) for all $1 > \varepsilon > 0$ implies relation (4.11) i.e. the remaining part of the proof of Theorem 2.

We shall prove relation (2.5) with some fixed $\varepsilon > 0$ and $x \ge 1$ for the sequence $\zeta_n(\omega) = \chi_n^{(1)}(\omega)$ by estimating the expression $\sum\limits_{j=N(n-1)+1}^{N(n)} E\zeta_j(\omega) = E\left|\sum\limits_{j=N(n-1)+1}^{N(n)} \zeta_j(\omega)\right|$.

To do this we need a good bound on the term $N(n)$. By the relation $B_n \sim n\mu(a_n)$ and the definition of the numbers $N(n)$ we have $N(n)\mu(a_{N(n)}) \sim 2^n$ for sufficiently large n. The relation $a^2_{N(n)}\dfrac{N(n)\mu(a_{N(n)})}{a^2_{N(n)}} \sim 2^n$ and the definition of the numbers a_n imply that $\lim\limits_{n\to\infty} 2^{-n/2}a_{N(n)} = 1$. Hence $N(n) \le \dfrac{2^{n+2}}{\mu(a_{N(n)})}$ and $2^{(n-1)/2} < a_{N(n)} < 2^{(n+1)/2}$ for large n. Now we can write with the help of relation (4.6)

$$\sum_{j=N(n-1)+1}^{N(n)} E\zeta_j(\omega) \le \bar\varepsilon N(n)\frac{\mu(a_{N(n)})}{a_{N(n)}} \le \bar\varepsilon\frac{2^{n+2}}{a_{N(n)}} \le \bar\varepsilon 2^{(n+5)/2} \le \varepsilon^2 2^{n/2}$$

if the number $\bar\varepsilon$ is sufficiently small (and the threshold $n_0(\bar\varepsilon)$ is sufficiently large). Hence the Markov inequality yields that

$$P\left(\sum_{j=N(n-1)+1}^{N(n)} \zeta_j(\omega) \ge \varepsilon x 2^{n/2}\right) \le \frac{\varepsilon^2 2^{n/2}}{\varepsilon x 2^{n/2}} = \varepsilon x^{-1}.$$

It follows from this estimate that the random variables $V_n(\omega)$ defined in (4.12) satisfy the estimate (2.5) with $\gamma = 1$. Theorem 2 is proved.

5. The proof of Theorem 3

Proof of Theorem 3. By the results quoted in Sections 1 and 2 it is enough to show that the sequences $(\xi_n(\omega) - a_n, \eta_n(\omega))$, $n = 1, 2, \dots$, satisfy Property A with an appropriate sequence of constants a_n and $\eta_n(\omega) = \bar L(n)^{1/\alpha}\bar\eta_n(\omega)$, $n = 1, 2, \dots$, where $\bar\eta_n(\omega)$ are i.i.d. random variables with the stable distribution $G(x)$ satisfying formula (1.5), and $\bar L(\cdot)$ is the slowly varying function defined in the formulation of Theorem 3 whose existence still has to be proved.

We shall prove Property A with the help of the Basic Lemma together with a quantile transform representation of the random variables $\xi_n(\omega)$, by means of the random variables $\eta_n(\omega)$, $n = 1, 2, \dots$, to be described below.

The random variables $\eta_n(\omega)$, $n = 1, 2, \dots$, have distribution function $G_n(x) = G\left(\dfrac{x}{\bar L(n)^{1/\alpha}}\right)$ with the stable distribution function $G(x)$ which satisfies relation (1.5). The distribution function $G_n(x)$ has a density function, hence the independent random variables $G_n(\eta_n(\omega))$, $n = 1, 2, \dots$, are uniformly distributed in the interval $[0, 1]$. Then, similarly to the construction in the proof of Theorem 1 we can construct the random variables $\xi_n(\omega)$, $n = 1, 2, \dots$. More explicitly, we define a sequence of i.i.d. random variables with the same distribution function $F(x)$ as the originally given sequence $\xi_n(\omega)$, $n = 1, 2, \dots$, by the formula $\xi_n(\omega) = F^{-1}(G_n(\eta(\omega))$, $n = 1, 2, \dots$, where $F^{-1}(x) = \sup\{u\colon F(u) < x\}$. We will show with the help of the Basic Lemma that this construction of the pairs $(\xi_n(\omega) - a_n, \eta_n(\omega))$, $n = 1, 2, \dots$, with

$$a_n = E\left(\xi_n(\omega) - \eta_n(\omega)\right) I\left(|\eta_n(\omega)| < n^{1/\alpha}\bar L(n)^{1/\alpha}\right)$$

satisfies Property A. (Observe that the construction which must satisfy certain property depending on a parameter $\varepsilon>0$ does not depend on the parameter ε.)

First we formulate and prove a Lemma which plays an important role in the proof.

Lemma 1. *Let us consider the function* $b(x)=\max\left\{u\colon \dfrac{L(u)x}{u^\alpha}\ge 1\right\}$, $x>0$, *introduced in the formulation of Theorem 3. This function $b(x)$ is a regularly varying function with parameter $1/\alpha$, hence $\bar L(x)=b(x)^\alpha x^{-1}$ is a slowly varying function. Define the distribution functions $G_n(x)$ and random variables $\eta_n(\omega)$ and $\xi_n(\omega)$, $n=1,2,\dots$, in the way described above with the help of this slowly varying function $\bar L(\cdot)$. The relation* $\lim\limits_{x\to\infty}\dfrac{xL(b(x))}{b(x)^\alpha}=1$ *holds. For all $\bar\varepsilon>0$ there exists an index $n_0=n_0(\bar\varepsilon)$ such that*

$$|\xi_n(\omega)-\eta_n(\omega)|<\bar\varepsilon|\eta_n(\omega)| \quad \textit{if } \bar\varepsilon n^{1/\alpha}\bar L(n)^{1/\alpha}<\eta_n(\omega)<\bar\varepsilon^{-1}n^{1/\alpha}\bar L(n)^{1/\alpha} \textit{ and } n\ge n_0(\bar\varepsilon). \tag{5.1}$$

Proof of Lemma 1. Since $L(x)$ is a slowly varying function we have $\lim\limits_{x\to\infty}b(x)=\infty$, and the relations $\lim\limits_{x\to\infty}\dfrac{L(b(x))x}{b(x)^\alpha}=1$ and $\lim\limits_{x\to\infty}\dfrac{L(c^{1/\alpha}b(x))cx}{(c^{1/\alpha}b(x))^\alpha}=1$ hold for all $1\le c\le K$, where $K>1$ is an arbitrary fixed constant. Moreover, the convergence in the second relation is uniform in the variable c as $x\to\infty$. It can be shown with the help of these properties that $\lim\limits_{x\to\infty}\dfrac{b(cx)}{c^{1/\alpha}b(x)}=1$, since they imply that for large $x>0$ the number $c^{1/\alpha}b(x)$ is a good approximation for $b(cx)$. This relation means that the function $b(x)$ is regularly varying with parameter $1/\alpha$. The regular varying property of the function $b(x)$ implies that the function $\bar L(x)=b(x)^\alpha x^{-1}$ is slowly varying.

We claim that for any $\bar\varepsilon>0$

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{\frac{\bar\varepsilon}{2}n^{1/\alpha}\bar L(n)^{1/\alpha}<x<2\bar\varepsilon^{-1}n^{1/\alpha}\bar L(n)^{1/\alpha}} x^\alpha\frac{1-F(x)}{L(x)}=C_1\\
&\lim_{n\to\infty}\sup_{\frac{\bar\varepsilon}{2}n^{1/\alpha}\bar L(n)^{1/\alpha}<x<2\bar\varepsilon^{-1}n^{1/\alpha}\bar L(n)^{1/\alpha}} x^\alpha\frac{F(-x)}{L(x)}=C_2,
\end{aligned}\tag{5.2}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{\frac{\bar\varepsilon}{2}n^{1/\alpha}\bar L(n)^{1/\alpha}<x<2\bar\varepsilon^{-1}n^{1/\alpha}\bar L(n)^{1/\alpha}} x^\alpha\frac{1-G_n(x)}{L(x)}=C_1\\
&\lim_{n\to\infty}\sup_{\frac{\bar\varepsilon}{2}n^{1/\alpha}\bar L(n)^{1/\alpha}<x<2\bar\varepsilon^{-1}n^{1/\alpha}\bar L(n)^{1/\alpha}} x^\alpha\frac{G_n(-x)}{L(x)}=C_2.
\end{aligned}\tag{5.2$'$}$$

Formula (5.2) follows from relation (1.6). Formula (5.2′) can be deduced from (1.5) and the definition of the function $G_n(x)$ if we show that

$$\lim_{n\to\infty}\sup_{\frac{\bar\varepsilon}{2}n^{1/\alpha}\bar L(n)^{1/\alpha}<x<2\bar\varepsilon^{-1}n^{1/\alpha}\bar L(n)^{1/\alpha}}\frac{\bar L(n)}{L(x)}=1.$$

To see the last statement observe that in the domain we are interested in $L(x) \sim L(n^{1/\alpha}\bar{L}(n)^{1/\alpha}))$, hence it can be reduced to the formula

$$\lim_{n\to\infty} \frac{\bar{L}(n)}{L(n^{1/\alpha}\bar{L}(n)^{1/\alpha})} = 1. \tag{5.3}$$

Relation (5.3) holds, since $L(n^{1/\alpha}\bar{L}(n)^{1/\alpha}) = L(b(n)) \sim \dfrac{b(n)^\alpha}{n} = \bar{L}(n)$.

Relations (5.2) and (5.2′) imply that for all $\bar{\varepsilon} > 0$ there is a threshold $n = n(\bar{\varepsilon})$ such that

$$\begin{aligned} 1 - F((1+\bar{\varepsilon})x) < 1 - G_n(x) &< 1 - F((1-\bar{\varepsilon})x) \\ F(-(1+\bar{\varepsilon})x) < G_n(-x) \quad &< F((1-\bar{\varepsilon})x) \\ &\text{if } \bar{\varepsilon}n^{1/\alpha}L(n)^{1/\alpha} < x < \bar{\varepsilon}^{-1}n^{1/\alpha}L(n)^{1/\alpha}. \end{aligned}$$

Hence

$$\begin{aligned} (1-\bar{\varepsilon})x < F^{-1}(G_n(x)) < (1+\bar{\varepsilon})x \\ -(1+\bar{\varepsilon})x < F^{-1}(G_n(-x)) < -(1-\bar{\varepsilon})x \end{aligned}$$

if $\bar{\varepsilon}n^{1/\alpha}L(n)^{1/\alpha} < x < \bar{\varepsilon}^{-1}n^{1/\alpha}L(n)^{1/\alpha}$. The last formula together with the definition of the quantile transform imply relation (5.1). Lemma 1 is proved.

Now we turn back to the proof of Theorem 3. We shall prove Property A with the help of the Basic Lemma with the (greatest possible) refinement $N(n,k)$ of the sequence $N(n)$, $n = 1, 2, \dots$, for which the numbers $N(n,k)$ contain all integers in the interval $[N(n-1), N(n)]$ for all $n = 1, 2, \dots$. With this choice of the sequence $N(n,k)$ condition (2.4) has not be checked in the application of the Basic Lemma. We only have to check formula (2.5), which states in the present case that

$$\begin{aligned} P\left(\sup_{N(n-1)<k\le N(n)} \left| \sum_{j=N(n-1)+1}^{k} (\xi_j(\omega) - a_j - \eta_j(\omega)) \right| > \varepsilon x 2^{n/\alpha} \right) < C\varepsilon x^{-\gamma} \\ \text{for all } x \ge 1 \text{ and } n \ge n_0(\varepsilon) \end{aligned} \tag{5.4}$$

with an appropriate $\gamma > 0$.

Let us make the following observation. Since $\bar{L}(n)$ is a slowly varying function the relation $\lim\limits_{n\to\infty} \dfrac{\bar{B}_n}{B_n} = 1$ holds with $\bar{B}_n = n\bar{L}(n)$. This relation together with formula (1.1) imply that for all large n the number $N(n)$ satisfies the relation $N(n)\bar{L}(N(n)) \sim 2^n$. Since $\bar{L}(n)$ is a slowly varying function this relation also implies that $N(n) \le 5N(n-1)$ if n is sufficiently large. Indeed, since $\dfrac{\bar{L}(N(n-1))}{\bar{L}(N(n))} \le 1.01 \left(\dfrac{N(n)}{N(n-1)}\right)^{1/2}$ for sufficiently large n, we have $\sqrt{5} \ge \dfrac{1.01N(n)\bar{L}(N(n))}{N(n-1)\bar{L}(N(n-1))} \ge \left(\dfrac{N(n)}{N(n-1)}\right)^{1/2}$ for sufficiently large

n, as we claimed. Hence $\bar L(k) \sim \bar L(N(n))$ and $(k\bar L(k))^{1/\alpha} \le \frac{9}{8}2^{n/\alpha}$ for $N(n-1) \le k \le N(n)$.

In the proof of the inequality (5.4) we shall bound separately the contribution of those indices j for which $\eta_j(\omega)$ is relatively small, more explicitly their values is much less than $j^{1/\alpha}\bar L(j)^{1/\alpha}$, the indices j of middle order terms for which $\eta_j(\omega)$ is of order $j^{1/\alpha}\bar L(j)^{1/\alpha}$ and finally those indices for which the terms $\eta_j(\omega)$ are much larger than $j^{1/\alpha}\bar L(j)^{1/\alpha}$. The contribution of the terms with small and middle order will be bounded by means of an estimate for the expected values and variances of their sum together with an application of Kolmogorov's inequality. The contribution of the large values can be estimated by means of the tail behaviour of the distribution functions $F(x)$ and $G_n(x)$. The separation between the large and middle values will depend also on the value of the parameter x in formula (5.4). In the estimates we shall fix a sufficiently small $\bar\varepsilon = \bar\varepsilon(\varepsilon)$ and apply the estimate (5.1).

To estimate the contribution of the small terms let us first bound the terms

$$E(\xi_j(\omega)-\eta_j(\omega))I\left(|\eta_j(\omega)|<\varepsilon^{4/(2-\alpha)}j^{1/\alpha}L(j)^{1/\alpha}\right)-a_j$$

and

$$E(\xi_j(\omega)-\eta_j(\omega))^2I\left(|\eta_j(\omega)|<\varepsilon^{4/(2-\alpha)}j^{1/\alpha}\bar L(j)^{1/\alpha}\right).$$

Fix a sufficiently small $\bar\varepsilon=\bar\varepsilon(\varepsilon)$ to be defined later, and consider indices $j>n_0(\bar\varepsilon)$. i.e. such indices for which the relation (5.1) is applicable. In this case, if $\bar\varepsilon(\varepsilon)$ is sufficiently small, we can write because of the definition of the norming constants a_n

$$\begin{aligned}
&\left|E(\xi_j(\omega)-\eta_j(\omega))I\left(|\eta_j(\omega)|<\varepsilon^{4/(2-\alpha)}j^{1/\alpha}\bar L(j)^{1/\alpha}\right)-a_j\right|\\
&\quad=\left|E(\xi_j(\omega)-\eta_j(\omega))I\left(\varepsilon^{4/(2-\alpha)}j^{1/\alpha}\bar L(j)^{1/\alpha}\le|\eta_j(\omega)|<j^{1/\alpha}\bar L(j)^{1/\alpha}\right)\right|\\
&\quad\le\bar\varepsilon E|\eta_j(\omega)|I\left(\varepsilon^{4/(2-\alpha)}j^{1/\alpha}L(j)^{1/\alpha}\le|\eta_j(\omega)|<j^{1/\alpha}\bar L(j)^{1/\alpha}\right)\\
&\quad=\bar\varepsilon\int_{\varepsilon^{4/(2-\alpha)}j^{1/\alpha}\bar L(j)^{1/\alpha}\le|u|\le j^{1/\alpha}\bar L(j)^{1/\alpha}}|u|G_j(\,du)\\
&\quad\le\text{const.}\,\bar\varepsilon\bar L(j)^{1/\alpha}\int_{\varepsilon^{4/(2-\alpha)}j^{1/\alpha}}^{j^{1/\alpha}}u\bar G(\,du)\le\frac{\varepsilon}{45}j^{(1-\alpha)/\alpha}\bar L(j)^{1/\alpha}
\end{aligned}\tag{5.5}$$

with $\bar G(u)=1-G(u)+G(-u)$. To estimate the second moment of these summands write

$$\begin{aligned}
&E(\xi_j(\omega)-\eta_j(\omega))^2I\left(|\eta_j(\omega)|<\varepsilon^{4/(2-\alpha)}j^{1/\alpha}\bar L(j)^{1/\alpha}\right)\\
&\qquad\le 2E\xi_j^2(\omega)I\left(|\xi_j(\omega)|<2\varepsilon^{4/(2-\alpha)}j^{1/\alpha}\bar L(j)^{1/\alpha}\right)\\
&\qquad\qquad+2E\eta_j^2(\omega)I\left(|\eta_j(\omega)|<\varepsilon^{4/(2-\alpha)}j^{1/\alpha}\bar L(j)^{1/\alpha}\right).
\end{aligned}$$

This inequality implies together with an asymptotic relation about the behaviour of the moments of a random variable with regularly varying distribution function, a result which follows for instance from Theorem 2, Part (i) in [13] Chapter VIII. Section 9 with the choice $\eta = 0$ and $\zeta = 2$, that

$$\begin{aligned}
&E(\xi_j(\omega) - \eta_j(\omega))^2 I\left(|\eta_j(\omega)| < \varepsilon^{4/(2-\alpha)} j^{1/\alpha} \bar L(j)^{1/\alpha}\right) \\
&\qquad \le \text{const.}\, \varepsilon^{8/(2-\alpha)} j^{2/\alpha} \bar L(j)^{2/\alpha} \left[\left(1 - F(\varepsilon^{4/(2-\alpha)} j^{1/\alpha} \bar L(j)^{1/\alpha})\right)\right. \\
&\qquad\qquad\qquad \left. + \left(1 - G_j(\varepsilon^{4/(2-\alpha)} j^{1/\alpha} \bar L(j)^{1/\alpha})\right)\right] \\
&\qquad \le \text{const.}\, \varepsilon^{8/(2-\alpha)} j^{2/\alpha} \bar L(j)^{2/\alpha} \varepsilon^{-4\alpha/(2-\alpha)} j^{-1} \bar L(j)^{-1} \left[L\left(j^{1/\alpha} \bar L(j)^{1/\alpha}\right) + \bar L(j)\right] \\
&\qquad \le \text{const.}\, \varepsilon^4 j^{-1+2/\alpha} \bar L(j)^{2/\alpha}
\end{aligned} \tag{5.6}$$

because of formula (5.3).

The estimates (5.5) and (5.6) together with the estimates $N(n)\bar L(N(n)) \sim 2^n$, $(k\bar L(k))^{1/\alpha} \le \frac{9}{8} 2^{n/\alpha}$ for $k \le N(n)$, $N(n) < 5N(n-1)$ and Kolmogorov's inequality imply that with the notation of the events $A_j = \{\omega\colon |\eta_j(\omega)| < \varepsilon^{4/(2-\alpha)} j^{1/\alpha} \bar L(j)^{1/\alpha}\}$ and their indicator functions $I(A_j(\omega))$, $j = 1, 2, \dots,$

$$\begin{aligned}
&\left| \sum_{j=N(n-1)+1}^{k} E(\xi_j(\omega) - \eta_j(\omega)) I(A_j(\omega)) - a_j \right| \le \sum_{j=N(n-1)+1}^{k} \frac{\varepsilon}{45} j^{(1-\alpha)/\alpha} \bar L(j)^{1/\alpha} \\
&\qquad \le \frac{\varepsilon}{40} 2^{n/\alpha} \sum_{j=N(n-1)+1}^{N(n)} \frac{1}{j} \le \frac{\varepsilon}{30} 2^{n/\alpha} \log \frac{N(n)}{N(n-1)} \le \frac{\varepsilon}{8} 2^{n/\alpha}
\end{aligned}$$

for all $N(n-1) < k \le N(n)$, and

$$\begin{aligned}
&P\left(\sup_{N(n-1) < k \le N(n)} \left| \sum_{j=N(n-1)+1}^{k} (\xi_j(\omega) - \eta_j(\omega)) I(A_j(\omega)) - a_j \right| > \frac{\varepsilon}{4} x 2^{n/\alpha} \right) \\
&\qquad \le \frac{\sum\limits_{j=N(n-1)+1}^{N(n)} E(\xi_j(\omega) - \eta_j(\omega))^2 I(A_j(\omega))}{\frac{\varepsilon^2}{64} x^2 2^{2n/\alpha}} \le \frac{\varepsilon^4 \left[N(n) \bar L(N(n))\right]^{2/\alpha}}{\frac{\varepsilon^2}{64} x^2 2^{2n/\alpha}} \\
&\qquad \le \frac{\text{const.}\, \varepsilon^2}{x^2} \quad \text{for all } x \ge 1 \text{ and } n \ge n_0(\bar\varepsilon)
\end{aligned} \tag{5.7}$$

We still have to bound the contribution of those terms $\xi_j(\omega) - \eta_j(\omega)$ in the sum (5.4) for which $\eta_j(\omega) \ge \varepsilon^{4/(2-\alpha)} j^{1/\alpha} \bar L(j)^{1/\alpha}$. We shall separate them to middle terms for which $|\eta_j(\omega)|$ is not too large and large terms for which $|\eta_j(\omega)|$ is large. This separation to the middle and large terms will be made differently for large and small values of the parameter x in formula (5.4).

Let us choose the number $\bar\varepsilon(\varepsilon)$ sufficiently small, and define the sets

$$B_j(x)=\left\{\omega\colon \varepsilon^{4/(2-\alpha)}j^{1/\alpha}\bar L(j)^{1/\alpha}\le|\eta_j(\omega)|<\varepsilon^{-1/\alpha}xj^{1/\alpha}\bar L(j)^{1/\alpha}\right\}$$

and

$$C_j(x)=\left\{\omega\colon |\eta_j(\omega)|\ge\varepsilon^{-1/\alpha}xj^{1/\alpha}\bar L(j)^{1/\alpha}\right\}$$

together with the indicator functions $I(B_j(x)(\omega))$ and $I(C_j(x)(\omega))$, $j=1,2,\dots,$ of these sets if $1\le x\le\bar\varepsilon^{-1/4}$. For sufficiently large j relation (5.1) can be applied on the sets $B_j(\omega)$. This implies that because of the factor $\bar\varepsilon$ in the upper bound given (5.1) we can prove similarly to the proof of relations (5.5) and (5.6) that

$$|E(\xi_j(\omega)-\eta_j(\omega))I(B_j(x)(\omega))|\le\frac{\varepsilon}{45}j^{(1-\alpha)/\alpha}\bar L(j)^{1/\alpha}. \tag{5.8}$$

and

$$E(\xi_j(\omega)-\eta_j(\omega))^2I(B_j(x)(\omega))\le\text{const.}\,\varepsilon^4j^{-1+2/\alpha}\bar L(j)^{2/\alpha} \tag{5.9}$$

for $x\le\bar\varepsilon^{-1/4}$. These estimates together with Kolmogorov's inequality and the asymptotic relations we have for $N(n)$ and $L(N(n)$ yield similarly to the estimate (5.7) the bound

$$\begin{aligned}\left|\sum_{j=N(n-1)+1}^{k}E(\xi_j(\omega)-\eta_j(\omega))I(B_j(\omega))\right|&\le\sum_{j=N(n-1)+1}^{k}\frac{\varepsilon}{45}j^{(1-\alpha)/\alpha}\bar L(j)^{1/\alpha}\\&\le\frac{\varepsilon}{9}\left(N(n)\bar L(N(n))\right)^{1/\alpha}\le\frac{\varepsilon}{8}2^{n/\alpha}\end{aligned}$$

for all $N(n-)+1\le k\le N(n)$ and

$$\begin{aligned}&P\left(\sup_{N(n-1)<k\le N(n)}\left|\sum_{j=N(n-1)+1}^{k}(\xi_j(\omega)-\eta_j(\omega))I(B_j(x)(\omega))\right|>\frac{\varepsilon}{4}x2^{n/\alpha}\right)\\&\quad\le\frac{\sum\limits_{j=N(n-1)+1}^{N(n)}E(\xi_j(\omega)-\eta_j(\omega))^2I(B_j(x)(\omega))}{\frac{\varepsilon^2}{64}x^22^{2n/\alpha}}\le\text{const.}\,\frac{\varepsilon^4\left[N(n)\bar L(N(n))\right]^{2/\alpha}}{\frac{\varepsilon^2}{64}x^22^{2n/\alpha}}\\&\quad\le\frac{\text{const.}\,\varepsilon^2}{x^2}\quad\text{for all }\bar\varepsilon^{-1/4}\ge x\ge1\text{ and }n\ge n_0(\bar\varepsilon).\end{aligned} \tag{5.10}$$

In the outer domain we can write for $\bar\varepsilon^{-1/4} \ge x$ because of the definition of the sets $C_j(x)$

$$\begin{aligned}
&P\left(\sup_{N(n-1)<k\le N(n)} \left|\sum_{j=N(n-1)+1}^{k} (\xi_j(\omega)-\eta_j(\omega))I(C_j(x)(\omega))\right| > \frac{\varepsilon}{2}x2^{n/\alpha}\right)\\
&\qquad \le P\left(\{\omega\colon \omega\in C_j(x) \quad \text{for some } N(n-1)<k\le N(n)\}\right)\\
&\qquad \le \sum_{j=N(n-1)+1}^{N(n)} P\left(|\eta_j(\omega)| \ge \varepsilon^{-1/\alpha} x j^{1/\alpha}\bar L(j)^{1/\alpha}\right)\\
&\qquad \le \text{const.}\, \varepsilon x^{-\alpha} \sum_{j=N(n-1)+1}^{N(n)} \frac{1}{j} \le \text{const.}\, \varepsilon x^{-\alpha}.
\end{aligned} \tag{5.11}$$

Since $A_j \cup B_j(x) \cup C_j(x) = \Omega$ for all j, relations (5.7), (5.10) and (5.11) imply relation (5.4) with $\gamma = \alpha$ in the case $x < \bar\varepsilon^{-1/4}$.

In the case $x \ge \bar\varepsilon^{-1/4}$ define the sets

$$B'_j(x) = \left\{\omega\colon \varepsilon^{4/(2-\alpha)} j^{1/\alpha}\bar L(j)^{1/\alpha} \le |\eta_j(\omega)| < x^{1/2} j^{1/\alpha}\bar L(j)^{1/\alpha}\right\}$$

and

$$C'_j(x) = \left\{\omega\colon |\eta_j(\omega)| \ge x^{1/2} j^{1/\alpha}\bar L(j)^{1/\alpha}\right\}$$

together with the indicator functions of these sets $I(B'_j(x)(\omega))$ and $I(C'_j(x)(\omega))$, $j = 1,2,\dots$. We remark that if $\bar\varepsilon(\varepsilon) > 0$ is very small, then $x^{1/2} \ll \varepsilon x$. In this case we can apply the estimation

$$L\left(x^{1/2} j^{1/\alpha}\bar L(j)^{1/\alpha}\right) \le C(t)x^t L(j^{1/\alpha}\bar L(j)^{1/\alpha}) \le C'(t)x^t \bar L(j) \tag{5.12}$$

for all $x \ge 1$ and $t > 0$ which holds, because $L(\cdot)$ is a slowly varying function. With the help of relation (5.12) the following (weaker) version of the estimate (5.11) can be proved which is appropriate for our purposes.

$$\begin{aligned}
&P\left(\sup_{N(n-1)<k\le N(n)} \left|\sum_{j=N(n-1)+1}^{k} (\xi_j(\omega)-\eta_j(\omega))I(C'_j(x)(\omega))\right| > \frac{\varepsilon}{2}x2^{n/\alpha}\right)\\
&\qquad \le \sum_{j=N(n-1)+1}^{N(n)+1} P\left(|\eta_j(\omega)| \ge x^{1/2} j^{1/\alpha}\bar L(j)^{1/\alpha}\right)\\
&\qquad \le \text{const.}\, x^{-\alpha/3} \sum_{j=N(n-1)+1}^{N(n)} \frac{1}{j} \le \text{const.}\, x^{-\alpha/3} < \varepsilon x^{-\alpha/4},
\end{aligned} \tag{5.13}$$

if $\bar\varepsilon$ is sufficiently small, and as a consequence x is sufficiently large.

To prove an estimate analogous to (5.10), to bound the contribution of the middle terms for large x, we give a (weaker) estimate on the first two moments of the random variables $(\xi_j(\omega)-\eta_j(\omega))I(B'_j(x))(\omega)$.

The estimation of the second moment is simpler. In this case we can argue similarly to the estimate (5.6). We can apply the result from Feller's book [13], Theorem 2, Part (i) Chapter VIII. Section 9 with the choice $\eta=0$ and $\zeta=2$. Then we get with the application of formula (5.12) that

$$\begin{aligned}
E(\xi_j(\omega)-\eta_j(\omega))^2 I(B'_j(x)(\omega)) &\le \text{const.}\, x j^{2/\alpha}\bar L(j)^{2/\alpha} \\
&\quad \left[\left(1-F(x^{1/2}j^{1/\alpha}\bar L(j)^{1/\alpha})\right)+\left(1-G_j(x^{1/2}j^{1/\alpha}\bar L(j)^{1/\alpha})\right)\right] \\
&\le \text{const.}\, x j^{2/\alpha}\bar L(j)^{2/\alpha} x^{-\alpha/2} j^{-1}\bar L(j)^{-1}\left[L\left(x^{1/2}j^{1/\alpha}\bar L(j)^{1/\alpha}\right)+\bar L(j)\right] \\
&\le \text{const.}\, x j^{-1+2/\alpha}\bar L(j)^{2/\alpha},
\end{aligned}$$

and

$$\operatorname{Var}\left(\sum_{j=N(n-1)+1}^{N(n)}(\xi_j(\omega)-\eta_j(\omega))I(B'_j(x))(\omega)\right)\le \text{const.}\, x2^{2n/\alpha}. \tag{5.14}$$

To get an appropriate estimate on the first moment let us observe that in the estimation in formula (5.5) we gave a good bound on the integral we want to estimate if the domain of integration is restricted to the interval $\varepsilon^{4/(2-\alpha)}j^{1/\alpha}\bar L(j)^{1/\alpha}\le|u|\le j^{1/\alpha}\bar L(j)^{1/\alpha}$. Hence, handling this part of the integral separately we get integrating by parts and applying (5.12) with $t=\alpha>0$

$$\begin{aligned}
\left|E(\xi_j(\omega)-\eta_j(\omega))I(B'_j(x)(\omega))\right| &\le \varepsilon j^{-1+1/\alpha}\bar L(j)^{1/\alpha} \\
&\quad +\int_{j^{1/\alpha}\bar L(j)^{1/\alpha}\le|u|\le x^{1/2}j^{1/\alpha}\bar L(j)^{1/\alpha}}|u|(F(\,du)+G_j(du)) \\
&\le \text{const.}\int_{j^{1/\alpha}\bar L(j)^{1/\alpha}\le|u|\le x^{1/2}j^{1/\alpha}\bar L(j)^{1/\alpha}}(F(u)+G_j(u))\,du \\
&\quad +\text{const.}\, x^{1/2}j^{-1+1/\alpha}\bar L(j)^{1/\alpha}\le \text{const.}\, x^{1/2}j^{-1+1/\alpha}\bar L(j)^{1/\alpha}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|E\sum_{j=N(n-1)+1}^{k}(\xi_j(\omega)-\eta_j(\omega))I(B'_j(x)(\omega))\right| \\
&\qquad \le \text{const.}\, x^{1/2}N(n)^{1/\alpha}\bar L(N(n))^{1/\alpha}\le \text{const.}\, x^{1/2}2^{n/\alpha}\le\frac{\varepsilon}{8}x2^{n/\alpha}
\end{aligned} \tag{5.15}$$

for all $N(n-1)\le k\le N(n)$ if $\bar\varepsilon=\bar\varepsilon(\varepsilon)$ is sufficiently small, hence $x>\bar\varepsilon^{-1/4}$ is sufficiently large. The estimates (5.14) and (5.15) together with Kolmogorov's inequality imply that

$$\begin{aligned}P&\left(\sup_{N(n-1)<k\le N(n)}\left|\sum_{j=N(n-1)+1}^{k}(\xi_j(\omega)-\eta_j(\omega))I(B'_j(x)(\omega))\right|>\frac{\varepsilon}{4}x2^{n/\alpha}\right)\\&\qquad\le\frac{x2^{2n/\alpha}}{\frac{\varepsilon^2}{64}x^2 2^{2n/\alpha}}\le\frac{\varepsilon}{x^{1/2}}\quad\text{for all } x\ge\bar\varepsilon^{-1/4}\text{ and } n\ge n_0(\bar\varepsilon)\end{aligned}\tag{5.16}$$

if $\bar\varepsilon$ is sufficiently small. Now relations (5.7), (5.13) and (5.16) imply relation (5.4) in remaining case $x>\bar\varepsilon^{-1/4}$ with $\gamma=\min\left(\frac{\alpha}{4},\frac12\right)=\frac{\alpha}{4}$. Since the number $\bar\varepsilon=\bar\varepsilon(\varepsilon)$ can be chosen in such a way that all inequalities needed in the proof are satisfied, the above calculations imply the almost sure functional limit theorem.

To complete the proof of Theorem 3 we still have to show that the random variables $\frac{1}{B_n}\sum\limits_{j=1}^{n}(\xi_j(\omega)-a_n)$ converge in distribution to the distribution function $G(x)$. We prove this if we show that $\frac{1}{n^{1/\alpha}\bar L(n)^{1/\alpha}}\sum\limits_{j=1}^{n}(\xi_j(\omega)-\eta_j(\omega)-a_n)\Rightarrow 0$ as $n\to\infty$, where $\Rightarrow$ denotes convergence in distribution. (The denominator B_n can be replaced by $n^{1/\alpha}\bar L(n)^{1/\alpha}\sim B_n$ in this formula.) Let us choose the representation

$$\begin{aligned}\xi_j(\omega)-\eta_j(\omega)&=(\xi_j(\omega))-\eta_j(\omega))I\left(|\eta_j(\omega)|<\varepsilon^{4/(2-\alpha)}n^{1/\alpha}\bar L(n)^{1/\alpha}\right)\\&\quad+(\xi_j(\omega))-\eta_j(\omega))I\left(\varepsilon^{4/(2-\alpha)}n^{1/\alpha}\bar L(n)^{1/\alpha})\le|\eta_j(\omega)|\le\varepsilon^{-1}n^{1/\alpha}\bar L(n)^{1/\alpha}\right)\\&\quad+(\xi_j(\omega))-\eta_j(\omega))I\left(|\eta_j(\omega)|\ge\varepsilon^{-1}n^{1/\alpha}\bar L(n)^{1/\alpha}\right),\end{aligned}\tag{5.17}$$

and apply a natural modification of relations (5.5), (5.6), (5.8) and (5.9) appropriate in the present case. Here the separation levels are chosen as $\varepsilon^{4/(2-\alpha)}n^{1/\alpha}\bar L(n)^{1/\alpha}$ instead of $\varepsilon^{4/(2-\alpha)}j^{1/\alpha}\bar L(j)^{1/\alpha}$ in relations (5.5) and (5.6), and $\varepsilon^{-1}n^{1/\alpha}\bar L(n)^{1/\alpha}$ instead of $xj^{1/\alpha}\bar L(j)^{1/\alpha}$ in relations (5.8) and (5.9). The main difference in the proof of the almost sure functional limit theorem and in the proof of the limit theorem for the distribution of the normalized partial sum is that now we make the same truncation for all indices $1\le j\le n$. Then we can bound the expression we get if we take the sum of the first two terms in (5.17) for indices $j=1,2,\dots,n$ by means of Chebishev's inequality. We get that

$$\begin{aligned}P&\left(\frac{1}{n^{1/\alpha}\bar L(n)^{1/\alpha}}\sum_{j=1}^{n}\left((\xi_j(\omega)-\eta_j(\omega))I(|\eta_j(\omega)|<\varepsilon^{-1}n^{1/\alpha}\bar L(n)^{1/\alpha})-a_n\right)>\varepsilon\right)\\&\qquad\le\text{const.}\,\varepsilon^2.\end{aligned}\tag{5.18}$$

On the other hand,

$$\sum_{j=1}^{n}P\left(|\eta_j(\omega)|\ge\varepsilon^{-1}n^{1/\alpha}\bar L(n)^{1/\alpha}\right)\le n\times\text{const.}\,\varepsilon^{\alpha}n^{-1}=\text{const.}\,\varepsilon^{\alpha}\tag{5.19}$$

by relations (1.5) and (5.3). Relations (5.18) and (5.19) imply the weak convergence. Theorem 3 is proved. (The functional limit theorem formulated in Remark 1 can be proved by some modification of the proof, in particular by applying the Kolmogorov inequality instead of the Chebishev inequality in the estimates.)

Proof of Theorem $3'$. The stable process $X_0(t,\omega)$ is not self-similar, because only the relation

$$X_0(tT,\omega) \stackrel{\Delta}{=} TX_0(t,\omega) + \gamma tT \log T$$

holds with $\gamma = C_1 - C_2$, where $\stackrel{\Delta}{=}$ denotes equation in distribution. In this case the process $X_0'(t,\omega) = X_0(t,\omega) - \gamma t \log t$ is self-similar with self-similarity parameter $\alpha = 1$. Indeed, the relation

$$\begin{aligned} X_0'(tT,\omega) &= X_0(tT,\omega) - \gamma tT \log tT \stackrel{\Delta}{=} TX_0(t,\omega) + \gamma tT(\log T - \log tT) \\ &= T(X_0(t,\omega) - \gamma t \log t) = TX_0'(t,\omega) \end{aligned}$$

holds for all $T > 0$. The results of Part I. can be applied for this process. They imply that the random variables $\eta_n(\omega) = \bar{L}(n)\bar{\eta}_n(\omega) - C_n$ satisfy the almost sure functional limit theorem with $C_n = \gamma(B_n \log B_n - B_{n-1} \log B_{n-1})$, where $\eta_j(\omega)$, $n = 1, 2, \dots$, are i.i.d. random variables with distribution function $G(x)$. Then checking the proof of Theorem 3 one can see that the estimates given there also hold in the case $\alpha = 1$. They imply that relation (5.4) also holds in this case. Hence Property A holds for the pairs $(\xi_n(\omega) - C_n, \eta_n(\omega) - C_n)$. This result implies that the sequence $\xi_n(\omega) - C_n$ or the sequence $\xi_n(\omega)$ with an appropriate (modified) shift a_n and weight function $B_n = \sum\limits_{k=1}^{n} \bar{L}(k)$ satisfies the almost sure functional limit theorem.

6. Discussion of the results, and some open problems

Our results state that for a sequence of random variables $\xi_n(\omega)$, $n=1,2,\dots,$ with some nice properties a result of the following type holds. Define a sequence of random broken lines $X_n(\cdot,\omega)$, $n=1,2,\dots,$ in the way described in formula (1.2) by means of these random variables $\xi_1(\omega),\xi_2(\omega),\dots,$ for all $n=1,2.\dots,$ and $\omega\in\Omega$, define the probability $\mu_N(\omega)$ in the function space $D([0,1])$ (or in the space $C([0,1])$ if this is possible) by attaching an appropriately defined probability $a_{k,N}$ to the appropriately normalized version of the trajectories $X_k(\cdot,\omega)$, $k=1,\dots,N$, $\sum\limits_{k=1}^N a_{k,N}=1$. Carrying out the above construction in an appropriate way, e.g. in the way described in this work, we get that the measures $\mu_N(\omega)$ weakly converge to an appropriate measure μ_0 for almost all $\omega\in\Omega$. In the first part of this work we proved such results for general self-similar processes. A weaker version of such results also appeared in the paper [9] of Csáki and Földes. In the second part we proved such results for processes which are close to some special self-similar processes. Actually the transformation which enables one to construct stationary processes by means of self-similar processes and vice versa was found already in Lamperti's paper [17]. This transformation which enabled us to study self-similar processes by means of "generalized Ornstein–Uhlenbeck processes" was applied by Lamperti to construct self-similar processes. Let us remark that this method in itself does not settle the problem of construction of self-similar processes. An important question is to construct such self-similar processes which are also stationary or have stationary increments. Such constructions demand new ideas.

One may ask how close these measures $\mu_N(\omega)$, $N=1,2,\dots,$ are to the limit measure μ_0. The following two questions seem to be natural problems in this direction.

- i.) Can a more precise estimate be given about the distance $d(\mu_N(\omega),\mu_0)$, where $d(\cdot,\cdot)$ is an appropriate metric on the space of probability measures which metrizes weak convergence? With which replacement of the weights $\dfrac{\log\frac{B_{k+1}}{B_k}}{\log\frac{B_N}{B_1}}$ in formula (1.3) do the measures $\mu_N(\omega)$, $N=1,2,\dots,$ have the same limit for almost all $\omega\in\Omega$ as the original measures in the definition of the almost sure invariance principle?
- ii.) The weak convergence of the measures $\mu_N(\omega)$ to μ_0 states that $\int\mathcal{F}(x)\mu_N(\omega)(\,dx)\to\int\mathcal{F}(x)\mu_0(\,dx)$ as $N\to\infty$ for all continuous and bounded functional $\mathcal{F}$ in the space $D([0,1])$ (or $C([0,1])$. For which larger classes of functionals $\mathcal{F}$ does this statement hold?

There are some results in the spirit of problem 1, see e.g. [14], but some further, deeper results in this direction would be welcome. The second problem is a natural version of the generalization of the Donsker theorem, and description of the so-called Donsker classes. This is a popular subject, (see e.g. [12]), but I do not know of any improvement of the almost sure functional limit theorem in this direction. Let us also remark that the basis of our proofs was the application of the ergodic theorem for the "generalized Ornstein–Uhlenbeck processes". In the case of general self-similar processes we cannot assume a stronger result, but in some important special cases, like in the case of (non-generalized) Ornstein–Uhlenbeck process there is a chance to improve the ergodic theorem and to get a non-trivial partial answer to the question (i).

Another natural generalization of the almost sure invariance principle is to prove the existence of limit of the appropriate weighted average of the variables $\mathcal{F}S_k(\cdot,\omega)$ for almost all $\omega\in\Omega$, where the trajectories $S_k(\cdot,\omega)$ are defined in formula (1.2), not only for bounded continuous functionals in the space $C([0,1])$ or $D([0,1])$, but also for such functionals $\mathcal{F}$ which satisfy certain moment conditions, but may be unbounded. The idea behind such a generalization is that the ergodic theorem which is in the background of the proofs requires only that certain moment condition be satisfied.

There are several papers about such problems. These papers consider such special functionals which depend only on $S_k(1,\omega)$. The deepest result in this direction I know about is contained in the paper [16] of Ibragimov and Lifshitz.

Roughly speaking, the second part of this work stated that the almost sure functional limit theorem holds for independent random variables under the conditions of the limit theorem for the distribution of the normalized partial sums of these random variables. Let us also remark that there are examples (see e.g. [4]) showing that the almost sure functional limit theorem also may hold in cases when the limit theorem for the normalized partial sums of these random variables does not hold.

The construction and proofs (the formulation and application of the Basic Lemma) in Part II. strongly exploited the independence of the random variables under consideration. The question arises what can be said in the dependent case. Does the almost sure limit theorem hold under general conditions? Are the conditions sufficient for a limit theorem for the distribution of the normalized partial sums sufficient also for the almost sure functional limit theorem? I would formulate as a ‘fist rule’ a positive answer to this question, but cannot supply a proof. It might be interesting to study such limit theorems where the limit is a self-similar process with dependent increments. The process constructed by Dobrushin in [10], and the papers proving limit theorem with this limit (see e.g. [11] or [19]) may be interesting in this respect.

We make a brief comparison between our results and results of earlier papers. A more detailed overview together with a comprehensive list of literature can be found in paper [3].

The most frequently studied problem in this subject is the case when the limit is Gaussian. Most works is restricted to a one-dimensional version of the almost sure functional limit theorem called the almost sure central limit theorem. This result states that under general condition the partial sums $S_n(\omega)=\sum\limits_{j=1}^{n}\xi_j(\omega)$, $n=1,2,\dots$, of independent random variables with expectation zero satisfy the relation

$$\lim_{n\to\infty}\frac{1}{\sum\limits_{k=1}^{n}b_k}\sum_{k=1}^{n}\frac{1}{b_k}P\left(S_k<x\sqrt{\operatorname{Var}S_k}\right)=\Phi(x)\quad\text{for almost all }\omega\in\Omega$$

with the choice of appropriate weights b_n, $n=1,2,\dots$, where $\Phi(\cdot)$ denotes the normal distribution function. The result of M. Atlagh [1] is the sharpest result in this direction among the results I know about. Atlagh assumed, similarly to our Theorem 1, that the Lindeberg condition holds, and he also formulated a very weak restriction about the growth of the variance of the summands. He chose the weights $b_k = \dfrac{E\xi_k^2}{\operatorname{Var} S_k}$ in his paper. Formally, this is a choice of weights different from ours. But since $\log \dfrac{\operatorname{Var} S_k}{\operatorname{Var} S_{k-1}} \sim \dfrac{E\xi_k^2}{\operatorname{Var} S_k}$, and this approximation is sufficiently good under the conditions formulated by Atlagh, it can be proved with the help of Theorem 4 in Part I. that the results with these two different weights are equivalent. More explicitly, it follows from this result that the two choices of weights are equivalent if $\sum\limits_{k=1}^{\infty} \left(\dfrac{E\xi_k^2}{\operatorname{Var} S_k} \right)^2 < \infty$, and this relation holds under the conditions of Atlagh's result. We omit the details of the proof.

We discuss the analogs of Theorems 2 and 3 more briefly. There are results (see e.g. [2], [5] and [6]) which supply a one-dimensional version of these results. Here the weight functions are different from ours. At this point it is important that these results only deal with one dimensional distribution and not with the random broken lines which contain 'the whole history' of the process. In these one dimensional problems some general theorems about averaging, see e.g. the paper Bingham and Roger [7], give a fairly big freedom in the choice of the weight functions. On the other hand, in the case of the almost sure functional limit theorem a radical change of the weight functions also modifies the points where the random broken lines have a jump, hence it may modify the shape of the broken lines. This means that in such a case we have less freedom in the choice of the weights. Let us remark that the class of weight functions for which the almost sure functional limit theorem holds could be enlarged. There is a possibility to generalize the class of possible weights given in Theorem 4 of Part I. to triangular arrays $B_{k,n}$, $1 \le k \le n$ under appropriate conditions. This would give a better possibility to compare our results with those of [2], [5] and [6], but we shall not discuss this problem.

Finally we remark that in Theorem 3 (and Theorem 3′) we have to give a 'shift parameter' a_n beside the weight functions B_n to define the random broken lines which satisfy the almost sure functional limit theorem. Here again we have certain freedom. As we showed in the Remark 2. in Section 1 this norming constants can be chosen in the same way as in the limit theorems for the distribution function of the normalized partial sums. This means that, by the limit theorems with a stable limit law we can choose $a_n = 0$ in the case $\alpha < 1$ and $a_n = E\xi_1(\omega)$ in the case $1 < \alpha < 2$. The choice of the norming constant a_n cannot be given in such a simple way in the case $\alpha = 1$.

The results of [5] and of the recent paper [15] also imply Theorem 3 and 3′. The method of these papers is different from ours.

After having finished this work I have learned about the recent paper [15] of Ibragimov. I. A. and Lifshitz, M. A. which has not yet appeared. The subject of this paper is similar to ours, but there are considerable differences both in the formulation of the results and method of the proofs. The aim of the authors of this work — similarly to the present paper — is to find the general principles and results in the theory of almost sure limit theorems. They prove both one dimensional and functional almost sure limit theorems. They show that the usual limit theorems for distribution functions, which they apply in an equivalent form expressed by means of characteristic functions, also imply the almost sure limit theorems. It is worth mentioning that the proof of the almost sure functional limit theorem in [15] formulated in Theorem 3.1 contains an interesting idea which also could help to considerably simplify the proof of Theorem 1 in Part I. of the present paper.

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