# Wiener–Itô integral representation in vector valued Gaussian stationary random fields

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*Abstract:* The subject of this work is the multivariate generalization of the theory of multiple Wiener–Itô integrals. In the scalar valued case this theory was described in paper [11]. Our proofs apply the technique of this work, but in the proof of some results new ideas were needed. The motivation for this study was a result in paper [1] of Arcones where he formulated the multivariate version of a non-central limit theorem for non-linear functionals of Gaussian stationary random fields presented in paper [6]. We found the proof in paper [1] incomplete and wanted to give a full proof. We did it in paper [13], but in that proof we needed a detailed description of the properties of non-linear functionals of vector valued stationary Gaussian fields. Here we provide the foundation needed to carry out that proof.

## 1 Introduction. An overview of the results.

Let $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, where $\mathbb{Z}^\nu$ denotes the lattice points with integer coordinates in the $\nu$-dimensional Euclidean space $\mathbb{R}^\nu$, be a $d$-dimensional real valued Gaussian stationary random field with expectation $EX(p) = 0$, $p \in \mathbb{Z}^\nu$. We define the notion of Gaussian property of a random field in the usual way, i.e. we demand that all finite sets $(X(p_1), \dots, X(p_k))$, $p_j \in \mathbb{Z}^\nu$, $1 \le j \le k$, be a Gaussian random vector, and we call a random field $X(p)$, $p \in \mathbb{Z}^\nu$, stationary if for all $m \in \mathbb{Z}^\nu$ the random field $X^{(m)}(p) = X(p+m)$, $p \in \mathbb{Z}^\nu$, has the same finite dimensional distributions as the original random field $X(p)$, $p \in \mathbb{Z}^\nu$. In most works only the case $\nu = 1$ is considered, but since we can prove our results without any difficulty for stationary random fields with arbitrary parameter $\nu \ge 1$ we consider such more general models.

Our goal is to work out a good calculus which provides such a representation of the non-linear functionals of our vector valued Gaussian stationary random field which helps us in the study of limit theorems for such functionals. To understand what kind of limit theorems we have in mind take the following example which is discussed in Section 8 of this paper.

Let us have a function $H(x_1, \dots, x_d)$ of $d$ variables, and define with the help of a $d$-dimensional vector valued Gaussian stationary random field

$$X(p) = (X_1(p), \dots, X_d(p)), \quad p \in \mathbb{Z}^\nu,$$

and this function the random variables $Y(p) = H(X_1(p), \dots, X_d(p))$ for all $p \in \mathbb{Z}^\nu$. Let us introduce for all $N = 1, 2, \dots$ the normalized sum

$$S_N = A_N^{-1} \sum_{p \in B_N} Y(p) \tag{1.1}$$

with an appropriate norming constant $A_N > 0$, where

$$B_N = \{p = (p_1, \dots, p_\nu) \colon \ 0 \le p_k < N \text{ for all } 1 \le k \le \nu\}.$$

We are interested in a limit theorem for these normalized sums $S_N$ with an appropriate norming constant $A_N$ as $N \to \infty$. We are interested in the case when there is a relatively strong dependence between the elements of the random field $X(p)$, $p \in \mathbb{Z}^\nu$, and new kind of limit theorems can appear.

A. M. Arcones studied a similar problem in paper [1]. He considered only the case $\nu = 1$, but this is not a serious restriction. The main point is that he investigated the non-linear functionals of *vector valued* stationary Gaussian random sequences, and such problems were not studied before. He was interested both in the question when the above defined normalized sums $S_N$ satisfy the central limit theorem with the usual normalization $A_N = \sqrt{N}$, and when we get a new kind of limit theorem with an unorthodox normalization.

First he proved a classical central limit theorem if the covariance function $r(p)$, $p \in \mathbb{Z}^1$, of the underlying vector valued Gaussian stationary random sequence tends to zero sufficiently fast at infinity, and the function $H(x_1, \dots, x_d)$ satisfies some nice properties. This result can be considered the multivariate generalization of the result in [3], and the proof is based on a refinement of the arguments of this paper. Then he presented in Theorem 6 a non-central limit theorem for $S_N$ which can be considered as the multivariate generalization of a non-central limit theorem for the random sums $S_N$ proved in [6] under appropriate conditions for a scalar valued Gaussian stationary random field, i.e. in the case $d = 1$. However, I found the proof (and formulation) of this result incomplete.

In the proof of [6] a crucial point was the representation of the random variables $S_N$ in the form of a multiple Wiener–Itô integral with respect to the random spectral measure of the underlying stationary random field. With its help we could carry out a limiting procedure that enabled us to express the limit by means of a multiple Wiener–Itô integral. We did this in the scalar valued case. A detailed explanation of this approach is given in [11]. But the results applied in this approach were proved only in the scalar valued case. In the proof of Arcones which is based on similar arguments one would need the vector valued version of the notions and results discussed in [11]. But they were not available.

I wanted to replace Arcones' discussion about Theorem 6 of his paper with a full proof of this result. It turned out that to do this first I have to work out the multivariate version of the theory presented in [11] which dealt with the scalar valued case. This is the subject of the present paper. It can be considered as a continuation of the research of Itô in [10] and of Dobrushin in [5].

Itô considered a Gaussian random field in [10] whose elements could be expressed as random integrals with respect to a Gaussian orthogonal random measure. He defined multiple random integrals (called later Wiener–Itô integrals in the literature) with respect to this orthogonal random measure, and expressed all square integrable random variables measurable with respect to the $\sigma$-algebra generated by the elements of the Gaussian random field as a sum of such multiple integrals. This notion turned out to be useful, because it helped in the study of the non-linear functionals of the Gaussian random field. In particular, he found a very useful relation between the multiple random integrals he defined and Hermite polynomials.

Later Dobrushin worked out a version of this theory in [5], where he studied the non-linear functionals of a stationary Gaussian random field. In such a case a spectral and a random spectral measure can be defined in such a way that the elements of the stationary Gaussian random field can be expressed as the Fourier transform of the random spectral measure. Dobrushin introduced the multiple random integral with respect to this random spectral measure, and studied its properties. He proved that this new integral has similar properties as the original multiple integral introduced by Itô. This new integral turned out to be useful, because it made possible to combine Itô's theory with the Fourier analysis. A detailed discussion of these theories can be found in my work [11].

We would like to apply Dobrushin's theory if we are dealing with vector valued stationary Gaussian random field. But we cannot do this in a direct way, because Dobrushin's theory applies some results which are valid in their original form only for scalar valued random fields. So we have to work out the vector valued version of this theory, and this is the subject of the present work.

First we have to find the multivariate version of the spectral and random spectral measure, and this is the subject of Sections 2 and 3. Moreover, we need later the notion of spectral measure and random spectral measure corresponding to stationary generalized random fields, and this is the subject of Section 4. (The precise definition of the notions mentioned in this introduction will be given in the more detailed discussion of the main text.)

Then I define the multiple Wiener–Itô integrals with respect to the coordinates of a vector valued random spectral measure in Section 5. In Section 6 I prove the diagram formula which enables us to express the product of two multiple Wiener–Itô integrals as the sum of appropriately defined multiple Wiener–Itô integrals. This result has an important role in Section 7 which has two main subjects. The first one is to create a useful relation between multiple Wiener–Itô integrals and the multivariate generalization of Hermite polynomials, the so-called Wick polynomials. The second subject of Section 7 is a useful representation of the shift transformation of a random variable given in the form of a multiple Wiener–Itô integral. These results enable us to work out a method in Section 8 to prove new type non-central limit theorems, in particular to give a correct formulation and proof of Theorem 6 of Arcones paper [1]. Next I give a more detailed description about the content of the subsequent sections.

## 1.1 A more detailed description of the results.

In working out the multivariate version of the theory in [11] first I characterize the distribution of the vector valued Gaussian stationary random fields $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, with expectation zero. This is the subject of the second section of this work. Because of the Gaussian and stationary property of such a random field its distribution is determined by the correlation function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$ for all $1 \le j, j' \le d$ and $p \in \mathbb{Z}^\nu$. We are interested in the description of those functions $r_{j,j'}(p)$ which can appear as the correlation function of a vector valued stationary random field.

In the scalar valued case a well-known result solves this problem. The correlation function $r(p) = EX(0)X(p)$, $p \in \mathbb{Z}^\nu$, of a stationary field $X(p)$, $p \in \mathbb{Z}^\nu$, can be represented in a unique way as the Fourier transform of a spectral measure, and the spectral measure can be characterized. Namely, we call the finite (non negative), even measures on the torus $[-\pi, \pi)^\nu$ spectral measures. For any correlation function $r(p)$ of a stationary field there is a unique spectral measure $\mu$ such that $r(p) = \int e^{i(p,x)}\mu(\,dx)$ for all $p \in \mathbb{Z}^\nu$, and for all spectral measures $\mu$ there is a (Gaussian) stationary random field whose correlation function equals the Fourier transform of this spectral measure $\mu$.

In Section 2 we prove a similar result for vector valued stationary random fields. In the case of a vector valued Gaussian stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, we have for all pairs of indices $(j, j')$, $1 \le j, j' \le d$, a unique complex measure $G_{j,j'}$ on the torus $[-\pi, \pi)^\nu$ with finite total variation such that $r_{j,j'}(p) = EX_j(0)X_{j'}(p) = \int e^{i(p,x)}G_{j,j'}(\,dx)$ for all $p \in \mathbb{Z}^\nu$. This can be interpreted so that the correlation function $r_{j,j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, is the Fourier transform of a matrix valued measure $(G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi, \pi)^\nu$. We want to give, similarly to the scalar valued case, a complete description of those matrix valued measures on the torus $[-\pi, \pi)^\nu$ for which the correlation function of a vector valued Gaussian stationary random field can be represented as its Fourier transform. Such matrix valued measures will be called matrix valued spectral measures.

As I have mentioned, the coordinates of a matrix valued spectral measure are complex measures with finite total variation. The scalar valued counterpart of this condition is the condition that the spectral measure of a scalar valued stationary random field must be finite. Another important property of a matrix valued spectral measure is that it must be positive semidefinite. The meaning of this property is explained before the formulation of Theorem 2.2, and Lemma 2.3 gives a different, equivalent characterization of this property. Let me remark that in the scalar valued case the spectral measure must be a measure (and not only a complex measure), and this fact corresponds to the above property of matrix valued spectral measures. Finally, a matrix valued spectral measure must be even. This means that its coordinates are even, i.e. for all $1 \le j, j' \le d$

and measurable sets $A$ on the torus $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$, where the overline indicates complex conjugate.

Theorem 2.2 states that the above properties characterize the matrix valued spectral measures. Let me remark that there are papers, (see e.g. [4], [8] or [15]) which contain the above results, although in a slightly different formulation at least in the case $\nu = 1$. Nevertheless, I worked out their proof, since I applied a different method which is used also in the later part of the paper.

In Section 3 I consider the vector valued random spectral measures which correspond to a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$. They are sets of vector valued random vectors $(Z_{G,1}(A), \dots, Z_{G,d}(A))$ defined for all measurable subsets $A \subset [-\pi, \pi)^\nu$ on the torus with some nice properties which enable us to define random integrals with respect to them. Let me remark that all random variables $Z_{G,j}(A)$, $1 \le j \le d$, $A \subset [-\pi, \pi)^\nu$, are complex valued. The correlation function of a vector valued Gaussian random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, can be expressed as the Fourier transform of its matrix valued spectral measure. One of the main results in Section 3 states that a random spectral measure $(Z_{G,1}, \dots, Z_{G,d})$ can be constructed in such a way that its Fourier transform expresses the random field itself. More explicitly, $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\,dx)$ for all $p \in \mathbb{Z}^\nu$ and $1 \le j \le d$. I have listed some properties of this random measure $(Z_{G,1}, \dots, Z_{G,d})$ which determine its distribution, and we call a vector valued random measure with such properties a vector valued random spectral measure corresponding to the matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$. We can prove that the Fourier transform of a vector valued random spectral measure corresponding to a matrix valued spectral measure is a vector valued Gaussian stationary random field with this matrix valued spectral measure.

Besides the result that for all matrix valued spectral measures $(G_{j,j'})$, $1 \le j, j' \le d$, there exists a vector valued random spectral measure corresponding to it I also proved some important properties of the random integrals with respect to a vector valued spectral measure in Section 3. I also proved that if a vector valued Gaussian stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, is given, we fix some parameter $1 \le j \le d$, and take the real Hilbert space consisting of the closure of finite linear combinations $\sum_k c_k X_j(p_k)$ with real number valued coefficient $c_k$ in the Hilbert space of square integrable random variables, then each element of this Hilbert space can be expressed as the integral of a function on the torus $[-\pi, \pi)^\nu$ with respect to the random spectral measure $Z_{G,j}$. The functions taking part in the representation of this Hilbert space also constitute a real Hilbert space. A more detailed formulation of this result is given in Lemma 3.2.

It may be worth discussing the relation of the results in Section 3 to their scalar valued correspondents. The results about the existence of random spectral measures for scalar valued Gaussian stationary random fields give a great help in proving the results in Section 3. In particular, these results provide the definition of the random spectral measures $Z_{G,j}$, and determine their distribution for all $1 \le j \le d$. The definition of $Z_{G,j}$, and the properties determining its distribution depend only on the measure $G_{j,j}$. On the other hand, we had to carry out some additional work to prove those properties of a vector valued spectral random measure which determine the joint distribution of their coordinates. The complex measures $G_{j,j'}$ with $j \neq j'$ appear at this point of the investigation.

The fourth section deals with a special subject, and our motivation to study it demands some explanation. Here we consider vector valued Gaussian stationary generalized random fields.

We could have considered the continuous time version of vector valued stationary random fields where the parameter set is $t \in \mathbb{R}^\nu$ and not $p \in \mathbb{Z}^\nu$. We did not discuss such models, we have considered instead vector valued Gaussian stationary generalized random fields. This means a set of random vectors $(X_1(\varphi), \dots, X_d(\varphi))$ with some nice properties which are indexed by an appropriately chosen class of functions. The precise definition of this notion is given in Section 4. We have constructed a large class of Gaussian stationary generalized random fields, presented their matrix valued spectral measures, and constructed the vector valued random spectral measures corresponding to them. In [11] the notion of Gaussian stationary generalized random fields was introduced and investigated in the scalar valued case. Some useful results were proved there. It was shown (with the help of some important results of Laurent Schwartz about generalized functions) that in the scalar valued case the class of Gaussian, stationary generalized random fields constructed in such a way as it was done in the present paper contains all Gaussian stationary generalized random fields. (Here I consider two random fields the same if their finite dimensional distributions agree.) This result is probably also valid in the case of vector valued generalized random fields, but I did not study this question, because I was interested in a different problem.

Although the theory of generalized random fields is an interesting subject in itself, I was interested not so much in their properties, I was investigating them for a different reason. I was interested in the matrix valued spectral measures of vector valued Gaussian stationary generalized random fields and the vector valued random spectral measures corresponding to them and not in the Gaussian, stationary generalized random fields which were needed for their construction. They behave similarly to the analogous objects corresponding to (non-generalized) Gaussian stationary random fields. We can work with them in the same way. Nevertheless, there is a difference between these new spectral and random spectral measures and the previously defined ones which is very important for us. Namely, the coordinates of a matrix valued spectral measure corresponding to a non-generalized random field are complex measures with finite total variation, while in the case of generalized random fields the matrix valued spectral measures need not satisfy this condition. It is enough to demand that the corresponding matrix valued measures have locally finite total variation, and the matrix valued spectral measures are semidefinite matrix valued measures with moderately increasing distribution at infinity. (The definition of these notions is contained in Section 4.)

The above facts mean that we can work with a much larger class of random spectral measures after the introduction of Gaussian stationary generalized random fields and random spectral measures corresponding to them. This is important for us, because in the limit theorems we are interested in the limit can be expressed by means of multiple Wiener–Itô integrals with respect to random spectral measures constructed with the help of vector valued Gaussian stationary generalized random fields.

Sections 2—4 contain the main results about the linear functionals of vector valued Gaussian stationary random fields we need in our investigation. They are also needed in the study of non-linear functionals of vector valued stationary Gaussian fields, which is the subject of the remaining part of this work. Here I work out some tools which are useful in the study of limit theorems with a new type of non-Gaussian limit.

In Section 5 multiple Wiener–Itô integrals are defined with respect to the coordinates of a vector valued random spectral measure $(Z_{G,1},\dots,Z_{G,d})$. We define for all numbers $n=1,2,\dots,$ and parameters $j_1,\dots,j_n$ such that $1\le j_k\le d$ for all $1\le k\le n$ and all functions $f\in\mathcal{K}_{n,j_1,\dots,j_n}$, where $\mathcal{K}_{n,j_1,\dots,j_n}$ is a real Hilbert space defined in Section 5 an $n$-fold Wiener–Itô integral

$$I_n(f|j_1,\dots,j_n)=\int f(x_1,\dots,x_n)Z_{G,j_1}(\,dx_1)\dots Z_{G,j_n}(\,dx_n),$$

and prove some of its basic properties. The definition and proofs are very similar to the definition and proofs in scalar valued case, only we have to apply the properties of vector valued random spectral measures.

There is one point where we have a weaker estimate than in the scalar valued case. We can give an upper bound on the second moment of a multiple Wiener–Itô integral with the help of the $L_2$ norm of the kernel function of this integral in the way as it is formulated in formula (5.6), but we can state here only an inequality and not an equality. The behaviour of Wiener–Itô integrals with respect to a scalar valued random spectral measure is different. If we integrate in this case a symmetric function, and we may restrict our attention to such integrals, then we have equality in the corresponding relation. This weaker form of the estimate (5.6) has the consequence that in certain problems we can get only weaker results for Wiener–Itô integrals with respect to the coordinates of a vector valued random spectral measure than for Wiener–Itô integrals with respect to scalar valued random spectral measures. But as it turns out we can work well with multiple Wiener–Itô integrals with respect to the coordinates of a vector valued spectral measure.

The multiple Wiener–Itô integrals were introduced in order to express a large class of random variables with their help. More precisely, we are interested in the following problem. Let us have a vector valued Gaussian stationary random field $X(p)=(X_1(p),\dots,X_d(p))$, $p\in\mathbb{Z}^\nu$. Their elements can be expressed as the Fourier transforms of a vector valued random spectral measure $Z_G=(Z_{G,1},\dots,Z_{G,d})$. Let us consider the real Hilbert space $\mathcal{H}$ defined in the second paragraph of Section 5 with the help of this vector valued stationary Gaussian random field. We would like to express the elements of this Hilbert space in the form of a sum of multiple Wiener–Itô integrals with respect to the coordinates of the vector valued spectral measure $Z_G$.

Let $\mathcal{H}_1$ denote the linear subspace of $\mathcal{H}$ generated by the finite linear combinations of the random variables $X_j(p)$, $p \in \mathbb{Z}^\nu$, $1 \le j \le d$, (with real valued coefficients). It follows from Lemma 3.2 that the random variables $\xi \in \mathcal{H}_1$ which can be written as the sum of one-fold Wiener–Itô integrals $\xi = \sum_{j=1}^d \int f_j(x) Z_{G,j}(\,dx_j)$ with some functions $f_j \in \mathcal{K}_{1,j}$, $1 \le j \le d$ constitute an everywhere dense linear subspace in $\mathcal{H}_1$. Besides, we can define for all $n = 0,1,2,\dots$ some appropriate orthogonal subspaces $\mathcal{H}_n$ of the Hilbert space $\mathcal{H}$ such that $\mathcal{H}$ is the direct sum $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \cdots$ of these subspaces. This result is described in Section 7 in more detail. In the case of scalar valued random fields it was proved (see [11]) that all elements of $\mathcal{H}_n$ can be expressed as an $n$-fold Wiener–Itô integral. We would like to prove a similar result for vector valued fields. This is one of the main subjects of the investigations in Sections 6 and 7.

In the case of vector valued Gaussian stationary random fields we can prove only a weaker result. We can write the elements of a dense linear subspace of $\mathcal{H}_n$ in the form of a sum $n$-fold Wiener–itô integrals. This subspace contains all polynomials of the random variables $X_j(p)$, $1 \le j \le d$, $p \in \mathbb{Z}^\nu$. On the other hand, we can write the presentation of these elements in a more or less explicit form. The reason for being able to do this only for the elements of a dense subspace of $\mathcal{H}_n$ is related to the weakness of the estimate (5.6) mentioned before.

In Section 6 I formulate and prove the multivariate version of a classical result. I describe the product of two multiple Wiener–Itô integrals as the sum of multiple Wiener–Itô integrals with respect to the coordinates of a vector valued random spectral measure. The formulation and proof of this result is similar to that of the corresponding result in the scalar valued case. In this result we define the kernel functions of the Wiener–Itô integrals appearing in the sum expressing the product of two Wiener–Itô integrals with the help of some diagrams. Hence this result got the name diagram formula. I wrote down the formulation of the diagram formula in the case of vector valued random spectral measures in detail. On the other hand, I gave only a sketch of proof of the diagram formula, because it is actually an adaptation of the original proof with a rather unpleasant notation. I concentrated on the points which explain why the diagram formula has such a form as we claim. Besides, I tried to explain those steps of the proof where we have to apply some new ideas. I hope that the interested reader can reconstruct the proof on the basis of these explanations by looking at the original proof.

Section 6 also contains a corollary of the diagram formula, where I formulate this result in a special case. I formulated this corollary, because in this work we need only this corollary of the diagram formula.

Section 7 has two main subjects. One of them is the multivariate version of Itô's formula which enables us to rewrite Wick polynomials in the form of multiple Wiener–Itô integrals, the other one is a useful formula which expresses the shift transformations of a random variable given in the form of a multiple Wiener–Itô integral. Although the introduction and investigation of the shift transformations were introduced only in Section 7, their investigation is a most important subject of the present work. We can prove the limit theorems we are interested in with the help of the representation of the shift transformations given in Section 7.

In the original version of Itô's formula the Hermite polynomial of a standard normal random variable or the product of Hermite polynomials of independent standard normal random variables is expressed in the form of a multiple Wiener–Itô integral. Its multivariate version is a similar result about Wick polynomials. Wick polynomials are the natural multivariate versions of Hermite polynomials of standard Gaussian random variables.

In Section 7 first I recall from [11] the definition of Wick polynomials together with their most important properties. Given a homogeneous polynomial of order $n$ of some random variables $U_p \in \mathcal{H}_1$ we define the Wick polynomial corresponding to it as it is written down in this section. In Theorem 7.2 and in its corollary we express a Wick polynomial of order $n$ as a sum of Wiener–Itô integrals of order $n$ if the random variables $U_p$ appearing in the definition of this Wick polynomial are given in the form $U_p = \int \varphi_p(x) Z_{G,j_p}(\,dx)$ with some parameter $1 \le j_p \le d$ and $\varphi_p \in \mathcal{K}_{1,j_p}$. The proof of the original version of Itô's formula was based on the diagram formula and the recursive relation $H_n(x) = xH_{n-1}(x) - (n-1)H_{n-2}(x)$ for Hermite polynomials. The multivariate version of Itô's formula is proved with the help of the corollary of the diagram formula formulated in Section 6 and a multivariate version of the above mentioned recursive formula for Wick polynomials which is proved in Proposition 7.1.

Next I recall the definition of the shift transformations and show how the shift transformations of a random variable given in the form of a multiple Wiener–Itô integral can be calculated. This result is interesting for us, because as the corollary of Theorem 7.2 and Proposition 7.3 indicate, a large class of random variables can be written in the form of a sum of multiple Wiener–Itô integrals.

The formula for the calculation of the shift transform for Wiener–Itô integrals is very similar to the analogous result about the shift transformations of a Wiener–Itô integral with respect to a scalar valued random spectral measure. This result, formulated in Proposition 7.4, states that if

$$Y = \int h(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(\,dx_n),$$

then its shift transformation $T_u$ equals

$$T_u Y = \int e^{i(u, x_1 + \dots + x_n)} h(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(\,dx_n).$$

for all $u \in \mathbb{Z}^\nu$.

In Section 8 a method for proving limit theorems for non-linear functionals of vector valued Gaussian stationary fields is discussed. The example mentioned at the start of this Introduction is considered. Some standard arguments show that in this problem we can restrict our attention to the case when the random variables $Y(p) = H(X_1(p), \dots, X_d(p))$ in the normalized sum we are investigating are Wick polynomials.

Thus we consider in Section 8 the above limit problem in the special case when $Y(p) = H(X_1(p), \dots, X_d(p))$ is a Wick polynomial (for all parameters $p \in \mathbb{Z}^\nu$). In this case we can rewrite the normalized sums $S_N$ in a useful form with the help of the multivariate version of Itô's formula and our result about the representation of the shift transformation of multiple Wiener–Itô integrals. We rewrite these normalized sums in such a form that provides a heuristic argument for the proof of a limit theorem.

Our goal is to find a result that enables us to make a precise proof on the basis of this heuristic argument. The main result of Section 8, Proposition 8.1 may help us to carry out such a program. Besides, we prove another result in Lemma 8.2 that simplifies a little bit the conditions of Proposition 8.1.

In a subsequent paper [13] I shall prove a multivariate version of the limit theorem in [6] with the help of the present paper. In that proof I shall apply Proposition 8.1, and the main problem is to check its conditions. The result in [6] was proved with the help of the scalar valued version of the results in this work. In that proof it had to be shown that the conditions of the scalar valued version of Proposition 8.1 hold under appropriate assumptions. We did it by proving a relation between the behaviour of the correlation function and spectral measure of a stationary random field. In the proof of the multivariate generalization of the result in paper [6] it will be shown that a similar relation holds also between the behaviour of the correlation function and spectral measure of a vector valued stationary random field.

# 2 Spectral representation of vector valued stationary random fields

Let $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, where $\mathbb{Z}^\nu$ denotes the lattice of points with integer coordinates in the $\nu$-dimensional Euclidean space $\mathbb{R}^\nu$, be a $d$-dimensional real valued Gaussian stationary random field with expected value $EX(p) = 0$, $p \in \mathbb{Z}^\nu$. Let us first characterize the covariance matrices $R(p) = (r_{j,j'}(p))$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$ of this $d$-dimensional stationary random field, where $r_{j,j'}(p) = EX_j(0)X_{j'}(p) = EX_j(m)X_{j'}(p+m)$, $1 \le j, j' \le d$, $p, m \in \mathbb{Z}^\nu$.

In the case $d = 1$ we can characterize the function $R(p) = EX(0)X(p)$, (in this case $j = j' = 1$, so we can omit these indices) as the Fourier transform of an even finite (and positive) measure $G$ on the torus $[-\pi, \pi)^\nu$, called the spectral measure. We are looking for the vector valued version of this result. Before discussing this problem I recall the definition of the torus $[-\pi, \pi)^\nu$.

The points of the torus $[-\pi, \pi)^\nu$ are those points $x = (x_1, \dots, x_\nu) \in \mathbb{R}^\nu$ for which $-\pi \le x_j \le \pi$ for all $1 \le j \le \nu$. But if a coordinate of $x$ in this set equals $\pi$, then we consider this point the same if we replace this coordinate by $-\pi$. In such a way we can identify all points of this set by a point of the set $[-\pi, \pi)^\nu \subset \mathbb{R}^\nu$. We define the topology on the torus on $[-\pi, \pi)^\nu$ as the topology induced by the metric $\rho(x,y)=\sum\limits_{j=1}^{\nu}(|x_j-y_j| \bmod 2\pi)$ if $x=(x_1,\dots,x_\nu)\in[-\pi,\pi)^\nu$ and $y=(y_1,\dots,y_\nu)\in[-\pi,\pi)^\nu$. These properties of the torus $[-\pi,\pi)^\nu$ must be taken into account when we speak of the set $-A=\{-x\colon\ x\in A\}$ for a set $A\subset[-\pi,\pi)^\nu$ or of a continuous function on the torus $[-\pi,\pi)^\nu$.

Later we shall speak also about the torus $[-A,A)^\nu$ for arbitrary $A>0$. This is defined in the same way, only the number $\pi$ is replaced by $A$ in the definition.

It is natural to expect that there is a natural definition of even positive semidefinite matrix valued measures also in the $d$-dimensional case, $d\ge 2$, and this takes the role of the spectral measure in the vector valued case. To define this notion first I prove a lemma. Before formulating it I recall the definition of a complex measure with finite total variation, since this notion appears in the formulation of the lemma. We say that a complex measure on a measurable space has finite total variation if both its real and imaginary part can be represented as the difference of two finite measures. I also recall Bochner's theorem, more precisely that version of this result that we shall apply in the proof.

**Bochner's theorem.** *Let $f(p)$, $p\in\mathbb{Z}^\nu$, be a positive definite function on $\mathbb{Z}^\nu$, i.e. such a function for which the inequality $\sum\limits_{j=1}^{N}\sum\limits_{j'=1}^{N} z_j\bar z_{j'} f(p_j-p_{j'})\ge 0$ holds for any set of points $p_j\in\mathbb{Z}^\nu$, and complex numbers $z_j$, $1\le j\le N$, with some number $N\ge 1$. Then there exists a unique finite measure $G$ on the torus $[-\pi,\pi)^\nu$ such that*

$$f(p)=\int_{[-\pi,\pi)^\nu} e^{i(p,x)}G(\,dx) \quad \text{for all } p\in\mathbb{Z}^\nu.$$

*If the function $f$ is real valued, then the measure $G$ is even, i.e. $G(-A)=G(A)$ for all $A\subset[-\pi,\pi)^\nu$.*

Next I formulate the following lemma.

**Lemma 2.1.** *Let $X(p)=(X_1(p),\dots,X_d(p))$, $p\in\mathbb{Z}^\nu$, be a $d$-dimensional stationary Gaussian random field with expectation zero. Then for all pairs $1\le j,j'\le d$ the correlation function $r_{j,j'}(p)=EX_j(0)X_{j'}(p)$, $p\in\mathbb{Z}^\nu$, can be written in the form*

$$r_{j,j'}(p)=EX_j(0)X_{j'}(p)=EX_j(m)X_{j'}(m+p)=\int_{[-\pi,\pi)^\nu} e^{i(p,x)}G_{j,j'}(\,dx) \tag{2.1}$$

*with a complex measure $G_{j,j'}$ on the torus $[-\pi,\pi)^\nu$ with finite total variation. The function $r_{j,j'}(p)$, $p\in\mathbb{Z}^\nu$, uniquely determines this complex measure $G_{j,j'}$ with finite total variation. It is even, i.e. $G_{j,j'}(-A)=\overline{G_{j,j'}(A)}$ for all measurable sets $A\subset[-\pi,\pi)^\nu$. The relation $G_{j',j}(A)=\overline{G_{j,j'}(A)}$ also holds for all $1\le j,j'\le d$ and $A\subset[-\pi,\pi)^\nu$.*

*Remark.* Let us remark that given a $d$-dimensional stationary random field with expectation zero, there exist also such $d$-dimensional stationary random fields with expectation zero which are Gaussian, and have the same correlation function. As a consequence, in Lemma 2.1 we could drop the condition that the stationary random field we are considering is Gaussian. The same can be said about the other results of Section 2. I imposed this condition, because later, as we work with random spectral measures and random integrals with respect to them the Gaussian property of the underlying random field is important.

*Proof of Lemma 2.1.* By Bochner's theorem we may write

$$r_{j,j}(p) = \int_{[-\pi,\pi)^\nu} e^{i(p,x)} G_{j,j}(\,dx), \quad p \in \mathbb{Z}^\nu,$$

for all $1 \le j \le d$ with some finite measure $G_{j,j}$ on $[-\pi,\pi)^\nu$. We find a good representation for $r_{j,j'}(n)$ if $j \ne j'$ with the help of following argument.

The function

$$\begin{aligned} q_{j,j'}(p) &= E[X_j(0) + iX_{j'}(0)][X_j(p) - iX_{j'}(p)] \\ &= E[X_j(0) + iX_{j'}(0)]\overline{[X_j(p) + iX_{j'}(p)]}, \end{aligned}$$

$p \in \mathbb{Z}^\nu$, is positive definite, hence it can be written in the form

$$E[X_j(0) + iX_{j'}(0)][X_j(p) - iX_{j'}(p)] = \int_{[-\pi,\pi)^\nu} e^{i(p,x)} H_{j,j'}(\,dx)$$

with some finite measure $H_{j,j'}$ on $[-\pi,\pi)^\nu$. Similarly,

$$E[X_j(0) + X_{j'}(0)][X_j(p) + X_{j'}(p)] = \int_{-[\pi,\pi)^\nu} e^{i(p,x)} K_{j,j'}(\,dx)$$

with some finite measure $K_{j,j'}$ on $[-\pi,\pi)^\nu$. Hence

$$\begin{aligned} EX_j(0)X_{j'}(p) &= \frac{i}{2} E[X_j(0) + iX_{j'}(0)][X_j(p) - iX_{j'}(p)] \\ &\quad + \frac{1}{2} E[X_j(0) + X_{j'}(0)][X_j(p) + X_{j'}(p)] \\ &\quad - \frac{(1+i)}{2}[EX_j(0)X_j(p) + EX_{j'}(0)X_{j'}(p)] \\ &= \int_{[-\pi,\pi)^\nu} e^{i(p,x)} G_{j,j'}(\,dx) \end{aligned}$$

with $G_{j,j'}(\,dx) = \frac{1}{2}[iH_{j,j'}(\,dx) + K_{j,j'}(\,dx)] - \frac{(1+i)}{2}[G_{j,j}(\,dx) + G_{j',j'}(\,dx)]$.

In such a way we have found complex measures $G_{j,j'}$ with finite total variation which satisfy relation (2.1). Since this relation holds for all $p \in \mathbb{Z}^\nu$, the function $r_{j,j'}(p)$, $p \in \mathbb{Z}^\nu$, determines the measure $G_{j,j'}$ uniquely.

Since $r_{j,j'}(p)$ is real valued, i.e. $r_{j,j'}(p) = \overline{r_{j,j'}(p)}$, it can be written both in the form

$$r_{j,j'}(p) = \int_{[-\pi,\pi)^\nu} e^{i(p,x)} G_{j,j'}(\,dx)$$

and

$$r_{j,j'}(p)=\int_{[-\pi,\pi)^\nu} e^{-i(p,x)}\overline{G_{j,j'}(\,dx)}=\int_{[-\pi,\pi)^\nu} e^{i(p,x)}\overline{G_{j,j'}(-\,dx)}.$$

Comparing these relations we get that $G_{j,j'}(A)=\overline{G_{j,j'}(-A)}$ for all measurable sets $A\subset[-\pi,\pi)^\nu$. Similarly, the relation $r_{j',j}(p)=r_{j,j'}(-p)$ implies that $G_{j',j}(A)=G_{j,j'}(-A)=\overline{G_{j,j'}(A)}$ for all measurable sets $A\subset[-\pi,\pi)^\nu$. Lemma 2.1 is proved.

Since all complex measures $G_{j,j'}$, $1\le j,j'\le d$, have finite total variation by Lemma 2.1 there is a finite measure $\mu$ on the torus $[-\pi,\pi)^\nu$ such that all these complex measures $G_{j,j'}$ are absolutely continuous with respect to $\mu$, and the absolute value of the Radon–Nikodym derivatives $g_{j,j'}(x)=\frac{dG_{j,j'}}{d\mu}(x)$ is integrable with respect to $\mu$. The properties of the measures $G_{j,j'}$ proved in Lemma 2.1 imply that the $d\times d$ matrix $(g_{j,j'}(x))$, $1\le j,j'\le d$, is Hermitian for almost all $x\in[-\pi,\pi)^\nu$ with respect to the measure $\mu$. We shall call the matrix valued measure $(G_{j,j'}(A))$, $A\subset[-\pi,\pi)^\nu$, positive semidefinite if the matrix $(g_{j,j'}(x))$, $1\le j,j'\le d$, is positive semidefinite for almost all $x\in[-\pi,\pi)^\nu$ with respect to $\mu$. More precisely, we introduce the following definition.

**Definition of positive semidefinite matrix valued, even measures on the torus.** *Let us have some complex measures $G_{j,j'}$, $1\le j,j'\le d$, with finite total variation on the $\sigma$-algebra of the Borel measurable sets of the torus $[-\pi,\pi)^\nu$. Let us consider the matrix valued measure $(G_{j,j'})$, $1\le j,j'\le d$. We call this matrix valued measure positive semidefinite if there exists a (finite) positive measure $\mu$ on $[-\pi,\pi)^\nu$ such that all complex measures $G_{j,j'}$, $1\le j,j'\le d$, are absolutely continuous with respect to it, and their Radon–Nikodym derivatives $g_{j,j'}(x)=\frac{dG_{j,j'}}{d\mu}(x)$, $1\le j,j'\le d$, constitute a positive semidefinite matrix $(g_{j,j'}(x))$, $1\le j,j'\le d$ for almost all $x\in\mathbb{Z}^\nu$ with respect to the measure $\mu$. We call this positive semidefinite matrix valued measure $(G_{j,j'})$, $1\le j,j'\le d$, on the torus even if $G_{j,j'}(-A)=\overline{G_{j,j'}(A)}$ for all measurable sets $A\subset[-\pi,\pi)^\nu$ and $1\le j,j'\le d$.*

*Later we shall speak also of positive semidefinite matrix valued, even measures on a torus $[-A,A)^\nu$ for arbitrary $A>0$ which is defined in the same way, only the complex measures $G_{j,j'}$ and the dominating measure $\mu$ are defined on $[-A,A)^\nu$.*

*Remark.* Here I am speaking about measures with finite total variation, although such (complex) measures are called generally bounded measures in the literature. Actually, we know by Stone's theorem that any bounded signed measure can be represented as the difference of two bounded measures (with disjoint support). Nevertheless, I shall remain at this name, because actually we prove directly the finite total variation of the measures we shall work with in this paper. Besides, (in Section 4) I shall define complex measures on $\mathbb{R}^\nu$ with locally finite total variation, and I prefer such a name which refers to the similarity of these objects. (The complex measures with locally finite total variation are not measures in the original meaning of this word, only their restrictions to compact sets are complex measures.)

The next theorem about the characterization of the correlation function of a $d$-dimensional stationary Gaussian random field with zero expectation states that the correlation functions $r_{j,j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, can be given in the form (2.1) with the help of a positive semidefinite matrix valued, even measure $(G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi, \pi)^\nu$. Moreover, it will be shown that we have somewhat more freedom in the definition of positive semidefinite matrix valued measures on the torus. If the coordinates of a matrix valued measure $(G_{j,j'})$, $1 \le j, k \le d$, are complex measures with finite total variation, and this matrix valued measure satisfies the definition of the positive semidefinite property with some measure $\mu$, then this measure $\mu$ can be replaced in the definition by any such finite measure on the torus with respect to which the complex measures $G_{j,j'}$ are absolutely continuous. More explicitly, the following result holds.

**Theorem 2.2.** *The covariance matrices of a $d$-dimensional stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, with expectation zero can be given in the following form. For all $1 \le j, j' \le d$ there exists a complex measure $G_{j,j'}$ with finite total variation on the $\nu$-dimensional torus $[-\pi, \pi)^\nu$ in such a way that for all $1 \le j, j' \le d$ the correlation function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $p \in \mathbb{Z}^\nu$, is given by formula (2.1) with this complex measure $G_{j,j'}$. The $d \times d$ matrix $G = (G_{j,j'})$, $1 \le j, j' \le d$, whose coordinates are the complex measures $G_{j,j'}$ has the following properties. This matrix is Hermitian, i.e. the measures $G_{j,j'}$ satisfy the relation $G_{j',j}(A) = \overline{G_{j,j'}(A)}$ for all pairs of indices $1 \le j, j' \le d$ and measurable sets $A \subset [-\pi, \pi)^\nu$, and the measures $G_{j,j'}$ are even, i.e. $G_{j,j'}(-A) = \overline{G_{j,j'}(-A)}$ for all $1 \le j, j' \le d$ and $A \subset [-\pi, \pi)^\nu$. For all pairs $(j, j')$, $1 \le j, j' \le d$, the function $r_{j,j'}(p)$, $p \in \mathbb{Z}^\nu$, defined by formula (2.1) uniquely determines the complex measure $G_{j,j'}$ with finite total variation. Besides, $G_{j,j'}$ has the following property.*

*Let us take a finite measure $\mu$ on the torus $[-\pi, \pi)^\nu$ such that all complex measures $G_{j,j'}$ are absolutely continuous with respect to it, (because of the finite total variation of the complex measures $G_{j,j'}$ there exist such measures), and put $g_{j,j'}(x) = g_{j,j',\mu}(x) = \frac{dG_{j,j'}}{d\mu}(x)$. Then the matrix $(g_{j,j'}(x))$, $1 \le j, j' \le d$, is positive semidefinite for almost all $x \in [-\pi, \pi)^\nu$ with respect to the measure $\mu$.*

*Conversely, if a class of complex measures $G_{j,j'}$ on $[-\pi, \pi)^\nu$, $1 \le j, j' \le d$, have finite total variation, and $(G_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite matrix valued, even measure on the torus, then there exists a $d$-dimensional stationary Gaussian field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, with expectation $EX_j(p) = 0$ and covariance $EX_j(p)X_{j'}(q) = r_{j,j'}(p - q)$, where the function $r_{j,j'}(p)$ is defined in (2.1) with the complex measure $G_{j,j'}$ for all parameters $1 \le j, j' \le d$ and $p, q \in \mathbb{Z}^\nu$.*

*Remark.* We shall call the positive semidefinite matrix valued, even measure $(G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi, \pi)^\nu$ with coordinates $G_{j,j'}$ satisfying relation (2.1) the matrix valued spectral measure of the correlation function

$r_{j,j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$. In general, we shall call an arbitrary positive semidefinite matrix valued, even measure on the torus $[-\pi,\pi)^\nu$ a matrix valued spectral measure on the torus $[-\pi,\pi)^\nu$. (More generally, later we shall call for any $A > 0$ a positive semidefinite matrix valued, even measure on the torus $[-A, A)^\nu$ a matrix valued spectral measure on this torus.) We have the right for such a terminology, since by Theorem 2.2 for an arbitrary positive semidefinite matrix valued, even measure on the torus $[-\pi,\pi)^\nu$ there is a vector valued stationary Gaussian random field on $\mathbb{Z}^\nu$ such that this positive semidefinite matrix valued, even measure is the spectral measure of its correlation function.

*Proof of Theorem 2.2.* The statements formulated in the first paragraph of Theorem 2.2 follow from Lemma 2.1. Next we prove that the matrix $(g_{j,j'}(x))$, $1 \le j, j' \le d$, whose elements are defined as the Radon–Nikodym derivatives of the complex measures $G_{j,j'}$ with respect to a measure $\mu$ satisfying the conditions of Theorem 2.2 is positive semidefinite for $\mu$ almost all $x$.

We prove this by first showing with the help of Weierstrass' second approximation theorem that

$$\int_{[-\pi,\pi)^\nu} v(x)g(x)v^*(x)\mu(\,dx) \ge 0 \tag{2.2}$$

for any continuous, $d$-dimensional vector-valued function $v(x) = (v_1(x), \dots, v_d(x))$ on the $\nu$-dimensional torus $[-\pi,\pi)^\nu$, where $g(x)$ denotes the $d \times d$ matrix $(g_{j,j'}(x))$, $1 \le j, j' \le d$, and $v^*(x)$ is the conjugate of the vector $v(x)$.

To prove (2.2) let us first observe that by Weierstrass' second approximation theorem for all $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ and $d$ trigonometrical polynomials of order $N$

$$v_{N,j}(x) = \sum_{\substack{s=(s_1,\dots,s_\nu)\\ -N\le s_k<N,\ 1\le k\le\nu}} a_{j,s_1,\dots,s_\nu} e^{i(s,x)}, \quad 1 \le j \le d, \quad x \in [-\pi,\pi)^\nu$$

for which

$$\sup_{x\in[-\pi,\pi)^\nu} |v_{N,j}(x) - v_j(x)| \le \varepsilon \quad \text{for all } 1 \le j \le d.$$

Let us also define the random vector $Y_N = (Y_{N,1}, \dots, Y_{N,d})$ with coordinates

$$Y_{N,j} = \sum_{\substack{s=(s_1,\dots,s_\nu)\\ -N\le s_k<N,\ 1\le k\le\nu}} a_{j,s_1,\dots,s_\nu} X_j(s), \quad 1 \le j \le d,$$

Then we have because of the relation $EX_j(s)X_{j'}(s') = \int e^{i(s-s',x)} g_{j,j'}(x)\mu(\,dx)$

$$0 \le E\left(\sum_{j=1}^d Y_{N,j}\right)\overline{\left(\sum_{j=1}^d Y_{N,j}\right)} = \sum_{j=1}^d \sum_{j'=1}^d \int_{[-\pi,\pi)^\nu} g_{j,j'}(x) v_{N,j}(x)\overline{v_{N,j'}(x)}\mu(\,dx).$$

Hence

$$\int_{[-\pi,\pi)^\nu} v_N(x)g(x)v_N^*(x)\mu(\,dx) \ge 0,$$

and we get relation (2.2) from it with the help of the limiting procedure $N\to\infty$.

Let us choose a vector $a=(a_1,\dots,a_d)\in\mathbb{R}^d$ and a non-negative continuous function $u(x)$ on the torus $[-\pi,\pi)^\nu$. Let us apply formula (2.2) with the choice of the function $v(x)=(a_1\sqrt{u(x)},\dots,a_d\sqrt{u(x)})$. With this choice formula (2.2) yields that

$$0\le\int_{[-\pi,\pi)^\nu} v(x)g(x)v^*(x)\mu(\,dx)=\int_{[-\pi,\pi)} u(x)h_a(x)\mu(\,dx)$$

with the function $h_a(x)=ag(x)a^*$. Since this inequality holds for all non-negative continuous functions this implies that $h_a(x)\ge0$ for almost all $x$ with respect to the measure $\mu$. Moreover, since $h_a(x)=ag(x)a^*$ is a continuous function of the parameter $a$ for a fixed number $x\in[-\pi,\pi)^\nu$ this also implies that $g(x)$ is a positive semidefinite matrix for almost all $x$ with respect to the measure $\mu$. We have proved that the covariance matrix of a vector valued stationary field has the properties stated in Theorem 2.2.

Next I show that if we have a class of complex measures $G_{j,j'}$ with finite total variation such that $(G_{j,j'})$ is a positive semidefinite matrix valued even measure on the torus, and the functions $r_{j,j'}(p)$, $p\in\mathbb{Z}^\nu$, are defined by formula (2.1) with these complex measures $G_{j,j'}$, then there exists a vector-valued stationary Gaussian field $X(p)=(X_1(p),\dots,X_d(p))$ with expectation zero and covariance function $EX_j(0)X_{j'}(p)=r_{j,j'}(p)$.

First I show that for all $N\ge1$ there is a set of Gaussian random vectors $X(p)=(X_1(p),\dots,X_d(p))$, with parameters $p=(p_1,\dots,p_\nu)$, $-N\le p_j\le N$ for all $j=1,\dots,d$, such that $EX_j(p)X_{j'}(q)=r_{j,j'}(p-q))$ for all $p=(p_1,\dots,p_\nu)$, $q=(q_1,\dots,q_\nu)$ with $-N\le p_s,q_s\le N$, $1\le s\le\nu$.

Let us observe that the covariances $r_{j,j'}(p)$ defined by (2.1) are real-valued, since $G_{j,j'}(A)=\overline{G_{j,j'}(-A)}$. To show that there exists a set of Gaussian random vectors with the desired covariance we have to check that the covariance matrix determined by the coordinates of these random vectors is positive semidefinite. This means that for all sets of complex numbers

$$\mathcal{A}_N=\{a_{j,p}=a_{j,p_1,\dots,p_\nu}\colon\ 1\le j\le d,\ -N\le p_s\le N,\ \text{for all } 1\le s\le\nu\}$$

$$I(\mathcal{A}_N)=\sum_{j=1}^d\sum_{j'=1}^d\sum_{\substack{p=(p_1,\dots,p_\nu)\\ -N\le p_s\le N,\ 1\le s\le\nu}}\ \sum_{\substack{q=(q_1,\dots,q_\nu)\\ -N\le q_s\le N,\ 1\le s\le\nu}} a_{j,p}\overline{a_{j',q}}r_{j,j'}(p-q)\ge0.$$

This inequality holds, since

$$I(\mathcal{A}_N) = \int \sum_{j=1}^d \sum_{j'=1}^d \frac{\left( \sum_{\substack{p=(p_1,\dots,p_\nu) \\ -N\le p_s\le N,\ 1\le s\le \nu}} a_{j,p} e^{i(p,x)} \right) g_{j,j'}(x)}{\left( \sum_{\substack{p=(p_1,\dots,p_\nu) \\ -N\le p_s\le N,\ 1\le s\le \nu}} a_{j',p} e^{i(p,x)} \right)} \mu(\,dx)$$

$$= \int \left( \sum_{j=1}^d \sum_{j'=1}^d b_j(x) g_{j,j'}(x) \overline{b_{j'}(x)} \right) \mu(\,dx) \ge 0,$$

where $b_j(x) = \sum_{\substack{p=(p_1,\dots,p_\nu) \\ -N\le p_s\le N,\ 1\le s\le \nu}} a_{j,p} e^{i(p,x)}$. This expression is really non-negative, since the matrix $g_{j,j'}(x)$ is positive semidefinite for $\mu$-almost all $x$, and this implies that the integrand at the right-hand side of this expression is non-negative for $\mu$-almost all $x$.

Since the distribution of the above sets of Gaussian random vectors are consistent for different parameters $N$ it follows from Kolmogorov's existence theorem for random processes with consistent finite distributions that there exists a Gaussian random field $X(p)$, $p\in\mathbb{Z}^\nu$, with $EZ_p=0$, $EX_j(p)X_{j'}(q) = r_{j,j'}(p-q)$, where $r_{j,j'}(p)$ is defined by formula (2.1) with our matrix valued spectral measure $G=(G_{j,j'})$, $1\le j,j'\le d$. In such a way we constructed a stationary Gaussian random field with the desired properties. Theorem 2.2 is proved.

In the next lemma I give a different characterization of positive semidefinite matrix valued, even measures on the torus $[-\pi,\pi)^\nu$.

**Lemma 2.3.** *Let us have a class of complex measures $G_{j,j'}$, $1\le j,j'\le d$, with finite total variation on the torus $[-\pi,\pi)^\nu$. Let us define with their help the following $\sigma$-additive matrix valued function on the measurable subsets of the torus $[-\pi,\pi)^\nu$. Define for all measurable sets $A\subset[-\pi,\pi)^\nu$ the $d\times d$ matrix $G(A)=(G_{j,j'}(A))$, $1\le j,j'\le d$. This matrix valued function is a positive semidefinite matrix valued, even measure on the torus $[-\pi,\pi)^\nu$ if and only if the matrix $(G_{j,j'}(A))$, $1\le j,j'\le d$, is positive semidefinite, and $G_{j,j'}(-A)=\overline{G_{j,j'}(A)}$ for all measurable sets $A\subset[-\pi,\pi)^\nu$ and $1\le j,j'\le d$.*

*Proof of Lemma 2.3.* It is clear that if $(G_{j,j'})$ is a positive semidefinite matrix valued, even measure, then the matrix $(G_{j,j'}(A))$ with

$$G_{j,j'}(A) = \int_A g_{j,j'}(x)\mu(\,dx), \quad 1\le j,j'\le d,$$

is a positive semidefinite matrix, and $G_{j,j'}(-A)=\overline{G_{j,j'}(A)}$ for all measurable sets $A\subset[-\pi,\pi)^\nu$ and $1\le j,j'\le d$.

On the other hand, it is not difficult to see that if the above properties hold, then $\sum\limits_{j=1}^{d}\sum\limits_{j'=1}^{d}\int v_j(x)\overline{v_{j'}(x)}G_{j,j'}(\,dx)\ge 0$ for all vectors $v(x)=(v_1(x),\dots,v_d(x))$, where $v_j(\cdot)$, $1\le j\le d$, is a continuous function on the torus $[-\pi,\pi)^\nu$. If $\mu$ is a finite measure on $[-\pi,\pi)^\nu$ such that all complex measures $G_{j,j'}$, $1\le j,j'\le d$, are absolutely continuous with respect to it with Radon–Nikodym derivative $g_{j,j'}(x)$, and we denote the matrix $(g_{j,j'}(x))$, $1\le j,j'\le d$, by $g(x)$, then the above inequality can be rewritten in the form $\int v(x)g(x)v^*(x)\mu(\,dx)\ge 0$. In the proof of Theorem 2.2 we have seen that this implies that $g(x)$ is a positive semidefinite matrix for $\mu$ almost all $x\in[-\pi,\pi)^\nu$. Lemma 2.3 is proved.

Let me also remark that the proof of Lemma 2.3 also implies that if the definition of positive semidefinite matrix valued, even measures holds with some finite measure $\mu$ on the torus with the property each the complex measure $G_{j,j'}$, $1\le j,j'\le d$, is absolutely continuous with respect to it, then the conditions of this definition also hold with any measure $\mu$ on the torus with the same properties.

Given a positive semidefinite matrix valued even measure $G=(G_{j,j'})$, $1\le j,j'\le d$, on the torus $[-\pi,\pi)^\nu$, there is a natural candidate for the choice of the measure $\mu$ on the torus $[-\pi,\pi)^\nu$ with respect to which all measures $G_{j,j'}$, $1\le j,j'\le d$, are absolute continuous. We shall prove an estimate in formula (3.2) which implies that the measure $\mu=\sum_{j=1}^d G_{j,j}$, i.e. the trace of the matrix valued measure $G$ has this property. Later this measure will be our choice for the measure $\mu$.

Let me remark that the proof of Lemma 2.3 yields another characterization of positive semidefinite matrix valued measures on the torus. I present it, although I shall not use it later.

A matrix valued measure $G=(G_{j,j'})$, $1\le j,j'\le d$, on the torus such that $G_{j,j}(A)=\overline{G_{j',j}(A)}$ for all $1\le j,j'\le d$ and measurable sets $A\subset[-\pi,\pi)^\nu$ is positive semidefinite if and only if

$$\sum_{j=1}^{d}\sum_{j'=1}^{d}\int_{[-\pi,\pi)^\nu}u_j(x)\overline{u_{j'}(x)}G_{j,j'}(\,dx)\ge 0$$

for all vectors $u(x)=(u_1(x),\dots,u_d(x))$ whose coordinates are continuous functions on the torus $[-\pi,\pi)^\nu$.

# 3 Random spectral measures in the multi-dimensional case

If $X(p)=(X_1(p),\dots,X_d(p))$, $p\in\mathbb{Z}^\nu$, is a $d$-dimensional stationary Gaussian random field with expectation zero, then its distribution is determined by its correlation functions $r_{j,j'}(p)=EX_j(0)X_{j'}(p)$, $1\le j,j'\le d$, $p\in\mathbb{Z}^\nu$. In Theorem 2.2 we described this correlation function as the Fourier transform of a matrix valued spectral measure $G=(G_{j,j'})$, $1\le j,j'\le d$. In the case of scalar valued stationary fields there is a continuation of this result. A so-called random spectral measure $Z_G$ can be constructed, and the elements of the stationary random field can be represented as an appropriate random integral with respect to it. This result can be interpreted so that the elements of a scalar valued stationary random field can be represented as the Fourier transforms of a random spectral measure. We want to find the multi-dimensional version of this result.

The results about scalar valued stationary fields also help in the study of vector valued stationary random fields. Indeed, since the $j$-th coordinates $X_j(p)$, of the random vectors $X(p)$, $p \in \mathbb{Z}^\nu$, define a scalar valued stationary random field we can apply for them the results known in the scalar valued case. This enables us to construct such a random spectral measure $Z_{G.j}$ for all $1 \le j \le d$ for which the identity $X_j(p) = \int_{[-\pi,\pi)^\nu} e^{i(p,x)} Z_{G,j}(\,dx)$ holds for all $p \in \mathbb{Z}^\nu$. The distribution of the random spectral measure $Z_{G,j}$ depends on the coordinate $G_{j,j}$ of the matrix valued spectral measure $G$, which is the spectral measure of the stationary random field $X_j(p)$, $p \in \mathbb{Z}^\nu$. For a fixed number $1 \le j \le d$ the properties of the random spectral measure $Z_{G,j}$ and the definition of the random integral with respect to it is worked out in the literature. I shall refer to my Lecture Note [11], where I described this theory.

Nevertheless, the results obtained in such a way are not sufficient for us. They describe the distribution of the random spectral measure $Z_{G,j}$ for each $1 \le j \le d$, but we need some additional results about their joint behaviour. To get them I recall the results in [11] which led to the construction of the random spectral measures $Z_{G,j}$, and then I extend them in order to get the results we need to describe their joint distribution.

I explain how we define simultaneously all random spectral measures $Z_{G,j}$, $1 \le j \le d$, by recalling the method of [11] with some necessary modifications in the notation to adapt this method to our case.

We construct the random spectral measure $Z_{G,j}$ for all $1 \le j \le d$ in the following way. First we introduce two Hilbert spaces $\mathcal{K}^c_{1,j}$ and $\mathcal{H}^c_{1,j}$, and define an appropriate norm-preserving invertible linear transformation $T_j$ from $\mathcal{K}^c_{1,j}$ to $\mathcal{H}^c_{1,j}$. (Here, and in the next discussion I apply the superscript $^c$ in the notation to emphasize that we are working in a complex, and not in a real Hilbert space.) The Hilbert space $\mathcal{K}^c_{1,j}$ consists of those complex valued functions $u(x)$ on the torus $[-\pi,\pi)^\nu$ for which $\int_{[-\pi,\pi)^\nu} |u(x)|^2 G_{j,j}(\,dx) < \infty$, and the norm is defined in this space by the formula $\|u\|^2_{0,j} = \int_{[-\pi,\pi)^\nu} |u(x)|^2 G_{j,j}(\,dx)$. The Hilbert space $\mathcal{H}^c_{1,j}$ is defined as the closure of the linear space consisting of the linear combinations $\sum c_{p_s} X_j(p_s)$ with some (complex valued) coefficients $c_{p_s}$ and parameters $p_s \in \mathbb{Z}^\nu$ in the Hilbert space $\mathcal{H}^c$. The Hilbert space $\mathcal{H}^c$ consists of the complex valued random variables with finite second moment, measurable with respect to the $\sigma$-algebra generated by the random variables $X_j(p)$, $1 \le j \le d$, $p \in \mathbb{Z}^\nu$, and the norm $\|\cdot\|_{1,j}$ in it is determined by the scalar product defined by the formula $\langle \xi, \eta\rangle = E\xi\bar\eta$, $\xi,\eta \in \mathcal{H}^c$. First we define the transformation $T_j$ only for finite trigonometrical sums in $\mathcal{K}^c_{1,j}$. We define it by the formula $T_j(\sum c_{p_s} e^{i(p_s,x)}) = \sum c_{p_s} X_j(p_s)$. We showed in [11] that we have defined in such a way a norm-preserving linear transformation from an everywhere dense subspace of $\mathcal{K}^c_{1,j}$ to an everywhere dense subspace of $\mathcal{H}^c_{1,j}$. This can be extended to a norm-preserving invertible linear transformation $T_j$ from $\mathcal{K}^c_{1,j}$ to $\mathcal{H}^c_{1,j}$ in a unique way. We define the random spectral measure $Z_{G,j}(A)$ for a measurable set $A \subset [-\pi,\pi)^\nu$ by the formula $Z_{G,j}(A) = T_j(\mathbb{I}_A(\cdot))$, where $\mathbb{I}_A(\cdot)$ denotes the indicator function of the set $A$.

It follows from the results of [11] that for any $1 \le j \le d$ the measure $G_{j,j}$ determines the distribution of the random spectral measure $Z_{G,j}$, (i.e. the joint distribution of the random variables $Z_{G,j}(A_1), \dots Z_{G,j}(A_N)$ for all $N \ge 1$ and measurable sets $A_k \subset [-\pi,\pi)^\nu$, $1 \le k \le N$). Next we shall study the joint distribution of the random fields $Z_{G,j}$ for all $1 \le j \le d$, i.e. the joint distribution of the random variables $Z_{G,j}(A_1), \dots Z_{G,j}(A_N)$ for all $N \ge 1$, measurable sets $A_k \subset [-\pi,\pi)^\nu$, $1 \le k \le N$ and $1 \le j \le d$. In particular, we shall show that the joint distribution of the random fields $Z_{G,j}$, $1 \le j \le d$, are determined by the matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$. The joint distribution of these random fields are determined by the matrix valued measure $G$, and not only by their diagonal elements $G_{j,j}$, $1 \le j \le d$.

To investigate the joint behaviour of the random spectral measures $Z_{G,j}$, $1 \le j \le d$, first we define two Hilbert spaces $\mathcal{K}^c_1$ and $\mathcal{H}^c_1$ together with a norm-preserving and invertible transformation between them. The elements of the Hilbert space $\mathcal{K}^c_1$ are the vectors $u = (u_1(x), \dots, u_d(x))$ with $u_j(x) \in \mathcal{K}^c_{1,j}$, $1 \le j \le d$. To define the (semi)-norm in $\mathcal{K}^c_1$ we introduce a positive semidefinite bilinear form $\langle \cdot, \cdot \rangle_0$ on it. To make some subsequent discussions simpler I make the following convention in the rest of the paper. Given a positive semidefinite matrix valued measure $(G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi,\pi)^\nu$, I fix a finite and even measure $\mu$ on $[-\pi,\pi)^\nu$ such that all complex measures $G_{j,j'}$ are absolutely continuous with respect to it, and I denote by $g_{j,j'}(x)$ their Radon–Nikodym derivative with respect to $\mu$. With the help of this notation we define $\langle \cdot, \cdot \rangle_0$ in the following way. If $u(x) = (u_1(x), \dots, u_d(x)) \in \mathcal{K}^c_1$ and $v(x) = (v_1(x), \dots, v_d(x)) \in \mathcal{K}^c_1$, then

$$\begin{aligned} \langle u(x), v(x) \rangle_0 &= \sum_{j=1}^d \sum_{j'=1}^d \int u_j(x) \overline{v_{j'}(x)} G_{j,j'}(\,dx) \qquad (3.1) \\ &= \sum_{j=1}^d \sum_{j'=1}^d \int g_{j,j'}(x) u_j(x) \overline{v_{j'}(x)} \mu(\,dx) \\ &= \int_{[-\pi,\pi)^\nu} u(x) g(x) v(x)^* \mu(\,dx) \end{aligned}$$

with the matrix $g(x) = (g_{j,j'}(x))$, $1 \le j, j' \le d$, where $v^*(x)$ denotes the column vector whose elements are the functions $\overline{v_k(x)}$, $1 \le k \le d$.

To show that the integral in the definition of $\langle u(x), v(x) \rangle_0$ is convergent let us observe that

$$|g_{j,j'}(x)|^2 \le g_{j,j}(x) g_{j',j'}(x) \text{ for almost all } x \text{ with respect to the measure } \mu \qquad (3.2)$$

for all $1 \le j, j' \le d$, because $g(x)$ is a positive semidefinite matrix for almost all $x$. This fact together with the Schwarz inequality imply that

$$\begin{aligned}
&\left| \int_{[-\pi,\pi)^\nu} u_j(x) g_{j,j'}(x) \overline{v_{j'}(x)} \mu(\,dx) \right| \\
&\quad \le \int_{[-\pi,\pi)^\nu} |u_j(x)| \sqrt{g_{j,j}(x) g_{j',j'}(x)} |v_{j'}(x)| \mu(\,dx) \\
&\quad \le \left( \int_{[-\pi,\pi)^\nu} |u_j(x)|^2 g_{j,j}(x) \mu(\,dx) \right)^{1/2} \left( \int_{[-\pi,\pi)^\nu} |v_{j'}(x)|^2 g_{j',j'}(x) \mu(\,dx) \right)^{1/2} \\
&\quad < \infty
\end{aligned}$$

for all pairs $1 \le j, j' \le d$ and $u_j \in \mathcal{K}^c_{1,j}$ and $v_{j'} \in \mathcal{K}^c_{1,k}$. This implies that the integral in (3.1) is finite. Moreover, the last inequality implies that

$$\begin{aligned}
\langle u(x), u(x) \rangle_0 &\le \left( \sum_{j=1}^d \left( \int_{[-\pi,\pi)^\nu} |u_j(x)|^2 G_{j,j}(\,dx) \right)^{1/2} \right)^2 \\
&\le d \sum_{j=1}^d \int_{[-\pi,\pi)^\nu} |u_j(x)|^2 G_{j,j}(\,dx) = d \sum_{j=1}^d \|u_j\|^2_{0,j} \qquad (3.3)
\end{aligned}$$

for all $u(x) = (u_1(x), \dots, u_d(x)) \in \mathcal{K}^c_1$.

Observe that $\langle u(x), u(x) \rangle_0 \ge 0$, because $g(x)$ is a positive semidefinite matrix, which implies that $u(x) g(x) u^*(x) \ge 0$ for almost all $x$ with respect to the measure $\mu$. In such a way we can define the norm $\|\cdot\|_0$ in $\mathcal{K}^c_1$ by the formula $\|u\|_0 = \langle u(x), u(x) \rangle_0$. We identify two elements $u$ and $v$ in $\mathcal{K}^c_1$ if $\|u - v\|_0 = 0$.

Next we define the Hilbert space $\mathcal{H}^c_1$ with the norm $\|\cdot\|_1$ on it. The elements of $\mathcal{H}^c_1$ are the vectors $\xi = (\xi_1, \dots, \xi_d)$, where $\xi_j \in \mathcal{H}^c_{1,j}$, $1 \le j \le d$, and we define the norm on it by the formula $\|\xi\|_1^2 = E \left| \sum_{j=1}^d \xi_j \right|^2$ if $\xi = (\xi_1, \dots, \xi_d) \in \mathcal{H}^c_1$. It is the norm induced by the scalar product $\langle \xi, \eta \rangle_1 = E \left( \sum_{j=1}^d \xi_j \right) \overline{\left( \sum_{j=1}^d \eta_j \right)}$ for $\xi = (\xi_1, \dots, \xi_d) \in \mathcal{H}^c_1$ and $\eta = (\eta_1, \dots, \eta_d) \in \mathcal{H}^c_1$. We identify two elements $\xi \in \mathcal{H}^c_1$ and $\eta \in \mathcal{H}^c_1$ if $\|\xi - \eta\|_1 = 0$.

Observe that

$$\begin{aligned}
\|\xi\|_1^2 &= E \left( \sum_{j=1}^d \xi_j \right) \overline{\left( \sum_{j'=1}^d \xi_{j'} \right)} \le \sum_{j=1}^d \sum_{j'=1}^d (E|\xi_j|^2)^{1/2} (E|\xi_{j'}|^2)^{1/2} \qquad (3.4) \\
&= \left( \sum_{j=1}^d (E(|\xi_j|^2)^{1/2} \right) \left( \sum_{j'=1}^d (E(|\xi|^2_{j'})^2)^{1/2} \right) \le d \sum_{j=1}^d E|\xi|^2_j = d \sum_{j=1}^k \|\xi_j\|^2_{1,j}
\end{aligned}$$

for a vector $\xi = (\xi_1, \dots, \xi_d) \in \mathcal{H}^c_1$

We define the operator $T$ mapping from $\mathcal{K}^c_1$ to $\mathcal{H}^c_1$ by the formula

$$Tu = T(u_1, \dots, u_d) = (T_1 u_1, \dots, T_d u_d)$$

for $u=(u_1,\dots,u_d)$, $u_j\in\mathcal{K}^c_{1,j}$, with the help of the already defined operators $T_j$, $1\le j\le d$. We show that $Tu=T(u_1,\dots,u_d)=(T_1u_1,\dots,T_du_d)$ for $u=(u_1,\dots,u_d)\in\mathcal{K}^c_1$ is a norm preserving and invertible transformation from $\mathcal{K}^c_1$ to $\mathcal{H}^c_1$. To prove this let us first observe that because of inequality (3.3) and Weierstrass' second approximation theorem the finite linear combinations

$$\left(\sum_{p\in A_N} c_{1,p}e^{i(p,x)},\dots,\sum_{p\in A_N} c_{d,p}e^{i(p,x)}\right),$$

where $A_N=\{p=(p_1,\dots,p_\nu)\colon\ -N\le p_s\le N,\ \text{for all } 1\le s\le\nu\}$, constitute an everywhere dense linear subspace in $\mathcal{K}^c_1$, and because of the inequality (3.4) the finite linear combinations

$$\begin{aligned}&\left(\sum_{p\in A_N} c_{1,p}X_1(p),\dots,\sum_{p\in A_N} c_{d,p}X_d(p)\right)\\ &\qquad =T\left(\sum_{p\in A_N} c_{1,p}e^{i(p,x)},\dots,\sum_{p\in A_N} c_{d,p}e^{i(p,x)}\right)\end{aligned}\tag{3.5}$$

constitute an everywhere dense linear subspace in $\mathcal{H}^c_1$ if $N=1,2,\dots$, and the coefficients $c_{j,p}$, $1\le j\le d$, $p\in A_N$, are arbitrary complex numbers. Hence the following calculation implies that $T$ is a norm preserving and invertible transformation from $\mathcal{K}^c_1$ to $\mathcal{H}^c_1$.

If

$$u(x)=\left(\sum_{p\in A_N} c_{1,p}e^{i(p,x)},\dots,\sum_{p\in A_N} c_{d,p}e^{i(p,x)}\right)$$

and

$$v(x)=\left(\sum_{p\in A_N} c'_{1,p}e^{i(p,x)},\dots,\sum_{p\in A_N}^{N} c'_{d,p}e^{i(p,x)}\right),$$

then

$$\begin{aligned}\langle u(x),v(x)\rangle_0&=\left\langle\left(\sum_{p\in A_N} c_{1,p}e^{i(p,x)},\dots,\sum_{p\in A_N} c_{d,p}e^{i(p,x)}\right),\right.\\ &\qquad\left.\left(\sum_{p\in A_N} c'_{1,p}e^{-i(p,x)},\dots,\sum_{p\in A_N} c'_{d,p}e^{-i(p,x)}\right)\right\rangle_0\\ &=\sum_{j=1}^d\sum_{j'=1}^d\sum_{s\in A_N}\sum_{t\in A_N} c_{j,s}\bar{c'}_{j',t}\int_{[-\pi,\pi)^\pi} g_{j,j'}(x)e^{i(s-t,x)}\mu(\,dx)\\ &=E\left(\sum_{j=1}^d\sum_{s\in A_N} c_{j,s}X_j(s)\right)\overline{\left(\sum_{j'=1}^d\sum_{t\in A_N} c'_{j',t}X_{j'}(t)\right)}=\langle Tu(x),Tv(x)\rangle_1.\end{aligned}$$

We shall define the random variables $Z_{G,j}(A)$ for all indices $1 \le j \le d$ and measurable sets $A \subset [-\pi,\pi)^\nu$, by the formula $Z_{G,j}(A) = T_j(\mathbb{I}_A(x))$ with the above defined operators $T_j$, $1 \le j \le d$, where $\mathbb{I}_A(\cdot)$ denotes the indicator function of the set $A \subset [-\pi,\pi)^\nu$. Next I formulate some properties of this class of random variables. These properties will appear in the definition of random spectral measures. All sets appearing in the next statements are measurable subsets of the torus $[-\pi,\pi)^\nu$.

(i) The random variables $Z_{G,j}(A)$ are complex valued, and their real and imaginary parts are jointly Gaussian, i.e. for any positive integer $N$ and sets $A_s$, $1 \le s \le N$, the random variables $\operatorname{Re} Z_{G,j}(A_s)$, $\operatorname{Im} Z_{G,j}(A_s)$, $1 \le s \le N$, $1 \le j \le d$, are jointly Gaussian.

(ii) $EZ_{G,j}(A) = 0$ for all $1 \le j \le d$ and $A$,

(iii) $EZ_{G,j}(A)\overline{Z_{G,j'}(B)} = G_{j,j'}(A \cap B)$ for all $1 \le j, j' \le d$ and sets $A, B$.

(iv) $\sum\limits_{s=1}^{n} Z_{G,j}(A_s) = Z_{G,j}\left(\bigcup\limits_{s=1}^{n} A_s\right)$ if $A_1, \dots, A_n$ are disjoint sets, $1 \le j \le d$.

(v) $Z_{G,j}(A) = \overline{Z_{G,j}(-A)}$ for all $1 \le j \le d$ and sets $A$.

Properties (i)–(v) were proved in the one-dimensional case e.g. in [11]. The only difference in checking its several dimensional version is that we have to apply the multi-dimensional operator $T$ from $\mathcal{K}_1^c$ to $\mathcal{H}_1^c$ to prove property (i), and to apply the same mapping $T$ in proving Property (iii). Here we exploit that $\langle u, v\rangle_0 = \langle Tu, Tv\rangle_1$. We apply this identity with the vector $u \in \mathcal{K}_1^c$ whose $j$-th coordinate is $\mathbb{I}_A(x)$, and the other coordinates are zero and the vector $v \in \mathcal{K}_1^c$ whose $k$-th coordinate is $\mathbb{I}_B(x)$ and the other coordinates are zero. Property (v) can be proved as the special case of the following more general relation.

(v′) $T_j(u) = \overline{T_j(u_-)}$ for all $1 \le j \le d$ and $u \in \mathcal{K}_j^c$, where $u_-(x) = \overline{u(-x)}$.

Property (v′) can be proved by first proving it in the special case when $u(x)$ is a trigonometrical polynomial, and then applying a limiting procedure.

Next we define the vector valued random spectral measures corresponding to a matrix valued spectral measure.

**Definition of vector valued random spectral measures on the torus.** *Let a matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, be given on the torus $[-\pi,\pi)^\nu$ together with a set of complex valued random variables indexed by pairs $(j, A)$, where $1 \le j \le d$, and $A$ is an element of the $\sigma$-algebra $\mathcal{A}$*

$$\mathcal{A} = \{A\colon\ A \subset [-\pi,\pi)^\nu \text{ is a Borel measurable set}\}$$

*of the Borel measurable sets of the torus whose joint distribution depends on the matrix valued spectral measure $G$. To recall this dependence we denote the random variable indexed by a pair $(j, A)$, $1 \le j \le d$, $A \in \mathcal{A}$, by $Z_{G,j}(A)$. We call* *the set of random variables $Z_{G,j}(A)$, $1 \le j \le d$, $A \in \mathcal{A}$, a $d$-dimensional vector valued random spectral measure corresponding to the matrix valued spectral measure $G$ on the torus $[-\pi,\pi)^\nu$ if this set of random variables satisfies properties (i)–(v) defined above. Given a fixed parameter $1 \le j \le d$ we call the set of random variables $Z_{G,j}(A)$, $A \in \mathcal{A}$, the $j$-th coordinate of this $d$-dimensional vector valued random spectral measure, and we denote it by $Z_{G,j}$. We denote the vector valued random spectral measure $Z_{G,j}(A)$, $1 \le j \le d$, $A \in \mathcal{A}$, by $Z_G = (Z_{G,1}, \dots, Z_{G,d})$.*

*More generally, if a matrix valued spectral measure $G$ is given on the torus $[-B,B)^\nu$ with some number $B > 0$ together with a set of complex valued random variables $Z_{G,j}(A)$, where $1 \le j \le d$, and $A$ is a Borel measurable set on the torus $[-B,B)^\nu$ which satisfies properties (i)–(v) defined above, then we call this set of random variables a $d$-dimensional vector valued random spectral measure corresponding to the spectral measure $G$. We call the set of random variables $Z_{G,j}(A)$, $A \in \mathcal{A}$, for a fixed $1 \le j \le d$ the $j$-th coordinate of this vector valued spectral measure, and denote it by $Z_{G,j}$. We denote the vector valued spectral measure by $Z_G = (Z_{G,1}, \dots, Z_{G,d})$.*

*Remark:* If $G = (G_{j,j'})$, $1 \le j,j' \le d$, is a matrix valued spectral measure, $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ is a vector valued spectral measure corresponding to it, then $G_{j,j}$ is a scalar valued spectral measure for any $1 \le j \le d$, and $Z_{G,j}$ is a scalar valued random spectral measure corresponding to it.

It follows from the above considerations that for any $d$-dimensional matrix valued spectral measure there exists a $d$-dimensional vector valued random spectral measure corresponding to it. We can define the random integral with respect to it by means of the method applied in the scalar valued case.

We shall define the random integrals of the functions $f \in \mathcal{K}^c_{1,j}$ with respect to the random spectral measure $Z_{G,j}$, $1 \le j \le d$. First we define these integrals for elementary functions. They are finite sums of the form $\sum_{s=1}^N c_s \mathbb{I}_{A_s}(x)$, where $A_1, \dots, A_N$ are disjoint sets in $[-\pi,\pi)^\nu$, and $c_s$, $1 \le s \le N$, are arbitrary complex numbers. Their integrals with respect to the random spectral measure $Z_{G,j}$, $1 \le j \le d$, are defined as

$$\int \left( \sum_{s=1}^N c_s \mathbb{I}_{A_s}(x) \right) Z_{G,j}(\,dx) = \sum_{s=1}^N c_s Z_{G,j}(A_s).$$

As it is remarked in [11], property (iv) implies that this definition is meaningful, the integral of an elementary function does not depend on its representation. Then a simple calculation with the help of (iii) shows that for two elementary functions $u$ and $v$

$$E\left( \int u(x) Z_{G,j}(\,dx) \overline{\int v(x) Z_{G,j}(\,dx)} \right) = \int u(x)\overline{v(x)} G_{j,j}(\,dx), \quad 1 \le j \le d. \tag{3.6}$$

This implies that the integral of the elementary functions with respect to the random spectral measure $Z_{G,j}$ define a norm preserving transformation from an everywhere dense subspace of the Hilbert space of $\mathcal{K}^c_{1,j}$ to an everywhere dense subspace of the Hilbert space of $\mathcal{H}^c_{1,j}$. This can be extended to a unitary transformation from $\mathcal{K}^c_{1,j}$ to $\mathcal{H}^c_{1,j}$ in a unique way, and this extension defines the integral of a function $u \in \mathcal{K}^c_{1,j}$. It is clear that relation (3.6) remains valid for general functions $u, v \in \mathcal{K}^c_{1,j}$. Moreover, it is not difficult to see with the help of (iii) that it can be generalized to the formula

$$E\left(\int u(x)Z_{G,j}(\,dx)\overline{\int v(x)Z_{G,j'}(\,dx)}\right) = \int u(x)\overline{v(x)}G_{j,j'}(\,dx) \tag{3.7}$$

if $u \in \mathcal{K}^c_{1,j}$ and $v \in \mathcal{K}^c_{1,j'}$, $1 \le j, j' \le d$.

It is clear that

$$E\int u(x)Z_{G,j}(\,dx) = 0 \quad \text{for all } u \in \mathcal{K}_{1,j}, \quad 1 \le j \le d. \tag{3.8}$$

Another important property of the random integrals with respect to $Z_{G,j}$ is that for all $1 \le j \le d$

$$\int u(x)Z_{G,j}(\,dx) \quad \text{is real valued if } u(-x) = \overline{u(x)} \text{ for } \mu \text{ almost all } x \in [-\pi,\pi)^\nu. \tag{3.9}$$

This relation holds, since $\int u(x)Z_{G,j}(\,dx) = \overline{\int u(x)Z_{G,j}(\,dx)}$ if $u(-x) = \overline{u(x)}$. We get this identity by means of the change of variables $x \to -x$ with the help of relation (v).

In the next Theorem I formulate the results we have about random spectral measures and random integrals with respect to them.

**Theorem 3.1** *Given a positive semidefinite matrix valued, even measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi,\pi)^\nu$ there exists a vector valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ corresponding to it. We have defined the random integrals $\int u(x)Z_{G,j}(\,dx)$ for all $1 \le j \le d$ and $u \in \mathcal{K}^c_{1,j}$. This is a linear operator which satisfies relations (3.7), (3.8), (3.9), and the formula*

$$X_j(p) = \int_{[-\pi,\pi)^\nu} e^{i(p,x)} Z_{G,j}(\,dx), \quad 1 \le j \le d, \;\; p \in \mathbb{Z}^\nu, \tag{3.10}$$

*defines a $d$-dimensional vector valued Gaussian stationary field whose matrix valued spectral measure is $G = (G_{j,j'})$, $1 \le j, j' \le d$. Moreover, if a $d$-dimensional vector valued Gaussian stationary random field is given with this matrix valued spectral measure, then the random integrals in formula (3.10) taken with respect to the random spectral measure that we have constructed with its help through an operator $T$ in this section equals this vector valued Gaussian stationary random field.*

*Proof of Theorem 3.1.* We have already proved the existence of the vector valued random spectral measure, and we constructed the random integral with respect to it. It satisfies formulas (3.7) and (3.8). The random variables

$X_j(p)$ defined in (3.10) are real valued by (3.9) and Gaussian with expectation zero. Hence we can show that they define a Gaussian stationary sequence by calculating their correlation function. We get by formula (3.7) that $EX_j(p)X_{j'}(q) = \int_{[-\pi,\pi)^\nu} e^{i(p-q,x)}G_{j,j'}(\,dx)$, and this had to be checked. If the random spectral measure is constructed in the way as we have done in this section, then a comparison of the random integral we have defined with its help and of the operator $T$ shows that $\int u(x)Z_{G,j}(\,dx) = T_j(u(x))$ for all $u \in \mathcal{K}^c_{1,j}$. In particular, $\int_{-[\pi,\pi)^\nu} e^{i(p,x)}Z_{G,j}(\,dx) = T_j(e^{i(p,x)}) = X_j(p)$. This identity implies the last relation of Theorem 3.1. Theorem 3.1 is proved.

Formula (3.9) and Theorem 3.1 make possible to define for all $1 \le j \le d$ a real Hilbert space $\mathcal{K}_{1,j}$ consisting of appropriate elements of $\mathcal{K}^c_{1,j}$ for which the operator $T_j$ is a norm preserving invertible transformation from $\mathcal{K}_{1,}j$ to the real Hilbert space $\mathcal{H}_{1,j}$ consisting of the real valued functions of the Hilbert space $\mathcal{H}^c_{1,j}$. More precisely, the following statement holds.

**Lemma 3.2.** *Let $(G_{j,j'})$, $1 \le j, j' \le d$, be a matrix valued spectral measure on the torus $[-\pi,\pi)^\nu$, and let $(Z_{G,1},\dots,Z_{G,d})$ be a vector valued spectral measure corresponding to it. Define the $d$-dimensional vector valued Gaussian stationary field $(X_1(p),\dots,X_p(d))$ by formula (3.10) with the help of this vector valued random spectral measure. Define for all $1 \le j \le d$ the set of complex valued functions $\mathcal{K}_{1,j}$ on the torus $[-\pi,\pi)^\nu$ as*

$$\mathcal{K}_{1,j} = \left\{ u\colon \int |u(x)|^2 G_{j,j}(\,dx) < \infty, \quad u(-x) = \overline{u(x)} \text{ for all } x \in [-\pi,\pi)^\nu \right\}.$$

*Then $\mathcal{K}_{1,j}$ is a real Hilbert space with the scalar product*

$$\langle u, v\rangle = \int u(x)\overline{v(x)}G_{j,j}(\,dx), \quad u, v \in \mathcal{K}_{1,j}.$$

*Let $\mathcal{H}_{1,j}$ be the real Hilbert space consisting of the closure of the finite linear combinations $\sum_{k=1}^N c_k X_j(p_k)$, $p_k \in \mathbb{Z}^\nu$, with real coefficients $c_k$ in the Hilbert space $\mathcal{H}$ of random variables with finite second moments in the probability space where the random spectral measures $Z_{G,j}$ exists. (We define the scalar product in $\mathcal{H}$ in the usual way.) Then the map $T_j(u) = \int u(x)Z_{G,j}(\,dx)$, $u \in \mathcal{K}_{1,j}$, is a norm preserving, invertible linear transformation from the real Hilbert space $\mathcal{K}_{1,j}$ to the real Hilbert space $\mathcal{H}_{1,j}$.*

*Proof of Lemma 3.2.* The space $\mathcal{K}_{1,j}$ is a real Hilbert space, since the change of variable $x \to -x$ in the integral $\langle u, v\rangle = \int u(x)\overline{v(x)}G_{j,j}(\,dx)$ implies that $\langle u, v\rangle = \overline{\langle u, v\rangle}$ for all $u, v \in \mathcal{K}_{1,j}$ because of the evenness of the measure $G_{j,j}$. Clearly $e^{i(p,x)} \in \mathcal{K}_{1,j}$ for all $p \in \mathbb{Z}^\nu$. The class of functions $\mathcal{K}_{1,j}$ agrees with the class of functions which have the form $u(x) = \frac{v(x)+\overline{v(-x)}}{2}$ with some $v \in \mathcal{K}^c_{1,j}$. As a consequence the set of finite trigonometrical polynomials $\sum c_k e^{i(p_k,x)}$, $p_k \in \mathbb{Z}^\nu$, with real valued coefficients $c_k$ is an everywhere dense subspace of $\mathcal{K}_{1,j}$. Since $T_j(\sum c_k e^{i(p_k,x)}) = \sum c_k X_j(p_k)$, the transformation $T_j$ maps an everywhere dense subspace of $\mathcal{K}_{1,j}$ to an everywhere dense subspace of $\mathcal{H}_{1,j}$. Because of formulas (3.7) and (3.9) $T_j$ is a norm preserving transformation in $\mathcal{K}_{1,j}$. Hence $T_j$ is an invertible, norm preserving transformation from $\mathcal{K}_{1,j}$ to $\mathcal{H}_{1,j}$. Lemma 3.2 is proved.

I would remark that the transformation $T_j$ on $\mathcal{K}_{1,j}$ defined in Lemma 3.2 is the restriction of the previously defined transformation $T_j$ on $\mathcal{K}^c_{1,j}$ to its subset $\mathcal{K}_{1,j}$. I make also the following remark.

**Lemma 3.3.** *The positive semidefinite matrix valued, even measure $G(A) = (G_{j,j'}(A))$, $1 \le j, j' \le d$, $A \in [-\pi, \pi)^\nu$, determines the distribution of a vector valued spectral random measure $Z_{G,j}$, $1 \le j \le d$, corresponding to it.*

To prove this lemma we have to show that for any collection of measurable sets $A_1,\dots, A_N$, the matrix valued measure $G(A)$ determines the joint distribution of the random vector consisting of the elements $\operatorname{Re} Z_{G,j}(A_s)$, $\operatorname{Im} Z_{G,j}(A_s)$, $1 \le s \le N$, $1 \le j \le d$. Since this is a Gaussian random vector with expectation zero, it is enough to check that the covariance of these random variables can be expressed by means of the matrix valued measure $G(A)$. Since $\operatorname{Re} Z_{G,j}(A) = \frac{Z_{G,j}(A)+\overline{Z_{G,j}(A)}}{2}$ and $\operatorname{Im} Z_{G,j}(A) = \frac{Z_{G,j}(A)-\overline{Z_{G,j}(A)}}{2i}$ we can calculate these covariances with the help of properties (iii) and (v) of vector valued random spectral measures.

Finally I prove an additional property of the vector valued random spectral measures which will be useful in the study of multiple Wiener–Itô integrals.

(vi) The random variables of the form $Z_{G,j}(A \cup (-A))$ are real valued. Let a set $A \cup (-A)$ be disjoint from some sets $B_1 \cup (-B_1)$,..., $B_n \cup (-B_n)$. Then for any indices $1 \le j, j' \le d$ the (complex valued) random vector $(Z_{G,j}(A), Z_{G,j'}(A))$, is independent of the random vector consisting of the elements $Z_{G,k}(B_s)$, $1 \le s \le n$, $1 \le k \le d$.

*Proof of property (vi).* It follows from property (v) that $Z_{G,j}(A \cup (-A)) = \overline{Z_{G,j}(A \cup (-A))}$, hence $Z_{G,j}(A \cup (-A))$ is real valued. To prove the second statement of (vi) it is enough to check that under its conditions the (real valued) random variables $\operatorname{Re} Z_{G,j}(A)$ and $\operatorname{Im} Z_{G,j}(A)$ are uncorrelated to all random variables $\operatorname{Re} Z_{G,k}(B_s)$, $\operatorname{Im} Z_{G,k}(B_s)$, $1 \le s \le n$, $1 \le k \le d$. This relation holds, since by the conditions of (vi) $(\pm A) \cap (\pm B_s) = \emptyset$, hence relation (iii) implies that $EZ_{G,j}(\pm A)\overline{Z_{G,j'}(\pm B_s)} = 0$ for all sets $B_s$, $1 \le s \le n$, and indices $1 \le j, j' \le d$. On the other hand, all covariances can be expressed as a linear combination of such expressions, since by relation (v) $\operatorname{Re} Z_{G,j}(\pm A) = \frac{Z_{G,j}(\pm A)+\overline{Z_{G,j}(\pm A)}}{2} = \frac{Z_{G,j}(\pm A)+Z_{G,j}(\mp A)}{2}$, and a similar relation holds also for $\operatorname{Im} Z_{G,j}(\pm A)$, $\operatorname{Re} Z_{G,j'}(\pm B_s)$ and $\operatorname{Im} Z_{G,j'}(\pm B_s)$, $1 \le s \le n$, $1 \le j' \le d$.

# 4 Spectral representation of vector valued stationary generalized random fields

In Sections 2 and 3 we discussed the properties of vector valued Gaussian stationary random fields with discrete parameters, which means a class of Gaussian random vectors $X(p)$, $p \in \mathbb{Z}^\nu$, with some nice properties. Similarly, we could have defined and investigated vector valued Gaussian stationary random fields with continuous parameters, where we consider a set of random vectors $X(t)$ indexed by $t \in \mathbb{R}^\nu$ which have some nice properties. But we do not discuss this topic here. Here we define and investigate instead so-called vector valued Gaussian stationary generalized random fields $X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi))$, parametrized with a nice linear space of functions $\varphi$.

Actually I am interested here in the vector valued Gaussian stationary generalized random fields not for their own sake. We shall construct a class of vector valued, Gaussian stationary generalized random fields. We shall show that their distribution can be described by means of a matrix valued spectral measure. We can also construct a vector valued random spectral measure in such a way that the elements of our vector valued generalized random field can be expressed in a form that can be considered as the Fourier transform of this random spectral measure. These matrix valued spectral measures and vector valued random spectral measures slightly differ from those defined in Sections 2 and 3, but since they are very similar to the corresponding objects defined for stationary random fields with discrete parameters it is natural to give them the same name.

The results that we shall prove are very similar to the results we got about vector valued random fields with discrete parameters. The main difference is that we can construct a larger class of matrix valued spectral measures and vector valued random spectral measures by means of generalized random fields. We shall need them, because in our later investigations we shall deal with such limit theorems where we can express the limit by means of these new, more general objects. On the other hand, these new vector valued random spectral measures behave similarly to the previous ones. In particular, the later results of this paper about multiple Wiener–Itô integrals also hold for this more general class of vector valued random spectral measures. Let me remark that we met a similar picture in the study of scalar valued Gaussian random fields in [11], so that here we actually generalize the results in that work to the multi-dimensional case.

In the definition of vector valued generalized random fields we shall choose the functions of the Schwartz space for the class of parameter set. So to define the vector valued generalized random fields first I recall the definition of the Schwartz space, (see [7]).

We define the Schwartz space $\mathcal{S}$ of real valued functions on $\mathbb{R}^\nu$ together with its version $\mathcal{S}^c$ consisting of complex valued functions on $\mathbb{R}^\nu$. The space $\mathcal{S}^c = (\mathcal{S}^\nu)^c$ consists of those complex valued functions of $\nu$ variables which decrease at infinity, together with their derivatives, faster than any polynomial degree. More explicitly, $\varphi\in\mathcal{S}^c$ for a complex valued function $\varphi$ defined on $\mathbb{R}^\nu$ if

$$\left|x_1^{k_1}\cdots x_\nu^{k_\nu}\frac{\partial^{q_1+\cdots+q_\nu}}{\partial x_1^{q_1}\dots\partial x_\nu^{q_\nu}}\varphi(x_1,\dots,x_\nu)\right|\le C(k_1,\dots,k_\nu,q_1,\dots,q_\nu)$$

for all points $x=(x_1,\dots,x_\nu)\in\mathbb{R}^\nu$ and vectors $(k_1,\dots,k_\nu)$, $(q_1,\dots,q_\nu)$ with non-negative integer coordinates with some constant $C(k_1,\dots,k_\nu,q_1,\dots,q_\nu)$ which may depend on the function $\varphi$. The elements of the space $\mathcal{S}$ are defined similarly, with the only difference that they are real valued functions.

To complete the definition of the spaces $\mathcal{S}$ and $\mathcal{S}^c$ we still have to define the topology in them. We introduce the following topology in these spaces.

Let a basis of neighbourhoods of the origin consist of the sets

$$U(k,p,\varepsilon)=\left\{\varphi\colon\ \varphi\in\mathcal{S},\ \max_{\substack{q=(q_1,\dots,q_\nu)\\ 0\le q_s\le p,\text{ for all }1\le s\le\nu}}\ \sup_x(1+|x|^2)^k|D^q\varphi(x)|<\varepsilon\right\}$$

with $k=0,1,2,\dots$, $p=1.2,\dots$ and $\varepsilon>0$, where $|x|^2=x_1^2+\cdots+x_\nu^2$, and $D^q=\frac{\partial^{q_1+\cdots+q_\nu}}{\partial x_1^{q_1}\dots\partial x_\nu^{q_\nu}}$ for $q=(q_1,\dots,q_\nu)$. A basis of neighbourhoods of an arbitrary function $\varphi\in\mathcal{S}^c$ (or $\varphi\in\mathcal{S}$) consists of sets of the form $\varphi+U(k,q,\varepsilon)$, where the class of sets $U(k,q,\varepsilon)$ is a basis of neighbourhood of the origin. Actually we shall use only the following property of this topology. A sequence of functions $\varphi_n\in\mathcal{S}^c$ (or $\varphi_n\in\mathcal{S}$) converges to a function $\varphi$ in this topology if and only if

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^\nu}(1+|x|^2)^k|D^q\varphi_n(x)-D^q\varphi(x)|=0.$$

for all $k=1,2,\dots$ and $q=(q_1,\dots,q_\nu)$. The limit function $\varphi$ is also in the space $\mathcal{S}^c$ (or in the space $\mathcal{S}$).

I shall define the notion of vector valued generalized random fields together with some related notions with the help of the notion of Schwartz spaces. A $d$-dimensional generalized random field is a random field whose elements are $d$-dimensional random vectors

$$(X_1(\varphi),\dots,X_d(\varphi))=(X_1(\varphi,\omega),\dots,X_d(\varphi,\omega))$$

defined for all functions $\varphi\in\mathcal{S}$, where $\mathcal{S}=\mathcal{S}^\nu$ is the Schwartz space. Before defining vector valued generalized random fields I write down briefly the idea of their definition. This is explained in [11] and [12] in more detail.

Given a vector valued Gaussian stationary random field

$$X(t)=(X_1(t),\dots,X_d(t)),\quad t\in\mathbb{R}^\nu,$$

we can define with its help the random field $X(\varphi)=(X_1(\varphi),\dots,X_d(\varphi))$, $\varphi\in\mathcal{S}^\nu$, $X_j(\varphi)=\int\varphi(t)X_j(t)\,dt$, $1\le j\le d$, indexed by the elements of the Schwartz space, and this determines the original random field. We define generalized random fields with elements indexed by $\varphi\in\mathcal{S}$ as such random fields which behave similarly to the random fields defined by means of such integrals.

**Definition of vector valued generalized random fields.** *We say that the set of random vectors $(X_1(\varphi),\dots,X_d(\varphi))$, $\varphi\in\mathcal{S}$, is a $d$-dimensional vector valued generalized random field over the Schwartz space $\mathcal{S}=\mathcal{S}^\nu$ of rapidly decreasing smooth functions if:*

(a) $X_j(a_1\varphi+a_2\psi)=a_1X_j(\varphi)+a_2X_j(\psi)$ *with probability 1 for the $j$-th coordinate of the random vectors $(X_1(\varphi),\dots,X_d(\varphi))$ and $(X_1(\psi),\dots,X_d(\psi))$. This relation holds for each coordinate $1\le j\le d$, all real numbers $a_1$ and $a_2$, and pair of functions $\varphi$, $\psi$ from the Schwartz space $\mathcal{S}$. (The exceptional set of probability 0 where this identity does not hold may depend on $a_1$, $a_2$, $\varphi$ and $\psi$.)*

(b) $X_j(\varphi_n)\Rightarrow X_j(\varphi)$ *stochastically for any $1\le j\le d$ if $\varphi_n\to\varphi$ in the topology of $\mathcal{S}$.*

We also introduce the following definition. In its formulation we use the notation $\stackrel{\Delta}{=}$ for equality in distribution.

**Definition of stationarity and Gaussian property for a vector valued generalized random field.** *The $d$-dimensional vector valued generalized random field $X=\{(X_1(\varphi)\dots,X_d(\varphi)),\ \varphi\in\mathcal{S}\}$ is stationary if*

$$(X_1(\varphi)\dots,X_d(\varphi))\stackrel{\Delta}{=}(X_1(T_t\varphi)\dots,X_d(T_t\varphi))$$

*for all $\varphi\in\mathcal{S}$ and $t\in\mathbb{R}^\nu$, where $T_t\varphi(x)=\varphi(x-t)$. It is Gaussian if $(X_1(\varphi),\dots,X_d(\varphi))$ is a Gaussian random vector for all $\varphi\in\mathcal{S}$. We call a vector valued generalized random field a vector valued generalized random field with zero expectation if $EX_j(\varphi)=0$ for all $\varphi\in\mathcal{S}$ and coordinates $1\le j\le d$.*

In the definition of stationarity and Gaussian property we imposed a condition for a single random vector. But because of the linearity property of generalized random fields formulated in property (a) of their definition and the fact that if we have $N$ random vectors $\xi_1,\dots,\xi_N$ and $\eta_1,\dots,\eta_N$ such that the linear combinations $\sum\limits_{k=1}^N a_k\xi_k$ and $\sum\limits_{k=1}^N a_k\eta_k$ have the same distribution for any coefficients $a_k$, $1\le k\le N$, then the joint distribution of the random vectors $\xi_1,\dots,\xi_N$ and $\eta_1,\dots,\eta_N$ agree imply that an analogous statement holds about the properties of the joint distribution of several random vectors in a vector valued stationary random field. Indeed, if we take $N$ random vectors $(X_1(\varphi_k),\dots,X_d(\varphi_k))$, $1\le k\le N$, then their joint distribution agrees with the joint distribution of their shifts $(X_1(T_t\varphi_k),\dots,X_d(T_t\varphi_k))$, $1\le k\le N$, for any $t\in\mathbb{R}^\nu$. This follows from the fact that

$$\sum_{k=1}^N a_k(X_1(\varphi_k),\dots,X_d(\varphi_k))\stackrel{\Delta}{=}\sum_{k=1}^N a_k(X_1(T_t\varphi_k),\dots,X_d(T_t\varphi_k))$$

for all $t\in\mathbb{R}^\nu$ and coefficients $a_k$, $1\le k\le N$, for a $d$-dimensional vector valued stationary generalized random field because of the linearity property of the generalized random fields and the properties of the operator $T_t$. A similar argument shows that the joint distribution of some vectors $(X_1(\varphi_k),\dots,X_d(\varphi_k))$, $1\le k\le N$, in a vector valued Gaussian generalized random field is Gaussian.

I shall construct a large class of $d$-dimensional vector valued Gaussian stationary generalized random fields with expectation zero. I shall construct them with the help of positive semidefinite matrix valued even measures on $\mathbb{R}^\nu$. In the next step I write down this definition. The main difference between the definition of this notion and its counterpart defined on the torus $[-\pi,\pi)^\nu$ is that now we consider such complex measures which may have non-finite total variation. We impose instead a less restrictive condition. We shall work with complex measures on $\mathbb{R}^\nu$ which have locally finite total variation. For the sake of completeness I give their definition.

**Definition of complex measures on $\mathbb{R}^\nu$ with locally finite total variation. The definition of their evenness property.** *A complex measure on $\mathbb{R}^\nu$ with locally finite total variation is such a complex valued function on the bounded, Borel measurable subsets of $\mathbb{R}^\nu$ whose restrictions to the measurable subsets of a cube $[-T,T]^\nu$ are complex measures with finite total variation for all $T>0$. We say that a complex measure $G$ on $\mathbb{R}^\nu$ with locally finite total variation is even, if $G(-A)=\overline{G(A)}$ for all bounded and measurable sets $A\subset\mathbb{R}^\nu$.*

Let me remark that not all complex measures with locally finite total variation can be extended to a complex measure on all measurable subsets of $\mathbb{R}^\nu$. On the other hand, this can be done if we are working with a (real, positive number valued) measure. Next I formulate the definition we need in our discussion.

**Definition of positive semidefinite matrix valued measures on $\mathbb{R}^\nu$ with moderately increasing distribution at infinity. The definition of their evenness property.** *A Hermitian matrix valued measure on $\mathbb{R}^\nu$ is a class of such Hermitian matrices $(G_{j,j'}(A))$, $1\le j,j'\le d$, defined for all bounded, measurable sets $A\subset\mathbb{R}^\nu$ for which all coordinates $G_{j,j'}(\cdot)$, $1\le j,j'\le d$, are complex measures on $\mathbb{R}^\nu$ with locally finite total variation. We call a Hermitian matrix valued measure $(G_{j,j'}(\cdot))$, $1\le j,j'\le d$, on $\mathbb{R}^\nu$ positive semidefinite if there exists a ($\sigma$-finite) positive measure $\mu$ on $\mathbb{R}^\nu$ such that for all numbers $T>0$ and indices $1\le j,j'\le d$ the restriction of the complex measures $G_{j,j'}$ to the cube $[-T,T]^\nu$ is absolutely continuous with respect to $\mu$, and the matrices $(g_{j,j'}(x))$, $1\le j,j'\le d$, defined with the help of the Radon–Nikodym derivatives $g_{j,j'}(x)=\frac{dG_{j,j'}}{d\mu}(x)$, $1\le j,j'\le d$, are Hermitian, positive semidefinite matrices for almost all $x\in\mathbb{R}^\nu$ with respect to the measure $\mu$. We call this Hermitian matrix valued measure $(G_{j,j'}(\cdot))$, $1\le j,j'\le d$, on $\mathbb{R}^\nu$ even if the complex measures $G_{j,j'}$ with locally finite variation are even for all $1\le j,j'\le d$.*

*We shall say that the distribution of a positive semidefinite matrix valued measure $(G_{j,j'}(\cdot))$, $1\le j,j'\le d$, on $\mathbb{R}^\nu$ is moderately increasing at infinity if*

$$\int(1+|x|)^{-r}G_{j,j}(\,dx)<\infty \qquad \textit{for all } 1\le j\le d \textit{ with some number } r>0. \quad (4.1)$$

*Remark.* We can give, similarly to Lemma 2.3, a different characterization of positive semidefinite matrix valued, even measures on $\mathbb{R}^\nu$. Let us have some complex measures $G_{j,j'}$, $1 \le j, j' \le d$, on the $\sigma$-algebra of the Borel measurable sets of $\mathbb{R}^\nu$ such that their restrictions to any cube $[-T,T]^\nu$, $T>0$, have finite total variation. Let us consider the matrix valued measure $(G_{j,j'}(A))$, $1 \le j, j' \le d$ on $\mathbb{R}^\nu$ for all bounded, measurable sets $A \subset \mathbb{R}^\nu$. This matrix valued measure is positive semidefinite and even if and only if it satisfies the following two conditions.

(i.) The $d\times d$ matrix $(G_{j,j'}(A))$, $1 \le j, j' \le d$, is Hermitian, positive semidefinite for all bounded, measurable sets $A \subset \mathbb{R}^\nu$.

(ii.) $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$, for all $1 \le j, j' \le d$ and bounded, measurable sets $A \subset \mathbb{R}^\nu$.

This statement has almost the same proof as Lemma 2.3. The only difference in the proof is that now we have to work with such vectors $v(x) = (v_1(x), \dots, v_d(x))$ whose coordinates $v_j(x)$ are continuous functions on $\mathbb{R}^\nu$ with bounded support, $1 \le j \le d$. Let me also remark that the following statement also follows from this proof. If a matrix valued measure $(G_{j,j'}(A))$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$ satisfies the conditions in the definition of positive semidefinite matrices with some $\sigma$-finite measure $\mu$ on $\mathbb{R}^\nu$ with respect to which all complex measures $G_{j,j}$ are absolutely continuous, then it satisfies these conditions with any $\sigma$-finite measure $\mu$ on $\mathbb{R}^\nu$ with the same property.

Before constructing a large class of vector valued Gaussian stationary generalized random fields I recall an important property of the Fourier transform of the functions in the Schwartz spaces $\mathcal{S}$ and $\mathcal{S}^c$, (see e.g. [7]). Actually this property of the Schwartz spaces made useful their choice in the definition of generalized fields.

The Fourier transform $f \to \tilde f$ is a bicontinuous map from $\mathcal{S}^c$ to $\mathcal{S}^c$. (This means that this transformation is invertible, and both the Fourier transform and its inverse are continuous maps from $\mathcal{S}^c$ to $\mathcal{S}^c$.) (The restriction of the Fourier transform to the space $\mathcal{S}$ of real valued functions is a bicontinuous map from $\mathcal{S}$ to the subspace of $\mathcal{S}^c$ consisting of those functions $f \in \mathcal{S}^c$ for which $f(-x) = \overline{f(x)}$ for all $x \in \mathbb{R}^\nu$.)

Next I formulate the following result.

**Theorem 4.1 about the construction of vector valued Gaussian stationary generalized random fields with zero expectation.** *Let $(G_{j,j'})$, $1 \le j, j' \le d$, be a positive semidefinite matrix valued, even measure on $\mathbb{R}^\nu$ whose distribution is moderately increasing at infinity.*

*Then there exists a vector valued, Gaussian stationary generalized random field $(X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}$, such that $EX_j(\varphi) = 0$ for all $\varphi \in \mathcal{S}$, and given two Shwartz functions $\varphi \in \mathcal{S}$ and $\psi \in \mathcal{S}$, the covariance function $r_{j,j'}(\varphi,\psi) = EX_j(\varphi)X_{j'}(\psi)$ is given by the formula*

$$r_{j,j'}(\varphi,\psi) = EX_j(\varphi)X_{j'}(\psi) = \int \tilde\varphi(x)\bar{\tilde\psi}(x) G_{j,j'}(\,dx) \quad \textit{for all } \varphi,\psi \in \mathcal{S}, \tag{4.2}$$

*where* $\tilde{}$ *denotes Fourier transform, and* $\bar{}$ *is complex conjugate.*

*Formula (4.2) and the identity* $EX_j(\varphi) = 0$ *for all* $\varphi \in \mathcal{S}$ *determine the distribution of the vector valued, Gaussian stationary random field* $(X_1(\varphi), \dots, X_d(\varphi))$.

*Contrariwise, for all* $1 \le j, j' \le d$ *the covariance function* $EX_j(\varphi)X_{j'}(\psi)$, $\varphi, \psi \in \mathcal{S}$, *determines the coordinate* $G_{j,j'}$ *of the positive semidefinite, even matrix* $(G_{j,j'})$. $1 \le j, j' \le d$, *with moderately increasing distribution at infinity for which identity (4.2) holds.*

Let me remark that the moderate decrease of the distribution of the positive semidefinite matrix $(G_{j,j'})$, $1 \le j, j' \le d$, together with inequality (3.2) and the fast decrease of the functions $\varphi \in \mathcal{S}$ at infinity guarantee that the integral in (4.2) is convergent.

Condition (4.1) which we wrote in the definition of moderately increasing positive semidefinite matrix valued measures appears in the theory of generalized functions in a natural way. Such a condition characterizes those measures which are generalized functions, i.e. continuous linear maps in the Schwartz space.

In [11] we have proved with the help of some important results of Laurent Schwartz about generalized functions that in the case of scalar valued models, i.e. if $d = 1$ the covariance function of every Gaussian stationary generalized random field with expectation zero agrees with the covariance function of a Gaussian stationary generalized random field constructed in the same way as we have done in Theorem 4.1. (In the case $d = 1$ the formulation of this result is simpler.) It seems very likely that a refinement of that argument would give the proof of an analogous statement in the general case. I did not investigate this question, because in the present paper we do not need such a result.

*Remark.* Similarly to the case of vector valued stationary fields with discrete parameter we shall introduce the following terminology. If $(G_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite, matrix valued even measure with moderately increasing distribution at infinity, and there is a stationary generalized random field $(X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}$, whose covariance function

$$r_{j,j'}(\varphi, \psi) = EX_j(\varphi)X_{j'}(\psi), \quad 1 \le j, j' \le d, \;\; \varphi, \psi \in \mathcal{S},$$

satisfies relation (4.2) with this matrix valued measure $G$, then we call $G$ the matrix valued spectral measure of this covariance function $r_{j,j'}(\varphi, \psi)$. In general, we shall call a positive semidefinite matrix valued even measure on $\mathbb{R}^\nu$ with moderately increasing distribution at infinity a matrix valued spectral measure on $\mathbb{R}^\nu$. We have the right for such a terminology, because by Theorem 4.1 for any such matrix valued measure there exists a Gaussian stationary generalized random field such that this matrix valued measure is the matrix valued spectral measure of its covariance function.

Let me remark that the diagonal elements $G_{j,j}$ of the matrix valued spectral measure of the correlation function $r_{j,j'}(\varphi, \psi)$ of a vector valued stationary random field may have non finite measure on $\mathbb{R}^\nu$, they have to satisfy only relation (4.1). As a consequence, we can find a much richer class of matrix valued spectral measures by working with generalized random fields than by working only with classical stationary random fields. As we shall see also vector valued random spectral measures corresponding to these matrix valued spectral measures can be constructed. Actually we discussed vector valued stationary generalized random fields in this paper in order to construct this larger class of matrix valued spectral and vector valued random spectral measures. We are interested in them, because they appear in the limit theorems we shall prove.

*Proof of Theorem 4.1.* Let us observe that the function $r_{j,j'}(\varphi,\psi)$ defined in (4.2) is real valued. This can be seen by applying the change of variables $x \to -x$ in this integral and by exploiting that $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$, and $\tilde\varphi(-x) = \bar{\tilde\varphi}(x)$, $\tilde\psi(-x) = \bar{\tilde\psi}(x)$, since this calculation yields that $r_{j,j'}(\varphi,\psi) = \overline{r_{j,j'}(\varphi,\psi)}$. Let us also remark that $r_{j,j'}(\varphi,\psi) = r_{j',j}(\psi,\varphi)$, since by formula (4.2) and the property $G_{j,j'}(A) = \overline{G_{j',j}(A)}$ of the matrix $(G_{j,j'}(A))$, $1 \le j,j' \le d$, for all measurable sets $A \subset \mathbb{R}^\nu$ we have $r_{j,j'}(\varphi,\psi) = \overline{r_{j',j}(\psi,\varphi)}$, and we know that both side of this identity is real valued.

First we show that for all positive integers $N$ and functions $\varphi_k \in \mathcal{S}$, $1 \le k \le N$, there are some Gaussian random vectors $(X_1(\varphi_k),\dots,X_d(\varphi_k))$, $1 \le k \le N$, with expectation zero and covariances $EX_j(\varphi_k)X_{j'}(\varphi_{k'}) = r_{j,j'}(\varphi_k,\varphi_{k'})$ for all $1 \le j,j' \le d$, $1 \le k,k' \le N$, on an appropriate probability space, where $r_{j,j'}(\varphi_k,\varphi_{k'})$ is defined at the right-hand side of formula (4.2) with our matrix valued measure $(G_{j,j'})$, $1 \le j,j' \le d$ and with the choice $\varphi = \varphi_k$, $\psi = \varphi_{k'}$.

We prove this statement if we show that the matrix with elements

$$d_{(j,k),(j',k')} = r_{j,j'}(\varphi_k,\varphi_{k'}), \quad 1 \le j,j' \le d, \ \ 1 \le k,k' \le N,$$

is positive semidefinite. To prove this result take any vector $(a_{j,k},\ 1 \le j \le d, 1 \le k \le N)$, and observe that

$$\begin{aligned}
&\sum_{j=1}^d \sum_{j'=1}^d \sum_{k=1}^N \sum_{k'=1}^N a_{j,k}\overline{a_{j',k'}} r_{j,j'}(\varphi_k,\varphi_{k'}) \\
&\qquad = \sum_{j=1}^d \sum_{j'=1}^d \sum_{k=1}^N \sum_{k'=1}^N \int (a_{j,k}\tilde\varphi_k(x))(\overline{a_{j',k'}\tilde\varphi_{k'}(x)}) g_{j,j'}(x)\mu(\,dx) \\
&\qquad = \sum_{j=1}^d \sum_{j'=1}^d \int \psi_j(x)\overline{\psi_{j'}(x)} g_{j,j'}(x)\mu(\,dx) = \int \psi(x)g(x)\overline{\psi(x)}\,\mu(\,dx) \ge 0,
\end{aligned}$$

where $\psi_j(x) = \sum\limits_{k=1}^N a_{j,k}\tilde\varphi_k(x)$, $1 \le j \le d$, $\psi(x) = (\psi_1(x),\dots,\psi_d(x))$, and $g(x)$ denotes the matrix $(g_{j,j'}(x))$, $1 \le j,j' \le d$. In this calculation we applied formula (4.2), the representation $G_{j,j'}(\,dx) = g_{j,j'}(x)\mu(\,dx)$ and finally the fact that $g(x)$ is a semidefinite matrix for $\mu$ almost all $x$.

Then it follows from Kolmogorov's existence theorem for random processes with consistent finite distributions that there is a Gaussian random field

$$(X_1(\varphi),\dots,X_d(\varphi)), \quad \varphi \in \mathcal{S},$$

with zero expectation such that $EX_j(\varphi)X_{j'}(\psi) = r_{j,j'}(\varphi,\psi)$ for all functions $\varphi \in \mathcal{S}$, $(\psi \in \mathcal{S}$ and $1 \le j, j' \le d$. Besides, the finite dimensional distributions of this random field are determined because of the Gaussian property. Next we show that this random field is a vector valued generalized random field.

Property (a) of the vector valued generalized random fields follows from the following calculation.

$$
\begin{aligned}
&E[a_1X_j(\varphi) + a_2X_j(\psi) - X_j(a_1\varphi + a_2\psi)]^2 \\
&\quad = \int \left(a_1\tilde\varphi(x) + a_2\tilde\psi(x) - (a_1\widetilde{\varphi + a_2}\psi)(x)\right) \\
&\qquad\qquad \left(\overline{a_1\tilde\varphi(x)} + \overline{a_2\tilde\psi(x)} - \overline{(a_1\widetilde{\varphi + a_2}\psi)(x)}\right) G_{j,j}(\,dx) = 0
\end{aligned}
$$

by formula (4.2) for all real numbers $a_1$, $a_2$, $1 \le j \le d$ and $\varphi, \psi \in \mathcal{S}$.

Property (b) of the vector valued generalized random fields also holds for this model. Actually it is proved in [11] that if $\varphi_n \to \varphi$ in the topology of the space $\mathcal{S}$, then $E[X_j(\varphi_n) - X_j(\varphi)]^2 = \int |\tilde\varphi_n(x) - \tilde\varphi(x)|^2 G_{j,j}(\,dx) \to 0$ as $n \to \infty$, hence property (b) also holds. (The proof is not difficult. It exploits that for a sequence of functions $\varphi_n \in \mathcal{S}^c$, $n = 0, 1, 2, \dots$, $\varphi_n \to \varphi_0$ as $n \to \infty$ in the topology of $\mathcal{S}^c$ if and only if $\tilde\varphi_n \to \tilde\varphi_0$ in the same topology. Besides, the measure $G_{j,j}$ satisfies inequality (4.1).)

It is also clear that the Gaussian random field constructed in such a way is stationary.

It remained to show that the covariance function $r_{j,j'}(\varphi,\psi) = EX_j(\varphi)X_{j'}(\psi)$ determines the complex measure $G_{j,j'}$. To show this we have to observe that inequality (3.2) holds also in this case, hence the Schwarz inequality implies that

$$
\int (1+|x|)^{-r} |g_{j,j'}(x)| \mu(\,dx) < \infty \qquad \text{for all } 1 \le j, j' \le d
$$

for a positive semidefinite matrix valued measure with moderately increasing distribution, i.e. this inequality holds not only for $j = j'$. Then it follows from the standard theory of Schwartz spaces that the class of Schwartz functions is sufficiently rich to guarantee that the function $r_{j,j'}(\varphi,\psi)$ determines the complex measure $G_{j,j'}$. Theorem 4.1 is proved.

Next we construct a vector valued random spectral measure corresponding to a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$. We argue similarly to Section 3, where the vector valued random spectral measures corresponding to matrix valued spectral measures on $[-\pi,\pi)^\nu$ were considered. In the construction we shall also refer to some results in [11].

Let us have a vector valued, Gaussian stationary generalized random field $X = (X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}$, $1 \le j \le d$, with a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$. First we define for all $1 \le j \le d$ some (complex) Hilbert spaces $\mathcal{K}^c_{1,j}$, $\mathcal{H}^c_{1,j}$ and a norm preserving, invertible linear transformation $T_j$ between them in the following way. $\mathcal{K}^c_{1,j}$ consists of those complex valued functions $u(x)$ on $\mathbb{R}^\nu$ for which $\int |u(x)|^2 G_{j,j}(\,dx) < \infty$ with the scalar product

$\langle u(x), v(x)\rangle = \int u(x)\overline{v(x)} G_{j,j}(\,dx)$. To define the Hilbert space $\mathcal{H}^c_{1,j}$ let us first introduce the Hilbert space $\mathcal{H} = \mathcal{H}^c$ of (complex valued) random variables with finite second moment on the probability space $(\Omega, \mathcal{A}, \mathcal{P})$ where our stationary generalized random field is defined. We define the Hilbert space $\mathcal{H}^c$ in the space consisting of these random variables with the usual scalar product $\langle \xi, \eta\rangle = E\xi\bar\eta$ in $\mathcal{H}^c$. The Hilbert space $\mathcal{H}^c_{1,j}$ is defined as the closure of the linear subspace of $\mathcal{H}^c$ consisting of the complex valued random variables $X_j(\varphi) + iX_j(\psi)$, $\varphi, \psi \in \mathcal{S}$.

First we define the operator $T_j$ for functions of the form $\widetilde{\varphi + i\psi}$, $\varphi, \psi \in \mathcal{S}$. We define it by the formula

$$T_j(\widetilde{\varphi + i\psi}) = X_j(\varphi) + iX_j(\psi), \quad \varphi,\, \psi \in \mathcal{S}. \tag{4.3}$$

Some calculation which was actually carried out in [11] shows that the set of functions $\widetilde{\varphi + i\psi}$, $\varphi, \psi \in \mathcal{S}$, is dense in $\mathcal{K}^c_{1,j}$, and the transformation $T_j$, defined in (4.3) can be extended to a norm preserving, invertible linear transformation from $\mathcal{K}^c_{1,j}$ to $\mathcal{H}^c_{1,j}$. (In the calculation leading to this statement we apply formula (4.2) with the choice $j' = j$.)

Then we can define the random spectral measure $Z_{G,j}(A)$, similarly to the case discussed in Section 3, by the formula $Z_{G,j}(A) = T_j \mathbb{I}_A(\cdot))$ for all bounded measurable sets $A \subset \mathbb{R}^\nu$. To determine the joint distribution of the spectral measures $Z_{G,j}$ we make the following version of the corresponding argument in Section 3.

We define the following two Hilbert spaces $\mathcal{K}^c_1$ and $\mathcal{H}^c_1$ together with a norm preserving linear transformation $T$ between them.

The elements of the Hilbert space $\mathcal{K}^c_1$ are the vectors $u = (u_1(x), \dots, u_d(x))$ with $u_j(x) \in \mathcal{K}^c_{1,j}$, $1 \le j \le d$. We define the scalar product on $\mathcal{K}^c_1$ with the help of the following positive semidefinite bilinear form $\langle \cdot, \cdot \rangle_0$. If $u(x) = (u_1(x), \dots, u_d(x)) \in \mathcal{K}^c_1$ and $v(x) = (v_1(x), \dots, v_d(x)) \in \mathcal{K}^c_1$, then

$$\begin{aligned}
\langle u(x), v(x)\rangle_0 &= \sum_{j=1}^d \sum_{j'=1}^d \int u_j(x)\overline{v_{j'}(x)} G_{j,j'}(\,dx) \\
&= \sum_{j=1}^d \sum_{j'=1}^d \int g_{j,j'}(x) u_j(x)\overline{v_{j'}(x)} \mu(\,dx) = \int u(x) g(x) v(x)^* \mu(\,dx)
\end{aligned}$$

with the matrix $g(x) = (g_{j,j'}(x))$, $1 \le j, j' \le d$, where $v^*(x)$ denotes the column vector whose elements are the functions $\overline{v_{j'}(x)}$, $1 \le j' \le d$. Actually here we simply copied the corresponding definition in Section 3 for the discrete time model, and we can also prove that $\mathcal{K}^c_1$ is a Hilbert space with the scalar $\langle \cdot, \cdot \rangle_0$ in the same way as it was done in Section 3.

The construction $\mathcal{H}^c_1$ and the proof of its properties is again a simple copying of argument made in Section 3. The elements of $\mathcal{H}^c_1$ are the vectors $\xi = (\xi_1, \dots, \xi_d)$, where $\xi_j \in \mathcal{H}^c_{1,j}$, $1 \le j \le d$, and we define the norm on it by means of the scalar product $\langle \xi, \eta\rangle_1 = E\left(\sum_{j=1}^d \xi_j\right) \overline{\left(\sum_{j=1}^d \eta_j\right)}$ for $\xi = (\xi_1, \dots, \xi_d) \in \mathcal{H}^c_1$

and $\eta = (\eta_1, \dots, \eta_d) \in \mathcal{H}_1^c$. We identify two elements $\xi \in \mathcal{H}_1^c$ and $\eta \in \mathcal{H}_1^c$ if $\|\xi - \eta\|_1 = 0$. Then the argument of Section 3 yields that $\mathcal{H}_1^c$ is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle_1$.

We define the operator $T$ from $\mathcal{K}_1^c$ to $\mathcal{H}_1^c$ again in the same way as in Section 3. We define it by the formula

$$Tu = T(u_1, \dots, u_d) = (T_1 u_1, \dots, T_d u_d)$$

for $u = (u_1, \dots, u_d)$, $u_j \in \mathcal{K}_{1,j}^c$, with the help of the already defined operators $T_j$, $1 \le j \le d$. We want to show that it is a norm preserving and invertible transformation from $\mathcal{K}_1^c$ to $\mathcal{H}_1^c$. Here again we apply a similar, but sightly different argument from that in Section 3. We exploit that if we take the class of vectors

$$W = \{w = (u_1 + iv_1, \dots, u_d + iv_d)\colon\ u_j \in \mathcal{S},\ v_j \in \mathcal{S} \text{ for all } 1 \le j \le d\}$$

then the class of vectors

$$\tilde W = \{(\widetilde{u_1 + iv_1}, \dots, \widetilde{u_d + iv_d})\colon\ (u_1 + iv_1, \dots, u_d + iv_d) \in W\}$$

is an everywhere dense subspace of $\mathcal{K}_1^c$. and the class of vectors

$$W(X) = \{((X_1(u_1 + iv_1), \dots, X_d(u_d + iv_d))\colon\ (u_1 + iv_1, \dots, u_d + iv_d) \in W\}$$

is an everywhere dense subspace of $\mathcal{H}_1^c$. (Here again the sign $\tilde{}$ denotes Fourier transform.)

Take two vectors $(u_{1,1}+iv_{1,1}, \dots, u_{d,1}+iv_{d,1}) \in W$ and $(u_{1,2}+iv_{1,2}, \dots, u_{d,2}+iv_{d,2}) \in W$. The desired property of the operator $T$ will follow from the following calculation.

$$\begin{aligned}
&\langle (\widetilde{u_{1,1} + iv_{1,1}}, \dots, \widetilde{u_{d,1} + v_{d,1}}), (\widetilde{u_{1,2} + iv_{1,2}}, \dots, \widetilde{u_{d,2} + v_{d,2}}) \rangle_0 \\
&\quad = \sum_{j=1}^d \sum_{j'=1}^d \int (\widetilde{u_{j,1}(x) + iv_{j,1}(x)}) \overline{(\widetilde{u_{j',2}(x) + iv_{j',2}(x)})} G_{j,j'}(\,dx) \\
&\quad = \sum_{j=1}^d \sum_{j'=1}^d E[X_j(u_{j,1}) + iX_j(v_{j,1})][X_{j'}(u_{j',2}) - iX_j(u_{j',2})] \\
&\quad = \langle (X_1(u_{1,1}) + iX_1(v_{1,1}), \dots, X_d(u_{d,1}) + iX_d(v_{d,1})), \\
&\qquad\qquad (X_1(u_{1,2}) + iX_1(v_{1,2}), \dots, X_d(u_{d,2}) + iX_d(v_{d,2})) \rangle_1,
\end{aligned}$$

i.e.

$$\begin{aligned}
&\langle (\widetilde{u_{1,1} + iv_{1,1}}, \dots, \widetilde{u_{d,1} + v_{d,1}}), (\widetilde{u_{1,2} + iv_{1,2}}, \dots, \widetilde{u_{d,2} + v_{d,2}}) \rangle_0 \\
&\quad = \langle (T_1(u_{1,1} + iv_{1,1}), \dots, T_d(u_{d,1} + iv_{d,1})), \\
&\qquad\qquad (T_1(u_{1,2} + iv_{1,2})), \dots, T_d(u_{d,2} + iv_{d,2})) \rangle_1.
\end{aligned}$$

This means that the operator $T$ maps the everywhere dense subspace $\tilde W$ of $\mathcal{K}_1^c$ to the everywhere dense subspace $W(X)$ of $\mathcal{H}_1^c$ in a norm preserving form. This implies that $T$ is a norm preserving, invertible transformation from $\mathcal{K}_1^c$ to $\mathcal{H}_1^c$.

Now we turn to the definition of the vector valued random spectral measures corresponding to a matrix valued spectral measure on $\mathbb{R}^\nu$.

Let a vector valued, Gaussian stationary generalized random field

$$X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi)), \quad \varphi \in \mathcal{S},$$

be given with a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$. (We take such generalized, stationary random fields which were constructed in Theorem 4.1.) Let us consider the operators $T_j$, $1 \le j \le d$, and $T$ constructed above with the help of these quantities. We define, similarly to the case of Gaussian stationary random fields with discrete parameters discussed in Section 3 the random variables $Z_{G,j}(A) = T_j(\mathbb{I}_A(x))$ for all $1 \le j \le d$ and bounded, measurable sets $A \subset \mathbb{R}^\nu$. (These functions $\mathbb{I}_A(\cdot)$ are clearly elements of the Hilbert space $\mathcal{K}^c_{1,j}$ for all $\le j \le d$). It can be proved with the help of the properties of the operator $T$ that these random functions satisfy properties (i)–(v) formulated in the definition of random spectral measures on the torus, considered in Section 3. The argument applied in Section 3 holds also in in this case. In particular, property (v) can be proved with the help of property (v$'$). Property (v$'$) can be proved with some work, and actually this was done in [11]. We prove (v$'$) by checking it first for functions $u \in \mathcal{S}^c$.

The above result makes natural the following definition of vector valued random spectral measures corresponding to a matrix valued spectral measure on $\mathbb{R}^\nu$. This is very similar to the definition of vector valued random spectral measures on the torus.

**Definition of vector valued random spectral measures on $\mathbb{R}^\nu$.** *Let $G = (G_{j,j'})$, $1 \le j, j' \le d$, be a matrix valued spectral measure on $\mathbb{R}^\nu$. We call a set of complex valued random variables $Z_{G,j}(A)$ depending on pairs $(j, A)$, where $1 \le j \le d$, $A \in \mathcal{A}$, and $\mathcal{A}$ is the algebra*

$$\mathcal{A} = \{A\colon\ A \textit{ is a bounded Borel measurable set in } \mathbb{R}^\nu\},$$

*a $d$-dimensional vector valued random spectral measure corresponding to the matrix valued spectral measure $G$ on $\mathbb{R}^\nu$ if this set of random variables $Z_{G,j}(A)$, $1 \le j \le d$, $A \in \mathcal{A}$, satisfies properties (i)–(v) introduced in Section 3 in the definition of vector valued random spectral measures on the torus. Given a fixed index $1 \le j \le d$, we call the set of random variables $Z_{G,j}(A)$, $A \in \mathcal{A}$, with this index $j$ the $j$-th coordinate of this matrix valued spectral measure, and we denote it by $Z_{G,j}$. We denote a $d$-dimensional vector valued random spectral measure corresponding to the matrix valued spectral measure $G$ by $Z_G = (Z_{G,1}, \dots, Z_{G,d})$.*

We have shown with the help of the arguments applied in Section 3 that for any $d$-dimensional matrix valued spectral measure on $\mathbb{R}^\nu$ there exists a $d$-dimensional vector valued random spectral measure corresponding to it.

We can define the random integral $\int f(x) Z_{G,j}(\,dx)$ of the functions $f \in \mathcal{K}^c_{1,j}$ with respect to the random spectral measure $Z_{G,j}$, $1 \le j \le d$, corresponding to the matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, of a Gaussian stationary generalized field in the same way as we defined these random integrals with respect to random spectral measures corresponding to a spectral measures on the torus $[-\pi,\pi)^\nu$ in Section 3. First we define these integrals for elementary functions which are defined in the same way as it was done in Section 3. Then following the calculation of that section we can define these integrals for a general function $f \in \mathcal{K}^c_{1,j}$, and it can be seen that formulas (3.7), (3.8) and (3.9) remain valid for them. In particular, the random integrals $\int \tilde\varphi(x) Z_{G,j}(\,dx)$ are (meaningful and) real valued random variables for all $\varphi \in \mathcal{S}$, and

$$E\left(\int \tilde\varphi(x) Z_{G,j}(\,dx) \int \bar{\tilde{\psi}}(x) Z_{G,j'}(\,dx)\right) = \int \tilde\varphi(x) \bar{\tilde{\psi}}(x) G_{j,j'}(\,dx)$$

for all $\varphi, \psi \in \mathcal{S}$ and $1 \le j, j' \le d$. This identity together with relation (3.7) and the fact that the above considered random integrals are linear operators imply that the set of random variables

$$X_j(\varphi) = \int \tilde\varphi(x) Z_{G,j}(\,dx), \quad \varphi \in \mathcal{S}, \ \ 1 \le j \le d, \tag{4.4}$$

constitute a vector valued Gaussian, stationary generalized random field with spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$.

This implies that the natural version of Theorem 3.1 remains valid if we consider a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$. Then there exists a random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ corresponding to it, and we have defined the random integrals $\int u(x) Z_{G,j}(\,dx)$, $1 \le j \le d$, with respect to it for all $u \in \mathcal{K}^c_{1,j}$. The class of random variables, $X_j(\varphi)$, $\varphi \in \mathcal{S}$, $1 \le j \le d$, defined in (4.4) constitute a vector valued, Gaussian stationary generalized random field with matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$. Moreover, if a $d$-dimensional vector valued Gaussian stationary random field is given with spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, then we can consider the random spectral measure $(Z_{G,1}, \dots, Z_{G,d})$ constructed in this section with the help of this random field. This random spectral measure has the property that the random field given by the random integrals defined in formula (4.4) with their help agrees with the original vector valued Gaussian stationary generalized random field.

We can formulate a natural version of Lemma 3.2 where we consider a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$ instead of a matrix valued spectral measure on the torus $[-\pi,\pi)^\nu$. In this version of Lemma 3.2 we define $\mathcal{K}_{1,j}$ as

$$\mathcal{K}_{1,j} = \left\{u\colon \int |u(x)|^2 G_{j,j}(\,dx) < \infty, \quad u(-x) = \overline{u(x)} \text{ for all } x \in \mathbb{R}^\nu \right\},$$

with the scalar product $\langle u, v\rangle = \int u(x)\overline{v(x)} G_{j,j}(\,dx)$, $u, v \in \mathcal{K}_{1,j}$, and $\mathcal{H}_{1,j}$ as the closure of the linear space consisting of the random variables $X_j(\varphi)$, $\varphi \in \mathcal{S}$, in the Hilbert space $\mathcal{H}$. This version of Lemma 3.2 states that $\mathcal{K}_{1,j}$ and $\mathcal{H}_{1,j}$

are real Hilbert spaces, and $T_j(u) = \int u(x) Z_{G,j}(\,dx)$ is a norm preserving and invertible transformation from $\mathcal{K}_{1,j}$ to $\mathcal{H}_{1,j}$.

The proof of this version of Lemma 3.2 is very similar to the proof of the original lemma. The main difference is that now we show that the class of functions $\tilde\varphi$ with $\varphi \in \mathcal{S}$ is a dense linear subspace of $\mathcal{K}_{1,j}$, and the transformation $T_j(\tilde\varphi) = \int \tilde\varphi(x) Z_{G,j}(\,dx) = X_j(\varphi)$, $\varphi \in \mathcal{S}$, is a norm preserving transformation from an everywhere dense subspace of $\mathcal{K}_{1,j}$ to an everywhere dense subspace of $\mathcal{H}_{1,j}$.

The natural version of Lemma 3.3 also holds. It states that a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$ determines the distribution of a vector valued random spectral measure $Z_{G,j}$, $1 \le j \le d$, corresponding to it. The proof of this version is the same as the proof of the original lemma. The only difference is that now we consider the random spectral measure $Z_{G,j}(A)$ for all measurable, bounded sets $A \subset \mathbb{R}^\nu$.

Finally I would remark that property (vi) of the random spectral measures also remains valid for this new class of random spectral measures, because its proof applies only properties (i)–(v) of random spectral measures.

# 5 Multiple Wiener–Itô integrals with respect to vector valued random spectral measures

Next we want to rewrite the random variables with finite second moments which are measurable with respect the $\sigma$-algebra generated by the elements of a vector valued Gaussian stationary random field in an appropriate form which enables us to rewrite also the random sums defined in (1.1) in a form that helps in the study of their limit behaviour. In the scalar valued case, i.e. when $d = 1$ we could do this with the help of the introduction of multiple Wiener–Itô integrals. We could rewrite with their help the random sums (1.1) in a form that provided great help in the study of the limit theorems we were interested in. Next we show that a similar method can be applied also in the case of vector valued Gaussian stationary fields. To do this first we have to define the multiple Wiener–Itô integrals also in the vector valued case. We start the definition of multiple Wiener–itô integrals in this case with the introduction of the following notation.

Let $X(p) = (X_1(p), \dots, X_d(p))$, $EX(p) = 0$, $p \in \mathbb{Z}^\nu$, be a vector valued stationary Gaussian random field with some matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$. Let $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ be a vector valued random spectral measure corresponding to it which is chosen in such a way that $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\,dx)$ for all $p \in \mathbb{Z}^\nu$ and $1 \le j \le d$. Let us consider the (real) Hilbert space $\mathcal{H}$ of square integrable random variables measurable with respect to the $\sigma$-algebra generated by the random vectors $X(p)$, $p \in \mathbb{Z}^\nu$. More generally, let us consider a (possibly generalized) matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, and a vector valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ corresponding to it, where the matrix valued spectral measures $G_{j,j'}$ and vector valued random spectral measures $Z_{G,j}$ are defined either on the torus $[-\pi, \pi)^\nu$ or on $\mathbb{R}^\nu$, and consider the (real) Hilbert space $\mathcal{H}$ of the square integrable (real valued) random variables, measurable with respect to the $\sigma$-algebra generated by the random variables of the vector valued random spectral measures $Z_G$ with the usual scalar product in this space. We would like to write the elements of the Hilbert space $\mathcal{H}$ in the form of a sum of multiple Wiener–Itô integrals with respect to the vector valued random spectral measure $Z_G$. I shall construct these Wiener–Itô integrals in this section, and I prove some of their important properties.

As a discussion in Section 7 will show we cannot write all elements of $\mathcal{H}$ in the form of a sum of Wiener–Itô integrals, but we can do this for the elements of an everywhere dense subspace of $\mathcal{H}$. In particular, if we consider finitely many random variables $X_j(p)$, $1 \le j \le d$, $p \in \mathbb{Z}^\nu$ of a discrete or $X_j(\varphi)$, $1 \le j \le d$, $\varphi \in \mathcal{S}^\nu$, of a generalized vector valued stationary Gaussian random field, then all polynomials of these random variables can be written as the sum of Wiener–Itô integrals. Such a result will be sufficient for our purposes. In the subsequent discussion I impose a technical condition about the properties of the matrix valued spectral measure $G = (G_{j,j'})$ I shall be working with. I assume that it is non-atomic. More precisely, I assume that we are working with such a dominating measure $\mu$ for the coordinates of the matrix valued spectral measures $G_{j,j'}$ for which $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^\nu$.

First I define for all $n = 1, 2, \dots$ and $1 \le j_s \le d$ for the indices $1 \le s \le n$ the $n$-fold multiple Wiener–Itô integral

$$I_n(f|j_1, \dots, j_n) = \int f(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(d\,x_n)$$

with respect to the coordinates of a vector valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$, corresponding to a matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$. We shall define these Wiener–Itô integrals functions with kernel functions $f \in \mathcal{K}_{n,j_1,\dots,j_n}$ in a (real) Hilbert space $\mathcal{K}_{n,j_1,\dots,j_n} = \mathcal{K}_{n,j_1,\dots,j_n}(G_{j_1,j_1}, \dots, G_{j_n,j_n})$ defined below.

We define $\mathcal{K}_{n,j_1,\dots,j_n} = \mathcal{K}_{n,j_1,\dots,j_n}(G_{j_1,j_1} \dots, G_{j_n,j_n})$ as the Hilbert space consisting of those complex valued functions $f(x_1, \dots, x_n)$ on $\mathbb{R}^{n\nu}$ which satisfy the following relations (a) and (b):

(a) $f(-x_1, \dots, -x_n) = \overline{f(x_1, \dots, x_n)}$ for all $(x_1, \dots, x_n) \in \mathbb{R}^{n\nu}$,

(b) $\|f\|^2 = \int |f(x_1, \dots, x_n)|^2 G_{j_1,j_1}(\,dx_1) \dots G_{j_n,j_n}(\,dx_n) < \infty$.

We define the scalar product in $\mathcal{K}_{n,j_1,\dots,j_n}$ in the following way. If $f$, $g \in \mathcal{K}_{n,j_1,\dots,j_n}$, then

$$\begin{aligned}\langle f, g \rangle &= \int f(x_1, \dots, x_n) \overline{g(x_1, \dots, x_n)} G_{j_1,j_1}(\,dx_1) \dots G_{j_n,j_n}(\,dx_n) \\ &= \int f(x_1, \dots, x_n) g(-x_1, \dots, -x_n) G_{j_1,j_1}(\,dx_1) \dots G_{j_n,j_n}(\,dx_n).\end{aligned}$$

Because of the symmetry $G_{j_s,j_s}(A) = G_{j_s,j_s}(-A)$ of the spectral measure $\langle f,g\rangle = \overline{\langle f,g\rangle}$, i.e. the scalar product $\langle f,g\rangle$ is a real number for all $f$, $g \in \mathcal{K}_{n,j_1,\dots,j_n}$. This means that $\mathcal{K}_{n,j_1,\dots,j_n}$ is a real Hilbert space, as I claimed. We also define the real Hilbert space $\mathcal{K}_0$ for $n = 0$ as the space of real constants with the norm $\|c\| = |c|$.

*Remark.* In the case $n = 1$ the above defined real Hilbert space $\mathcal{K}_{1,j}$ agrees with the real Hilbert space $\mathcal{K}_{1,j}$ introduced in Lemma 3.2.

Similarly to the scalar valued case, first we introduce so-called simple functions and define the multiple integrals for them. We prove some properties of this integral which enable us to extend its definition by means of an $L_2$ extension for all functions $f \in \mathcal{K}_{j_1,\dots,j_n}$. We define the class of simple functions together with the notion of regular systems.

**Definition of regular systems and the class of simple functions.** *Let*

$$\mathcal{D} = \{\Delta_k,\ k = \pm 1, \pm 2, \dots, \pm N\}$$

*be a finite collection of bounded, measurable sets in $\mathbb{R}^\nu$ indexed by the integers $\pm 1,\dots,\ \pm N$ with some positive integer $N$. We say that $\mathcal{D}$ is a regular system if $\Delta_k = -\Delta_{-k}$, and $\Delta_k \cap \Delta_l = \emptyset$ if $k \neq l$ for all $k, l = \pm 1, \pm 2, \dots, \pm N$. A function $f \in \mathcal{K}_{n,j_1,\dots,j_n}$ is adapted to this system $\mathcal{D}$ if $f(x_1,\dots,x_n)$ is constant on the sets $\Delta_{k_1} \times \Delta_{k_2} \times \dots \times \Delta_{k_n}$, $k_l = \pm 1,\dots,\pm N$, $l = 1, 2, \dots, n$, it vanishes outside these sets, and it also vanishes on those sets of the above form for which $k_l = \pm k_{l'}$ for some $l \neq l'$.*

*A function $f \in \mathcal{K}_{n,j_1,\dots,j_n}$ is in the class $\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ of simple functions if it is adapted to some regular system $\mathcal{D} = \{\Delta_k,\ k = \pm 1,\dots,\pm N\}$.*

**Definition of Wiener–Itô integrals of simple functions.** *Let a simple function $f \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ be adapted to some regular system*

$$\mathcal{D} = \{\Delta_k,\ k = \pm 1,\dots,\pm N\}.$$

*Its $n$-fold Wiener–Itô integral with respect to $Z_G = (Z_{G,1},\dots,Z_{G,d})$ with parameters $j_1,\dots,j_n$, $1 \le j_k \le d$ for all $1 \le k \le n$, is defined as*

$$\begin{aligned}\int f(x_1,\dots,x_n) & Z_{G,j_1}(\,dx_1)\dots Z_{G,j_n}(\,dx_n) \qquad (5.1)\\ &= I_n(f|j_1,\dots,j_n)\\ &= \sum_{\substack{k_l=\pm 1,\dots,\pm N\\ l=1,2,\dots,n}} f(u_{k_1},\dots,u_{k_n}) Z_{G,j_1}(\Delta_{k_1})\cdots Z_{G,j_n}(\Delta_{k_n}),\end{aligned}$$

*where $u_k \in \Delta_k$, $k = \pm 1,\dots,\pm N$.*

Although the regular system $\mathcal{D}$ to which $f$ is adapted is not uniquely determined (e.g. the elements of $\mathcal{D}$ can be divided to smaller sets), the integral defined in (5.1) is meaningful, i.e. its value does not depend on the choice of $\mathcal{D}$. This can be proved with the help of property (iv) of vector valued random spectral measures defined in Section 3 in the same way as it was done in the scalar valued case in [11]. (Let me also remark that here I defined the random integral $I_n(f|j_1,\dots,j_n)$ with a normalization different from the normalization of the corresponding expression $I_G(f)$ introduced in [11]. Here I omitted the norming term $\frac{1}{n!}$.)

Because of the definition of simple functions the sum in (5.1) does not change if we allow in it summation only for such sequences $k_1,\dots,k_n$ for which $k_l\neq\pm k_{l'}$ if $l\neq l'$. This fact will be exploited in the subsequent considerations.

Next I formulate some important properties about the Wiener–Itô integrals of simple functions. Later we shall see that these properties remain valid in the general case.

$$I_n(f|j_1,\dots,j_n) \text{ is a real valued random variable for all } f\in\hat{\mathcal{K}}_{n,j_1,\dots,j_n}. \tag{5.2}$$

Indeed, $I_n(f|j_1,\dots,j_n)=\overline{I_n(f|j_1,\dots,j_n))}$ by Property (a) of the functions in $\mathcal{K}_{n,j_1,\dots,j_n}$ and property (v) of the random spectral measures defined in Section 3, hence (5.2) holds. It is also clear that $\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ is a linear space, and the mapping $f\to I_n(f|j_1,\dots,j_n)$ is a linear transformation on it.

The relation

$$EI_n(f|j_1,\dots,j_n)=0 \quad \text{for } f\in\hat{\mathcal{K}}_{n,j_1,\dots,j_k} \quad \text{if } n\neq 0 \tag{5.3}$$

also holds. (In the non-zero terms of the sum in (5.1) we have the product of independent random variables with expectation zero by property (vi) of the random spectral measures described also in Section 3.) Next I express the covariance between random variables of the form $I_n(f|j_1,\dots,j_n)$. To do this first I introduce the following notation. Let $\Pi(n)$ denote the set of all permutations of the set $\{1,\dots,n\}$, and let $\pi=(\pi(1),\dots,\pi(n))$ denote one of its element.

Let us have a positive integer $n\ge 1$, and two sequences $j_1,\dots,j_n$ and $j'_1,\dots,j'_n$, $1\le j_s,j'_s\le d$ for all $1\le s\le d$. Let $f\in\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ and $h\in\hat{\mathcal{K}}_{n,j'_1,\dots,j'_n}$. I shall show that

$$\begin{aligned} &EI_n(f|j_1,\dots,j_n)I_n(h(|j'_1,\dots,j'_n) \\ &\qquad=\sum_{\pi\in\Pi(n)}\int f(x_1,\dots x_n)\overline{h(x_{\pi(1)},\dots,x_{\pi(n)})} \\ &\qquad\qquad G_{j_1,j'_{\pi^{-1}(1)}}(\,dx_1)\dots G_{j_n,j'_{\pi^{-1}(n)}}(\,dx_n). \end{aligned} \tag{5.4}$$

On the other hand, if $n\neq n'$, and $f\in\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$, $h\in\hat{\mathcal{K}}_{n',j'_1,\dots,j'_{n'}}$, then

$$EI_n(f|j_1,\dots,j_n)I_{n'}(h(|j'_1,\dots,j'_{n'})=0. \tag{5.5}$$

Next I show the following inequality with the help of formula (5.4).

$$\begin{aligned} E|I_n(f|j_1,\dots,j_n)|^2 &\le n!\int|f(x_1,\dots x_n)|^2G_{j_1,j_1}(\,dx_1)\dots G_{j_n,j_n}(\,dx_n) \\ &= n!\|f_{n,j_1,\dots,j_n}\|^2 \end{aligned} \tag{5.6}$$

for all $f\in\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$.

Indeed we get by applying (5.4) for $f=h\in\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ together with relation (3.2) that

$$E|I_n(f|j_1,\dots,j_n)|^2\le\sum_{\pi\in\Pi(n)}\int|f(x_1,\dots x_n)||f(x_{\pi(1)},\dots,x_{\pi(n)})| \tag{5.7}$$
$$\prod_{s=1}^n\left(g_{j_s,j_s}(x_s)g_{j_{\pi^{-1}(s)},j_{\pi^{-1}(s)}}(x_s)\right)^{1/2}\mu(\,dx_1)\dots\mu(\,dx_n).$$

On the other hand, we get with the help of the Schwarz inequality that

$$\int|f(x_1,\dots x_n)||f(x_{\pi(1)},\dots,x_{\pi(n)})|\prod_{s=1}^n\left(g_{j_s,j_s}(x_s)g_{j_{\pi^{-1}(s)},j_{\pi^{-1}(s)}}(x_s)\right)^{1/2}$$
$$\mu(\,dx_1)\dots\mu(\,dx_n) \tag{5.8}$$
$$\le\left(\int|f(x_1,\dots x_n)|^2\prod_{s=1}^n g_{j_s,j_s}(x_s)\mu(\,dx_1)\dots\mu(\,dx_n)\right)^{1/2}$$
$$\left(\int|f(x_{\pi(1)},\dots,x_{\pi(n)})|^2\prod_{s=1}^n g_{j_{\pi^{-1}(s)},j_{\pi^{-1}(s)}}(x_s)\mu(\,dx_1)\dots\mu(\,dx_n)\right)^{1/2}$$

for all $\pi\in\Pi(n)$. Let us also observe that the map $T$ from $\mathbb{R}^{n\nu}$ to $\mathbb{R}^{n\nu}$, defined as

$$T(x_1,\dots,x_n)=(x_{\pi(1)},\dots,x_{\pi(n)})$$

is a bijection, and it is a measure preserving transformation from

$$(\mathbb{R}^{n\nu},G_{j_1,j_1}\times\dots\times G_{j_n,j_n})=(\mathbb{R}^{n\nu},g_{j_1,j_1}(x_1)\cdots g_{j_n,j_n}(x_n)\mu(\,dx_1)\dots\mu(\,dx_n)\,)$$

to

$$\begin{aligned}&(\mathbb{R}^{n\nu},G_{j_{\pi^{-1}(1)},j_{\pi^{-1}(1)}}\times\dots\times G_{j_{\pi^{-1}(n)},j_{\pi^{-1}(n)}})\\&\qquad=(\mathbb{R}^{n\nu},g_{j_{\pi^{-1}(1)},j_{\pi^{-1}(1)}}(x_1)\cdots g_{j_{\pi^{-1}(n)},j_{\pi^{-1}(n)}}(x_n)\mu(\,dx_1)\dots\mu(\,dx_n)\,).\end{aligned}$$

To see this it is enough to check that if $A=A_1\times\dots\times A_n$, then

$$(G_{1,1}\times\dots\times G_{n,n})(A)=\prod_{l=1}^n G_{l,l}(A_l),$$

$TA=A_{\pi^{-1}(1)}\times\dots\times A_{\pi^{-1}(n)}$,

$$\begin{aligned}&(G_{j_{\pi^{-1}(1)},j_{\pi^{-1}(1)}}\times\dots\times G_{j_{\pi^{-1}(n)},j_{\pi^{-1}(n)}})(TA)\\&\qquad=\prod_{l=1}^n G_{j_{\pi^{-1}(l)},j_{\pi^{-1}(l)}}(A_{\pi^{-1}(l)})=(G_{1,1}\times\dots\times G_{n,n})(A).\end{aligned}$$

The last identity together with the bijective property of $T$ imply that it is measure preserving.

Because of the measure preserving property of the operator $T$ we can write that

$$\int |f(x_1,\dots x_n)|^2 \prod_{s=1}^n g_{j_s,j_s}(x_s)\mu(\,dx_1)\dots\mu(\,dx_n) \tag{5.9}$$
$$= \int |f(x_{\pi(1)},\dots,x_{\pi(n)})|^2 \prod_{s=1}^n g_{j_{\pi^{-1}(s)},j_{\pi^{-1}(s)}}(x_s)\mu(\,dx_1)\dots\mu(\,dx_n).$$

Relation (5.6) follows from relations (5.7), (5.8) and (5.9).

To prove formulas (5.4) and (5.5) first we prove the following relations. Let a regular system $\mathcal{D}=\{\Delta_k,\ k=\pm1,\pm2,\dots,\pm N\}$ be given, choose an integer $n\ge1$, some numbers $j_1,\dots,j_n$ and $j'_1\dots,j'_n$ such that $1\le j_s,j'_s\le d$, $1\le s\le d$, together with two sequences of numbers $k_1,\dots,k_n$ and $l_1,\dots,l_n$ such that $k_s,l_s\in\{\pm1,\dots,\pm N\}$ for all $1\le s\le n$, and they also satisfy the relation $k_s\ne\pm k_{s'}$, and $l_s\ne\pm l_{s'}$ if $s\ne s'$. I claim that under these conditions

$$EZ_{G,j_1}(\Delta_{k_1})\cdots Z_{G,j_n}(\Delta_{k_n})\overline{Z_{G,j'_1}(\Delta_{l_1})\cdots Z_{G,j'_n}(\Delta_{l_n})}=0 \tag{5.10}$$

if $\{k_1,\dots,k_n\}\ne\{l_1,\dots,l_n\}$. On the other hand, if

$$l_p=k_{\pi(p)} \text{ for all } 1\le p\le n \tag{5.11}$$

with some permutation $\pi\in\Pi(n)$, then

$$EZ_{G,j_1}(\Delta_{k_1})\cdots Z_{G,j_n}(\Delta_{k_n})\overline{Z_{G,j'_1}(\Delta_{l_1})\cdots Z_{G,j'_n}(\Delta_{l_n})}$$
$$=G_{j_1,j'_{\pi^{-1}(1)}}(\Delta_{k_1})\cdots G_{j_n,j'_{\pi^{-1}(n)}}(\Delta_{k_n}). \tag{5.12}$$

Let me remark that there cannot be two different permutations $\pi\in\Pi(n)$ satisfying relation (5.11), since by our assumption also elements of the set $\{k_1,\dots,k_n\}$ are different, and the same relation holds for the set $\{1_1,\dots,l_n\}$.

To prove (5.10) we show that under its conditions the product

$$Z_{G,j_1}(\Delta_{k_1})\cdots Z_{G,j_n}(\Delta_{k_n})\overline{Z_{G,j'_1}(\Delta_{l_1})\cdots Z_{G,j'_n}(\Delta_{l_n})}$$

can be written in the form of a product of two independent terms in such a way that one of them has expectation zero.

Since $\{k_1,\dots,k_n\}\ne\{l_1,\dots,l_n\}$, there is such an element $k_s$ for which $k_s\ne l_t$ for all $1\le t\le n$, and also the relation $k_s\ne\pm k_t$ if $s\ne t$, holds. If the relation $k_s\ne\pm l_t$ also holds for all $1\le t\le n$, then $Z_{G,j_s}(\Delta_{k_s})$ is independent of the product of the product of the remaining terms in this product because of property (vi) of vector valued random spectral measures given in Section 3, and $EZ_{G,j_s}(\Delta_{k_s})=0$. Hence relation (5.10) holds in this case.

In the other case, there is an index $s'$ such that $l_{s'}=-k_s$. In this case the vector

$$\begin{aligned}(Z_{G,j_s}(\Delta_{k_s}),\overline{Z_{G,j_{s'}}(\Delta_{l_{s'}})}) &= (Z_{G,j_s}(\Delta_{k_s}),Z_{G,j_{s'}}(-\Delta_{l_{s'}}))\\ &= (Z_{G,j_s}(\Delta_{k_s}),Z_{G,j_{s'}}(\Delta_{k_s}))\end{aligned}$$

is independent of the remaining terms, (because of property (vi) of the vector valued random spectral measures). In last the relation we exploited that $-\Delta_{l_{s'}} = \Delta_{k_s}$). Hence

$$EZ_{G,j_s}(\Delta_{k_s})\overline{Z_{G,j_{s'}}(\Delta_{l_{s'}})} = EZ_{G,j_s}(\Delta_{k_s})\overline{Z_{G,j_{s'}}(-\Delta_{k_s})} = 0,$$

and relation (5.10) holds in this case, too.

To prove (5.12) let us observe that under its condition the investigated product can be written in the form

$$\begin{aligned} &Z_{G,j_1}(\Delta_{k_1})\cdots Z_{G,j_n}(\Delta_{k_n})\overline{Z_{G,j'_1}(\Delta_{l_1})\cdots Z_{G,j'_n}(\Delta_{l_n})} \\ &\qquad = \prod_{p=1}^n Z_{G,j_p}(\Delta_{k_p})\overline{Z_{G,j'_{\pi^{-1}(p)}}(\Delta_{k_p})}. \end{aligned}$$

The terms in the product at the right-hand side are independent for different indices $s$, and $EZ_{G,j_p}(\Delta_{k_p})\overline{Z_{G,j'_{\pi^{-1}(p)}}(\Delta_{k_p})} = G_{j_p,j'_{\pi^{-1}(p}}(\Delta_{k_p})$. Formula (5.12) follows from these relations and the independence between the terms in the last product. (Here we use again property (vi) of the random spectral measures.)

To prove formula (5.4) let us take a regular system

$$\mathcal{D} = \{\Delta_k,\ k = \pm 1, \dots, \pm N\}$$

such that both functions $f$ and $h$ are adapted to it. This can be done by means of a possible refinement of the original regular systems corresponding to the functions $f$ and $h$. Then we can write by exploiting (5.2) and (5.10) that

$$\begin{aligned} &EI_n(f|j_1,\dots,j_n)I_n(h(|j'_1,\dots,j'_n) = EI_n(f|j_1,\dots,j_n)\overline{I_n(h(|j'_1,\dots,j'_n)} \\ &\quad = \sum_{\pi\in\Pi(n)} \sum_{\substack{(k_1,\dots k_n),\ (l_1,\dots l_n) \\ k_p=\pm 1,\dots,\pm N,\ p=1,\dots,n \\ l_p=k_{\pi(p)}\ p=1,\dots,n}} f(u_{k_1},\dots u_{k_n})\overline{h(u_{k_{\pi(1)}},\dots,u_{k_{\pi(n)}})} \\ &\qquad\qquad EZ_{G,j_1}(\Delta_{k_1})\cdots Z_{G,j_n}(\Delta_{k_n})\overline{Z_{G,j'_1}(\Delta_{l_1})\cdots Z_{G,j'_n}(\Delta_{l_n})}, \end{aligned}$$

where $u_k \in \Delta_k$ for all $k = \pm 1, \dots, \pm N$.

The expected value of the product at the right-hand side of this identity can be calculated with the help of (5.12), and this yields that

$$\begin{aligned} &EI_n(f|j_1,\dots,j_n)I_n(h(|j'_1,\dots,j'_n) \\ &\quad = \sum_{\pi\in\Pi(n)} \sum_{\substack{(k_1,\dots k_n),\ (l_1,\dots l_n) \\ k_p=\pm 1,\dots,\pm N,\ p=1,\dots,n \\ l_p=k_{\pi(p)},\ p=1,\dots,n}} f(u_{k_1},\dots u_{k_n})\overline{h(u_{l_1},\dots,u_{l_n})} \\ &\qquad\qquad G_{j_1,j'_{\pi^{-1}(1)}}(\Delta_{k_1})\cdots G_{j_n,j'_{\pi^{-1}(n)}}(\Delta_{k_n}) \\ &\quad = \sum_{\pi\in\Pi(n)} \int f(x_1,\dots x_n)\overline{h(x_{\pi(1)},\dots,x_{\pi(n)})} \\ &\qquad\qquad G_{j_1,j'_{\pi^{-1}(1)}}(\,dx_1)\dots G_{j_n,j'_{\pi^{-1}(n)}}(\,dx_n). \end{aligned}$$

Formula (5.4) is proved.

The proof of (5.5) is based on a similar idea, but it is considerably simpler. It can be proved similarly to relation (5.10) that for $n \neq n'$

$$EZ_{G,j_1}(\Delta_{k_1})\cdots Z_{G,j_n}(\Delta_{k_n})\overline{Z_{G,j'_1}(\Delta_{l_1})\cdots Z_{G,j'_{n'}}(\Delta_{l_{n'}})}=0 \tag{5.13}$$

if we define this expression by means a regular system

$$\mathcal{D}=\{\Delta_k,\ k=\pm1,\pm2,\dots,\pm N\},$$

some numbers $j_1,\dots,j_n$ and $j'_1\dots,j'_{n'}$, all of them between 1 and $d$, together with two sequences of numbers $k_1,\dots,k_n$ and $l_1,\dots,l_{n'}$ such that $k_s,l_s\in\{\pm1,\dots,\pm N\}$ for all these numbers, and they satisfy the relation $k_s\neq\pm k_{s'}$, and $l_s\neq\pm l_{s'}$ if $s\neq s'$. Then, if we express

$$EI_n(f|j_1,\dots,j_n)I_{n'}(h(|j'_1,\dots,j'_{n'})=EI_n(f|j_1,\dots,j_n)\overline{I_{n'}(h(|j'_1,\dots,j'_{n'})}$$

similarly as we have done in the proof of (5.12) we get such a sum where all terms equal zero because of (5.13). This implies relation (5.5).

To define the Wiener–Itô integral for all functions $f\in\mathcal{K}_{n,j_1,\dots,j_n}$ we need the following result.

**Lemma 5.1.** *The class of simple functions $\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ is a dense linear subspace of the (real) Hilbert space $\mathcal{K}_{n,j_1,\dots,j_n}$.*

Lemma 5.1 is the multivariate version of Lemma 4.1 in [11]. (A more transparent proof of this result was given in the Appendix of [12].) Actually, we do not have to prove Lemma 5.1, because it simply follows from Lemma 4.1 of [11]. By applying this result for $G=\sum_{j=1}^n G_{j,j}$ we get that all bounded functions of $\mathcal{K}_{n,j_1,\dots,j_n}$ are in the closure of $\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$. But this implies that all functions of $\mathcal{K}_{n,j_1,\dots,j_n}$ are in this closure.

Let us take the $L_2$ norm in the Hilbert space $\mathcal{H}$. Then we have for all $f\in\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ $I_n(f|j_1,\dots,j_n)\in\mathcal{H}$, and by formula (5.6)

$$\|I_n(f|j_1,\dots,j_n)\|=\left[E(I_n(f|j_1,\dots,j_n)^2)\right]^{1/2}\le\sqrt{n!}\|f_{n,j_1,\dots,j_n}\|.$$

Hence Lemma 5.1 enables us to extend the Wiener–Itô integral $I_n(f|j_1,\dots,j_n)$ for all $f\in\mathcal{K}_{n,j_1,\dots,j_n}$. Moreover, relations (5.2)—(5.6) remain valid in the Hilbert space $\mathcal{K}_{n,j_1,\dots,j_n}$ after this extension.

*Remark.* In (5.6) we have given an upper bound for the second moment of a multiple Wiener–Itô integral, but we cannot write equality in this formula. In the scalar-valued case we had an identity in the corresponding relation. At least this was the case if we took the Wiener–Itô integral of a symmetric function. On the other hand, working only with Wiener–Itô integrals of symmetric functions did not mean a serious restriction. This relative weakness of formula (5.6) (the lack of identity) is the reason why we cannot represent such a large class of random variables in the form of a sum of Wiener–Itô integrals as in the scalar valued case. (See the discussion in Section 7 about this problem.)

I would mention that there is a slightly stronger version of Lemma 5.1 which is useful in the study of the last section of this paper, when we are interested in the question under what conditions we can state that a sequence of Wiener–Itô integrals converges to a Wiener–Itô integral. Here is this result.

**Lemma 5.2.** *For all functions $f \in \mathcal{K}_{n,j_1,\dots,j_n}$ and numbers $\varepsilon > 0$ there is such a simple function $g \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ for which $\|f-g\| \le \varepsilon$ in the norm of the Hilbert space $\mathcal{K}_{n,j_1,\dots,j_n}$, and there is a regular system $\mathcal{D} = \{\Delta_k,\ k = \pm 1, \pm 2, \dots, \pm N\}$ to which the function $g$ is adapted, and the boundary of all sets $\Delta_k \in \mathcal{D}$ has zero $\mu$-probability with the measure $\mu$ we chose as the dominating measure for the complex measures $G_{j,j'}$ in our considerations.*

Lemma 5.2 also follows from the results of [11] or [12].

Finally, I would make the following small remark. If we define a new function by reindexing the variables of a function of $h \in \mathcal{K}_{n,j_1,\dots,j_n}$ by means of a permutation of the indices, and we change the indices of the spectral measure $Z_{G,j_s}$ in the Wiener-Itô integral $I_n(h|j_1,\dots,j_n)$ in an appropriate way, then we get a new Wiener–Itô integral whose value agrees with the original integral $I_n(h|j_1,\dots,j_n)$. More explicitly, the following result holds.

**Lemma 5.3.** *Given a function $h \in \mathcal{K}_{n,j_1,\dots,j_n}$ and a permutation $\pi \in \Pi(n)$ define the function $h_\pi(x_1,\dots,x_n) = h(x_{\pi(1)},\dots,x_{\pi(n)})$. The following identity holds.*

$$\begin{aligned} &\int h(x_1,\dots,x_n) Z_{G,j_1}(\,dx_1)\dots Z_{G,j_n}(\,dx_n) \\ &\qquad = \int h_\pi(x_1,\dots,x_n) Z_{G,j_{\pi(1)}}(\,dx_1)\dots Z_{G,j_{\pi(n)}}(\,dx_n). \end{aligned} \tag{5.14}$$

*(In particular, $h_\pi \in \mathcal{K}_{n,j_{\pi(1)},\dots,j_{\pi(n)}}$, thus the integrals on both sides of the identity are meaningful.)*

*Proof of Lemma 5.3.* This identity can be simply checked if $h$ is a simple function. It is enough to observe that if $h(x_1,\dots,x_n) = h_1(x_1)\cdots h_n(x_n)$ with some $x_l \in \Delta_{k_l}$, $g(_l(\cdot)$ is some function on $\mathbb{R}^\nu$, $1 \le l \le n$, then

$$\int h(x_1,\dots,x_n) Z_{G,j_1}(\,dx_1)\dots Z_{G,j_n}(\,dx_n) = \prod_{l=1}^n h_l(x_l) Z_{G,j_l}(\Delta_{k_l}),$$

$h_\pi(x_1,\dots,x_l) = h_1(x_{\pi_1})\cdots h_n(x_{\pi_n})$,

$$\int h_\pi(x_1,\dots,x_n) Z_{G,j_{\pi(1)}}(\,dx_1)\dots Z_{G,j_{\pi(n)}}(\,dx_n) = \prod_{l=1}^n h(x_{\pi_l}) Z_{G,j_{\pi_l}}(\Delta_{k_{\pi(l)}}),$$

and the last two Wiener–Itô integrals equal. Then a simple limiting procedure implies it in the general case. Lemma 5.3 is proved.

We saw in [11] that in the scalar valued case the value of a Wiener–Itô integral $\int f(x_1,\dots,x_n) Z_G(\,dx_1)\dots Z_G(\,dx_n)$ does not change if we replace the kernel function $f$ by the function we get by permuting its variables $x_1, \dots, x_n$ in an arbitrary way. Lemma 5.3 is the generalization of this result to the case when we integrate with respect to the coordinates of a vector-valued random spectral measure.

*Remark.* A consequence of the result of Lemma 5.3 shows an essential difference between the behaviour of multiple Wiener–Itô integrals with respect to scalar and vector valued random spectral measures. It follows from the scalar valued version of Lemma 5.3 that in the scalar valued case the Wiener–Itô integral of a kernel function agrees with the Wiener–itô integral of the symmetrization of this kernel function. This has the consequence that in the scalar valued case we can restrict our attention to the Wiener–Itô integrals of symmetrical functions which do not change their values by any permutation of their variables. It can be seen that any random variable which can be written as the sum of Wiener–Itô integrals can be written in a unique form as a sum of Wiener–Itô integrals of different multiplicity with symmetric kernel functions. The analogous result does not hold in the vector valued case. Indeed, if there is some linear dependence among the the coordinates of the underlying vectors in a vector valued stationary random field, then such functions $f_j$ can be found for which $\sum_{j=1}^d \int f_j(x) Z_{G,j}(\,dx) \equiv 0$, and not all kernel functions $f_j$ disappear in the above sum. This shows that the unique representation of the random variables by means of a sum of Wiener–Itô integrals may not hold in vector valued models.

# 6 The diagram formula for the product of multiple Wiener–Itô integrals

Let us consider a vector valued random spectral measure $(Z_{G,1}, \dots, Z_{G,d})$ corresponding to the matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, of a vector valued stationary Gaussian random field with expectation zero (either to a discrete random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, or to a generalized one $X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}^\nu$). Let us assume that the spectral measure $G_{j,j'}$, $1 \le j, j' \le d$, is non-atomic, and take two Wiener–Itô integrals

$$I_n(h_1|j_1, \dots, j_n) = \int h_1(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(dx_n) \tag{6.1}$$

and

$$I_m h_2|j'_1, \dots, j'_m) = \int h_2(x_1, \dots, x_m) Z_{G,j'_1}(\,dx_1) \dots Z_{G,j'_m}(dx_m) \tag{6.2}$$

with some kernel functions $h_1 \in \mathcal{K}_{n,j_1,\dots,j_n}$ and $h_2 \in \mathcal{K}_{m,j'_1,\dots,j'_m}$, where $j_s, j'_t \in \{1, \dots, d\}$ for all $1 \le s \le n$ and $1 \le t \le m$.

Actually we formulate our problems in a slightly different form which is more appropriate for our discussion. We take two functions $h_1(x_1, \dots, x_n)$ and

$h_2(x_{n+1},\dots,x_{n+m})$ in the space $\mathbb{R}^{(n+m)\nu}$, and define the function $h_2^{(0)}(x_1,\dots,x_m)$ by the identity

$$h_2^{(0)}(x_1,\dots,x_m)=h_2(x'_{n+1},\dots,x'_{n+m})) \text{ if } (x_1,\dots,x_m)=(x'_{n+1},\dots,x'_{n+m}).$$

We assume that $h_1\in\mathcal{K}_{n,j_1,\dots,j_n}$, $h_2^{(0)}\in\mathcal{K}_{m,j'_1,\dots,j'_m}$. Then we define the Wiener–Itô integrals (6.1) and (6.2) with the kernel functions $h_1$ and $h_2^{(0)}$. In formula (6.2) we should have written the function $h_2^{(0)}$, but we omitted the superscript $^{(0)}$.

I shall present a result in which we express the product of these two Wiener–Itô integrals as a sum of Wiener–Itô integrals with different multiplicities. This result is called the diagram formula, since the kernel functions of the Wiener–Itô integrals appearing in this sum are expressed by means of some diagrams. This result is a multivariate version of the diagram formula proved in Chapter 5 of [11]. In that work also the product of more than two Wiener–Itô integrals is expressed in the form of a sum of Wiener–integrals. But actually the main point of the proof is to show the validity of the diagram formula for the product of two Wiener–Itô integrals, and we shall need only this result. So I restrict my attention only to this case. Actually we need the diagram formula only in a special case. The result in this special case will be given in a corollary.

To express the product of the two Wiener–Itô integrals in formulas (6.1) and (6.2) as a sum of Wiener–Itô integrals first I introduce a class of coloured diagrams $\Gamma=\Gamma(n,m)$ that will be used in the definition of the Wiener–Itô integrals we shall be working with. A coloured diagram $\gamma\in\Gamma$ is a graph whose vertices are the pairs of integers $(1,s)$, $1\le s\le n$, and $(2,t)$, $1\le t\le m$. Each vertex is coloured with one of the numbers $1,\dots,d$. The colour of the vertex $(1,s)$ is $j_s$, $1\le s\le n$, and the colour of the vertex $(2,t)$ is $j'_t$, $1\le t\le m$. The set of vertices of the form $(1,s)$ will be called the first row and the set of vertices of the form $(2,t)$ will be called the second row of a diagram $\gamma\in\Gamma$. The coloured diagrams $\gamma\in\Gamma$ are those undirected graphs with the above coloured vertices for which edges can go only between vertices of the first and second row, and from each vertex there starts zero or one edge. Given a coloured diagram $\gamma\in\Gamma$ we shall denote the number of its edges by $|\gamma|$.

I shall define for all coloured diagrams $\gamma\in\Gamma$ a multiple Wiener–Itô integral depending on $\gamma$. The diagram formula states that the product of the Wiener–Itô integrals in (6.1) and (6.2) equals the sum of these Wiener–Itô integrals.

In the formulation of the diagram formula I shall work with the functions $h_1(x_1,\dots,x_n)$ and $h_2(x_{n+1},\dots,x_{n+m})$ in $\mathbb{R}^{n+m}$. The function $h_2(x_{n+1},\dots,x_{n+m})$ is the function which appeared in the definition of the kernel function $h_2^{(0)}(x_1,\dots,x_m)$ in the Wiener–Itô integral in (6.2). We define with their help the function

$$H(x_1,\dots,x_{n+m})=h_1(x_1,\dots,x_n)h_2(x_{n+1},\dots,x_{n+m}). \tag{6.3}$$

We shall define the kernel functions appearing in the Wiener–itô integrals in the diagram formula with the help of the functions $H(x_1,\dots,x_{n+m})$. In the definition of these kernel functions I shall apply the following natural bijection $S$ between the coordinates of the vectors in $\mathbb{R}^{n+m}$, i.e. the set $\{1,\dots,n+m\}$ and the vertices of the diagrams of $\gamma\in\Gamma$.

$$S((1,k))=k \text{ for } 1\le k\le n, \quad \text{and} \quad S((2,k))=n+k \text{ for } 1\le k\le m. \tag{6.4}$$

To simplify the formulation of the diagram formula I shall introduce the following notation with the help of the colours of the diagrams.

$$J(1,k)=j_k,\ \ 1\le k\le n \quad \text{and} \quad J(2,l)=j'_l,\ \ 1\le l\le m. \tag{6.5}$$

First I give the formal definition of the Wiener–Itô integrals that appear in the diagram formula, and then I give an informal explanation of this definition by briefly indicating the picture behind it. Then I describe the diagram formula with the help of the Wiener–Itô integrals corresponding to the diagrams $\gamma\in\Gamma$. To explain this result better I shall present an example after its formulation, where the product of two Wiener–Itô integrals is considered, and I show how to calculate a typical term in the sum of Wiener–Itô integrals which appears if we apply the diagram formula for this product.

Let us fix some diagram $\gamma\in\Gamma$. I explain how to define the the Wiener–Itô integral corresponding to $\gamma$ in the diagram formula. First I define a function $H_\gamma(x_1,\dots,x_{n+m})$ which we get by means of an appropriate permutation of the indices of the function $H$ defined in (6.3). This permutation of the indices depends on the diagram $\gamma$.

To define this permutation of the indices first I define a map $T_\gamma$ which maps the set $\{1,\dots,n+m\}$ to the elements in the rows of the diagrams. This map depends on the diagram $\gamma$.

To define this map first I introduce the following sets depending on the diagram $\gamma$:

$$\begin{aligned} A_1=A_1(\gamma) \quad &= \quad \{r_1,\dots,r_{n-|\gamma|}\colon\ 1\le r_1<r_2<\cdots<r_{n-|\gamma|}\le n \\ &\qquad \text{no edge of } \gamma \text{ starts from } (1,r_k), \quad 1\le k\le n-|\gamma|\}, \end{aligned} \tag{6.6}$$

$$\begin{aligned} A_2=A_2(\gamma) \quad &= \quad \{t_1,\dots,t_{m-|\gamma|}\colon\ 1\le t_1<t_2<\cdots<t_{m-|\gamma|}\le m \\ &\qquad \text{no edge of } \gamma \text{ starts from } (2,t_k), \quad 1\le k\le m-|\gamma|\} \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} B=B(\gamma) \quad &= \quad \{(v_1,w_1),\dots,(v_{|\gamma|},w_{|\gamma|}))\colon\ 1\le v_1<v_2<\cdots v_{|\gamma|}\le n \\ &\qquad ((1,v_k),(2,w_k)) \text{ is an edge of } |\gamma|,\ \ 1\le k\le|\gamma|\}. \end{aligned} \tag{6.8}$$

Let us also define with the help of the set $B$ the sets

$$B_1=B_1(\gamma)=\{v_1,\dots,v_{|\gamma|}\}, \quad B_2=B_2(\gamma)=\{w_1,\dots,w_{|\gamma|}\} \tag{6.9}$$

with the numbers $v_k$ and $w_l$ appearing in the set

$$B=B(\gamma)=\{(v_1,w_1)),\dots,(v_{|\gamma|},w_{|\gamma|}))\}.$$

Now, I define the map $T_\gamma$ in the following way.

$$
\begin{aligned}
&T_\gamma(k)=(1,r_k) \text{ for } 1\le k\le n-|\gamma|, \\
&T_\gamma(n-|\gamma|+k)=(2,t_k) \text{ for } 1\le k\le m-|\gamma|, \\
&T_\gamma(n+m-2|\gamma|+k)=(1,v_k) \text{ for } 1\le k\le |\gamma|, \\
&T_\gamma(n+m-|\gamma|+k)=(2,w_k) \text{ for } 1\le k\le |\gamma|.
\end{aligned} \tag{6.10}
$$

In formula (6.10) we worked with the numbers $r_k$, $t_k$, $v_k$ and $w_k$ defined in (6.6)—(6.9). It has the following meaning. The first $n-|\gamma|$ indices are given in increasing order to the vertices from the first row from which no edge starts. Then the vertices of the second row from which no edge starts get the next $m-|\gamma|$ indices also in increasing order. Then the $|\gamma|$ vertices from the first row from which an edge starts get the subsequent $|\gamma|$ indices in increasing order. The remaining $|\gamma|$ vertices from the second row from which an edge starts get the indices between $n+m-|\gamma|+1$ and $|n+m|$. They are indexed in such a way that if two vertices $(1,v_k)$ and $(2,w_k)$ are connected by en edge then the index of $(2,w_k)$ is obtained if we add $|\gamma|$ to the index of $(1,v_k)$.

I define with the help of the function $T_\gamma$ and the map $S(\cdot)$ defined in (6.4) the permutation

$$
\pi_\gamma(k)=S(T_\gamma(k)), \quad 1\le k\le n+m
$$

of the set $\{1,\dots,n+m\}$. Next I introduce the Euclidean space $\mathbb{R}^{n+m}_\gamma$ with elements $x(\gamma)=(x(\gamma)_1,\dots,x(\gamma)_{n+m})$ by reindexing the arguments of the Euclidean space $\mathbb{R}^{n+m}$, where the functions $h_1(x_1,\dots,x_n)$ and $h_2(x_{n+1},\dots,x_{n+m})$ are defined in the following way.

$$
(x(\gamma)_1,\dots,x(\gamma)_{n+m})=(x_{\pi_\gamma(1)},\dots,x_{\pi_\gamma(n+m)})
$$

with $(x(\gamma)_1,\dots,x(\gamma)_{n+m})\in\mathbb{R}^{n+m}_\gamma$ and $(x_1,\dots,x_{n+m})\in\mathbb{R}^{n+m}$. It will be simpler to define the quantities needed in the definition of the Wiener–Itô integral corresponding to the diagram $\gamma$ as functions defined in the space $R^{n+n}_\gamma$. First we define the function $H_\gamma$ as

$$
\begin{aligned}
&H_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m}) \\
&\quad=H(x(\gamma)_1,\dots,x(\gamma)_{n-|\gamma|},x(\gamma)_{n+m-2|\gamma|+1},\dots,x(\gamma)_{n+m-|\gamma|}, \\
&\qquad\qquad x(\gamma)_{n-|\gamma|+1},\dots,x(\gamma)_{n+m-2|\gamma|+1},x(\gamma)_{(n+m-|\gamma|+1},\dots,x(\gamma)_{n+m}) \\
&\quad=h_1(x(\gamma)_1,\dots,x(\gamma)_{n-|\gamma|},x(\gamma)_{\pi_\gamma(n+m-2|\gamma|+1)},\dots,x(\gamma)_{n+m-|\gamma|}) \\
&\qquad\quad h_2(x(\gamma)_{n-|\gamma|+1},\dots,x(\gamma)_{n+m-2|\gamma|+1},x(\gamma)_{n+m-|\gamma|+1},\dots,x(\gamma)_{n+m}).
\end{aligned} \tag{6.11}
$$

Next I define the function $\bar h_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-|\gamma|}))$ which we get by replacing the arguments $x(\gamma)_{n+m-|\gamma|+k}$ by $-x(\gamma)_{n+m-2|\gamma|+k})$ in the function $H_\gamma$

defined in formula (6.11) for all $1 \le k \le \gamma$, i.e. I define

$$
\begin{aligned}
&\bar h_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-|\gamma|}) \qquad (6.12)\\
&\quad = H_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-|\gamma|},-x(\gamma)_{n+m-2|\gamma|+1},\dots,-x(\gamma)_{n+m-|\gamma|})\\
&\quad = H(x(\gamma)_1,\dots,x(\gamma)_{n-|\gamma|},x(\gamma)_{n+m-2|\gamma|+1},\dots,x(\gamma)_{n+m-|\gamma|},\\
&\qquad\qquad x(\gamma)_{n-|\gamma|+1},\dots,x(\gamma)_{n+m-2|\gamma|+1},\\
&\qquad\qquad\qquad -x(\gamma)_{n+m-2|\gamma|+1},\dots,-x(\gamma)_{n+m-|\gamma|})\\
&\quad = h_1(x(\gamma)_1,\dots,x(\gamma)_{n-|\gamma|},x(\gamma)_{n+m-2|\gamma|+1},\dots,x(\gamma)_{n+m-|\gamma|})\\
&\qquad h_2(x(\gamma)_{n-|\gamma|+1},\dots,x(\gamma)_{n+m-2|\gamma|+1},\\
&\qquad\qquad\qquad -x(\gamma)_{n+m-2|\gamma|+1},\dots,-x(\gamma)_{n+m)-|\gamma|}).
\end{aligned}
$$

In the next step I define the function $\bar{\bar h}_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-2|\gamma|})$. This will be the kernel function of the Wiener–Itô integral which corresponds to the diagram $\gamma$ in the diagram formula if we express it as a Wiener–Itô integral with respect to the variables $x(\gamma)_1,\dots,x(\gamma)_{n+m-2|\gamma|}$.

$$
\begin{aligned}
\bar{\bar h}_\gamma(x_\gamma)_1,\dots,x(\gamma)_{n+m-2|\gamma|}) &= \int \bar h_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-|\gamma|}) \qquad (6.13)\\
&\quad \prod_{k=1}^{|\gamma|} G_{J(S^{-1}(n+m-2|\gamma|+k)),J(S^{-1}(n+m-|\gamma|+k))}(\,dx(\gamma)_{n+m-2|\gamma|+k})\\
&= \int \bar h_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-|\gamma|}) \prod_{k=1}^{|\gamma|} G_{j_{v_k},j'_{w_k}}(\,dx(\gamma)_{n+m-2|\gamma|+k})
\end{aligned}
$$

with the function $J(\cdot)$ defined in (6.5), the indices $v_k$ and $w_k$ defined in (6.8) and the function $T_\gamma$ defined in (6.10).

I shall show that the Wiener–Itô integrals

$$
\begin{aligned}
&I_{n+m-2|\gamma|}(\bar{\bar h}_\gamma|j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}}) \qquad (6.14)\\
&\quad = \int \bar{\bar h}_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-2|\gamma|}) \prod_{k=1}^{n+m-2|\gamma|} Z_{G,J(S^{-1}(k))}(dx(\gamma)_k)\\
&\quad = \int \bar{\bar h}_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-2|\gamma|})\\
&\qquad\qquad \prod_{k=1}^{n-|\gamma|} Z_{G,j_{r_k}}(\,dx(\gamma)_k) \prod_{l=1}^{m-|\gamma|} Z_{G,j'_{t_l}}(\,dx(\gamma)_{l+n-|\gamma|})
\end{aligned}
$$

exist for all $\gamma\in\Gamma$, and these Wiener–Itô integrals appear in the diagram formula. The numbers $r_k$ and $t_l$ in this formula were defined in (6.6) and (6.7).

In formula (6.14) we integrated with respect to the coordinates $x(\gamma)_s$, $1 \le s \le n+m$, of the vectors in the Euclidean space $\mathbb{R}^{n+m}_\gamma$. If we replace the variables $x(\gamma)_s$ by $x_s$ in (6.14), then we get a Wiener–itô integral in the space

$\mathbb{R}^{n+m}$ which has the same value. This means that the following relation holds.

$$\begin{aligned}
&I_{n+m-2|\gamma|}(\bar{\bar{h}}_\gamma|j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}}) \\
&\quad = I_{n+m-2|\gamma|}(h_\gamma|j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}}) \\
&\quad = \int h_\gamma(x_1,\dots,x_{n+m-2|\gamma|}) \\
&\qquad\qquad \prod_{k=1}^{n-|\gamma|} Z_{G,j_{r_k}}(\,dx_k) \prod_{l=1}^{m-|\gamma|} Z_{G,j'_{t_l}}(\,dx_{l+n-|\gamma|})
\end{aligned} \tag{6.15}$$

with

$$\begin{aligned}
h_\gamma(x_1,\dots,x_{n+m-2|\gamma|}) &= \bar{\bar{h}}_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-2|\gamma|}) \\
&= \bar{\bar{h}}_\gamma(x_{\pi_\gamma(1)},\dots,x_{\pi_\gamma(n+m-2|\gamma|)}).
\end{aligned}$$

Before describing the diagram formula I explain the content of the above defined formulas.

Let us fix a diagram $\gamma\in\Gamma$, and let us call a vertex of it from which no edge starts open, and a vertex from which an edge starts closed. We listed the open vertices from the first row in increasing order as $(1,r_1),\dots,(1,r_{n-|\gamma|})$, and the open vertices from the second row as $(2,t_1),\dots,(2,t_{m-|\gamma|})$. We listed the closed vertices from the first row in increasing order as $(1,v_1),\dots,(1,v_\gamma)$. Finally we listed the closed vertices from the second row as $(2,w_1),\dots,(2,w_\gamma)$, and we indexed them in such a way that the vertices $(1,v_k)$ and $(2,w_k)$ are connected by an edge for all $1\le k\le\gamma$.

In formula (6.10) we defined the map $T_\gamma$ from the set $\{1,\dots,n+m\}$ to the set of vertices of the diagram $\gamma$ with the help of the above listing of the vertices. First we considered the open vertices from the first row, then the open vertices from the second row, and then we finished with the closed vertices first from the first and then from the second row. We defined in (6.11) the permutation $\pi_\gamma$ of the set $\{1,\dots,n+m\}$ by applying first the map the map $T_\gamma$ and then the map $S$ defined (6.4). We defined the function $H_\gamma$ in (6.12) with the help of this permutation. We have introduced a Euclidean space $\mathbb{R}^{n+m}_\gamma$ whose elements we get by rearranging the indices of the coordinates of the Euclidean space $\mathbb{R}^{n+m}$ where we are working with the help of the permutation $\pi_\gamma$, and we have defined our functions in this space.

We defined the function $H_\gamma$ on the space $\mathbb{R}^{n+m}_\gamma$ as the product of the functions $h_1$ and $h_2$ with reindexed variables. In the function $h_1$ first we took the variables $x(\gamma)_s=x_{\pi_\gamma(s)}$ with those indices $\pi_\gamma(s)$ which correspond to the open vertices of the first row, and then the variables with indices corresponding to the closed vertices of the first row. We defined the reindexation of the variables in the second row similarly. First we took those variables whose indices correspond to the open vertices and then the variables whose indices correspond to the closed vertices of the second row.

The variables

$$x(\gamma)_{n+m-2|\gamma|+k}=x_{\pi_\gamma(n+m-2|\gamma|+k)} \text{ and } x(\gamma)_{n+m-|\gamma|+k}=x_{\pi_\gamma(n+m-|\gamma|+k)}$$

in the function $H_\gamma$ are variables with indices corresponding to vertices connected by an edge. So in the definition of the function $\bar h_\gamma$ in (6.13) I replaced in $H_\gamma$ the variable corresponding to the endpoint of an edge from the second row of the diagram $\gamma$ by the variable corresponding to the other endpoint of this edge, and multiplied this variable by $-1$. Thus the variables $x(\gamma)_{n+m-2|\gamma|+k} = x_{\pi_\gamma(n+m-2|\gamma|+k)}$, $1 \le k \le |\gamma$, of the function $\bar h_\gamma$ correspond to the edges of the diagram $\gamma$. I defined the function $\bar{\bar h}_\gamma$ by integrating the function $\bar h_\gamma$ by these variables. The variable $x(\gamma)_{n+m-2|\gamma|+k} = x_{\pi_\gamma(n+m-2|\gamma|+k)}$ corresponds to the $k$-th edge of the diagram, and we integrate this variable with respect to the measure $G_{j_{v_k},j'_{w_k}}$, that is with respect to the measure $G_{u,v}$ whose coordinates are the colours of the endpoints of the $k$-th edge.

Finally we define the Wiener–Itô integral corresponding to the diagram $\gamma$ with kernel function $\bar{\bar h}_\gamma$. We integrate the argument $x(\gamma)_k$ with respect to that random spectral measure $Z_{G,j}$ whose parameter agrees with the colour of the vertex corresponding to this variable. Thus we choose $Z_{G,j_{r_k}}(\,dx(\gamma)_k)$ for $1 \le k \le n-|\gamma|$ and $Z_{G_{j'_{t_{k-n+|\gamma}}}}(\,dx(\gamma)_k)$ if $n-|\gamma|+1 \le k \le n+m-2|\gamma|$. We can replace this Wiener–Itô integral defined in (6.14) with kernel function $\bar{\bar h}_\gamma$ by the Wiener–Itô integral defined in (6.15) with kernel function $h_\gamma$.

Next I formulate the diagram formula.

**Theorem 6.1. The diagram formula.** *Let us consider the Wiener–Itô integrals $I_n(h_1|j_1,\dots,j_n)$ and $I_m(h_2|j'_1,\dots,j'_m)$ introduced in formulas (6.1) and (6.2). The following results hold.*

(A) *The function $\bar{\bar h}_\gamma$ defined in (6.13) satisfies the relations*

$$\bar{\bar h}_\gamma \in \mathcal{K}_{n+m-2|\gamma|,j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}}},$$

*and $\|\bar{\bar h}_\gamma\| \le \|h_1\|\|h_2\|$ for all $\gamma \in \Gamma$. Here the norm of the function $h_1$ in $\mathcal{K}_{n,j_1,\dots,j_n}$, the norm of $\bar{\bar h}_2$ in $\mathcal{K}_{m,j'_1,\dots,j'_m}$, and the norm of $\bar{\bar h}_\gamma$ in $\mathcal{K}_{n+m-2|\gamma|,j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}}}$ is taken.*

(B)

$$\begin{aligned} &I_n(h_1|j_1,\dots,j_n)I_m(h_2|j'_1,\dots,j'_m) \\ &\qquad = \sum_{\gamma\in\Gamma} I_{n+m-2|\gamma|}(\bar{\bar h}_\gamma|j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}}). \end{aligned} \tag{6.16}$$

*The terms in the sum at the right-hand side of formula (6.16) were defined in formulas (6.11)—(6.14). The Wiener–Itô integral*

$$I_{n+m-2|\gamma|}(\bar{\bar h}_\gamma|j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}})$$

*in formula (6.16) can be replaced by the Wiener–Itô integral*

$$I_{n+m-2|\gamma|}(h_\gamma|j_{r_1},\dots,j_{r_{n-|\gamma|}},j'_{t_1},\dots,j'_{t_{m-|\gamma|}})$$

*defined in (6.15).*

To understand the formulation of the diagram formula better let us consider the following example. We take a five dimensional stationary Gaussian random field with some spectral measure $G_{j,j'}(x)$, $1 \le j, j' \le 5$, and random spectral measure $Z_{G,j}(\,dx)$, $1 \le j \le 5$, corresponding to it. Let us understand how we define the Wiener–Itô integral corresponding to a typical diagram when we apply the diagram formula in the following example. Take the product of two Wiener–Itô integrals of the following form:

$$I_3(h_1|2,3,5) = \int h_1(x_1,x_2,x_3) Z_{G,2}(\,dx_1) Z_{G,3}(\,dx_2) Z_{G,5}(\,dx_3)$$

and

$$\begin{aligned} I_4(h_2|1,5,4,1) \quad &= \quad \int h_2(x_1,x_2,x_3,x_4) \\ &\qquad Z_{G.1}(\,dx_1) Z_{G,5}(\,dx_2) Z_{G,4}(\,dx_3) Z_{G,2}(\,dx_4), \end{aligned}$$

and let us write it in the form of a sum of Wiener–Itô integrals with the help of the diagram formula.

First I give the vertices of the coloured diagrams we shall be working with together with their colours.

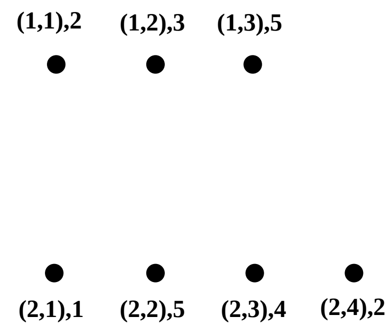


Figure 1: the vertices of the diagrams together with their colours

Next I consider a diagram $\gamma$ which yields one of the terms in the sum expressing the product of these two Wiener–Itô integrals. I take the diagram which has two edges, one edge connecting the vertices $(1,2)$ and $(2,4)$, and another edge connecting the vertices $(1,3)$ and $(2,1)$. Let us calculate which Wiener–Itô integral corresponds to this diagram $\gamma$.

Next I take this diagram $\gamma$, and I show not only the indices and colours of its vertices, but for each vertex I also tell which value $T_\gamma(k)$ it equals. Here $T_\gamma(k)$ is the function defined in formula (6.10).

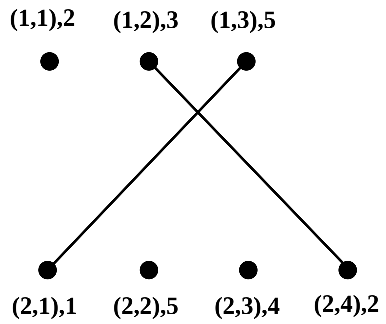


Figure 2: a typical diagram

To define the the Wiener–Itô integral corresponding to this diagram let us first consider the function

$$H(x_1,\dots,x_7)=h_1(x_1,x_2,x_3)h_2(x_4,x_5,x_6,x_7)$$

defined in (6.3). Simple calculation shows that the function $\pi_\gamma(\cdot)=S(T_\gamma(\cdot))$ has the following form in this example. $\pi_\gamma(1)=1$, $\pi_\gamma(2)=5$, $\pi_\gamma(3)=6$, $\pi_\gamma(4)=2$, $\pi_\gamma(5)=3$, $\pi_\gamma(6)=7$, $\pi_\gamma(7)=4$. This also means that the coordinates of the vectors in the Euclidean space $\mathbb{R}^7_\gamma$ which we get by reindexing the coordinates of the vectors in $\mathbb{R}^7$ have the form

$$(x(\gamma)_1,x(\gamma)_2,x(\gamma)_3,x(\gamma)_4,x(\gamma)_5,x(\gamma)_6,x(\gamma)_7)=(x_1,x_5,x_6,x_2,x_3,x_7,x_4).$$

Then we can write the function $\bar H_\gamma$ and $\bar h_\gamma$ defined in (6.11) and (6.12) as

$$H_\gamma(x(\gamma)_1,\dots,x(\gamma)_7)=h_1(x(\gamma)_1,x(\gamma)_4,x(\gamma)_5)h_2(x(\gamma)_2,x(\gamma)_3,x(\gamma)_6,x(\gamma)_7),$$

and

$$\bar h_\gamma(x(\gamma)_1,\dots,x(\gamma)_5)=h_1(x(\gamma)_1,x(\gamma)_4,x(\gamma)_5)h_2(x(\gamma)_2,x(\gamma)_3,-x(\gamma)_4,-x(\gamma)_5).$$

Then we have

$$\bar{\bar h}_\gamma(x(\gamma)_1,x(\gamma)_2,x(\gamma)_3)=\int \bar h_\gamma(x(\gamma)_1,\dots,x(\gamma)_5)G_{3,2}(\,dx(\gamma)_4)G_{5,1}(\,dx(\gamma)_5),$$

and

$$I_3(\bar{\bar h}_\gamma|2,5,4) \\ =\int \bar{\bar h}_\gamma(x(\gamma)_1,x(\gamma)_2,x(\gamma)_3)Z_{G,2}(\,dx(\gamma)_1)Z_{G,5}(\,dx(\gamma)_2)Z_{G,4}(\,dx(\gamma)_3)$$

is the multiple Wiener–Itô integral corresponding to the diagram $\gamma$ in the diagram formula. To understand the definition of the function $\bar{\bar h}_\gamma$ and of the

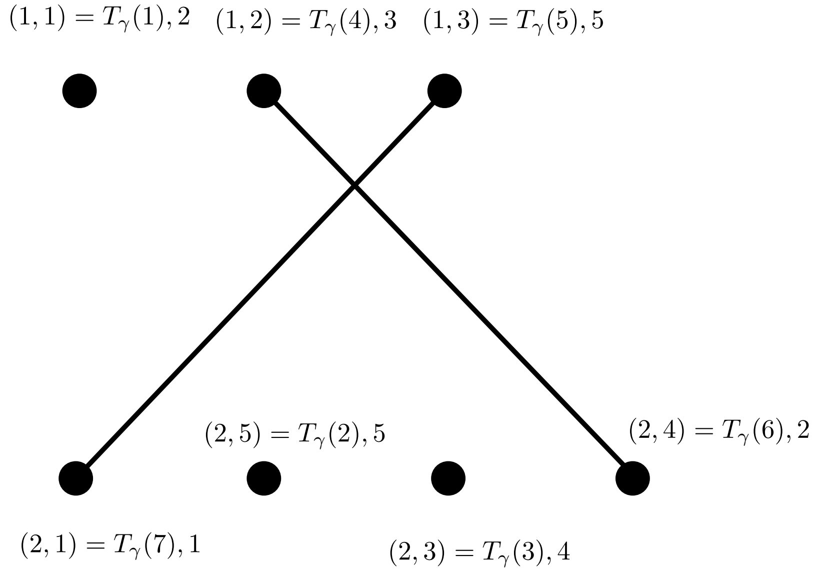


Figure 3: the previous diagram and the enumeration of their vertices with the help of the function $T_\gamma$

Wiener–Itô integral $I_3(\bar{\bar{h}}_\gamma)$ let us observe that the first edge of the diagram connects the vertices $(1,2)$ and $(2,4)$ with colours 3 and 2, hence in the definition of $\bar{\bar{h}}_\gamma$ we integrate the argument $x(\gamma)_4$ by $G_{3,2}(\,dx(\gamma)_4)$, the second edge connects the vertices $(1,3)$ and $(2,1)$ with colours 5 and 1, hence we integrate the variable $x(\gamma)_5$ by $G_{5,1}(\,dx(\gamma)_5)$. In the definition of the Wiener integral the variable $x(\gamma)_1$ corresponds to the vertex $S^{-1}(\pi_\gamma(1))=(1,1)$ which has colour 2, hence we integrate the variable $x(\gamma)_1)$ by $Z_{G,2}(\,dx(\gamma)_1)$. Similarly, we define the variable $x(\gamma)_2$ by the measure determined by the colour of $S^{-1}(\pi_\gamma(2))=(2,2)$ which is 5, i.e. we integrate by $Z_{G,5}(\,dx(\gamma)_2)$. Finally $S^{-1}(\pi_\gamma(3))=(2,3)$ has colour 4, and we integrate the variable $x(\gamma)_3$ by $Z_{G,4}(\,dx(\gamma)_3)$.

The Wiener–Itô integral $I_3(\bar{\bar{h}}_\gamma|3,1,3)$ can be rewritten with the help of formula (6.15) in the following form.

$$I_3(\bar{\bar{h}}_\gamma|2,5,4)=I_3(h_\gamma|2,5,4)=\int h_\gamma(x_1,x_2,x_3)Z_{G,2}(\,dx_1)Z_{G,5}(\,dx_2)Z_{G,4}(\,dx_3)$$

with

$$h_\gamma(x_1,x_2,x_3)=\int h_1(x_1,x_4,x_5)h_2(x_2,x_3,-x_4,-x_5)G_{3,2}(\,dx_4)G_{5,1}(\,dx_5).$$

This expression can be calculated similarly to $I_3(\bar{\bar{h}}_\gamma|2,5,4)$, only we have to replace $x(\gamma)_s$ everywhere by $x_s$ in the calculation.

I formulate a Corollary of the diagram formula in which I consider that special case of this result when the second Wiener–Itô integral defined in formula (6.2) is a one-fold integral. In this case it has the simpler form

$$I_1(h_2|j'_1) = \int h_2(x_1) Z_{G,j'_1}(\,dx_1) \quad \text{with } h_2 \in \mathcal{K}_{1,j'_1}. \tag{6.17}$$

Here again we formulate the problem in the following way. We take a pair of functions $h_1(x_1,\dots,x_n)$ and $h_2(x_{n+1})$ on $\mathbb{R}^{(n+1)\nu}$. Then we define a function $h_2^{(0)}(x_1)$ on $\mathbb{R}^1$ by the formula $h_2^{(0)}(x_1) = h_2(x_{n+1})$ if $x_1 = x_{n+1}$. We integrate the function $h_2^{(0)}(x)$ in formula (6.17), but we omit the superscript $^{(0)}$ in our notation. We assume that $h_1 \in \mathcal{K}_{n,j_1,\dots,j_n}$, and $h_2 \in \mathcal{K}_{1,j'_1}$.

In the next Corollary I express the product of the Wiener–Itô integrals given in (6.1) and (6.17) as a sum of Wiener–Itô integrals. This formula will be needed in the proof of the multivariate version of Itô's formula in the next section.

The diagram formula in this case has a simpler form, since the second row of the diagrams we are working with consists only of one point $(2,1)$. Hence there are only the diagram $\gamma_0 \in \Gamma$ that contains no edges, and the diagrams $\gamma_p \in \Gamma$, $1 \le p \le n$, which contain one edge that connects the vertices $(1,p)$ and $(2,1)$.

**Corollary of Theorem 6.1.** *The product of the Wiener–Itô integrals*

$$I_n(h_1|j_1,\dots,j_n) \quad \textit{and} \quad I_1(h_2|j'_1)$$

*introduced in formulas (6.1) and (6.17) satisfy the identity*

$$\begin{aligned} &I_n(h_1|j_1,\dots,j_n) I_1(h_2|j'_1) \\ &\quad = \int h_{\gamma_0}(x_1,\dots,x_{n+1}) Z_{G,j_1}(\,dx_1)\cdots Z_{G,j_n}(\,dx_n) Z_{G,j'_1}(\,dx_{n+1}) \\ &\qquad + \sum_{p=1}^n \int h_{\gamma_p}(x_1,\dots,x_{n-1}) \prod_{s=1}^{p-1} Z_{G,j_s}(\,dx_s) \prod_{s=p}^{n-1} Z_{G,j_{s+1}}(\,dx_s) \\ &\quad = I_{n+1}(h_{\gamma_0}|j_1,\dots,j_n,j'_1) + \sum_{p=1}^n I_{n-1}(h_{\gamma_p}|j_1,\dots,j_{p-1},j_{p+1},\dots,j_n), \end{aligned} \tag{6.18}$$

*where* $h_{\gamma_0}(x_1,\dots,x_{n+1}) = h_1(x_1,\dots,x_n) h_2(x_{n+1})$*, and for* $1 \le p \le n$

$$h_{\gamma_p}(x_1,\dots,x_{n-1}) = \int h_{1,\gamma_p}(x_1,\dots,x_n) \overline{h_2(x_n)} G_{j_p,j'_1}(\,dx_n)$$

*with* $h_{1,\gamma_p}(x_1,\dots,x_n) = h_1(x_{\pi_p(1)},\dots,x_{\pi_p(n)})$*, where* $\pi_p(k) = k$ *if* $1 \le k \le p-1$*,* $\pi_p(p) = n$*, and* $\pi_p(k) = k-1$ *if* $p+1 \le k \le n$*.*

*To make the definition of formula (6.18) complete I remark that for* $p=1$ *we put* $\prod\limits_{s=1}^{0} Z_{G,j_s}(\,dx_s) \equiv 1$ *and for* $p=n$ $\prod\limits_{s=n}^{n-1} Z_{G,j_s}(\,dx_s) \equiv 1$*.*

*Proof of the Corollary.* We get the result of the corollary by applying Theorem 6.1 in the special case when the second Wiener–Itô integral is defined by formula (6.17) instead of (6.2). We have to check that in this case the function $h_{\gamma_0}$ corresponding to the diagram $\gamma_0$ agrees with the function $h_{\gamma_0}$ defined in the corollary, and to calculate the functions $h_{\gamma_p}$ defined in (6.13) for the remaining diagrams $\gamma_p$, $1 \le p \le n$. In this case $\pi_{\gamma_p}(k) = k$ for $1 \le k \le p-1$, $\pi_{\gamma_p}(k) = k+1$ for $p \le k \le n-1$, $\pi_{\gamma_p}(n) = p$, $\pi_{\gamma_p}(n+1) = n+1$, hence

$$(x(\gamma_p)_1, \dots, x(\gamma_p)_{n+1}) = (x_1, \dots, x_{p-1}, x_{p+1}, \dots, x_n, x_p, x_{n+1}),$$

and

$$\bar{h}_{\gamma_p}(x(\gamma_p)_1, \dots, x(\gamma_p)_{n+1}) = h_1(x(\gamma_p)_1, \dots, x(\gamma_p)_n) h_2(-x(\gamma_p)_n)$$

for $1 \le p \le n$. On the other hand, $h_2(-x) = \overline{h_2(x)}$, since $h_2 \in \mathcal{K}_{1,j_1'}$. Thus

$$\begin{aligned}&\bar{\bar{h}}_{\gamma_p}(x(\gamma_p)_1, \dots, x(\gamma_p)_{n-1}) \\ &\qquad = \int h_1(x(\gamma_p)_1, \dots, x(\gamma_p)_{n-1}, x(\gamma_p)_n) \overline{h_2(x(\gamma_p)_n)} G_{j_p, j_1'}(\,dx(\gamma_p)_n).\end{aligned}$$

Then simple calculation shows that for $\gamma = \gamma_p$ the kernel function $h_\gamma = h_{\gamma_p}$ in formula (6.15) agrees with the function $h_{\gamma_p}$ defined in the corollary for all $1 \le p \le n$, and Theorem 6.1 yields identity (6.18) under the conditions of the corollary. The corollary is proved.

The proof of Theorem 6.1 is similar to the proof of the diagram formula (Theorem 5.3 in [11]). It applies the same method, only the notation becomes more complicated than the also rather complicated notation of the original proof, since we have to work with spectral measures of the form $G_{j_s, j_t'}$ and random spectral measures of the form $Z_{G,j_s}$ or $Z_{G,j_t'}$ instead of the spectral measure $G$ and random spectral measure $Z_G$. Hence I decided not to describe the complete proof, I only concentrate on its main ideas and the formulas that explain why such a formula appears in the diagram formula. The interested reader can reconstruct the proof by means of a careful study of the proof of Theorem 5.3 in [11].

*A sketch of proof for Theorem 6.1.* The proof of Part A is relatively simple. One can check that the function $h_\gamma$ satisfies relation (a) in the definition of the functions in $\mathcal{K}_{n+m-2|\gamma|, j_{r_1}, \dots, j_{r_{n-|\gamma|}}, j'_{t_1}, \dots, j'_{t_{m-|\gamma|}}}$ given in Section 5 by exploiting formula (6.13), the similar property of the functions $h_1$ and $h_2$ together with the symmetry property $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$ for all $1 \le j, j' \le d$ and sets $A$ of the spectral measure $G$.

To prove the inequality formulated in Part A let us first rewrite the definition of $h_\gamma$ in (6.13) by replacing all measures of the form $G_{j.j'}(dx)$ by $g_{j,j'}(x)\mu(\,dx) = G_{j,j'}(\,dx)$, where $\mu$ is a dominating measure for all complex measures $G_{j,j'}$, $g_{j,j'}$ is the Radon–Nikodym derivative of $G_{j,j'}$ with respect to $\mu$, and observe that the inequality (3.2) and formula (6.12) and (6.13) imply that

$$
\begin{aligned}
&|\bar{\bar{h}}_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-2|\gamma|})| \\
&\qquad \le \int h_1(x_{\pi_\gamma(1)},\dots,x_{\pi_\gamma(n-|\gamma|)},x_{\pi_\gamma(n+m-2|\gamma|+1)},\dots,x_{\pi_\gamma(n+m-|\gamma|+1)}) \\
&\qquad\quad h_2(x_{\pi_\gamma(n-|\gamma|+1)},\dots,x_{\pi_\gamma(n+m-2|\gamma|)}, \\
&\qquad\qquad -x_{\pi_\gamma(n+m-|2\gamma|+1)},\dots,-x_{\pi_\gamma(n+m-|\gamma|)}) \\
&\qquad\qquad \prod_{k=1}^{|\gamma|}\sqrt{g_{j_{v_k},j_{v_k}}(x_{\pi_\gamma(n+m-2|\gamma|+k)})}\sqrt{g_{j'_{w_k},j'_{w_k}}(x_{\pi_\gamma(n+m-2|\gamma|+k)})} \\
&\qquad\qquad\quad \mu(\,dx_{\pi_\gamma(n+m-2|\gamma|+k)}).
\end{aligned}
$$

We get by applying the Schwarz inequality, the evenness of the measures $G_{j,j}$ and by replacing the measures of the form $g_{j,j}(x)\mu(\,dx)$ or $g_{j',j'}(x)\mu(\,dx)$ by the measures of the form $G_{j,j}(\,dx)$ and $G_{j',j'}(\,dx)$ that

$$
\begin{aligned}
&|\bar{\bar{h}}_\gamma(x(\gamma)_1,\dots,x(\gamma)_{n+m-2|\gamma|})|^2 \\
&\qquad \le \int |h_1(x_{\pi_\gamma(1)},\dots,x_{\pi_\gamma(n-|\gamma|)},x_{\pi_\gamma(n+m-2|\gamma|+1)},\dots,x_{\pi_\gamma(n+m-|\gamma|+1)})|^2 \\
&\qquad\qquad \prod_{k=1}^{|\gamma|} G_{j_{v_k},j_{v_k}}(\,dx_{\pi_\gamma(n+m-2|\gamma|+k)}) \\
&\qquad\quad \int |h_2(x_{\pi_\gamma(n-|\gamma|+1)},\dots,x_{\pi_\gamma(n+m-2|\gamma|)}, \\
&\qquad\qquad\qquad -x_{\pi_\gamma(n+m-|2\gamma|+1)},\dots,-x_{\pi_\gamma(n+m-|\gamma|)})|^2 \\
&\qquad\qquad\quad \prod_{k=1}^{|\gamma|} G_{j'_{w_k},w_k}(\,dx_{\pi_\gamma(n+m-2|\gamma|+k)}).
\end{aligned}
$$

Let us integrate the last inequality with respect to the product measure

$$
\begin{aligned}
&\prod_{k=1}^{n-|\gamma|} G_{j_{r_k},j_{r_k}}(\,dx(\gamma)_k)\prod_{l=1}^{m-|\gamma|} G_{j'_{t_l},j'_{t_l}}(\,dx(\gamma)_{n-|\gamma|+l}) \\
&\qquad =\prod_{k=1}^{n-|\gamma|} G_{j_{r_k},j_{r_k}}(\,dx_{\pi_\gamma(k)})\prod_{l=1}^{m-|\gamma|} G_{j'_{t_l},j'_{t_l}}(\,dx_{\pi_\gamma(n-|\gamma|+l)}).
\end{aligned}
$$

A careful analysis shows that the inequality we get in such a way agrees with the inequality formulated in Part A of Theorem 6.1. Indeed, we get at the left-hand side of this inequality $\|\bar{\bar{h}}_\gamma\|$ with the norm formulated in Part A of Theorem 6.1, and the right-hand side equals the product $\|h_1\|\|h_2\|$. We got the same integrals as the integrals defining these norms, only we integrate by the variables of the functions $h_1$ and $h_2$ in a different order. We also have to exploit that the measures $G_{j,j}$ are symmetric, hence the value of the integrals we are investigating does not change if we replace the coordinate $x_k$ by $-x_k$ in the kernel function for certain coordinates $k$.

Next I turn to the proof of Part B of Theorem 6.1. First we prove this result, i.e. identity (6.16) in the special case when both $h_1$ and $h_2$ are simple functions. We may also assume that they are adapted to the same regular system

$$\mathcal{D} = \{\Delta_p,\ p = \pm 1, \pm 2, \dots, \pm N\},$$

and by a possible further division of the sets $\Delta_p$ we may also assume that the elements of $\mathcal{D}$ are very small. More explicitly, first we choose such a measure $\mu$ on $\mathbb{R}^\nu$ which has finite value on all compact sets, all complex measures $G_{k,l}$, $1 \le k, l \le d$ are absolutely continuous with respect to $\mu$, and their Radon–Nikodym derivatives satisfy the inequality $|\frac{dG_{k,l}}{d\mu}(x)| \le 1$ for all $x \in \mathbb{R}^\nu$. Fix a small number $\varepsilon > 0$. We may achieve, by splitting up the sets $\Delta_p$ into smaller sets if it is necessary, that $\mu(\Delta_p) \le \varepsilon$ for all $\Delta_p \in \mathcal{D}$. Let us fix a number $u_p \in \Delta_p$ in all sets $\Delta_p \in \mathcal{D}$. We can express the product $I_n(h_1|j_1, \dots, j_n) I_m(h_2|j'_1, \dots, j'_m)$ as

$$I = I_n(h_1|j_1, \dots, j_n) I_m(h_2|j'_1, \dots, j'_m) = \sum{}' h_1(u_{p_1}, \dots, u_{p_n}) h_2(u_{q_1}, \dots, u_{q_m})$$
$$Z_{G,j_1}(\Delta_{p_1}) \cdots Z_{G,j_n}(\Delta_{p_n}) Z_{G,j'_1}(\Delta_{q_1}) \cdots Z_{G,j'_m}(\Delta_{q_m}).$$

The summation in the sum $\sum'$ goes through all pairs $((p_1, \dots, p_n), (q_1, \dots, q_m))$ such that $p_k,\ q_l \in \{\pm 1, \dots, \pm N\}$, $k = 1, \dots, n$, $l = 1, \dots, m$, and $p_k \ne \pm p_{\bar{k}}$, if $k \ne \bar{k}$, and $q_l \ne \pm q_{\bar{l}}$ if $l \ne \bar{l}$.

Write

$$I = \sum_{\gamma \in \Gamma} \sum{}^{\gamma} h_1(u_{p_1}, \dots, u_{p_n}) h_2(u_{q_1}, \dots, u_{q_m})$$
$$Z_{G,j_1}(\Delta_{p_1}) \cdots Z_{G,j_n}(\Delta_{p_n}) Z_{G,j'_1}(\Delta_{q_1}) \cdots Z_{G,j'_n}(\Delta_{q_m}).$$

where $\sum^\gamma$ contains those terms of $\sum'$ for which $p_k = q_l$ or $p_k = -q_l$ if the vertices $(1,k)$ and $(2,l)$ are connected in $\gamma$, and $p_k \ne \pm q_l$ if $(1,k)$ and $(2,l)$ are not connected in $\gamma$.

Let us introduce the notation

$$\Sigma^\gamma = \sum{}^{\gamma} h_1(u_{p_1}, \dots, u_{p_n}) h_2(u_{q_1}, \dots, u_{q_m})$$
$$Z_{G,j_1}(\Delta_{p_1}) \cdots Z_{G,j_n}(\Delta_{p_n}) Z_{G,j'_1}(\Delta_{q_1}) \cdots Z_{G,j'_n}(\Delta_{q_m}).$$

for all $\gamma \in \Gamma$.

We want to show that for small $\varepsilon > 0$ (where $\varepsilon$ is an upper bound for the measure $\mu$ of the sets $D_p \in \mathcal{D}$) the expression $\Sigma^\gamma$ is very close to

$$I_\gamma = I_{n+m-2|\gamma|}(\bar{\bar{h}}_\gamma | j_{v_1}, \dots, j_{v_{(n-|\gamma|}}, j'_{w_1}, \dots, j'_{w_{m-|\gamma|}}) \tag{6.19}$$

for all $\gamma \in \Gamma$. For this goal we make the decomposition $\Sigma^\gamma = \Sigma_1^\gamma + \Sigma_2^\gamma$ of $\Sigma^\gamma$

with

$$\Sigma_1^\gamma = \sum\nolimits^\gamma h_1(u_{p_1},\dots,u_{p_n})h_2(u_{q_1},\dots,u_{q_m}) \prod_{k\in A_1} Z_{G,j_k}(\Delta_{p_k}) \prod_{l\in A_2} Z_{G,j'_l}(\Delta_{q_l}) \cdot \prod_{(k,l)\in B} E\left(Z_{G,j_k}(\Delta_{p_k})Z_{G,j'_l}(\Delta_{q_l})\right)$$

and

$$\Sigma_2^\gamma = \Sigma^\gamma - \Sigma_1^\gamma,$$

where the sets $A_1$, $A_2$ and $B$ were defined in formulas (6.6), (6.7) and (6.8).

It is not difficult to check that both $\Sigma_1^\gamma$ and $\Sigma_2^\gamma$ are real valued random variables. We want to show that $\Sigma_1^\gamma$ is close to the random variable $I_\gamma$ introduced in (6.19) while $\Sigma_2^\gamma$ is a small error term. To understand the behaviour of $\Sigma_1^\gamma$ observe that

$$E(Z_{G,j_k}(\Delta_{p_k})Z_{G,j'_l}(\Delta_{q_l}) = E(Z_{G,j_k}(\Delta_{p_k})\overline{Z_{G,j'_l}(-\Delta_{q_l})} = 0$$

if $\Delta_{p_k} = \Delta_{q_l}$ (and as a consequence if $\Delta_{p_k}\cap(-\Delta_{q_l})=\emptyset$), and

$$E(Z_{G,j_k}(\Delta_{p_k})Z_{G,j'_l}(\Delta_{q_l}) = E(Z_{G,j_k}(\Delta_{p_k})\overline{Z_{G,j'_l}(-\Delta_{q_l})} = G_{j_k,j'_l}(\Delta_{p_k})$$

if $\Delta_{p_k} = -\Delta_{q_l}$. In the case $(k,l)\in B$ one of these possibilities happens.

These relations make possible to rewrite $\Sigma_1^\gamma$ in a simpler form. It can be rewritten in the form of a Wiener–Itô integral of order $n+m-2|\gamma|$ with integration with respect to the random measure $\prod\limits_{k\in A_1} Z_{G,j_k}(\,dx_k) \prod\limits_{l\in A_2} Z_{G,j'_l}(\,dx_l)$. Then we can rewrite this integral, by reindexing its variables in a right way to an integral very similar to the Wiener–Itô integral (6.14) (with the same parameter $\gamma$). The difference between these two expressions is that the kernel function $h'_\gamma$ of the Wiener–Itô integral expressing $\Sigma_1^\gamma$ is slightly different from the kernel function $\bar{\bar{h}}_\gamma$ appearing in the other integral. The main difference between these two kernel functions is that there is a small set in the domain of integration where $h'_\gamma$ disappears, while $\bar{\bar{h}}_\gamma$ may not disappear. But the two Wiener–Itô integrals are very close to each other. An adaptation of the argument in the proof of Theorem 5.3 in [11] shows that

$$E(\Sigma_1^\gamma - I_\gamma)^2 \le C\varepsilon$$

with an appropriate constant $C>0$.

We also want to show that $\Sigma_2^\gamma$ is a negligibly small error term. To get a good upper bound on $E(\Sigma_2^\gamma)^2$ we write it in the form

$$E(\Sigma_2^\gamma)^2 = \sum\nolimits_2^\gamma h_1(u_{p_1},\dots,u_{p_n})h_2(u_{q_1},\dots,u_{q_m}) h_1(u_{\bar p_1},\dots,u_{\bar p_n})h_2(u_{\bar q_1},\dots,u_{\bar q_m}) \Sigma_3^\gamma(p_k,q_l,p_{\bar k},q_{\bar l},\ k,\bar k\in\{1,\dots,n\},\ l,\bar l\in\{1,\dots,m\})$$

with

$$
\begin{aligned}
&\Sigma_3^\gamma(p_k, q_l, p_{\bar k}, q_{\bar l},\ k,\bar k\in\{1,\dots,n\},\ l,\bar l\in\{1,\dots,m\})\\
&\quad = E\Biggl(\Biggl(\prod_{k\in A_1} Z_{G,j_k}(\Delta_{p_k})\prod_{l\in A_2} Z_{G,j'_l}(\Delta_{q_l})\prod_{\bar k\in A_1} Z_{G,j_{\bar k}}(\Delta_{p_{\bar k}})\prod_{\bar l\in A_2} Z_{G,j'_{\bar l}}(\Delta_{q_{\bar l}})\Biggr)\\
&\qquad\left[\prod_{(k,l)\in B} Z_{G,j_k}(\Delta_{p_k})Z_{G,j'_l}(\Delta_{q_l}) - E\prod_{(k,l)\in B} Z_{G,j_k}(\Delta_{p_k})Z_{G,j'_l}(\Delta_{q_l})\right]\\
&\qquad\left[\prod_{(\bar k,\bar l)\in B} Z_{G,j_{\bar k}}(\Delta_{p_{\bar k}})Z_{G,j'_{\bar l}}(\Delta_{q_{\bar l}}) - E\prod_{(\bar k,\bar l)\in B} Z_{G,j_{\bar k}}(\Delta_{p_{\bar k}})Z_{G,j'_{\bar l}}(\Delta_{q_{\bar l}})\right]\Biggr),
\end{aligned}
$$

where we sum up in $\sum_2^\gamma$ for such sequences of indices $p_k$, $q_l$, $p_{\bar k}$, $q_{\bar l}$, $k,\bar k\in\{1,\dots,n\}$, $l,\bar l\in\{1,\dots,m\}$, $p_k,p_{\bar k},q_l,q_{\bar l}\in\{\pm1,\dots,\pm N\}$ which satisfy the following properties. For all indices $k,l,\bar k$ and $\bar l$, $p_k=q_l$ or $p_k=-q_l$ if $(k,l)\in B$, and similarly $p_{\bar k}=q_{\bar l}$ or $p_{\bar k}=-q_{\bar l}$ if $(\bar k,\bar l)\in B$. Otherwise all numbers $\pm p_k$ and $\pm q_l$ are different, and similarly otherwise all $\pm p_{\bar k}$ and $\pm q_{\bar l}$ are different.

We get a good estimate on $E(\Sigma_2^\gamma)^2$ by giving a good bound on all terms

$$
\Sigma_3^\gamma(p_k, q_l, p_{\bar k}, q_{\bar l},\ k,\bar k\in\{1,\dots,n\},\ l,\bar l\in\{1,\dots,m\}) \tag{6.20}
$$

in the formula expressing it. This can be done by adapting the corresponding argument in Theorem 5.3 of [11]. This argument shows that for most sets of parameters $p_k,q_k,p_{\bar k},q_{\bar l}$ the term in (6.20) equals zero. More explicitly, it is equal to zero if $\mathcal{A}\neq-\bar{\mathcal{A}}$ with

$$
\mathcal{A}=\{p_k\colon\ k\in A_1\}\cup\{q_l\colon\ l\in A_2\},\quad\text{and}\quad \bar{\mathcal{A}}=\{p_{\bar k}\colon\ \bar k\in A_1\}\cup\{q_{\bar l}\colon\ \bar l\in A_2\},
$$

and it also equals zero if $\mathcal{F}\cup(-\mathcal{F})$ and $\bar{\mathcal{F}}\cup(-\bar{\mathcal{F}})$ are disjoint, where

$$
\mathcal{F}=\bigcup_{(k,l)\in B}\{p_k,q_l\}\quad\text{and}\quad \bar{\mathcal{F}}=\bigcup_{(\bar k,\bar l)\in B}\{p_{\bar k},q_{\bar l}\}.
$$

These statements can be proved by adapting the corresponding argument in Theorem 5.3 of [11]. More precisely, in the proof of the first statement we still need the following additional observation. If $(X,Y,Z)$ is a three-dimensional Gaussian vector with $EX=EY=EZ=0$, then $EXYZ=0$. (In the proof of Theorem 5.3 in [11] we needed this statement only in a special case when it trivially holds.)

To prove this statement let us apply the following orthogonalization for the random variables $X$, $Y$ and $Z$. Write $Y=\alpha X+\eta$, $Z=\beta_1X+\beta_2\eta+\zeta$, where $X,\eta,\zeta$ are orthogonal, (jointly) Gaussian random variables with expectation zero. Then they are also independent, hence $EXYZ=EX(\alpha X+\eta)(\beta_1X+\beta_2\eta+\zeta)=0$.

In the remaining cases the expression in (6.20) can be estimated (again by adapting the argument of Theorem 5.3 in [11]) in the following way.

$$\begin{aligned}&\Sigma_3^\gamma(p_k, q_l, p_{\bar k}, q_{\bar l},\ k,\bar k\in\{1,\dots,n\},\ l,\bar l\in\{1,\dots,m\})\\&\qquad \le C\varepsilon {\prod}' \mu(\Delta_{p_k})\mu(\Delta_{l_q})\mu(\Delta_{p_{\bar k}})\mu(\Delta_{q_{\bar l}})\end{aligned}$$

with some constant $C$ (not depending on $\varepsilon$) and the measure $\mu$ dominating the complex measures $G_{j,k}$ with the properties we demanded at the start of the proof. The sign $'$ in the product ${\prod}'$ means that first we take the sets $\Delta_{p_k}$, $\Delta_{q_l}$, $\Delta_{p_{\bar k}}$, $\Delta_{q_{\bar l}}$ for all parameters $k,\bar k\in\{1,\dots,n\}$ and $l,\bar l\in\{1,\dots,m\}$, then if a set $\Delta$ appears twice in the sequence of these sets we omit one of them. Then if both the sets $\Delta$ and $-\Delta$ appear for some set $\Delta$, then we omit one of them from this sequence. Then we take in ${\prod}'$ the product of the terms $\mu(\Delta)$ with the sets $\Delta$ in the remaining sequence.

It can be proved with the help of the estimates on the terms in (6.20) (see again Theorem 5.3 in [11]) that

$$E(\Sigma_2^\gamma)^2 \le C\varepsilon.$$

It is not difficult to prove part B of Theorem 6.1 with the help of the estimates on $E(\Sigma_1^\gamma - I_\gamma)^2 \le C\varepsilon$ and $E(\Sigma_2^\gamma)^2 \le C\varepsilon$ if $h_1$ and $h_2$ are simple functions. One only has to make an appropriate limiting procedure with $\varepsilon\to 0$. Then we can complete the proof of Theorem 6.1 similarly to the proof of Theorem 5.3 in [11] by means of an appropriate approximation of Wiener–Itô integrals with Wiener–Itô integrals of simple functions. In this approximation we have to apply Lemma 5.1 and the properties of the Wiener–Itô integrals, in particular the already proved Part A of Theorem 6.1.

# 7 Wick polynomials and their relation to Wiener–Itô integrals

In the case of scalar valued stationary Gaussian random fields (i.e. if $d = 1$) there is a so-called Itô formula (see Theorem 4.3 in [11]) which shows an important relation between Wiener–Itô integrals and Hermite polynomials. Here I shall present its multivariate version, where Wick polynomials take the role of the Hermite polynomials. Wick polynomials are the natural multi-dimensional generalizations of Hermite polynomials. I shall also discuss an important consequence of the multivariate version of the Itô formula. It enables us to present the shift transforms of a random variable given in the form of a sum of multiple Wiener–Itô integrals in such a way that helps us in the study of limit theorems for non-linear functionals of a vector valued stationary Gaussian field.

First I recall the definition of Wick polynomials and some results about their most important properties. Here I follow the discussion in Section 2 of [11].

Let $X_t$, $t\in T$, be a set of jointly Gaussian random variables indexed by a parameter set $T$, and such that $EX_t = 0$ for all $t\in T$. We define the following real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}$. A square integrable (real valued) random variable is in $\mathcal{H}$ if and only if it is measurable with respect to the $\sigma$-algebra $\mathcal{B} = \mathcal{B}(X_t,\ t \in T)$, and the scalar product in $\mathcal{H}$ is defined as $\langle \xi, \eta \rangle = E\xi\eta$, $\xi, \eta \in \mathcal{H}$. The Hilbert space $\mathcal{H}_1 \subset \mathcal{H}$ is the subspace of $\mathcal{H}$ generated by the finite linear combinations $\sum c_j X_{t_j}$, $t_j \in T$, with real coefficients. We consider only such sets of Gaussian random variables $X_t$ for which $\mathcal{H}_1$ is separable. Otherwise $X_t$, $t \in T$, can be arbitrary, but the most interesting case for us is when $T = \mathbb{Z}^\nu \times \{1, \dots, d\}$, and the original Gaussian random variables we are working with are the coordinates $X_j(p)$, $j \in \{1, \dots, d\}$, $p \in \mathbb{Z}^\nu$ of a vector-valued Gaussian stationary field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$.

To define the Wick polynomials and to get their most important properties we need the following result formulated in Theorem 2.1 of [11].

**Theorem 7A.** *Let $Y_1, Y_2, \dots$ be an orthonormal basis in the Hilbert space $\mathcal{H}_1$ defined above with the help of a set of Gaussian random variables $X_t$, $t \in T$. Then the set of all possible finite products $H_{j_1}(Y_{l_1}) \cdots H_{j_k}(Y_{l_k})$ is a complete orthogonal system in the Hilbert space $\mathcal{H}$ defined above. (Here, and in the subsequent discussion $H_j(\cdot)$ denotes the $j$-th Hermite polynomial with leading coefficient 1.)*

Let $\mathcal{H}_{\le n} \subset \mathcal{H}$, $n = 1, 2, \dots$, (with the previously introduced Hilbert space $\mathcal{H}$) denote the linear subspace of the Hilbert space $\mathcal{H}$ which is the closure of the linear space consisting of the elements $P_n(X_{t_1}, \dots, X_{t_m})$, where $P_n$ runs through all polynomials of degree less than or equal to $n$, and the integer $m$ and indices $t_1, \dots, t_m \in T$ are arbitrary. Let $\mathcal{H}_0 = \mathcal{H}_{\le 0}$ consist of the constant functions, and let $\mathcal{H}_n = \mathcal{H}_{\le n} \ominus \mathcal{H}_{\le n-1}$, $n = 1, 2, \dots$, where $\ominus$ denotes orthogonal completion. It is clear that the Hilbert space $\mathcal{H}_1$ given in this definition agrees with the previously defined Hilbert space $\mathcal{H}_1$. If $\xi_1, \dots, \xi_m \in \mathcal{H}_1$, and $P_n(x_1, \dots, x_m)$ is a polynomial of degree $n$, then $P_n(\xi_1, \dots, \xi_m) \in \mathcal{H}_{\le n}$. Then Theorem 7A implies that

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \cdots , \tag{7.1}$$

where $+$ denotes direct sum. Now I present the definition of Wick polynomials.

**Definition of Wick polynomials.** *Let $P(x_1, \dots, x_m)$ be a homogeneous polynomial of degree $n$, and let a set of (jointly Gaussian) random variables $\xi_1, \dots, \xi_m \in \mathcal{H}_1$ be given. The Wick polynomial $:P(\xi_1, \dots, \xi_m):$ determined by them is defined as the orthogonal projection of the random variable $P(\xi_1, \dots, \xi_m)$ to the above defined subspace $\mathcal{H}_n$ of the Hilbert space $\mathcal{H}$.*

In the sequel we shall use the notation $:P(\xi_1, \dots, \xi_m):$ for the Wick polynomial corresponding to a homogeneous polynomial $P(x_1, \dots, x_m)$ with arguments $\xi_1, \dots, \xi_m$, $\xi_j \in \mathcal{H}_1$ for all $1 \le j \le m$. It may happen that a random variable $\zeta$ can be expressed in two different forms as a homogeneous polynomial of some random variables from $\mathcal{H}_1$, i.e. $\zeta = P_1(\xi_1, \dots, \xi_m)$, and $\zeta = P_2(\xi_1, \dots, \xi_m)$, and $P_1 \ne P_2$. But in such a case $:P_1(\xi_1, \dots, \xi_m): \ = :P(\xi_1, \dots, \xi_m):$ , i.e. the value of a Wick polynomial does not depend on its representation.

It is clear that Wick polynomials of different degree are orthogonal. Given some $\xi_1, \dots, \xi_m \in \mathcal{H}_1$ define the subspaces $\mathcal{H}_{\le n}(\xi_1, \dots, \xi_m) \subset \mathcal{H}_{\le n}$, $n =$

$1, 2, \dots,$ as the set of all polynomials of the random variables $\xi_1, \dots, \xi_m$ with degree less than or equal to $n$. Let $\mathcal{H}_{\le 0}(\xi_1, \dots, \xi_m) = \mathcal{H}_0(\xi_1, \dots, \xi_m) = \mathcal{H}_0$, and $\mathcal{H}_n(\xi_1, \dots, \xi_m) = \mathcal{H}_{\le n}(\xi_1, \dots, \xi_m) \ominus \mathcal{H}_{\le n-1}(\xi_1, \dots, \xi_m)$. With the help of this notation I formulate the following result given in Proposition 2.2 of [11].

**Theorem 7B.** *Let $P(x_1, \dots, x_m)$ be a homogeneous polynomial of degree $n$. Then $:P(\xi_1, \dots, \xi_m):$ equals the orthogonal projection of $P(\xi_1, \dots, \xi_m)$ to $\mathcal{H}_n(\xi_1, \dots, \xi_m)$.*

This result has the following important consequences formulated in Corollaries 2.3 and 2.4 in [11].

**Corollary 7C.** *Let $\xi_1, \dots, \xi_m$ be an orthonormal system in $\mathcal{H}_1$, and let*

$$P(x_1, \dots, x_m) = \sum c_{j_1, \dots, j_m} x^{j_1} \cdots x_m^{j_m}$$

*be a homogeneous polynomial, i.e. let $j_1 + \dots + j_m = n$ with some fixed number $n$ for all sets $(j_1, \dots, j_m)$ appearing in this summation. Then*

$$:P(\xi_1, \dots, \xi_m): \; = \sum c_{j_1, \dots, j_m} H_{j_1}(\xi_1) \cdots H_{j_m}(\xi_m).$$

*In particular,*

$$:\xi^n: \; = H_n(\xi) \quad \text{if } \xi \in \mathcal{H}_1, \text{ and } E\xi^2 = 1.$$

**Corollary 7D.** *Let $\xi_1, \xi_2, \dots$ be an orthonormal basis in $\mathcal{H}_1$. Then the random variables $H_{j_1}(\xi_1) \cdots H_{j_k}(\xi_k)$, $k = 1, 2, \dots$, $j_1 + \dots + j_k = n$, form a complete orthogonal basis in $\mathcal{H}_n$.*

In the proof of the Itô formula for scalar valued stationary random fields we needed, besides the diagram formula, the following important recursive formula for Hermite polynomials which is contained e.g. in Lemma 5.2 of [11].

$$H_n(x) = xH_{n-1}(x) - (n-1)H_{n-2}(x) \quad \text{for } \; n = 1, 2, \dots, \tag{7.2}$$

with the notation $H_{-1}(x) \equiv 0$ in the case $n = 1$.

In the next result I formulate a multivariate version of this formula for Wick polynomials.

**Proposition 7.1.** *Let $U_1, \dots, U_{n+1}$, $n \ge 1$, be elements in $\mathcal{H}_1$. Then*

$$\begin{aligned} &:U_1 \cdots U_n: U_{n+1} \\ &\quad = :U_1 \cdots U_n U_{n+1}: \; + \sum_{s=1}^n :U_1 \cdots U_{s-1} U_{s+1} \cdots U_n U_{n+1}: EU_s U_{n+1}. \end{aligned} \tag{7.3}$$

*In the special case $n = 1$ this formula is meant as $U_1 U_2 = :U_1 U_2: + EU_1 U_2$.*

*Proof of Proposition 7.1.* Formula (7.3) clearly holds if all random variables $U_j$, $1 \le j \le n+1$ agree, and $EU_1^2 = 1$, since in this case the left-hand side of (7.3) equals $U_1H_n(U_1)$, while its right-hand side equals $H_{n+1}(U_1)+nH_{n-1}(U_1)$ by Corollary 7C, and these two expressions are equal by formula (7.2). A somewhat more complicated, but similar argument shows that this formula also holds if the sequence $U_1,\dots,U_n$ consists of some independent random variables $V_1\dots,V_k$ with standard normal distribution, the random variable $V_p$ is contained in the sequence $U_1,\dots,U_n$ with multiplicity $l_p$, $1\le p\le k$, and finally $U_{n+1}$ is either one of these random variables $V_p$, $1\le p\le k$, or it is a random variable $V_{k+1}$ with standard normal distribution which is independent of all of them.

Indeed, if $U_{n+1}=V_p$ with some $1\le p\le k$, then the left-hand side of (7.3) equals

$$H_{l_1}(V_1)\cdots H_{l_k}(V_k)V_p,$$

while the right-hand side equals

$$\begin{aligned}&H_{l_1}(V_1)\cdots H_{l_{p-1}}(V_{p-1})H_{l_p+1}(V_p)H_{l_{p+1}}(V_{p+1})\cdots H_{l_k}(V_k)\\&\qquad+l_p\,H_{l_1}(V_1)\cdots H_{l_{p-1}}(V_{p-1})H_{l_p-1}(V_p)H_{l_{p+1}}(V_{p+1})\cdots H_{l_k}(V_k)\end{aligned}$$

by Corollary 7C. A comparison of these expressions together with relation (7.2) imply that identity (7.3) holds in this case. If $U_{n+1}=V_{k+1}$, then the left-hand side of (7.3) equals

$$H_{l_1}(V_1)\cdots H_{l_k}(V_k)V_{k+1},$$

and the right-hand side also equals $H_{l_1}(V_1)\cdots H_{l_k}(V_k)V_{k+1}$. Hence formula (7.3) holds in this case, too.

In the general case we can choose some independent Gaussian random variables $Z_1,\dots,Z_m$ in $\mathcal{H}_1$ with variance 1 in such a way that our random variables $U_1,\dots,U_{n+1}$ can be expressed as their linear combination, i.e. $U_p=\sum_{l=1}^m c_{p,l}Z_l$ with some coefficients $c_{l,m}$. We have already seen that formula (7.3) is valid in the special case when all random variables $U_p$ equal one of the random variables $Z_j$, i.e. if $U_p=Z_{j(p)}$ with some $1\le j(p)\le m$ for all $1\le p\le n+1$. Since the expressions of both sides of (7.3) are multi-linear functionals on the $n$-fold direct product $\mathcal{H}_1\times\cdots\times\mathcal{H}_1$, this implies that formula (7.3) also holds for the random variables $U_1,\dots,U_{n+1}$. Proposition 7.1 is proved.

We can prove the following multivariate version of Itô's formula with the help of Proposition 7.1 and the diagram formula for multiple Wiener–Itô integrals for vector-valued Gaussian stationary fields.

**Theorem 7.2. Multivariate version of Itô's formula.** *Let us have some vector valued Gaussian stationary random field with a vector valued random spectral measure $Z_G=(Z_{G,1},\dots,Z_{G,d})$. Let us consider some functions functions $\varphi_p\in\mathcal{K}_{1,j_p}$, $1\le p\le n$, $1\le j_p\le d$, and define with their help the random variables $U_p=\int\varphi_p(x)Z_{G,j_p}(\,dx)\in\mathcal{H}_1$, $1\le p\le n$. The identity*

$$:U_1\cdots U_n:\;=\int\varphi_1(x_1)\varphi_2(x_2)\cdots\varphi_n(x_n)Z_{G,j_1}(\,dx_1)Z_{G,j_2}(\,dx_2)\cdots Z_{G,j_n}(\,dx_n)\tag{7.4}$$

*holds.*

*Proof of Theorem 7.2.* Relation (7.4) clearly holds for $n = 1$. We prove by induction that it holds for $n+1$ if it holds for $k \le n$. In the proof we apply the Corollary of Theorem 6.1 with the choice

$$\begin{aligned} h_1(x_1,\dots,x_n) &= \varphi_1(x_1)\cdots\varphi_n(x_n) \\ h_2(x) &= \varphi_{n+1}(x), \end{aligned}$$

and the random spectral measure $Z_{G,j_1'}$ is chosen as $Z_{G,j_1'} = Z_{G,j_{n+1}}$, where $Z_{G,j_{n+1}}$ is the random spectral measure appearing in the definition of $U_{n+1}$. We can write with this choice

$$\begin{aligned} &\int \varphi_1(x_1)\cdots\varphi_n(x_n) Z_{G,j_1}(\,dx_1)\cdots Z_{G,j_n}(\,dx_n) \int \varphi_{n+1}(x) Z_{G,j_{n+1}}(\,dx) \\ &\quad = \int \varphi_1(x_1)\cdots\varphi_n(x_n)\varphi_{n+1}(x_{n+1}) Z_{G,j_1}(\,dx_1)\cdots Z_{G,j_{n+1}}(\,dx_{n+1}) \\ &\qquad + \sum_{p=1}^n EU_pU_{n+1} \int \varphi_1(x_1)\cdots\varphi_{p-1}(x_{p-1})\varphi_{p+1}(x_p)\cdots\varphi_n(x_{n-1}) \\ &\qquad\quad Z_{G,j_1}(\,dx_1)\cdots Z_{G,j_{p-1}}(\,dx_{p-1}) Z_{G,j_{p+1}}(\,dx_p)\cdots Z_{G,j_n}(\,dx_{n-1}), \end{aligned}$$

since formula (6.18) gives this identity with the above choice of $h_1$ and $h_2$. To see this observe that

$$\begin{aligned} h_{\gamma_p}(x_1,\dots,x_{n-1}) &= \varphi_1(x_1)\cdots\varphi_{p-1}(x_{p-1})\varphi_{p+1}(x_p)\cdots\varphi_n(x_{n-1}) \\ &\qquad \int \varphi_p(x_n)\overline{\varphi_n(x_{n+1})} G_{j_p,j_{n+1}}(\,dx_n) \\ &= \varphi_1(x_1)\cdots\varphi_{p-1}(x_{p-1})\varphi_{p+1}(x_p)\cdots\varphi_n(x_{n-1}) EU_pU_{n+1} \end{aligned}$$

for $1 \le p \le n$, since by formula (3.7) and the relation $U_{n+1} = \overline{U}_{n+1}$

$$\int \varphi_p(x)\overline{\varphi_{n+1}(x)} G_{j_p,j_{n+1}}(\,dx) = EU_pU_{n+1},$$

and

$$h_{\gamma_0}(x_1,\dots,x_{n+1}) = \varphi_1(x_1)\cdots\varphi_n(x_n)\varphi_{n+1}(x_{n+1}).$$

This formula together with our induction hypothesis imply that

$$\begin{aligned} &\int \varphi_1(x_1)\cdots\varphi_n(x_n)\varphi_{n+1}(x_{n+1}) Z_{G,j_1}(\,dx_1)\cdots Z_{G,j_{n+1}}(\,dx_{n+1}) \\ &\qquad =: U_1\cdots U_n : U_{n+1} - \sum_{p=1}^n : U_1\cdots U_{p-1}U_{p+1}\cdots U_n : EU_pU_{n+1}. \end{aligned}$$

In the case $n = 1$ this formula means that

$$\int \varphi_1(x_1)\varphi_2(x_2) Z_{G,j_1}(\,dx_1) Z_{G,j_2}(\,dx_2) = U_1U_2 - EU_1U_2.$$

By comparing this formula with (7.3) we get that the statement of Theorem 7.2 holds also for $n+1$. Theorem 7.2 is proved.

In Theorem 7.2 we rewrote some Wick polynomials of special form as multiple Wiener–Itô integrals. This enables us to express a sum of such Wick polynomials as the sum of Wiener–Itô integrals. This implies that all Wick polynomials of random variables from some $\mathcal{H}_{1,j}$, $1 \le j \le d$, can be written in the form of a sum of Wiener–Itô integrals. In the next simple corollary of Theorem 7.2 I describe this result in a more explicit form.

To formulate this result let us introduce the following notation. Let us fix some numbers $n \ge 1$ (the order of the homogeneous polynomial we are considering), $m \ge 1$ and some functions $\varphi_{j,k}(x) \in \mathcal{K}_{1,j}$, $1 \le j \le d$, $1 \le k \le m$, and define the random variables

$$\xi_{j,k} = \int \varphi_{j,k}(x) Z_{G,j}(\,dx), \quad 1 \le j \le d,\ 1 \le k \le m.$$

Then $\xi_{j,k} \in \mathcal{H}_{1,j}$. (We defined the real Hilbert space $\mathcal{H}_{1,j}$ in the formulation of Lemma 3.2. Lemma 3.2 stated that the elements of $\mathcal{H}_{1,j}$ can be given in the form of the above integral.)

In the next corollary we consider homogeneous polynomials of these random variables $\xi_{j,k}$, and express the Wick polynomials corresponding to them in the form of a sum of multiple Wiener–Itô integrals.

**Corollary of Theorem 7.2.** *Let us consider a homogeneous polynomial*

$$\begin{aligned}&P(x_{j_s,k_s},\ 1 \le j_s \le d,\ 1 \le k_s \le m \textit{ for all } 1 \le s \le n)\\ &\qquad = \sum_{\substack{1\le j_s\le d \textit{ for all } 1\le s\le n\\ 1\le k_s\le m \textit{ for all } 1\le s\le n}} a_{j_1,k_1,\dots j_n,k_n} x_{j_1,k_1} x_{j_2,k_2} \cdots x_{j_n,k_n}\end{aligned}$$

*of order n of the variables $x_{j_s,k_s}$ with indices $1 \le j_s \le d$ and $1 \le k_s \le m$ for all $1 \le s \le n$ and real coefficients $a_{j_1,k_1,\dots,j_n,k_n}$.*

*If we replace the variables $x_{j_s,k-s}$ by the random variables*

$$\xi_{j_s,k_s} = \int \varphi_{j_s,k_s}(x) Z_{G,j_s}(\,dx)$$

*in this polynomial (we assumed that $\varphi_{j,k} \in \mathcal{K}_{1,j}$), then we get the following homogeneous polynomial of some jointly Gaussian random variables.*

$$\begin{aligned}&P(\xi_{j_s,k_s},\ 1 \le j_s \le d,\ 1 \le k_s \le m \textit{ for all } 1 \le s \le n)\\ &\qquad = \sum_{\substack{1\le j_s\le d \textit{ for all } 1\le s\le n\\ 1\le k_s\le m \textit{ for all } 1\le s\le n}} a_{j_1,k_1,\dots j_n,k_n} \xi_{j_1,k_1} \xi_{j_2,k_2} \cdots \xi_{j_n,k_n}.\end{aligned}$$

*With the help of this expression we can define the Wick polynomial*

$$:P(\xi_{j_s,k_s},\ 1 \le j_s \le d,\ 1 \le k_s \le m \textit{ for all } 1 \le s \le n):.$$

*This Wick polynomial can be expressed as a sum of Wiener–Itô integrals in the following way.*

*Let us consider for all sequences of indices $\{(j_s,k_s),:\ 1\le s\le n\}$ with $1\le j_s\le d$, $1\le k_s\le d$ for all $1\le s\le n$ the function*

$$f_{j_1,k_1,\dots,j_n,k_n}(x_1,\dots,x_n)=\varphi_{j_1,k_1}(x_1)\cdots\varphi_{j_n,k_n}(x_n)\in\mathcal{K}_{n,j_1,\dots,j_n}$$

*and the Wiener–Itô integral*

$$\begin{aligned}&I_n(f_{j_1,k_1,\dots,j_n,k_n}|j_1,\dots,j_n)\\&\qquad=\int f_{j_1,k_1,\dots,j_n,k_n}(x_1,\dots,x_n)Z_{G,j_1}(\,dx_1)\dots Z_{G,j_n}(\,dx_n).\end{aligned}$$

*The identity*

$$\begin{aligned}&:P(\xi_{j_s,k_s},\ 1\le j_s\le d,\ 1\le k_s\le m \textit{ for all } 1\le s\le n):\\&\quad=\sum_{\substack{1\le j_s\le d \textit{ for all } 1\le s\le n\\1\le k_s\le m \textit{ for all } 1\le s\le n}}a_{j_1,k_1,\dots j_n,k_n}I_n(f_{j_1,k_1,\dots,j_n,k_n}|j_1,\dots,j_n)\end{aligned}$$

*holds.*

*Remark.* Theorem 4.7 of [11] contains a version of this result for scalar valued fields.

*Proof of the Corollary of Theorem 7.2.* By Theorem 7.2 we have

$$a_{j_1,k_1,\dots j_n,k_n}:\xi_{j_1,k_1}\xi_{j_2,k_2}\cdots\xi_{j_n,k_n}:\ =a_{j_1,k_1,\dots j_n,k_n}I_n(f_{j_1,k_1,\dots,j_n,k_n}|j_1,\dots,j_n)$$

for all sequences of indices $(j_s,k_s)$, $1\le s\le n$. By summing up these inequalities for all sequences of indices we get the proof of the corollary.

With the help of the above corollary we prove the following result.

**Proposition 7.3.** *For all $n\ge1$ and functions $f\in\mathcal{K}_{n,j_1,\dots,j_n}$ with some indices $1\le j_s\le d$, $1\le s\le n$, $I_n(f|j_1,\dots,j_n)\in\mathcal{H}_n$ for the $n$-fold Wiener–Itô integral $I_n(f|j_1,\dots,j_n)$. Besides, the set of all sums of $n$-fold Wiener–Itô integrals i.e. the set of all sums of the form*

$$\sum_{1\le j_s\le d \textit{ for all } 1\le s\le n}I_n(f_{j_1,\dots,j_n}|j_1,\dots,j_n),$$

*where $f_{j_1,\dots,j_n}\in\mathcal{K}_{n,j_1,\dots,j_n}$ constitute an everywhere dense linear subspace of $\mathcal{H}_n$.*

*Proof of Proposition 7.3.* We shall prove Proposition 7.3 by induction with respect to $n$. By Lemma 3.2 Proposition 7.3 holds for $n=1$. Indeed, by this result every random variable of the form $\xi=\sum_{j=1}^d\xi_j$ with some $\xi_j\in\mathcal{H}_{1,j}$ can be written as the sum of one-fold Wiener–Itô integrals, and the random variables of this form constitute an everywhere dense linear subspace of $\mathcal{H}_1$.

If the statements of Proposition 7.3 hold for all $m < n$, then we can say for one part that $I_n(f|j_1, \dots, j_n) \in \mathcal{H}_{\le n}$, because this relation holds if $f$ is a simple function, i.e. if $f \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$, and since $\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ is dense in $\mathcal{K}_{n,j_1,\dots,j_n}$, and we defined the Wiener–Itô integral by the extension of a bounded operator in the general case, the above property remains valid for general functions $f \in \mathcal{K}_{n,j_1,\dots,j_n}$. Moreover, we know that $I_n(f|j_1, \dots, j_m)$ is orthogonal to any Wiener–Itô integral of the form $I_m(h|j'_1, \dots, j'_m)$ with $m < n$ because of relation (5.5). Then $I_n(f|j_1, \dots, j_n)$ is also orthogonal to any linear combination of such integrals. But these linear combinations constitute an everywhere dense set in $\mathcal{H}_m$ by our inductive hypothesis. Hence $I_n(f|j_1, \dots, j_n)$ is orthogonal to the whole space $\mathcal{H}_m$ for all $0 \le m \le n-1$, and this implies that it is contained in the Hilbert subspace $\mathcal{H}_n$. It follows from the corollary of Theorem 7.2 that the sums of Wiener–Itô integrals considered in Proposition 7.3 are dense in $\mathcal{H}_n$, and they constitute a linear subspace. Indeed, this corollary implies that all Wick polynomials of order $n$ can be expressed as a sum of such integrals, and the Wick polynomials of order $n$ are dense in $\mathcal{H}_n$. Proposition 7.3 is proved.

*Remark.* In Proposition 7.3 we expressed a dense subset of $\mathcal{H}_n$ as a sum of $n$-fold Wiener–Itô integrals, but we did not express all elements of $\mathcal{H}_n$ in such a form. But even this weaker result suffices for our purposes.

In the case of scalar valued stationary random fields the situation is different. In that case we can express all elements of $\mathcal{H}_n$ as an $n$-fold Wiener–Itô integral, and actually we can say somewhat more. There is a so-called Fock space representation of all elements $h \in \mathcal{H}$, which represents the elements $h \in \mathcal{H}$ in the form of a sum of multiple Wiener–Itô integrals of different multiplicity. Moreover, this result has some useful consequences about the properties of this representation.

We cannot prove a similar result in the vector valued case. The main problem is that while in the case of scalar valued models if have a sequence of random variables $h_N \in \mathcal{H}_n$, $N = 0, 1, 2, \dots$, such that $h_N \to h_0$ with some $h_0 \in \mathcal{H}_n$ in the norm of $\mathcal{H}_n$ as $N \to \infty$, then these random variables can be expressed as the $n$-fold Wiener–Itô integrals of some functions $k_N \in \mathcal{K}_n$ for which $k_N \to k_0$ in the norm of $\mathcal{K}_n$, in the case of vector valued models we do not have a similar result.

Next we consider a stationary Gaussian field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, whose elements can be written in the form $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\,dx)$ by means of the random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ of this random field for all $p \in \mathbb{Z}^\nu$ and $1 \le j \le d$. Let us consider a random variable $Y \in \mathcal{H}_n$ which can be represented as the $n$-fold Wiener–Itô integral of some function $h \in \mathcal{K}_{n,j_1,\dots,j_n}$, i.e.

$$Y = \int h(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(\,dx_n). \tag{7.5}$$

I shall express the shift transforms $T_u Y$, $u \in \mathbb{Z}^\nu$, of $Y$ given in formula (7.5) in a form which can be considered as a Fourier type random integral.

To do this first I recall the definition of shift transforms $T_u$, $u \in \mathbb{Z}^\nu$, in a stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$.

Given some element $X_j(m)$, $m \in \mathbb{Z}^\nu$, $1 \le j \le d$, of the random field, and $u \in \mathbb{Z}^\nu$, we define the shift transform of $X_j(m)$ by $T_u$ as $T_u X_j(m) = X_j(u+m)$. More generally, given any measurable function $h(X_j(m), m \in \mathbb{Z}^\nu, 1 \le j \le d)$, we define the shift transform of the random variable $Y = h(X_j(m), m \in \mathbb{Z}^\nu, 1 \le j \le d)$, by the formula $T_u Y = h(X_j(m+u), m \in \mathbb{Z}^\nu, 1 \le j \le d)$. This transformation was discussed in the scalar valued case in [11]. It can be seen, (similarly to the argument in that work) that the definition of this transformation is meaningful, (i.e. the value of $T_u Y$ does not depend on the choice of the function $h$ for which $Y = h(X_j(m), m \in \mathbb{Z}^\nu, 1 \le j \le d)$), and we have defined in such a way unitary (linear) transformations $T_u$, $u \in \mathbb{Z}^\nu$, on $\mathcal{H}$ for which $T_u T_v = T_{u+v}$.

In Lemma 3.2 I have shown that each random variable $U_j \in \mathcal{H}_{1,j}$ can be written in the form $U_j = \int h(x) Z_{G,j}(\,dx)$ with some function $h(x) \in \mathcal{K}_{1,j}$. On the other hand, I claim that for all $u \in \mathbb{Z}^\nu$

$$T_u U_j = \int e^{i(u,x)} h(x) Z_{G,j}(\,dx) \quad \text{if} \quad U_j = \int h(x) Z_{G,j}(\,dx) \tag{7.6}$$

with some $h \in \mathcal{K}_{1,j}$. Indeed, relation (7.6) clearly holds if $h(x) = e^{i(p,x)}$ with some $p \in \mathbb{Z}^\nu$, since in this case $U_j = X_j(p)$ and $T_u U_j = X_j(p+u)$. But this implies that relation (7.6) holds for all finite trigonometrical polynomials of the form $h(x) = \sum c_k e^{i(p_k,x)}$, and for the closure of these functions with respect to the $L_2$ norm determined by the measure $G_{j,j}$, i.e. for all $h \in \mathcal{K}_{1,j}$.

In Proposition 7.4 I present a similar formula about the shift transform of a random variable $Y$ given by formula (7.5). This result is useful in the study of limit theorems related to non-linear functionals of a stationary Gaussian field.

**Proposition 7.4 about the representation of shift transformations.** *Let a vector valued stationary Gaussian random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, be given with a vector valued random spectral measure*

$$Z_G = (Z_{G,1}, \dots, Z_{G,d}) \text{ such that } X_j(p) = \int e^{i(p,x)} Z_{G,j}(\,dx)$$

*for all $p \in \mathbb{Z}^\nu$ and $1 \le j \le d$. Let $Y \in \mathcal{H}_n$ be the random variable defined in formula (7.5) with the help of this vector valued random spectral measure $Z_G$ and some function $h \in \mathcal{K}_{n,j_1,\dots,j_n}$. Then*

$$T_u Y = \int e^{i(u,x_1+\cdots+x_n)} h(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(\,dx_n) \tag{7.7}$$

*for all $u \in \mathbb{Z}^\nu$.*

*Proof of Proposition 7.4.* Formula (7.7) holds in the special case if $n = 1$, and $h(x) \in \mathcal{K}_{1,j}$, since in this case $Y = \int h(x) Z_{G,j}(\,dx)$, and

$$T_u Y = \int e^{i(u,x)} h(x) Z_{G,j}(\,dx)$$

by formula (7.6).

I claim that formula (7.7) also holds in the case when the random variable $Y$ is given by formula (7.5) with a kernel function of the form $h(x_1,\dots,x_n)=\varphi_1(x_1)\cdots\varphi_n(x_n)$ defined with the help of some functions $\varphi_s(x)\in\mathcal{K}_{1,j_s}$, $1\le s\le n$. Indeed, in this case $Y=:U_1\cdots U_n:$ with $U_s=\int\varphi_s(x)Z_{G,j_s}(\,dx)$, $1\le s\le n$, because of Theorem 7.2. On the other hand, I claim that

$$T_u\!:U_1\cdots U_n:\;=:(T_uU_1)\cdots(T_uU_n):.$$

To see this let us observe that by Theorem 7B $:U_1\cdots U_n:$ is the orthogonal projection of the product $U_1\cdots U_n$ to the Hilbert subspace $\mathcal{H}_n(U_1,\dots,U_n)$. Similarly, $:(T_uU_1)\cdots(T_uU_n):$ is the orthogonal projection of $(T_uU_1)\cdots(T_uU_n)$ to the Hilbert subspace $\mathcal{H}_n(T_uU_1,\dots,T_uU_n)$. Since the vectors $(U_1,\dots,U_n)$ and $(T_uU_1,\dots,T_uU_n)$ have the same distribution, and the Wick polynomial corresponding to their product can be calculated in the same way this implies that if $:U_1\cdots U_n:\;=g(U_1,\dots,U_n)$ with some function $g$, then

$$:(T_uU_1)\cdots(T_uU_n):\;=g(T_uU_1,\dots,T_uU_n)$$

with the same function $g$. (In the present case $g(x_1,\dots,x_n)$ is a polynomial of order $n$.) On the other hand, $T_u\!:U_1\cdots U_n:\;=T_ug(U_1,\dots,U_n)=g(T_uU_1,\dots,T_uU_n)$ in this case. The above argument implies the desired identity.

Thus we can state that if $Y$ is defined by formula (7.5) with a function

$$h(x_1,\dots,x_n)=\varphi_1(x_1)\cdots\varphi_n(x_n)$$

with the above properties, then

$$\begin{aligned}T_uY&=\;:(T_uU_1)\cdots(T_uU_n):\\&=\int e^{i(u,x_1+\cdots+x_n)}h(x_1,\dots,x_n)Z_{G,j_1}(\,dx_1)\dots Z_{G,j_n}(\,dx_n)\end{aligned}$$

because of Theorem 7.2 and the relation $T_uU_s=\int e^{i(u,x)}\varphi_s(x)Z_{G,j_s}(\,dx)$ for all indices $1\le s\le n$.

From the result in the previous case follows that relation (7.7) also holds if $Y$ is defined by (7.5) with a function $h(x_1,\dots,x_n)$ of the form of a finite sum

$$h(x_1,\dots,x_n)=\sum_k\varphi_{1,k}(x_1)\varphi_{2,k}(x_2)\cdots\varphi_{n,k}(x_n)$$

with $\varphi_{s,k}\in\mathcal{K}_{1,j_s}$, $1\le s\le n$.

Since functions of the above form are dense in $\mathcal{K}_{n,j_1,\dots,j_n}$, $T_u$ is a unitary operator, and both (linear) transformations

$$h(x_1,\dots,x_n)\to e^{i(u,x_1+\cdots+x_n)}h(x_1,\dots,x_n)$$

and $h\to I_n(h|j_1,\dots,j_n)$ are of bounded norm, it is not difficult to see that Proposition 7.4 holds in the general case. Proposition 7.4 is proved.

# 8 On the proof of limit theorems for non-linear functionals of vector valued Gaussian stationary random fields

Let $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, be a $d$-dimensional vector valued Gaussian stationary field, and let a function $H(x_1, \dots, x_d)$, $H\colon\ \mathbb{R}^d \to \mathbb{R}^1$, of $d$ variables be given. Let us define with their help the random variables $Y(p) = H(X_1(p), \dots, X_d(p))$ for all $p \in \mathbb{Z}^\nu$, and introduce for all $N = 1, 2, \dots$ the normalized random sum

$$S_N = A_N^{-1} \sum_{p \in B_N} Y(p) \tag{8.1}$$

with an appropriate norming constant $A_N > 0$, where

$$B_N = \{p = (p_1, \dots, p_\nu)\colon\ 0 \le p_k < N \text{ for all } 1 \le k \le \nu\}. \tag{8.2}$$

Let us also fix the vector valued random spectral measure $(Z_{G,1}, \dots, Z_{G,d})$ on the torus $[-\pi, \pi)^\nu$ for which $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\,dx)$, $1 \le j \le d$, $p \in \mathbb{Z}^\nu$. We are interested in the question what kind of limit theorems may hold for the normalized sums $S_N$ defined in (8.1) as $N \to \infty$ with appropriate norming constants $A_N$. Here we are interested in the case when the correlation functions $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $1 \le j, j' \le d$, tend to zero slowly as $|p| \to \infty$. This means strong dependence of the random variables in the stationary random fields. In such cases we can get limit theorems with a non-Gaussian limit.

We have studied the above problem in [6] for scalar valued stationary random fields, i.e. in the case $d = 1$, and we have proved some new kind of limit theorems. Let me remark that at the same time M. Taqqu also proved similar results with the help of a different method, see [16]. I do not discuss Taqqu's work, because here I am interested in the question how to generalize the method in [6] to prove limit theorems also for non-linear functionals of vector valued Gaussian stationary random fields.

In Theorem 6 of [1] M. A. Arcones formulated a limit theorem about non-linear functionals of vector valued Gaussian stationary fields, and he referred in his proof to paper [6], which is explained in [11] in more detail. However, as I explained in the introduction, I do consider Arcones' proof satisfactory, and I want to give a correct proof. The goal of this paper is to work out the tools needed to apply the method of [6] in the multivariate case.

The previous part of this paper was about the discussion of the notions and results we need to adopt the method of [11] when we are working with multivariate models. In this section I explain how to generalize the method of [11] to prove limit theorems when we are working with functionals of vector valued Gaussian stationary fields. I shall give the proof of the results formulated by Arcones in paper [13]. Here I work out the tools we need in the proof of these results. They are a rather direct adaptation of some results in [11] to the multivariate case.

In the first step of this discussion I rewrite the limit problem we are interested in in a different form. Let us observe that we have $X_j(p) = T_p X_j(0)$ with the shift transformation $T_p$ for all $p \in \mathbb{Z}^\nu$ and $1 \le j \le d$, hence $Y(p) = T_p Y(0)$, and we can rewrite the sum in (8.1) in the form

$$S_N = A_N^{-1} \sum_{p \in B_N} T_p Y(0). \tag{8.3}$$

As it will turn out the crucial point in the investigation of our limit theorems is the study of limit theorems in the special case when $Y(0)$ is a Wick polynomial, and here we restrict our attention to this case.

Let $Y(0)$ be a Wick polynomial of order $n$ which has the form

$$Y(0) = \ :\sum_{\substack{(k_1,\dots,k_d)\\ k_1+\cdots+k_d=n}} a_{k_1,\dots,k_d} X_1(0)^{k_1} \cdots X_d(0)^{k_d}:$$

with some real coefficients $a_{k_1,\dots,k_d}$. Then by the corollary of Theorem 7.2 and the identities $X_j((0) = \int \mathbb{I}_1(x) Z_{G,j}(\,dx)$, $1 \le j \le d$, where $\mathbb{I}_1(\cdot)$ denotes the indicator function of the torus $[-\pi,\pi)^\nu$, the random variable $Y(0)$ can be written in the form

$$Y(0) = \sum_{\substack{(k_1,\dots,k_d)\\ k_1+\cdots+k_d=n}} a_{k_1,\dots,k_d} \int \mathbb{I}_1(x_1)\dots\mathbb{I}_1(x_n) \prod_{j=1}^d \left( \prod_{t=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} Z_{G,j}(\,dx_t)\right),$$

where for $j=1$ we define $\prod\limits_{t=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} Z_{G,j}(\,dx_t) = \prod\limits_{t=1}^{k_1} Z_{G,1}(\,dx_t)$, and if $k_j = 0$ for some $1 \le j \le d$, then the product $\prod\limits_{t=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} Z_{G_j}(\,dx_t)$ is omitted from this expression.

By Proposition 7.4 we can write

$$T_p Y(0) = \sum_{\substack{(k_1,\dots,k_d)\\ k_1+\cdots+k_d=n}} a_{k_1,\dots,k_d} \int e^{i(p,x_1+\cdots+x_n)} \prod_{j=1}^d \left( \prod_{t=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} Z_{G,j}(\,dx_t)\right) \tag{8.4}$$

for all indices $p \in \mathbb{Z}^\nu$.

We get by summing up formula (8.4) for all $p \in B_N$ with our choice of $Y(0)$ that

$$S_N = A_N^{-1} \sum_{\substack{(k_1,\dots,k_d)\\ k_1+\cdots+k_d=n}} a_{k_1,\dots,k_d} \int \prod_{l=1}^{\nu} \frac{e^{iN(x_1^{(l)}+\cdots+x_n^{(l)})}-1}{e^{i(x_1^{(l)}+\cdots+x_n^{(l)})}-1} \prod_{j=1}^d \left( \prod_{t=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} Z_{G,j}(\,dx_t)\right),$$

where we write $x=(x^{(1)},\dots,x^{(\nu)})$ for all $x\in[-\pi,\pi)^\nu$. (The set $B_N$ was defined in (8.2).) We can rewrite the above integral with the change of variables $y_l=Nx_l$, $1\le l\le n$, in the following form.

$$S_N=\sum_{\substack{(k_1,\dots,k_d)\\ k_1+\cdots+k_d=n}}\int h^N_{k_1,\dots,k_d}(y_1,\dots,y_n)\prod_{j=1}^d\left(\prod_{t=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} Z_{G^{(N)},j}(\,dy_t)\right), \tag{8.5}$$

where

$$h^N_{k_1,\dots,k_d}(y_1,\dots,y_n)=a_{k_1,\dots,k_d}\prod_{l=1}^{\nu}\frac{e^{i(y_1^{(l)}+\cdots+y_n^{(l)})}-1}{N(e^{i(y_1^{(l)}+\cdots+y_n^{(l)})/N}-1)}$$

is a function on $[-N\pi,N\pi)^\nu$, and $Z_{G^{(N)},j}(A)=N^{\nu/n}A_N^{-1/n}Z_{G,j}(\frac{A}{N})$ is defined for all measurable sets $A\subset[-N\pi,N\pi)^\nu$ and $j=1,\dots,d$. Here we use the notation $y_s=(y_s^{(1)},\dots,y_s^{(\nu)})$, $1\le s\le n$. Let us observe that $(Z_{G^{(N)},1},\dots,Z_{G^{(N)},d})$ is a vector valued random spectral measure on the torus $[-N\pi,N\pi)^\nu$ corresponding to the matrix valued spectral measure $G^{(N)}=(G^{(N)}_{j,j'})$, $1\le j,j'\le d$, on the torus $[-N\pi,N\pi)^\nu)$, defined by the formula

$$G^{(N)}_{j,j'}(A)=N^{2\nu/n}A_N^{-2/n}G_{j,j'}(\frac{A}{N}),\quad 1\le j,j'\le d,$$

for all measurable sets $A\subset[-N\pi,N\pi)^\nu$, where $G=(G_{j,j'})$, $1\le j,j'\le d$, is the matrix valued spectral measure of the original vector valued stationary random field $X(p)=(X_1(p),\dots,X_d(p))$, $p\in\mathbb{Z}^\nu$. On the other hand, $h^N_{k_1,\dots,k_d}\in\mathcal{K}_{n,j_1,\dots,j_n}(G^{(N)}_{j_1,j_1},\dots,G^{(N)}_{j_n,j_n})$ with $j_p=s$ if $k_1+\cdots+k_{s-1}<p\le k_1+\cdots+k_s$, $1\le s\le d$. (For $s=1$ we define $k_1+\cdots+k_{s-1}=0$.)

Let us observe that

$$\lim_{N\to\infty}h^N_{k_1,\dots,k_d}(y_1,\dots,y_n)=h^0_{k_1,\dots,k_d}(y_1,\dots,y_n)$$

with the function

$$h^0_{k_1,\dots,k_d}(y_1,\dots,y_n)=a_{k_1,\dots,k_d}\prod_{l=1}^{\nu}\frac{e^{i(y_1^{(l)}+\cdots+y_n^{(l)})}-1}{i(y_1^{(l)}+\cdots+y_n^{(l)})}$$

defined on $\mathbb{R}^{n\nu}$, and this convergence is uniform in all bounded subsets of $\mathbb{R}^{n\nu}$.

It is natural to expect that if the matrix valued spectral measures $G^{(N)}=(G^{(N)}_{j,j'})$, $1\le j,j'\le d$, converge to a matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1\le j,j'\le d$, defined on $\mathbb{R}^\nu$ in an appropriate sense, then a limiting procedure in formula (8.5) supplies the limit theorem

$$\begin{aligned}S_N &\to S_0\\ &=\sum_{\substack{(k_1,\dots,k_d)\\ k_1+\cdots+k_d=n}}\int h^0_{k_1,\dots,k_d}(y_1,\dots,y_n)\prod_{j=1}^d\left(\prod_{t=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} Z_{G^{(0)},j}(\,dy_t)\right),\end{aligned}$$

in distribution as $N \to \infty$, where $(Z_{G^{(0)},1}, \dots, Z_{G^{(0)},d})$ is a vector valued random spectral measure on $\mathbb{R}^\nu$ corresponding to the matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$.

Next I explain how to work out a precise method to prove limit theorems on the basis of the above heuristic argument. In the scalar valued case this was done in Lemma 8.3 of [11], and here I prove the vector valued variant of this result.

In the formulation of Lemma 8.3 of [11] we had to introduce a generalization of the notion of weak convergence of measures when we work with locally finite measures, i.e. with measures whose restriction to any compact set is finite. Here I introduce a slight generalization of this notion, called vague convergence in [11], to the case when we are working with complex measures of locally finite total variation. (The definition of complex measures on $\mathbb{R}^\nu$ with locally finite total variation was explained in Section 4.)

**Definition of vague convergence of complex measures on $\mathbb{R}^\nu$ with locally finite total variation.** *Let $G_N$, $N = 1, 2, \dots$, be a sequence of complex measures on $\mathbb{R}^\nu$ with locally finite total variation. We say that the sequence $G_N$ vaguely converges to a complex measure $G_0$ on $\mathbb{R}^\nu$ with locally finite total variation (in notation $G_N \xrightarrow{v} G_0$) if*

$$\lim_{N\to\infty} \int f(x)\, G_N(\,dx) = \int f(x)\, G_0(\,dx) \tag{8.6}$$

*for all continuous functions $f$ on $\mathbb{R}^\nu$ with a bounded support.*

I shall take a sequence of sums of $n$-fold Wiener–Itô integrals, and then I formulate Proposition 8.1 which states that under some appropriate conditions these random integrals have a limit, and this limit is also expressed explicitly in Proposition 8.1. Besides the representation of non-linear functionals of vector valued Gaussian stationary fields by means of Wiener–Itô integrals this is our main tool to prove the limit theorems with non-Gaussian limit for non-linear functionals of vector valued Gaussian stationary fields.

For all $N = 1, 2, \dots$ take a sequence of matrix valued non-atomic spectral measures $(G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-A_N\pi, A_N\pi)^\nu$ with parameter $A_N$ such that $A_N \to \infty$ as $N \to \infty$. Let us also take some functions

$$h^N_{j_1,\dots,j_n}(x_1, \dots, x_n) \in \mathcal{K}_{n,j_1,\dots,j_n} = \mathcal{K}_{n,j_1,\dots,j_n}(G^{(N)}_{j_1,j_1}, \dots, G^{(N)}_{j_n,j_n})$$

on the torus $[-A_N\pi, A_N\pi)^\nu$ for all $1 \le j_k \le d$, $1 \le k \le n$, and $N = 1, 2, \dots$. For all $N = 1, 2, \dots$ fix a vector valued random spectral measure

$$(Z_{G^{(N)},1}, \dots, Z^{(N)}_{G^{(N)},d})$$

on the torus $[-A_N\pi, A_N\pi)^\nu$ corresponding to the matrix valued spectral measure $(G^{(N)}_{j,j'})$, $1 \le j, j' \le d$. Let us define with the help of these quantities the sums of $n$-fold Wiener–Itô integrals

$$Z_N=\sum_{\substack{(j_1,\dots,j_n)\\ 1\le j_k\le d \text{ for all } 1\le k\le n}}\int h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_n}(\,dx_n), \tag{8.7}$$

$N=1,2,\dots$. In the next result I show that under appropriate conditions these random variables $Z_N$ converge in distribution to a random variable $Z_0$ expressed in the form of a sum of multiple Wiener–Itô integrals.

**Proposition 8.1.** *Let us consider the sums of $n$-fold Wiener–Itô integrals $Z_N$ defined in formula (8.7) with the help of certain vector valued random spectral measures $(Z_{G^{(N)},1},\dots,Z_{G^{(N)},d})$ corresponding to some non-atomic matrix valued spectral measures $(G^{(N)}_{j,j'})$, $1\le j,j'\le d$, defined on tori $[-A_N,A_N)^\nu$ such that $A_N\to\infty$ as $N\to\infty$, and functions*

$$h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)\in\mathcal{K}_{n,j_1,\dots,j_n}(G^{(N)}_{j_1,j_1},\dots,G^{(N)}_{j_n,j_n}).$$

*Let the coordinates $G^{(N)}_{j,j'}$, $1\le j,j'\le d$, of the matrix valued spectral measures $(G^{(N)}_{j,j'})$, $1\le j,j'\le d$, converge vaguely to the coordinates $G^{(0)}_{j,j'}$ of a non-atomic matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1\le j,j'\le d$, on $\mathbb{R}^\nu$ for all $1\le j,j'\le d$ as $N\to\infty$, and let $(Z_{G^{(0)},1},\dots,Z_{G^{(0)},d})$ be a vector valued random spectral measure on $\mathbb{R}^\nu$ corresponding to the matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1\le j,j'\le d$. Let us also have some functions $h^0_{j_1,\dots,j_n}$ for all $1\le j_k\le d$, $1\le k\le n$, such that these functions and matrix valued spectral measures satisfy the following conditions.*

(a) *The functions $h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)$ are continuous on $\mathbb{R}^{n\nu}$ for all $1\le j_k\le d$, $1\le k\le n$, and for all $T>0$ and indices $1\le j_k\le d$, $1\le k\le n$, the functions $h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)$ converge uniformly to the function $h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)$ on the cube $[-T,T]^{n\nu}$ as $N\to\infty$.*

(b) *For all $\varepsilon>0$ there is some $T_0=T_0(\varepsilon)>0$ such that*

$$\int_{\mathbb{R}^{n\nu}\setminus[-T,T]^{n\nu}}|h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)|^2G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_n,j_n}(dx_n)<\varepsilon^2 \tag{8.8}$$

*for all $1\le j_k\le d$, $1\le k\le n$, and $N=1,2\dots$ if $T>T_0$.*

*Then inequality (8.8) holds also for $N=0$,*

$$h^0_{j_1,\dots,j_n}\in\mathcal{K}_{n,j_1,\dots,j_n}=\mathcal{K}_{n,j_1,\dots,j_n}(G^{(0)}_{j_1,j_1},\dots G^{(0)}_{j_n,j_n}), \tag{8.9}$$

*the sum of random integrals*

$$Z_0=\sum_{\substack{(j_1,\dots,j_n)\\ 1\le j_k\le d, \text{ for all } 1\le k\le n}}\int h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)Z_{G^{(0)},j_1}(\,dx_1)\dots Z_{G^{(0)},j_n}(\,dx_n) \tag{8.10}$$

*exists, and the random variables $Z_N$ defined in (8.7) satisfy the relation $Z_N \xrightarrow{\mathcal{D}} Z_0$ as $N \to \infty$, where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.*

*Remark 1.* A complex measure $G^{(N)}_{j,j'}$ with finite total variation defined on the torus $[-A_N\pi, A_N\pi)^\nu$ can be identified in a natural way with a complex measure on $\mathbb{R}^\nu$ which is concentrated on its subset $[-A_N\pi, A_N\pi)^\nu$. We take this identification of $G^{(N)}_{j,j'}$ with a complex measure on $\mathbb{R}^\nu$ when we give meaning to formula (8.6) with $G_N = G^{(N)}_{j,j'}$ and $G_0 = G^{(0)}_{j,j'}$ in the definition of the vague convergence of the complex measures $G^{(N)}_{j,j'}$ to $G^{(0)}_{j,j'}$ as $N \to \infty$.

*Remark 2.* Proposition 8.1 together with the previous consideration suggest when we can expect the appearance of a non-central limit theorems for non-linear functionals of vector valued stationary Gaussian random fields and how to prove such a result.

To prove such limit theorems first we present the non-linear functionals we are working with in the form of a finite sum of multiple Wiener–Itô integrals and then we rewrite them with an appropriate rescaling. We try to find such a rescaling where the random spectral measures which appear in the random integrals correspond to such spectral measures which have a (vague) limit, and also the kernel functions in these integrals have a nice limit satisfying condition (a) of Proposition 8.1. We can prove a non-central limit theorem if the functions and measures appearing in the definition of this multiple Wiener–Itô integrals satisfy a compactness type condition formulated in condition (b) of Proposition 8.1. This condition plays a very important role in this result. If it does not hold, then a different situation arises. In such cases we can get a central limit theorem with standard normalization, see e.g. Theorem 4 in [1], [3] or the discussion of this problem in a more general setting in the book [14]. It may be worth mentioning also the paper [9], where Ho, H. C. and Sun, T. C. proved an interesting result about the limit distribution of a linear functional of a stationary Gaussian process into $\mathbb{R}^2$, where the first coordinate of the limit was Gaussian and the second coordinate was non-Gaussian. It may be interesting to find the natural multivariate generalization of this result.

I shall adapt the proof of Lemma 6.6 in [12] to the multivariate case. This is a simplified version of the proof of Lemma 8.3 in [11]. The latter result is actually a slightly more general scalar valued version of Proposition 8.1. We could have proved also the generalization of Lemma 8.3 in [11] to the multivariate case. But the formulation of Proposition 8.1 in the present form is sufficient for our purposes.

*Proof of Proposition 8.1.* First I show that relation (8.8) holds also for $N = 0$. To see this let us first show that the measures $\mu^{(N)}_{j_1,\dots,j_n}$, $N = 1, 2, \dots$, defined as

$$\mu^{(N)}_{j_1,\dots,j_n}(A) = \int_A |h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)|^2 G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_n,j_n}(dx_n), \quad A \subset R^{n\nu},$$

converge vaguely to the locally finite measure $\mu^{(0)}_{j_1,\dots,j_n}$ defined as

$$\mu^{(0)}_{j_1,\dots,j_n}(A)=\int_A |h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)|^2 G^{(0)}_{j_1,j_1}(\,dx_1)\dots G^{(0)}_{j_n,j_n}(dx_n),\quad A\subset R^{n\nu},$$

if $N\to\infty$.

Indeed, it follows from the vague convergence of the measures $G^{(N)}_{j,j}$ to $G^{(0)}_{j,j}$ as $N\to\infty$ and the continuity of the function $h^{(0)}_{j_1,\dots,j_n}$ that this relation holds if we replace the kernel function $|h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)|^2$ by $|h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)|^2$ in the definition of the measures $\mu^{(N)}_{j_1,\dots,j_n}$. Then condition (a) of Proposition 8.1 implies that this relation also holds with the original definition of the measures $\mu^{(N)}_{j_1,\dots,j_n}$.

Next I state that the measure $\mu^{(0)}_{j_1,\dots,j_n}$ is finite, and the measures $\mu^{(N)}_{j_1,\dots,j_n}$ converge to it not only vaguely but also weakly. Indeed, condition (b) implies that the sequence of measures $\mu^{(N)}_{j_1,\dots,j_n}$ is compact with respect to the topology defining the weak convergence of finite measures, hence any subsequence of it has a convergent sub-subsequence. But the limit of these sub-subsequences can be only its limit with respect to the vague convergence, i.e. it is $\mu^{(0)}_{j_1,\dots,j_n}$. This implies that $\mu^{(0)}_{j_1,\dots,j_n}$ is a finite measure, and the sequence of measures $\mu^{(N)}_{j_1,\dots,j_n}$ converges also weakly to it.

Finally the properties of the functions $h^N_{j_1,\dots,j_n}$, and their convergence to $h^0_{j_1,\dots,j_n}$ formulated in condition (a) imply that also the symmetry property $h^0_{j_1,,\dots,j_n}(-x_1,\dots,-x_n)=\overline{h^0_{j_1,,\dots,j_n}(x_1,\dots,x_n)}$ holds, hence relation (8.9) is valid, and the random integral $Z_0$ defined in (8.10) is meaningful.

Then I reduce the proof of the relation $Z_N\stackrel{\mathcal D}{\to}Z_0$ to the proof of the following statement.

Under the conditions of Proposition 8.1

$$\begin{aligned}\sum_{\substack{(j_1,\dots,j_n)\\ 1\le j_k\le d,\text{ for all }1\le k\le d}} &\int h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)\chi_T(x_1,\dots,x_n) \\ &\qquad Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_n}(\,dx_n)\\ \stackrel{\mathcal D}{\to}\sum_{\substack{(j_1,\dots,j_n)\\ 1\le j_k\le d,\text{ for all }1\le k\le d}} &\int h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)\chi_T(x_1,\dots,x_n)\\ &\qquad Z_{G^{(0)},j_1}(\,dx_1)\dots Z_{G^{(0)},j_n}(\,dx_n),\end{aligned}\tag{8.11}$$

as $N\to\infty$, where $\chi_T(x_1,\dots,x_n)$ is the indicator function of the cube $[-T,T]^{n\nu}$. We state formula (8.11) for all such $T>0$ for which the boundary of the cube $[-T,T]^{n\nu}$ has zero measure with respect to the measure $\underbrace{\mu_0\times\dots\times\mu_0}_{n\text{ times}}$, where $\mu_0$ is the dominating measure of the complex measures $G^{(0)}_{j,j'}$, $1\le j,j'\le d$, with locally finite total variation which appears in the definition of semidefinite matrix valued even measures introduced in Section 4 if this definition is applied for the matrix valued spectral measure $(G^{(0)}_{j,j'})$.

To prove this reduction let us observe that by formulas (5.6) and (8.8)

$$
\begin{aligned}
E\biggl[\int [1-\chi_T(x_1,\dots,x_n)]h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)\\
Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_n}(\,dx_n)\biggr]^2\\
\le n!\int_{\mathbb{R}^{k\nu}\setminus[-T,T]^{n\nu}} |h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)|^2 G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_n,j_n}(\,dx_n)<n!\varepsilon^2
\end{aligned}
$$

for all sequences $(j_1,\dots,j_n)$, $1\le j_k\le d$, $1\le k\le n$, and $N=0,1,2,\dots$ if $T>T_0(\varepsilon)$. Hence

$$
\begin{aligned}
E\biggl[\sum_{\substack{(j_1,\dots,j_n)\\ 1\le j_k\le d \text{ for all } 1\le k\le n}}\int [1-\chi_T(x_1,\dots,x_n)]h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)\\
Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_n}(\,dx_n)\biggr]^2\le d^n n!\varepsilon^2 \qquad (8.12)
\end{aligned}
$$

for all $N=0,1,\dots$ if $T>T_0(\varepsilon)$.

Since $G^{(N)}_{j,j}\stackrel{v}{\to}G^{(0)}_{j,j}$ for all $1\le j\le d$ as $N\to\infty$, hence for all $T>0$ there is some number $C(T)$ such that $G^{(N)}_{j,j}([-T,T])\le C(T)$ for all $N=1,2,\dots$ and $1\le j\le d$. Because of this estimate and the uniform convergence $h^N_{j_1,\dots,j_n}\to h^0_{j_1,\dots,j_n}$ on any cube $[-T,T]^{n\nu}$ we have

$$
\begin{aligned}
E\biggl[\int [h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)-h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)]\chi_T(x_1,\dots,x_n)\\
Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_n}(\,dx_n)\biggr]^2\\
\le n!\int_{[-T,T]^{n\nu}} |h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)-h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)|^2\\
G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_n,j_n}(\,dx_n)<\varepsilon^2
\end{aligned}
$$

for all $T>0$ and $(j_1,\dots,j_n)$, $1\le j_k\le d$, $1\le k\le n$, if $N>N_1$ with some $N_1=N_1(T,\varepsilon)$. Hence

$$
\begin{aligned}
E\biggl[\sum_{1\le j_1,\dots,j_n\le d}\int [h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)-h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)] \qquad (8.13)\\
\chi_T(x_1,\dots,x_n)Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_n}(\,dx_n)\biggr]^2\le d^n\varepsilon^2
\end{aligned}
$$

for all $T>0$ if $N>N_1$ with some $N_1=N_1(T,\varepsilon)$.

Let us define the quantities

$$U_N = U_N(T) = \sum_{1\le j_1,\dots,j_n\le d} \int \qquad h^N_{j_1,\dots,j_n}(x_1,\dots,x_n)\chi_T(x_1,\dots,x_n)$$
$$Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_n}(\,dx_n),$$

and

$$V_N = V_N(T) = \sum_{1\le j_1,\dots,j_n\le d} \int \qquad h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)\chi_T(x_1,\dots,x_n)$$
$$Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_n}(\,dx_n),$$

$N = 1,2,\dots$. We introduce the definition of $V_N = V_N(T)$ also for $N = 0$, where we replace the spectral measures $Z_{G^{(N)},j}$, $N\ge 1$, $1\le j\le d$, by $Z_{G^{(0)},j}$, $1\le j\le d$, in the definition. We can reduce the proof of the relation $Z_N \stackrel{\mathcal{D}}{\to} Z_0$ to formula (8.11) in the following way. By formula (8.12) we can state that

$$\begin{aligned} |E(e^{itZ_N}-e^{itU_N})| &\le E|(1-e^{it(Z_n-U_N)})| \le E|(t(Z_N-U_N)| \\ &\le |t|(E(Z_N-U_N)^2)^{1/2} \le |t|(d^n n!)^{1/2}\varepsilon. \end{aligned}$$

for all $t\in\mathbb{R}^1$ with the random variable $Z_N$ defined in (8.7) if $T>T_0$ and $N>N_0(\varepsilon)$. Similarly, $|E(e^{itU_N}-e^{itV_N})| \le |t|(E(U_N-V_N)^2)^{1/2} \le |t|d^{n/2}\varepsilon$ for all $t\in\mathbb{R}^1$ and $N>N_0$ by inequality (8.13). Finally, $Ee^{itV_N}\to Ee^{itZ_0}$ for all $t\in\mathbb{R}^1$ with $Z_0$ defined in (8.10) if relation (8.11) holds. These relations together imply that $|Ee^{itZ_N}-Ee^{itZ_0}|\le C(t)\varepsilon$ if $N>N_0(t,\varepsilon)$ with some numbers $C(t)$ and $N_0(t,\varepsilon)$. Since this inequality holds for all $\varepsilon>0$, it implies that $Z_N \stackrel{\mathcal{D}}{\to} Z_0$. (In formula (8.11) we imposed a condition on the parameter $T>0$. We demanded that the boundary of $[-T,T]^{n\nu}$ must have measure zero with respect to the product measure of $\mu_0$. It causes no problem that we can apply the above argument only for parameters $T$ with this property.)

We shall prove (8.11) with the help of some statements formulated below. To formulate them let us first fix a number $T>0$ such that the boundary of the cube $[-T,T]^{n\nu}$ has zero measure with respect to the measure $\underbrace{\mu_0\times\dots\times\mu_0}_{n\text{ times}}$.

Observe that

$$h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)\chi_T(x_1,\dots,x_n)\in\mathcal{K}_{n,j_1,\dots,j_n}(G^{(N)}_{j_1,j_1},\dots,G^{(N)}_{j_n,j_n})$$

for all $T>0$ and $N=0,1,2,\dots$. I claim that for all $\varepsilon>0$ and $(j_1,\dots,j_n)$, $1\le j_k\le d$, $1\le k\le n$, there is a simple function

$$f^\varepsilon_{j_1,\dots,j_n}\in\hat{\mathcal{K}}_{n,j_1,\dots,j_n}(G^{(0)}_{j_1,j_1},\dots,G^{(0)}_{j_n,j_n})$$

such that it is adapted to such a regular system $\mathcal{D}=\{\Delta_k,\ k=\pm1,\dots,\pm M\}$ whose elements have boundaries with zero $\mu_0$ measure, i.e. $\mu_0(\partial\Delta_k)=0$ for all $1\le|k|\le M$, $\Delta_k\subset[-T,T]^\nu$ for all $1\le|k|\le M$, (we choose a regular system

$\mathcal{D}$ in such a way that all functions $f^\varepsilon_{j_1,\dots,j_n}$ with a fixed parameter $\varepsilon > 0$ are adapted to it), and the functions $f^\varepsilon_{j_1,\dots,j_n}$ satisfy the following inequalities.

$$\int |h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)\chi_T(x_1,\dots,x_n) - f^\varepsilon_{j_1,\dots,j_n}(x_1,\dots,x_n)|^2 \\ G^{(0)}_{j_1,j_1}(\,dx_1)\dots G^{(0)}_{j_n,j_n}(\,dx_n) < \varepsilon^2 \tag{8.14}$$

for all $1 \le j_k \le d$, $1 \le k \le n$, and also the inequalities

$$\int |h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)\chi_T(x_1,\dots,x_n) - f^\varepsilon_{j_1,\dots,j_n}(x_1,\dots,x_n)|^2 \\ G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_n,j_n}(\,dx_n) < \varepsilon^2 \tag{8.15}$$

hold for all $1 \le j_k \le d$, $1 \le k \le n$, and $N \ge N_0$ with some $N_0 = N_0(\varepsilon, T)$.

I also claim that

$$Y_N \stackrel{\mathcal{D}}{\to} Y_0 \tag{8.16}$$

as $N \to \infty$, where

$$\begin{aligned} Y_N &= Y_N(\varepsilon,T) \\ &= \sum_{\substack{(j_1,\dots,j_n)\\ 1\le j_k\le d \text{ for all } 1\le k\le n}} \int f^\varepsilon_{j_1,\dots,j_n}(x_1,\dots,x_n) Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_n}(\,dx_n) \end{aligned}$$

for $N = 0, 1, 2, \dots$.

Let me remark that the condition that all simple functions $f^\varepsilon_{j_1,\dots,j_n}$ (with a fixed parameter $\varepsilon > 0$) are adapted to the same regular system $\mathcal{D}$ causes no problem. By an appropriate refinement of those regular systems to which they are adapted we get such a regular system. Moreover, this refinement can be done in such a way that the sets $\Delta_k$ in the new regular system preserve the property that their boundaries have zero $\mu_0$ measure.

Relation (8.11) can be proved with the help of relations (8.14), (8.15) and (8.16) similarly to the reduction of the relation $Z_N \stackrel{\mathcal{D}}{\to} Z_0$ to formula (8.11). Indeed, one gets from inequalities (8.14), (5.6) and the definition of the quantities $V_N$ and $Y_0$, by applying an argument similar to the proof of relation (8.12) that

$$E(V_0 - Y_0)^2 \le n! n^d \varepsilon^2,$$

and also

$$E(V_N - Y_N)^2 \le n^d n! \varepsilon^2$$

if $N > N_0(\varepsilon, T)$ by (8.15) and (5.6).

Then we can show with the help of these relations similarly to the reduction of the relation $Z_N \stackrel{\mathcal{D}}{\to} Z_0$ to formula (8.11) that $|Ee^{itV_N} - Ee^{itY_N}| \le \varepsilon$, $|Ee^{itY_N} - Ee^{itY_0}| \le \varepsilon$, and $|Ee^{itY_0} - Ee^{itV_0}| \le \varepsilon$ if $N > N_0(\varepsilon, t, T)$ with some threshold index $N_0(\varepsilon, t, T)$. Here in the first and third inequality we apply the last two inequalities which were consequences of (8.14) and (8.15), while the second inequality follows from (8.16). Since these relations hold for all $\varepsilon > 0$ they imply that $Ee^{itV_N} \to Ee^{itV_0}$ for all $t \in \mathbb{R}^1$ as $N \to \infty$, i.e. $V_N \xrightarrow{\mathcal{D}} V_0$ as $N \to \infty$, and this is formula (8.11) written with a different notation.

It remains to prove (8.14), (8.15) and (8.16). Relation (8.14) is a direct consequence of Lemma 5.2. Then formula (8.15) follows from some classical results about vague (and weak) convergence of measures. Since we are working in the proof of (8.15) in a cube $[-T,T]^{n\nu}$ it is enough to know the results about weak convergence to carry out our arguments.

Let us first observe that since the restrictions of the measures $G^{(N)}_{j,j}$ to $[-T,T]^\nu$ tend weakly to the restriction of the measure $G^{(0)}_{j,j}$ to the cube $[-T,T]^\nu$ as $N \to \infty$, we can also say that the restrictions of the product measures $G^{(N)}_{j_1,j_1} \times \cdots \times G^{(N)}_{j_n,j_n}$ to the cube $[-T,T]^{n\nu}$ converge weakly to the restriction of the product measure $G^{(0)}_{j_1,j_1} \times \cdots \times G^{(0)}_{j_n,j_n}$ on the cube $[-T,T]^{n\nu}$, as $N \to \infty$. On the other hand, the function

$$
\begin{aligned}
&H^0_{j_1,\dots j_n}(x_1,\dots,x_n)\\
&\qquad = |h^0_{j_1,\dots,j_n}(x_1,\dots,x_n)\chi_T(x_1,\dots,x_n) - f^\varepsilon_{j_1,\dots,j_n}(x_1,\dots,x_n)|^2
\end{aligned}
$$

is almost everywhere continuous with respect to the measure $G^{(0)}_{j_1,j_1} \times \cdots \times G^{(0)}_{j_n,j_n}$. By the general theory about convergence of measures these properties imply that

$$
\begin{aligned}
&\int H^0_{j_1,\dots j_n}(x_1,\dots,x_n) G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_n,j_n}(\,dx_n)\\
&\qquad \to \int H^0_{j_1,\dots j_n}(x_1,\dots,x_n) G^{(0)}_{j_1,j_1}(\,dx_1)\dots G^{(0)}_{j_n,j_n}(\,dx_n)
\end{aligned}
$$

as $N \to \infty$. (Such a convergence is proved for probability measures e.g. in [2]. A careful analysis shows that this result remains valid for sequences of finite but not necessarily probability measures. Let me remark that here we are working with (real, non-negative) measures.) The last relation together with (8.14) imply (8.15).

To prove relation (8.16) first we show that $G^{(N)}_{j,j'}(\Delta_k) \to G^{(0)}_{j,j'}(\Delta_k)$ as $N \to \infty$ for all $1 \le j,j' \le d$ and $\Delta_k \in \mathcal{D}$ with the regular system $\mathcal{D}$ we are working with. (Let me recall that the boundary of all sets $\Delta_k \in \mathcal{D}$ has zero $\mu_0$ measure.)

This relation immediately follows if $j = j'$ from the facts that $G^{(N)}_{j,j} \xrightarrow{v} G^{(0)}_{j,j}$, $G^{(0)}_{j,j}(\partial\Delta_k) = 0$ for all $1 \le |k| \le M$, and $G^{(N)}_{j,j}$ is a locally finite measure for all $N = 0,1,2,\dots$. If $j \ne j'$, then we have to apply a more refined argument, since in this case we only know that $G^{(N)}_{j,j'}$ is a complex measure with locally finite total variation. In this case we will exploit that the matrix valued measures $(G^{(N)}_{j.j'})$, $1 \le j,j' \le d$, are positive semidefinite. This implies that the Radon–Nikodym derivatives $g^{(N)}_{j,j'}$ of the complex measures $G^{(N)}_{j,j'}$ with respect to the dominating measure $\mu_N$ have the following property. For all $N = 0,1,2,\dots$

and $1 \le j, j' \le d$ such that $j \ne j'$ the $2 \times 2$ matrices

$$g^{(N)}(x|j,j') = \begin{pmatrix} g^{(N)}_{j,j}(x), & g^{(N)}_{j,j'}(x) \\ g^{(N)}_{j',j}(x), & ng^{(N)}_{j',j'}(x) \end{pmatrix}$$

are positive semidefinite for $\mu_N$ almost all $x \in \mathbb{R}^\nu$. Let us define for all non-negative functions $v(x)$, $x \in \mathbb{R}^\nu$ the vector $S(x|v) = (\sqrt{v(x)}, \sqrt{v(x)})$. By exploiting that the matrices $g^{(N)}(x|j,j')$ are positive semidefinite we get that

$$\begin{aligned} &\int v(x)[G^{(N)}_{j,j}(\,dx) + G^{(N)}_{j,j'}(\,dx) + G^{(N)}_{j',j}(\,dx) + G^{(N)}_{j',j'}(\,dx)] \\ &\qquad = \int S(x|v) g^{(N)}(x|j,j') S(x|v)^* \mu_N(\,dx) \ge 0 \end{aligned}$$

for all functions $v$ such that $v(x) \ge 0$, $x \in \mathbb{R}^\nu$. Hence $H^{(N)}_{j.j'} = [G^{(N)}_{j,j} + G^{(N)}_{j,j'} + G^{(N)}_{j',j} + G^{(N)}_{j',j'}]$ is a locally finite measure on $\mathbb{R}^\nu$. Moreover $H^{(N)}_{j,j'} \stackrel{v}{\to} H^{(0)}_{j,j'}$ as $N \to \infty$. This implies that $H^{(N)}_{j,j'}(\Delta_k) \to H^{(0)}_{j,j'}(\Delta_k)$, therefore $G^{(N)}_{j,j'}(\Delta_k) + G^{(N)}_{j',j}(\Delta_k) \to G^{(0)}_{j,j'}(\Delta_k) + G^{(0)}_{j',j}(\Delta_k)$ as $N \to \infty$ for all $\Delta_k \in \mathcal{D}$.

We get similarly by working with the vectors $R(x|v) = (\sqrt{v(x)}, i\sqrt{v(x)})$ instead of the vectors $S(x|v) = (\sqrt{v(x)}, \sqrt{v(x)})$ for all functions $v(x) \ge 0$, $x \in \mathbb{R}^\nu$, that $K^{(N)}_{j.j'} = [G^{(N)}_{j,j} + iG^{(N)}_{j,j'} - iG^{(N)}_{j',j} + G^{(N)}_{j',j'}]$ is a a locally finite measure for all $N = 0, 1, 2, \dots$, and $K^{(N)}_{j,j} \stackrel{v}{\to} K^{(0)}_{j,j'}$ as $N \to \infty$. Thus $K^{(N)}_{j,j'}(\Delta_k) \to K^{(0)}_{j,j'}(\Delta_k)$, therefore $G^{(N)}_{j,j'}(\Delta_k) - G^{(N)}_{j',j}(\Delta_k) \to G^{(0)}_{j,j'}(\Delta_k) - G^{(0)}_{j',j}(\Delta_k)$ as $N \to \infty$ for all $\Delta_k \in \mathcal{D}$. These relations imply that $G^{(N)}_{j,j'}(\Delta_k) \to G^{(0)}_{j,j'}(\Delta_k)$ for all $\Delta_k \in \mathcal{D}$.

Let us define for all $N = 0, 1, 2, \dots$ and our regular system $\mathcal{D} = \{\Delta_k,\ 1 \le |k| \le M\}$ the Gaussian random vector

$$Z_N(\mathcal{D}) = \left(\operatorname{Re} Z_{G^{(N)},j}(\Delta_k), \operatorname{Im} Z_{G^{(N)},j}(\Delta_k), \quad |k| \le M,\ \ 1 \le j \le d\right)$$

I claim that the elements of the covariance matrices of the random vectors $Z_N(\mathcal{D})$ can be expressed by means of the numbers $G^{(N)}_{j,j'}(\Delta_k)$, $1 \le |k| \le M$ and $1 \le j, j' \le d$, and the covariance matrices of $Z_N(\mathcal{D})$ converge to the covariance matrix of $Z_0(\mathcal{D})$ as $N \to \infty$.

To prove these relations observe that

$$\begin{aligned} \operatorname{Re} Z_{G^{(N)},j}(\Delta_k) &= \frac{Z_{G^{(N)},j}(\Delta_k) + \overline{Z_{G^{(N)},j}(\Delta_k)}}{2}, \\ \operatorname{Im} Z_{G^{(N)},j}(\Delta_k) &= \frac{Z_{G^{(N)},j}(\Delta_k) - \overline{Z_{G^{(N)},j}(\Delta_k)}}{2i}, \end{aligned}$$

and $\overline{Z_{G^{(N)},j}(\Delta_k)} = Z_{G^{(N)},j}(-\Delta_k) = Z_{G^{(N)},j}(\Delta_{-k})$. In the last identity we exploited also the properties of the regular systems $\mathcal{D}$. Also the properties of the regular systems imply that if $\Delta_k, \Delta_l \in \mathcal{D}$, then we have either $\Delta_k \cap \Delta_l = \Delta_k$ or $\Delta_k \cap \Delta_l = \emptyset$. The first identity holds if $l = k$ and the second one if $l \ne k$. Hence we have either $EZ_{G^{(N)},j}(\Delta_k)\overline{Z_{G^{(N)},j'}(\Delta_l)} = G^{(N)}_{j,j'}(\Delta_k)$ if $k = l$

or $EZ_{G^{(N)},j}(\Delta_k)\overline{Z_{G^{(N)},j'}(\Delta_l)} = 0$ if $k \neq l$. These relations imply that we can express all covariances

$$E\mathrm{Re}\, Z_{G^{(N)},j}(\Delta_k)\mathrm{Re}\, Z_{G^{(N)},j'}(\Delta_l), \quad E\mathrm{Re}\, Z_{G^{(N)},j}(\Delta_k)\mathrm{Im}\, Z_{G^{(N)},j'}(\Delta_l)$$
$$\text{and } E\mathrm{Im}\, Z_{G^{(N)},j}(\Delta_k)\mathrm{Im}\, Z_{G^{(N)},j'}(\Delta_l)$$

with the help of the quantities $G^{(N)}_{j,j'}(\Delta_k)$, $1 \le j, j' \le d$, $1 \le |k| \le M$. The convergence of the numbers $G^{(N)}_{j,j'}(\Delta_k)$ to $G^{(0)}_{j,j'}(\Delta_k)$ also implies that the covariance matrices of $Z_N(\mathcal{D})$ converge to the covariance matrix of $Z_0(\mathcal{D})$ as $N \to \infty$.

The convergence of the covariance matrices of the Gaussian random vectors $Z_N(\mathcal{D})$ with expectation zero also implies that the distributions of $Z_N(\mathcal{D})$ converge weakly to the distribution of $Z_0(\mathcal{D})$ as $N \to \infty$. But then the same can be told about any continuous functions of the coordinates of the random vectors $Z_N(\mathcal{D})$. Because of the definition of the Wiener–Itô integrals of simple functions the random variables $Y_N$ in formula (8.16) are polynomials, hence continuous functions of the coordinates of the random vectors $Z_N(\mathcal{D})$. Besides, these polynomials do not depend on the parameter $N$. Hence the previous results imply that formula (8.16) holds. Proposition 8.1 is proved.

To simplify the application of Proposition 8.1 we also prove the following lemma.

**Lemma 8.2.** *Let us have a sequence of matrix valued spectral measures $(G^{(N)}_{j,j'})$, $N = 1, 2, \dots$, $1 \le j, j' \le d$, on the torus $[-A_N\pi, A_N\pi]^\nu$ such that $A_N \to \infty$, and $G^{(N)}_{j,j'} \xrightarrow{v} G^{(0)}_{j,j'}$ with some complex measure $(G^{(0)}_{j,j'})$ with locally finite total variation for all $1 \le j, j' \le d$ as $N \to \infty$. Then $G^{(0)} = (G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite matrix valued even measure on $\mathbb{R}^\nu$.*

*Remark.* Lemma 8.2 can be considered as the multidimensional version of the statement that the limit of locally finite measures on $\mathbb{R}^\nu$ with respect to the vague convergence is a locally finite measure.

*Proof of Lemma 8.2* We have to show that $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite matrix valued measure. To do this take a vector $v(x) = (v_1(x), \dots, v_d(x))$ whose coordinates $v_k(x)$, $1 \le k \le d$, are continuous functions with compact support. We have

$$\lim_{N\to\infty} \sum_{j=1}^{d} \sum_{j'=1}^{d} \int v_j(x) v_{j'}(x) G^{(N)}_{j,j'}(\,dx) = \sum_{j=1}^{d} \sum_{j'=1}^{d} \int v_j(x) v_{j'}(x) G^{(0)}_{j,j'}(\,dx) \ge 0. \tag{8.17}$$

The identity in (8.17) holds, since $G^{(N)}_{j,j'} \xrightarrow{v} G^{(0)}_{j,j'}$ for all $1 \le j, j' \le d$. The inequality at the end of (8.17) also holds, because $(G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite matrix valued measure for all $N = 1, 2, \dots$, and this implies that the left-hand side of (8.17) is non-negative for all $N = 1, 2, \dots$. Thus we got that if $g^{(0)}_{j,j'}(x)$ is the Radon–Nikodym derivative of $G^{(0)}_{j,j'}$ with respect to some dominating measure $\mu_0$ in the point $x \in \mathbb{R}^\nu$ for all $1 \le j, j' \le d$, we take the $d \times d$ matrix $g^{(0)}(x) = (g^{(0)}_{j,j'}(x))$, $1 \le j, j' \le d$, and the coordinates of the vector $v(x) = (v_1(x), \dots, v_d(x))$ are continuous functions with compact support, then

$$\int v(x) g^{(0)}(x) v^*(x) \mu_0(\,dx) \ge 0.$$

In the proof of Theorem 2.2 we have shown that this relation implies that $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite matrix valued measure.

We still have to show that the complex measure $G^{(0)}_{j,j'}$ with locally finite variation is even for all $1 \le j, j' \le d$. To do this fix a pair $j, j'$ of indices, $1 \le j, j' \le d$, and define for all $N = 0, 1, 2, \dots$ the complex measure $(G')^{(N)}_{j,j}$ by the relation $(G')^{(N)}_{j,j'}(A) = \overline{G^{(N)}_{j,j'}(-A)}$ for all bounded, measurable sets $A \subset \mathbb{R}^\nu$. It is not difficult to see that not only $G^{(N)}_{j,j'} \stackrel{v}{\to} G^{(0)}_{j,j'}$, but also $(G')^{(N)}_{j,j'} \stackrel{v}{\to} (G')^{0}_{j,j'}$ as $N \to \infty$. The evenness of the measures $G^{(N)}_{j,j'}$ for $N = 1, 2, \dots$ means that $G^{(N)}_{j,j'} = (G')^{(N)}_{j,j'}$ for all $N = 1, 2, \dots$. By taking the limit $N \to \infty$ we get that $G^{(0)}_{j,j'} = (G')^{(0)}_{j,j'}$. This means that $G^{(0)}_{j,j}$ is an even complex measure with locally finite variation. Lemma 8.2 is proved.

Proposition 8.1 is a useful tool to prove non-central limit theorems for non-linear functionals of vector valued Gaussian stationary random fields. In the application of this result we have to check some properties of the matrix valued spectral measure of a Gaussian stationary random field with some nice properties. In [6] we met a similar problem about scalar valued Gaussian stationary random fields. In that paper we have proved that if the correlation function of a scalar valued Gaussian stationary random field has some nice properties, then its spectral measure satisfies the properties needed in the proof of our non-central limit theorem. In the generalization of this result to vector valued stationary random fields we must get a good control on the behaviour of a matrix valued spectral measure of a vector valued stationary random field if its correlation function has some nice properties. Such a result is proved in paper [13], where we prove the multivariate version of the result in paper [6].

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