THE THEORY OF WIENER–ITÔ INTEGRALS IN VECTOR-VALUED GAUSSIAN STATIONARY RANDOM FIELDS. PART II

PÉTER MAJOR

ABSTRACT. This work is the continuation of my paper in Moscow Math. Journal Vol. 20, No. 4 in 2020. In that paper the existence of the spectral measure of a vector-valued stationary Gaussian random field is proved and the vector-valued random spectral measure corresponding to this spectral measure is constructed. The most important properties of this random spectral measure are formulated, and they enable us to define multiple Wiener–Itô integrals with respect to it. Then an important identity about the products of multiple Wiener–Itô integrals, called the diagram formula is proved. In this paper an important consequence of this result, the multivariate version of Itô's formula is presented. It shows a relation between multiple Wiener–Itô integrals with respect to vector-valued random spectral measures and Wick polynomials. Wick polynomials are the multivariate versions of Hermite polynomials. With the help of Itô's formula the shift transforms of a random variable given in the form of a multiple Wiener–Itô integral can be written in a useful form. This representation of the shift transforms makes possible to rewrite certain non-linear functionals of a vector-valued stationary Gaussian random field in such a form which suggests a limiting procedure that leads to new limit theorems. Finally, this paper contains a result about the problem when this limiting procedure may be carried out, i.e., when the limit theorems suggested by our representation of the investigated non-linear functionals are valid.

1. Introduction. The main results of the paper.

This work deals with the properties of vector-valued stationary Gaussian random fields. In particular, one of its subjects is the problem how to prove non-central limit theorems for certain non-linear functionals of such random fields. It is the continuation of paper [10]. These two papers deal with a generalized version of the problems studied in [8]. In Lecture Note [8] scalar-valued stationary random fields were investigated. It contains a good representation of non-linear functionals of a stationary Gaussian random field with the help of multiple Wiener–Itô integrals with respect to the random spectral measure of this Gaussian random field. In this work and in

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paper [10] the multivariate version of these integrals is investigated. They present the natural multivariate version of the results in [8] and provide a useful representation of non-linear functionals of a vector-valued stationary Gaussian random field with the help of (vector-valued) multiple Wiener–Itô integrals with respect to the random spectral measure of the underlying Gaussian field. In a subsequent paper [11] I will prove the natural multivariate version of the non-central limit theorems formulated in [8] with their help.

In Theorem 6 of his paper [1] Arcones formulated the multivariate version of the non-central limit theorems proved in [8]. But his proof contains some gaps. He assumed that the natural multivariate versions of some results in [8] hold true. But he did not formulate precisely the results he applied, and even the definitions of the notions needed in the formulation of his results are missing. We want to present correct formulation and a complete proof of Arcones' result. But to do this we need the results of [10] and of this work.

Next, I briefly describe the content of the present work. To get a better understanding about it its results will be compared with those of [8] where the one-dimensional case is considered, and also some results in [10] will be discussed.

In [8] the behavior of a scalar-valued stationary Gaussian random field $X(p)$, $p \in \mathbb{Z}^\nu$, with expectation $EX(p) = 0$ is investigated on the ν-dimensional integer lattice of the Euclidean space $\mathbb{R}^\nu$. For this goal it is useful to introduce the Hilbert space $\mathcal{H}$ consisting of those random variables with finite second moment which are measurable with respect to the σ-algebra generated by the random variables $X(p)$, $p \in \mathbb{Z}^\nu$, of our random field. Here the usual scalar product $\langle \xi, \eta \rangle = E\xi\eta$ is applied. The shift transforms in the underlying stationary random field induce a group of unitary operators in the Hilbert space $\mathcal{H}$ in a natural way, and we shall call the elements of this group shift transforms. We want to get a good description of this Hilbert space together with the shift transforms on it.

In [8] it is proved that this Hilbert space $\mathcal{H}$ has a natural representation as the direct sum $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \cdots$ of orthogonal subspaces $\mathcal{H}_n$, $0 \le n < \infty$, which are invariant subspaces of the shift transforms in the underlying stationary Gaussian random field, and the subspace $\mathcal{H}_n$ consists of those random that can be written in the form of an n-fold Wiener–Itô integral with respect to the random spectral measure of the underlying Gaussian random field. (Actually, [8] gives a more detailed description of the structure of the space $\mathcal{H}$ with the help of the so-called Fock space representation. The definition of the Fock space is given on page 28 of [8], and it is denoted by $\mathrm{Exp}\, \mathcal{H}_G$ there.) The proof of the above representation of the Hilbert space $\mathcal{H}$ is based on the so-called Itô formula which shows a relation between Hermite polynomials and multiple Wiener–Itô integrals. Another important result in [8], whose proof is also based on Itô's formula is a useful formula that

enables us to calculate the shift transforms of a random variable presented in the form of a multiple Wiener–Itô integral.

The above results help us to understand the most important properties of one-dimensional stationary Gaussian random fields. Vector-valued stationary Gaussian random fields show a similar behaviour, and their description is the main object of paper [10] and of this work.

In paper [10] the spectral measure and the random spectral measure of a vector-valued stationary Gaussian random field are defined. A class of *g*eneralized stationary Gaussian fields are also defined together with their spectral and random spectral measures. These measures play an important role in the limit theorems we want to study. The natural modifications of the results about scalar valued stationary Gaussian random fields mentioned in [8] remain valid in the vector-valued case both for classical and generalized stationary Gaussian random fields. In particular, the multiple Wiener–Itô integrals with respect to random spectral measures are defined in [10] also in the vector-valued case, and some important results are proved about them. One of these properties is the so-called diagram formula which enables us to rewrite the product of multiple Wiener–Itô integrals in the form of a sum of appropriately defined multiple Wiener–Itô integrals.

Now I turn to the discussion of the results of the present paper.

In Section 2 some properties of vector-valued stationary Gaussian random fields are proved with the help of the results in [10]. First, the formulation and proof of the multivariate version of Itô's formula are discussed. In this result, Wick polynomials, the multivariate generalizations of Hermite polynomials take the role of Hermite polynomials. At the start of Section 2 the definition and most important properties of Wick polynomials are recalled from [8]. Then the multivariate version of Itô's formula is proved by means of the adaptation of the method in the proof of its one-dimensional version. I discuss this proof in Appendix A. Here I explain the picture behind the definition of Wick polynomials, the idea of the proof of the Itô formula in the multivariate case, and why Wick polynomials appear in its formulation. Appendix A also contains a discussion about the proof of the remaining results of Section 2 with the help of Itô's formula.

In [8] a Hilbert space $\mathcal{H}$ was introduced with the help of a scalar valued stationary Gaussian random field. In Section 2 of this paper its version is defined when vector-valued stationary Gaussian random fields $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, with expectation $EX(p) = 0$, $p \in \mathbb{Z}^\nu$, are considered. This Hilbert space $\mathcal{H}$ is defined similarly to the case of scalar-valued random fields. It is the Hilbert space consisting of those random variables with finite second moment which are measurable with respect to the σ-algebra generated by the random vectors $X(p)$, $p \in \mathbb{Z}^\nu$, of our random field. Similarly to the one-dimensional case, there is a decomposition of this Hilbert space $\mathcal{H}$ to the direct product $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \cdots$ of orthogonal subspaces, $\mathcal{H}_n$, $0 \le n < \infty$, which are invariant subspaces of

the shift transformations in the underlying stationary random field. But in the case of vector-valued stationary Gaussian random fields we can prove only a weaker result about the behavior of the subspaces $\mathcal{H}_n$ than in the scalar-valued case. It is proved in Proposition 2.3 that the elements of an everywhere dense linear subspace of $\mathcal{H}_n$ can be presented in the form of a finite sum of multiple Wiener–Itô integrals of order n. More explicitly, the Wick polynomials of order n can be written as a finite sum of multiple Wiener–Itô integrals of order n, and they constitute an everywhere dense linear subspace of $\mathcal{H}_n$. On the other hand, we cannot represent all elements of $\mathcal{H}_n$ in such a form. But even this weaker result is sufficient for our purposes.

The last result of Section 2, Proposition 2.4, contains a useful formula about the calculation of the shift transforms of a random variable given in the form of a multiple Wiener–Itô integral. This formula is similar to the analogous result in the case of scalar-valued stationary random fields.

In Section 3 we deal with the question how the previously proved results can be applied in the investigation of limit theorems for non-linear functionals of vector-valued stationary Gaussian random fields.

In several interesting cases (and the problem investigated in paper [11] belongs to them) the limit problem we are interested in can be reformulated with the help of Itô's formula and an appropriate rescaling of certain multiple Wiener–Itô integrals we are working with to the study of a sequence of random variables presented in a very special form, and in Section 3 we are investigating limit theorems for such sequences of random variables.

We consider a sequence of random variables Z_N, $N = 1, 2, \dots$, presented as a finite sum of k-fold Wiener–Itô integrals with respect to a d-dimensional random spectral measure. We are interested in the behavior of such a sequence of random variables Z_N, $N = 1, 2, \dots$, whose elements are defined by formula (3.9) with the help of random spectral measures $Z_{G^{(N)}} = (Z_{G^{(N)},1}, \dots, Z_{G^{(N)},d})$ which correspond to some spectral measures $G^{(N)} = (G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, defined on some torus $[-A_N, A_N)^\nu$, where $A_N \to \infty$ as $N \to \infty$, and we integrate some kernel functions $h^{(N)}_{j_1,\dots,j_n}(x_1, \dots, x_d)$ with respect to them. Let us remark that the random variables Z_N introduced in (3.9) are sums of finitely many multiple Wiener–Itô integrals. Each element of this sum is indexed by some vector $(j_1, \dots, j_k)$, and the set of these indices does not depend on the parameter N. We expect that if both the spectral measures $G^{(N)}$ and the kernel functions $h^N_{(j_1,\dots,j_k)}$ appearing in the definition of the random variables Z_N behave nicely, then these random variables have a limit as $N \to \infty$. Somewhat more explicitly, we expect that if the spectral measures $G^{(N)}$ converge to a spectral measure $G^{(0)}$ of a d-dimensional (generalized) stationary Gaussian random field, and the kernel functions $h^N_{(j_1,\dots,j_k)}(x_1, \dots, x_d)$ converge to some nice functions $h^0_{(j_1,\dots,j_k)}(x_1, \dots, x_d)$ as $N \to \infty$, then the random variables Z_N converge in

distribution to the random variable Z_0 defined in (3.12) with the help of multiple Wiener–Itô integrals of the kernel functions $h^0_{(j_1,\dots,j_k)}(x_1,\dots,x_d)$ with respect to the random spectral measure $Z_{G^{(0)}}$ corresponding to the spectral measure $G^{(0)}$. Naturally, in the formulation of such a result we have to clarify what kind of limit should hold for the spectral measures $G^{(N)}$ and for the kernel functions $h^N_{(j_1,\dots,j_k)}(x_1,\dots,x_d)$.

The main result of Section 3 is Proposition 3.1. First the random variables Z_N, $N = 0, 1, 2, \dots$, mentioned in the above discussion are defined in formulas (3.9) and (3.12), and then Proposition 3.1 yields a sufficient condition for the weak convergence of the random variables Z_N to Z_0 as $N \to \infty$. This condition consists of two parts. The conditions of the first part demand that the spectral measures $G^{(N)}$ should converge to the spectral measure $G^{(0)}$ and the kernel functions $h^N_{(j_1,\dots,j_k)}(x_1,\dots,x_d)$ should converge to $h^0_{(j_1,\dots,j_k)}(x_1,\dots,x_d)$ as $N \to \infty$ in an appropriate way.

But to satisfy the desired weak convergence still another condition must be formulated. This is condition (b) of Proposition 3.1 which is a compactness type condition. Heuristically saying it demands the existence of a compact set in $\mathbb{R}^{k\nu}$ such that all multiple Wiener–Itô integrals appearing in the definition of the random variables Z_N, $N = 1, 2, \dots$, are almost completely concentrated in this compact set. In Appendix B the role of this condition is explained with the help of some examples.

The condition on the limiting behavior of the functions $h^N_{(j_1,\dots,j_k)}(x_1,\dots,x_d)$ is formulated in condition (a) of Proposition 3.1. The condition on the convergence of the spectral measures $G^{(N)}$ is also formulated there, and this deserves special attention.

In the conditions of Proposition 3.1 we demand the existence of a (generalized) spectral measure $G^{(0)} = (G^{(0)}_{j,j'})$ such that the coordinates $G^{(N)}_{j,j'}$ of the spectral measures $G^{(N)} = (G^{(N)}_{j,j'})$, $N = 0, 1, 2, \dots$, $1 \le j, j' \le d$, converge to the corresponding coordinates of $G^{(0)}$ in an appropriate way. We also demand that $G^{(0)}$ should be the spectral measure of such a stationary random field which belongs to the class of generalized stationary Gaussian random fields constructed in Section 4 of [10]. This implies in particular that the coordinates $G^{(0)}_{j,j'}$ of the spectral measure $G^{(0)}$ are complex measures on $\mathbb{R}^\nu$ with locally finite total variation. (This notion was defined in Section 4 of [10]). In Proposition 3.1 we demand that the coordinates $G^{(N)}_{j,j'}$ of the matrices $G^{(N)}$ should vaguely converge to the coordinate $G^{(0)}_{j,j'}$ of $G^{(0)}$ as $N \to \infty$ for all indices $1 \le j, j' \le d$ in the space of complex measures on $\mathbb{R}^\nu$ with locally finite total variation. The vague convergence in the space complex measures with locally finite total variation is defined in Section 3 of this paper.

In Lemma 8.3 of [8], where the scalar-valued version of Proposition 3.1 is formulated the notion of vague convergence also appears. But in that case

vague convergence is defined in the space of locally finite (non-negative) measures. The definitions of vague convergence in these two cases are very similar. But there is some difference between them that may be useful to explain.

In the definition of [8] the vague limit is a locally finite measure, which is a (possibly infinite) measure on the σ-algebra of the measurable sets in the space $\mathbb{R}^\nu$. In the definition of this paper the limit is a complex measure with locally finite total variation. It may happen that this limit cannot be extended to a (complex valued) measure on the σ-algebra of the measurable sets in the space $\mathbb{R}^\nu$.

Section 3 contains still another result. This is Lemma 3.2 which may be useful in the applications of Proposition 3.1. It states that if all coordinates $G^{(N)}_{j,j'}$ of a sequence of spectral measures $G^{(N)} = (G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, $N = 1, 2, \dots$, converge vaguely to some complex measures $G^{(0)}_{j,j'}$ with finite total variation, then also $G^{(0)} = (G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite matrix valued even measure on $\mathbb{R}^\nu$.

This result is useful, because it helps us to decide when the limit matrix $G^{(0)} = (G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, is the spectral measure of a (generalized) stationary Gaussian random field. (See the remark after the formulation of Lemma 3.2).

In this paper together with [10] and also in the work [8] a version of the multiple Wiener–Itô integrals introduced by Itô in his paper [7] is applied. (Itô called these integrals multiple Wiener integrals in his paper.) Here a version of this random integral introduced by Dobrushin in [4] is applied and its generalization for vector-valued stationary Gaussian random fields. At the end of this introduction I give a short explanation why it was useful to work with this version of the random integrals introduced by Itô in [7], and I also briefly mention another example in which a multidimensional time generalization of Itô's integral was introduced in order to study some problems.

Itô considered in his paper [7] a measure space $(T, \mathcal{B}, m)$, and a set of (jointly) Gaussian random variables $\beta(E)$ indexed by such sets $E \in \mathcal{B}$ for which $m(E) < \infty$, and the joint distribution of these random variables is determined by the relations $E\beta(E) = 0$, and $E\beta(E)\beta(E') = m(E \cap E')$. He also imposed the following continuity property for the measure m. For any set $E \in \mathcal{B}$ with $m(E) < \infty$ and $\varepsilon > 0$ there exists a finite decomposition $E = \sum_{i=1}^n E_i$ such that $m(E_i) < \varepsilon$ for all $i = 1, \dots, n$. Itô defined the p-fold Wiener integrals of square integrable functions (with respect to the product measure m^p) with respect to the random measure $\beta(E, \omega)$ for all $p = 0, 1, \dots$ In the definition of the random measure $\beta(E, \omega)$ the parameter set consists of those measurable sets E for which $m(E) < \infty$. He gave a useful representation of all square integrable random variables measurable

with respect to the σ-algebra generated by the random variables $\beta(E, \omega)$ as a sum of multiple Wiener integrals with different multiplicity.

To give such a representation Itô proved some useful results about the properties of multiple Wiener integrals. In particular, he proved an identity that was later called Itô's formula. He showed with its help an important relation between multiple Wiener integrals and Hermite polynomials.

In paper [4] Dobrushin introduced for the sake of the investigation of non-linear functionals of stationary Gaussian random fields such a version of the random integrals in Itô's paper [7], where he worked in the Euclidean space $(\mathbb{R}^\nu, \mathcal{B})$, and he replaced the random measure $\beta(E, \omega)$ by the random spectral measure of a stationary Gaussian random field. Dobrushin's results are described in more detail in [8]. This paper together with [10] yield a generalization of these results when vector-valued stationary Gaussian random fields are considered. They show that results similar to those of the paper [7] hold if we work with random spectral measures instead of the random measure $\beta(E, \omega)$ applied in [7].

I try to explain the advantage of this approach. By working with random spectral measures instead of the random measures $\beta(\cdot)$ applied in [7] some useful Fourier analysis type results can be proved. Proposition 2.4 of this paper is an example for it. Here the shift transform T_u of a random variable given in the form of a multiple Wiener–Itô integral is expressed in a useful form which shows some similarity to the Fourier transform. Formulas (2.6) and (2.8) in Theorem 2.4 show how to express the shift transform of a random variable given in form of a multiple Wiener–Itô integral. This result together with Itô's formula which enables us very often to rewrite the random variables we are working with as sums of multiple Wiener–Itô integrals may help in the study of limit theorems. The discussion at the beginning of Section 3 is an example for it.

The application of multiple stochastic integrals turned out to be useful also in the investigation of some other problems. I briefly mention the contribution of Wong–Zakai–Yor to the theory of multiple stochastic integrals as an example for it. They discussed the following problem. The study of Itô integrals is closely related to the study of martingales. For example, any continuous-time square integrable martingale with continuous trajectories adapted to a Wiener process has a canonical representation as an Itô integral with respect to the underlying Wiener process. The above-mentioned mathematicians were looking for the multidimensional time version of this result. Naturally, to formulate it first the multidimensional time Wiener processes and martingales had to be defined. But these objects are defined in the literature. I would remark that the definition of the multidimensional time martingales demands special attention.

A multidimensional version of the result about the canonical representation of square-integrable martingales can be proved. But in that representation, not only the Itô integrals (integrals with multidimensional time which also have to be defined) but also multiple Itô integrals appear. The precise

formulation of this result would demand a long explanation, hence I omit it. The interested reader can find it in paper [6] and in its list of references. This work also discusses with the help of this result a statistical problem, where the main point of the solution is the calculation of a Radon–Nikodym derivative.

2. Wick polynomials and their relation to multiple Wiener–Itô integrals

In the case of scalar-valued stationary Gaussian random fields (i.e., if $d = 1$) there is a so-called Itô formula (see Theorem 4.3 in [8]) which shows an important relation between multiple Wiener–Itô integrals and Hermite polynomials. Here I present its multivariate version, where Wick polynomials take the role of the Hermite polynomials. Wick polynomials are the natural multi-dimensional generalizations of Hermite polynomials. I shall also discuss an important consequence of the multivariate version of the Itô formula. This formula enables us to present a large class of random variables in the form of a sum of multiple Wiener–Itô integrals. Besides, there is a useful formula for the calculation of the shift transforms of such random variables which are given in the form of a sum of multiple Wiener–Itô integrals. As we shall see, this formula is very useful in the study of limit theorems for non-linear functionals of a vector-valued stationary Gaussian field. I shall explain in the first part of the Appendix the relation between the proof of Itô's formula in the scalar and in the vector-valued case. In that explanation I also write about the definition of the Wick polynomials and their role in the proof.

First I recall the definition of Wick polynomials and some results about their most important properties. This explanation is based on the results in Section 2 of [8].

Let X_t, $t \in T$, be a set of jointly Gaussian random variables indexed by a parameter set T, and such that $EX_t = 0$ for all $t \in T$. We define the following real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}$. A square integrable (real valued) random variable is in $\mathcal{H}$ if and only if it is measurable with respect to the σ-algebra $\mathcal{B} = \mathcal{B}(X_t,\ t \in T)$, and the scalar product in $\mathcal{H}$ is defined as $\langle \xi, \eta \rangle = E\xi\eta$, $\xi, \eta \in \mathcal{H}$. The Hilbert space $\mathcal{H}_1 \subset \mathcal{H}$ is the subspace of $\mathcal{H}$ generated by the finite linear combinations $\sum c_j X_{t_j}$, $t_j \in T$, with real coefficients. We consider only such sets of Gaussian random variables X_t for which $\mathcal{H}_1$ is separable. Otherwise X_t, $t \in T$, can be arbitrary, but the most interesting case for us is when $T = \mathbb{Z}^\nu \times \{1, \dots, d\}$, and the original Gaussian random variables we are working with are the coordinates $X_j(p)$, $j \in \{1, \dots, d\}$, $p \in \mathbb{Z}^\nu$, of a vector-valued stationary Gaussian random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$.

To define the Wick polynomials and to get their most important properties we need the following result formulated in Theorem 2.1 of [8].

Theorem 2A. *Let $Y_1, Y_2, \dots$ be an orthonormal basis in the Hilbert space $\mathcal{H}_1$ defined above with the help of a set of Gaussian random variables X_t, $t \in T$. Then the set of all possible finite products $H_{j_1}(Y_{l_1}) \cdots H_{j_k}(Y_{l_k})$ is a complete orthogonal system in the Hilbert space $\mathcal{H}$ defined above. (Here, and in the subsequent discussion $H_j(\cdot)$ denotes the j-th Hermite polynomial with leading coefficient 1.)*

Let $\mathcal{H}_{\le n} \subset \mathcal{H}$, $n = 1, 2, \dots$ (with the previously introduced Hilbert space $\mathcal{H}$) denote the linear subspace of the Hilbert space $\mathcal{H}$ which is the closure of the linear space consisting of the elements $P_n(X_{t_1}, \dots, X_{t_m})$, where P_n runs through all polynomials of degree less than or equal to n, and the integer m and indices $t_1, \dots, t_m \in T$ are arbitrary. Let $\mathcal{H}_0 = \mathcal{H}_{\le 0}$ consist of the constant functions, and let $\mathcal{H}_n = \mathcal{H}_{\le n} \ominus \mathcal{H}_{\le n-1}$, $n = 1, 2, \dots$, where $\ominus$ denotes orthogonal completion. It is clear that the Hilbert space $\mathcal{H}_1$ given in this definition agrees with the previously defined Hilbert space $\mathcal{H}_1$. If $\xi_1, \dots, \xi_m \in \mathcal{H}_1$, and $P_n(x_1, \dots, x_m)$ is a polynomial of degree n, then $P_n(\xi_1, \dots, \xi_m) \in \mathcal{H}_{\le n}$. Theorem 2A implies that

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \cdots , \tag{2.1}$$

where + denotes direct sum. Now I present the definition of Wick polynomials.

Definition of Wick polynomials. *Let $P(x_1, \dots, x_m)$ be a homogeneous polynomial of degree n, and let a set of (jointly Gaussian) random variables $\xi_1, \dots, \xi_m \in \mathcal{H}_1$ be given. The Wick polynomial $:P(\xi_1, \dots, \xi_m):$ determined by them is the orthogonal projection of the random variable $P(\xi_1, \dots, \xi_m)$ to the above-defined subspace $\mathcal{H}_n$ of the Hilbert space $\mathcal{H}$. The Wick polynomial of a homogeneous polynomial of degree n will be called a Wick polynomial of order n.*

In the sequel we shall use the notation $:P(\xi_1, \dots, \xi_m):$ for the Wick polynomial corresponding to a homogeneous polynomial $P(x_1, \dots, x_m)$ with arguments $\xi_1, \dots, \xi_m$, $\xi_j \in \mathcal{H}_1$ for all $1 \le j \le m$. It may happen that a random variable ζ can be expressed in two different forms as a homogeneous polynomial of some random variables from $\mathcal{H}_1$, i.e., $\zeta = P_1(\xi_1, \dots, \xi_m)$, and $\zeta = P_2(\xi_1, \dots, \xi_m)$, and $P_1 \neq P_2$. But in such a case

$$:P_1(\xi_1, \dots, \xi_m): \; =: P_2(\xi_1, \dots, \xi_m): ,$$

i.e., the value of a Wick polynomial $:P(\xi_1, \dots, \xi_m):$ does not depend on the representation of the random variable $P(\xi_1, \dots, \xi_m)$.

It is clear that Wick polynomials of different degrees are orthogonal. Given some $\xi_1, \dots, \xi_m \in \mathcal{H}_1$ define the subspaces $\mathcal{H}_{\le n}(\xi_1, \dots, \xi_m) \subset \mathcal{H}_{\le n}$, $n = 1, 2, \dots$, as the set of all polynomials of the random variables $\xi_1, \dots, \xi_m$ with degree less than or equal to n. Let $\mathcal{H}_{\le 0}(\xi_1, \dots, \xi_m) = \mathcal{H}_0(\xi_1, \dots, \xi_m) = \mathcal{H}_0$, and $\mathcal{H}_n(\xi_1, \dots, \xi_m) = \mathcal{H}_{\le n}(\xi_1, \dots, \xi_m) \ominus \mathcal{H}_{\le n-1}(\xi_1, \dots, \xi_m)$. With the help of this notation I formulate the following result given in Proposition 2.2 of [8].

Theorem 2B. *Let $P(x_1,\dots,x_m)$ be a homogeneous polynomial of degree n. Then $:P(\xi_1,\dots,\xi_m):$ equals the orthogonal projection of $P(\xi_1,\dots,\xi_m)$ to $\mathcal{H}_n(\xi_1,\dots,\xi_m)$.*

This result has the following important consequences formulated in Corollaries 2.3 and 2.4 in [8].

Corollary 2C. *Let $\xi_1,\dots,\xi_m$ be an orthonormal system in $\mathcal{H}_1$, and let*

$$P(x_1,\dots,x_m)=\sum c_{j_1,\dots,j_m}x^{j_1}\cdots x_m^{j_m}$$

be a homogeneous polynomial, i.e., let $j_1+\cdots+j_m=n$ with some fixed number n for all sets $(j_1,\dots,j_m)$ appearing in this summation. Then

$$:P(\xi_1,\dots,\xi_m):\ =\sum c_{j_1,\dots,j_m}H_{j_1}(\xi_1)\cdots H_{j_m}(\xi_m).$$

In particular,

$$:\xi^n:\ =H_n(\xi)\quad \text{if } \xi\in\mathcal{H}_1,\ \text{and } E\xi^2=1.$$

Corollary 2D. *Let $\xi_1,\xi_2,\dots$ be an orthonormal basis in $\mathcal{H}_1$. Then the random variables $H_{j_1}(\xi_1)\cdots H_{j_k}(\xi_k)$, $k=1,2,\dots$, $j_1+\cdots+j_k=n$, form a complete orthogonal basis in $\mathcal{H}_n$.*

In the proof of the Itô formula for scalar-valued stationary random fields we needed, besides the diagram formula, the following important recursive formula for Hermite polynomials which is contained for example in Lemma 5.2 of [8].

$$H_n(x)=xH_{n-1}(x)-(n-1)H_{n-2}(x)\quad \text{for}\ \ n=1,2,\dots, \tag{2.2}$$

with the notation $H_{-1}(x)\equiv 0$ in the case $n=1$.

In the next result I formulate a multivariate version of this formula for Wick polynomials.

Proposition 2.1. *Let $U_1,\dots,U_{n+1}$, $n\ge 1$, be elements in $\mathcal{H}_1$. Then*

$$\begin{aligned}&:U_1\cdots U_n:U_{n+1}\\ &\quad =:U_1\cdots U_nU_{n+1}:\ +\sum_{s=1}^{n}:U_1\cdots U_{s-1}U_{s+1}\cdots U_n:EU_sU_{n+1}.\end{aligned}\tag{2.3}$$

In the special case $n=1$ this formula is meant as $U_1U_2=:U_1U_2:\ +EU_1U_2$.

Proof of Proposition 2.1. Formula (2.3) clearly holds if all random variables U_j, $1\le j\le n+1$ agree, and $EU_1^2=1$, since in this case the left-hand side of (2.3) equals $U_1H_n(U_1)$, while its right-hand side equals $H_{n+1}(U_1)+nH_{n-1}(U_1)$ by Corollary 2C, and these two expressions are equal by formula (2.2). A somewhat more complicated, but similar argument shows that this formula also holds if the sequence $U_1,\dots,U_n$ consists of some independent random variables $V_1\dots,V_k$ with standard normal distribution, the random variable V_p is contained in the sequence $U_1,\dots,\ U_n$ with multiplicity l_p, $1\le p\le k$, and finally U_{n+1} is either one of these random variables V_p,

$1 \le p \le k$, or it is a random variable V_{k+1} with standard normal distribution which is independent of all of them.

Indeed, if $U_{n+1} = V_p$ with some $1 \le p \le k$, then the left-hand side of (2.3) equals

$$H_{l_1}(V_1) \cdots H_{l_k}(V_k) V_p,$$

while the right-hand side equals

$$\begin{aligned} &H_{l_1}(V_1) \cdots H_{l_{p-1}}(V_{p-1}) H_{l_p+1}(V_p) H_{l_{p+1}}(V_{p+1}) \cdots H_{l_k}(V_k) \\ &\quad + l_p\, H_{l_1}(V_1) \cdots H_{l_{p-1}}(V_{p-1}) H_{l_p-1}(V_p) H_{l_{p+1}}(V_{p+1}) \cdots H_{l_k}(V_k) \end{aligned}$$

by Corollary 2C. A comparison of these expressions together with relation (2.2) imply that identity (2.3) holds in this case. If $U_{n+1} = V_{k+1}$, then the left-hand side of (2.3) equals

$$H_{l_1}(V_1) \cdots H_{l_k}(V_k) V_{k+1},$$

and the right-hand side also equals $H_{l_1}(V_1) \cdots H_{l_k}(V_k) V_{k+1}$. Hence formula (2.3) holds in this case, too.

In the general case we can choose some independent Gaussian random variables $Z_1, \dots, Z_m$ in $\mathcal{H}_1$ with variance 1 in such a way that our random variables $U_1, \dots, U_{n+1}$ can be expressed as their linear combination, i.e., $U_p = \sum_{l=1}^m c_{p,l} Z_l$ with some coefficients $c_{l,m}$. We have already seen that formula (2.3) is valid in the special case when all random variables U_p equal one of the random variables Z_j, i.e., if $U_p = Z_{j(p)}$ with some $1 \le j(p) \le m$ for all $1 \le p \le n+1$. Since the expressions of both sides of (2.3) are multi-linear functionals on the n-fold direct product $\mathcal{H}_1 \times \cdots \times \mathcal{H}_1$, this implies that formula (2.3) also holds for the random variables $U_1, \dots, U_{n+1}$. Proposition 2.1 is proved. □

We can prove the multivariate version of Itô's formula with the help of Proposition 2.1 and the diagram formula for multiple Wiener–Itô integrals for vector-valued stationary Gaussian random fields formulated in Section 6 of [10].

Before its formulation, I make a remark about the notation in this section.

In the formulation of Itô's formula the notation $\mathcal{K}_{1,j}$ appears. This notion was introduced in Lemma 3.2 of [10]. It is a real Hilbert space, and it contains those functions u on the torus $[-\pi, \pi)^\nu$ for which we defined the random integral $\int u(x) Z_{G,j}(\,dx)$ with respect to the j-th coordinate $Z_{G,j}$ of the random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$, and the value of this integral is a real valued random variable. In Section 4 of [10] this Hilbert space is defined also in the case of generalized random spectral measure. Later, at the beginning of Section 5 of [10] its multidimensional generalization, the real Hilbert space $\mathcal{K}_{n,j_1,\dots,j_n} = \mathcal{K}_{n,j_1,\dots,j_n}(G_{j_1,j_1} \dots, G_{j_n,j_n})$ is defined for all $n = 1, 2, \dots$. It consists of those functions $f(x_1, \dots, x_n)$ for which the n-fold (real valued) Wiener–Itô integral

$$I_n(f|j_1, \dots, j_n) = \int f(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(d\,x_n)$$

is defined. The Hilbert space $\mathcal{K}_{1,j}$ is a special case of these Hilbert spaces with $n = 1$. Later the Hilbert spaces $\mathcal{K}_{n,j_1,\dots,j_n}$ also appear in this work. At some points I shall also work with the class of simple functions $\hat{\mathcal{K}}_{n,j_1,\dots,j_n} \subset \mathcal{K}_{n,j_1,\dots,j_n}$ defined also in Section 5 of [10]. The multiple Wiener–Itô integrals were first defined for simple functions which are adapted to some regular system (defined also in Section 5 of [10]), and the multiple Wiener–Itô integrals were defined in the general case by means of a good approximation of the functions $f \in \mathcal{K}_{n,j_1,\dots,j_n}$ by simple functions $f \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$.

Theorem 2.2. Multivariate version of Itô's formula. *Let us have some vector-valued stationary Gaussian random field with a vector-valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$. Let us consider some functions $\varphi_p \in \mathcal{K}_{1,j_p}$, $1 \le p \le n$, $1 \le j_p \le d$, and define with their help the random variables $U_p = \int \varphi_p(x) Z_{G,j_p}(\,dx) \in \mathcal{H}_1$, $1 \le p \le n$. The identity*

$$
\begin{aligned}
&:U_1 \cdots U_n: \qquad (2.4)\\
&\quad = \int \varphi_1(x_1)\varphi_2(x_2)\cdots\varphi_n(x_n) Z_{G,j_1}(\,dx_1) Z_{G,j_2}(\,dx_2)\cdots Z_{G,j_n}(\,dx_n)
\end{aligned}
$$

holds.

Proof of Theorem 2.2. Relation (2.4) clearly holds for $n = 1$. We prove by induction that it holds for $n+1$ if it holds for $k \le n$. In the proof we apply the Corollary of Theorem 6.1 from [10] (i.e., the corollary of the diagram formula from that paper) with the choice

$$
\begin{aligned}
h_1(x_1, \dots, x_n) &= \varphi_1(x_1)\cdots\varphi_n(x_n),\\
h_2(x) &= \varphi_{n+1}(x),
\end{aligned}
$$

and the random spectral measure $Z_{G,j_1'}$ is chosen as $Z_{G,j_1'} = Z_{G,j_{n+1}}$, where $Z_{G,j_{n+1}}$ is the random spectral measure appearing in the definition of U_{n+1}. We can write with this choice the identity

$$
\begin{aligned}
&\int \varphi_1(x_1)\cdots\varphi_n(x_n) Z_{G,j_1}(\,dx_1)\cdots Z_{G,j_n}(\,dx_n) \int \varphi_{n+1}(x) Z_{G,j_{n+1}}(\,dx) \qquad (2.5)\\
&\quad = \int \varphi_1(x_1)\cdots\varphi_n(x_n)\varphi_{n+1}(x_{n+1}) Z_{G,j_1}(\,dx_1)\cdots Z_{G,j_{n+1}}(\,dx_{n+1})\\
&\qquad + \sum_{p=1}^{n} EU_pU_{n+1} \int \varphi_1(x_1)\cdots\varphi_{p-1}(x_{p-1})\varphi_{p+1}(x_p)\cdots\varphi_n(x_{n-1})\\
&\qquad\quad Z_{G,j_1}(\,dx_1)\cdots Z_{G,j_{p-1}}(\,dx_{p-1}) Z_{G,j_{p+1}}(\,dx_p)\cdots Z_{G,j_n}(\,dx_{n-1}),
\end{aligned}
$$

since formula (6.19) in [10] gives this identity with our choice of h_1 and h_2. To see this observe that with these functions h_1 and h_2 the function h_{γ_p} in

the formulation of the corollary of Theorem 6.1 in [10] equals for $p \neq 0$

$$\begin{aligned} h_{\gamma_p}(x_1, \dots, x_{n-1}) &= \varphi_1(x_1) \cdots \varphi_{p-1}(x_{p-1}) \varphi_{p+1}(x_p) \cdots \varphi_n(x_{n-1}) \\ &\qquad \int \varphi_p(x_n) \overline{\varphi_{n+1}(x_n)} G_{j_p, j_{n+1}}(\, dx_n) \\ &= \varphi_1(x_1) \cdots \varphi_{p-1}(x_{p-1}) \varphi_{p+1}(x_p) \cdots \varphi_n(x_{n-1}) E U_p U_{n+1} \end{aligned}$$

since $U_{n+1} = \overline{U_{n+1}}$, and by formula (3.7) in [10] which expresses the scalar product of two one-fold random integrals

$$\begin{aligned} E U_p U_{n+1} &= E U_p \overline{U_{n+1}} = E \left(\int \varphi_p(x) Z_{G, j_p}(\, dx) \overline{\int \varphi_{n+1}(x) Z_{G, j_{n+1}}(\, dx)} \right) \\ &= \int \varphi_p(x_n) \overline{\varphi_{n+1}(x_n)} G_{j_p, j_{n+1}}(\, dx_n), \end{aligned}$$

and for $p = 0$

$$h_{\gamma_0}(x_1, \dots, x_{n+1}) = \varphi_1(x_1) \cdots \varphi_n(x_n) \varphi_{n+1}(x_{n+1}).$$

Corollary of Theorem 6.1 in [10] with the above form of the kernel functions h_{γ_p}, $0 \le p \le n$, in it imply formula (2.5). Formula (2.5) together with our induction hypothesis imply that

$$\begin{aligned} &\int \varphi_1(x_1) \cdots \varphi_n(x_n) \varphi_{n+1}(x_{n+1}) Z_{G, j_1}(\, dx_1) \cdots Z_{G, j_{n+1}}(\, dx_{n+1}) \\ &\qquad =: U_1 \cdots U_n :\, U_{n+1} - \sum_{p=1}^{n} : U_1 \cdots U_{p-1} U_{p+1} \cdots U_n :\, E U_p U_{n+1}. \end{aligned}$$

In the case $n = 1$ this formula means that

$$\int \varphi_1(x_1) \varphi_2(x_2) Z_{G, j_1}(\, dx_1) Z_{G, j_2}(\, dx_2) = U_1 U_2 - E U_1 U_2.$$

By comparing the last formula with (2.3) we get that the statement of Theorem 2.2 holds also for $n + 1$. Theorem 2.2 is proved. □

In Theorem 2.2 we rewrote some Wick polynomials of special form as multiple Wiener–Itô integrals. This enables us to express a sum of such Wick polynomials as the sum of multiple Wiener–Itô integrals. This implies that all Wick polynomials of random variables from some $\mathcal{H}_{1,j}$, $1 \le j \le d$, can be written in the form of a sum of multiple Wiener–Itô integrals. (The real Hilbert space $\mathcal{H}_{1,j}$, the real part of the Hilbert space $\mathcal{H}^c_{1,j}$ was defined in Section 3 of [10] together with $\mathcal{K}_{1,j}$.) In the next simple corollary of Theorem 2.2 I describe this result in a more explicit form.

To formulate this result let us introduce the following notation. Let us fix some numbers $n \ge 1$ (the order of the homogeneous polynomial we are considering), $m \ge 1$ and some functions $\varphi_{j,k}(x) \in \mathcal{K}_{1,j}$, $1 \le j \le d$, $1 \le k \le m$, and define the random variables

$$\xi_{j,k} = \int \varphi_{j,k}(x) Z_{G,j}(\, dx), \quad 1 \le j \le d,\ 1 \le k \le m.$$

Then $\xi_{j,k} \in \mathcal{H}_{1,j}$. (We defined the real Hilbert space $\mathcal{H}_{1,j}$ in the formulation of Lemma 3.2 of [10]. This Lemma 3.2 stated that the elements of $\mathcal{H}_{1,j}$ can be given in the form of the above integral.)

In the next corollary, we consider homogeneous polynomials of these random variables $\xi_{j,k}$, and express the Wick polynomials corresponding to them in the form of a sum of multiple Wiener–Itô integrals.

Corollary of Theorem 2.2. *Let us consider a homogeneous polynomial*

$$P(x_{j_s,k_s},\ 1 \le j_s \le d,\ 1 \le k_s \le m \text{ for all } 1 \le s \le n) = \sum_{\substack{1 \le j_s \le d \text{ for all } 1 \le s \le n \\ 1 \le k_s \le m \text{ for all } 1 \le s \le n}} a_{j_1,k_1,\dots j_n,k_n} x_{j_1,k_1} x_{j_2,k_2} \cdots x_{j_n,k_n}$$

of order n of the variables x_{j_s,k_s} with indices $1 \le j_s \le d$ and $1 \le k_s \le m$ for all $1 \le s \le n$ and real coefficients $a_{j_1,k_1,\dots,j_n,k_n}$.

If we replace the variables x_{j_s,k_s} with the random variables

$$\xi_{j_s,k_s} = \int \varphi_{j_s,k_s}(x) Z_{G,j_s}(\,dx)$$

in this polynomial (we choose a function $\varphi_{j,k} \in \mathcal{K}_{1,j}$ in the definition of $\xi_{j,k}$), then we get the following homogeneous polynomial of some jointly Gaussian random variables.

$$P(\xi_{j_s,k_s},\ 1 \le j_s \le d,\ 1 \le k_s \le m \text{ for all } 1 \le s \le n) = \sum_{\substack{1 \le j_s \le d \text{ for all } 1 \le s \le n \\ 1 \le k_s \le m \text{ for all } 1 \le s \le n}} a_{j_1,k_1,\dots j_n,k_n} \xi_{j_1,k_1} \xi_{j_2,k_2} \cdots \xi_{j_n,k_n}.$$

With the help of this expression we can define the Wick polynomial

$$:P(\xi_{j_s,k_s},\ 1 \le j_s \le d,\ 1 \le k_s \le m \text{ for all } 1 \le s \le n):\,.$$

This Wick polynomial can be expressed as a sum of multiple Wiener–Itô integrals in the following way.

Let us consider for all sequences of indices $\{(j_s,k_s)\colon\ 1 \le s \le n\}$ with $1 \le j_s \le d$, $1 \le k_s \le d$ for all $1 \le s \le n$ the function

$$f_{j_1,k_1,\dots,j_n,k_n}(x_1,\dots,x_n) = \varphi_{j_1,k_1}(x_1) \cdots \varphi_{j_n,k_n}(x_n) \in \mathcal{K}_{n,j_1,\dots,j_n}$$

and the multiple Wiener–Itô integral

$$I_n(f_{j_1,k_1,\dots,j_n,k_n} | j_1,\dots,j_n) = \int f_{j_1,k_1,\dots,j_n,k_n}(x_1,\dots,x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(\,dx_n).$$

The identity

$$:P(\xi_{j_s,k_s},\ 1 \le j_s \le d,\ 1 \le k_s \le m \text{ for all } 1 \le s \le n): = \sum_{\substack{1 \le j_s \le d \text{ for all } 1 \le s \le n \\ 1 \le k_s \le m \text{ for all } 1 \le s \le n}} a_{j_1,k_1,\dots j_n,k_n} I_n(f_{j_1,k_1,\dots,j_n,k_n} | j_1,\dots,j_n)$$

holds.

Remark. Theorem 4.7 of [8] contains a version of this result for scalar-valued stationary Gaussian random fields.

Proof of the Corollary of Theorem 2.2. By Theorem 2.2 we have

$$a_{j_1,k_1,\dots j_n,k_n} : \xi_{j_1,k_1}\xi_{j_2,k_2}\cdots\xi_{j_n,k_n} : \; = a_{j_1,k_1,\dots j_n,k_n} I_n(f_{j_1,k_1,\dots,j_n,k_n}|j_1,\dots,j_n)$$

for all sequences of indices (j_s,k_s), $1 \le s \le n$. By summing up these inequalities for all sequences of indices we get the proof of the corollary. □

With the help of the above corollary, we prove the following result.

Proposition 2.3. *For all $n \ge 1$ and functions $f \in \mathcal{K}_{n,j_1,\dots,j_n}$ with some indices $1 \le j_s \le d$, $1 \le s \le n$, $I_n(f|j_1,\dots,j_n) \in \mathcal{H}_n$ for the n-fold Wiener–Itô integral $I_n(f|j_1,\dots,j_n)$. Besides, the set of all sums of n-fold Wiener–Itô integrals i.e., the set of all sums of the form*

$$\sum_{1\le j_s\le d \text{ for all } 1\le s\le n} I_n(f_{j_1,\dots,j_n}|j_1,\dots,j_n),$$

where $f_{j_1,\dots,j_n} \in \mathcal{K}_{n,j_1,\dots,j_n}$ constitute an everywhere dense linear subspace of $\mathcal{H}_n$.

Proof of Proposition 2.3. We shall prove Proposition 2.3 by induction with respect to n. By Lemma 3.2 of [10] Proposition 2.3 holds for $n = 1$. Indeed, by this result every random variable of the form $\xi = \sum_{j=1}^d \xi_j$ with some $\xi_j \in \mathcal{H}_{1,j}$ can be written as the sum of one-fold Wiener–Itô integrals, and the random variables of this form constitute an everywhere dense linear subspace of $\mathcal{H}_1$.

If the statements of Proposition 2.3 hold for all $m < n$, then we can say for one part that $I_n(f|j_1,\dots,j_n) \in \mathcal{H}_{\le n}$, because this relation holds if f is a simple function, i.e., if $f \in \hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ with the space $\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ defined in Section 5 of [10], and since $\hat{\mathcal{K}}_{n,j_1,\dots,j_n}$ is dense in $\mathcal{K}_{n,j_1,\dots,j_n}$, and we defined the multiple Wiener–Itô integral by the extension of a bounded operator in the general case, the above property remains valid for general functions $f \in \mathcal{K}_{n,j_1,\dots,j_n}$. Moreover, we know that $I_n(f|j_1,\dots,j_n)$ is orthogonal to all multiple Wiener–Itô integrals of the form $I_m(h|j'_1,\dots,j'_m)$ with $m < n$ because of relation (5.5) in [10]. Then $I_n(f|j_1,\dots,j_n)$ is also orthogonal to any linear combination of such integrals. But these linear combinations constitute an everywhere dense set in $\mathcal{H}_m$ by our inductive hypothesis. Hence $I_n(f|j_1,\dots,j_n)$ is orthogonal to the whole space $\mathcal{H}_m$ for all $0 \le m \le n-1$, and this implies that it is contained in the Hilbert subspace $\mathcal{H}_n$ (and not only in $\mathcal{H}_{\le n}$). It follows from the corollary of Theorem 2.2 that the sums of multiple Wiener–Itô integrals considered in Proposition 2.3 are dense in $\mathcal{H}_n$, and they constitute a linear subspace. Indeed, this corollary implies that a large class of Wick polynomials of order n can be expressed as a sum of such integrals, and the class of these Wick polynomials of order n is dense in $\mathcal{H}_n$. Proposition 2.3 is proved. □

Remark. In Proposition 2.3 we expressed a dense subset of $\mathcal{H}_n$ as a sum of n-fold Wiener–Itô integrals, but we did not express all elements of $\mathcal{H}_n$ in such a form. But even this weaker result suffices for our purposes.

In the case of scalar-valued stationary random fields, we have a stronger result. In that case, we can express all elements of $\mathcal{H}_n$ as an n-fold Wiener–Itô integral, and actually, we can say somewhat more. There is a so-called Fock space representation of all elements $h \in \mathcal{H}$, which represents the elements $h \in \mathcal{H}$ in the form of a sum of multiple Wiener–Itô integrals of different multiplicity. (See Theorem 4.2 of [8] together with the definition of Fock spaces on page 28 of [8].) Moreover, this result has some useful consequences about the properties of this representation.

We cannot prove a similar result in the vector-valued case. This difference appears because of the following reason. If a sequence of random variables $h_N \in \mathcal{H}_n$, $N = 0, 1, 2, \dots$, has the property that $h_N \to h_0$ with some $h_0 \in \mathcal{H}_n$ in the norm of $\mathcal{H}_n$ as $N \to \infty$ in the scalar-valued case, then these random variables h_N can be expressed as n-fold Wiener–Itô integrals of such functions $k_N \in \mathcal{K}_n$ for which $k_N \to k_0$ in the norm of $\mathcal{K}_n$. On the other hand, in the case of vector-valued models we do not have a similar result.

Next, we consider a vector-valued stationary Gaussian random field

$$X(p) = (X_1(p), \dots, X_d(p)), \quad p \in \mathbb{Z}^\nu,$$

whose elements can be written in the form $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\,dx)$ by means of the random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ of this random field for all $p \in \mathbb{Z}^\nu$ and $1 \le j \le d$. Let us consider a random variable $Y \in \mathcal{H}_n$ which can be represented as the n-fold Wiener–Itô integral of some function $h \in \mathcal{K}_{n,j_1,\dots,j_n}$, i.e.,

$$Y = \int h(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(\,dx_n). \tag{2.6}$$

In the next result the shift transforms $T_u Y$, $u \in \mathbb{Z}^\nu$, of Y given in formula (2.6) will be expressed in a useful form which shows some similarity to the Fourier transform of a function.

To do this let us first recall the definition of the shift transforms T_u, $u \in \mathbb{Z}^\nu$, in a stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$.

Given some element $X_j(m)$, $m \in \mathbb{Z}^\nu$, $1 \le j \le d$, of the random field, and $u \in \mathbb{Z}^\nu$, we define the shift transform of $X_j(m)$ by T_u as $T_u X_j(m) = X_j(u + m)$. More generally, given any measurable function $h(X_j(m), m \in \mathbb{Z}^\nu, 1 \le j \le d)$, we define the shift transform of the random variable $Y = h(X_j(m),\, m \in \mathbb{Z}^\nu,\, 1 \le j \le d)$, by the formula $T_u Y = h(X_j(m + u),\, m \in \mathbb{Z}^\nu,\, 1 \le j \le d)$. This transformation was discussed in the scalar-valued case in [8]. It can be seen, (similarly to the argument in that work) that the definition of this transformation is meaningful (i.e., the value of $T_u Y$ does not depend on the choice of the function h for which $Y = h(X_j(m),\, m \in \mathbb{Z}^\nu,\, 1 \le j \le d)$), and we have defined in such a way unitary (linear) transformations T_u, $u \in \mathbb{Z}^\nu$, on $\mathcal{H}$ for which $T_u T_v = T_{u+v}$.

In Lemma 3.2 of [10] it was shown that each random variable $U_j \in \mathcal{H}_{1,j}$ can be written in the form $U_j = \int h(x) Z_{G,j}(\,dx)$ with some function $h(x) \in \mathcal{K}_{1,j}$. On the other hand, I claim that for all $u \in \mathbb{Z}^\nu$ and $h \in \mathcal{K}_{1,j}$

$$T_u U_j = \int e^{i(u,x)} h(x) Z_{G,j}(\,dx) \quad \text{for} \quad U_j = \int h(x) Z_{G,j}(\,dx). \tag{2.7}$$

Indeed, relation (2.7) clearly holds if $h(x) = e^{i(p,x)}$ with some $p \in \mathbb{Z}^\nu$, since in this case $U_j = X_j(p)$ and $T_u U_j = X_j(p+u)$. But this implies that relation (2.7) holds for all finite trigonometrical polynomials of the form $h(x) = \sum c_k e^{i(p_k,x)}$, and for the closure of these functions with respect to the L_2 norm determined by the measure $G_{j,j}$, i.e., for all $h \in \mathcal{K}_{1,j}$.

In Proposition 2.4 a similar formula is presented about the shift transforms of a random variable Y given by formula (2.6). This result is useful in the study of limit theorems related to non-linear functionals of a stationary Gaussian field.

Proposition 2.4 about the representation of shift transformations. *Let a vector-valued stationary Gaussian random field*

$$X(p) = (X_1(p), \dots, X_d(p)), \quad p \in \mathbb{Z}^\nu,$$

be given with a vector-valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ such that $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\,dx)$ for all $p \in \mathbb{Z}^\nu$ and $1 \le j \le d$. Let $Y \in \mathcal{H}_n$ be the random variable defined in formula (2.6) with the help of this vector-valued random spectral measure Z_G and some function $h \in \mathcal{K}_{n,j_1,\dots,j_n}$. Then

$$T_u Y = \int e^{i(u,x_1+\dots+x_n)} h(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(\,dx_n) \tag{2.8}$$

for all $u \in \mathbb{Z}^\nu$.

Proof of Proposition 2.4. Formula (2.8) holds in the special case if $n = 1$, and $h(x) \in \mathcal{K}_{1,j}$, since in this case $Y = \int h(x) Z_{G,j}(\,dx)$, and

$$T_u Y = \int e^{i(u,x)} h(x) Z_{G,j}(\,dx)$$

by formula (2.7).

I claim that formula (2.8) also holds in the case when the random variable Y is given by formula (2.6) with a kernel function of the form $h(x_1, \dots, x_n) = \varphi_1(x_1) \cdots \varphi_n(x_n)$ defined with the help of some functions $\varphi_s(x) \in \mathcal{K}_{1,j_s}$, $1 \le s \le n$. Indeed, in this case $Y =: U_1 \cdots U_n :$ with $U_s = \int \varphi_s(x) Z_{G,j_s}(\,dx)$, $1 \le s \le n$, because of Theorem 2.2. On the other hand, I claim that

$$T_u :U_1 \cdots U_n: \; =: (T_u U_1) \cdots (T_u U_n) : .$$

To see this let us observe that by Theorem 2B $:U_1 \cdots U_n:$ is the orthogonal projection of the product $U_1 \cdots U_n$ to the Hilbert subspace $\mathcal{H}_n(U_1, \dots, U_n)$. Similarly, $:(T_u U_1) \cdots (T_u U_n):$ is the orthogonal projection of the vector $(T_u U_1) \cdots (T_u U_n)$ to the Hilbert subspace $\mathcal{H}_n(T_u U_1, \dots, T_u U_n)$. Since the vectors $(U_1, \dots, U_n)$ and $(T_u U_1, \dots, T_u U_n)$ have the same distribution, and

the Wick polynomial corresponding to their product can be calculated in the same way this implies that if $:U_1 \cdots U_n: \; = g(U_1, \dots, U_n)$ with some function g, then

$$:(T_u U_1) \cdots (T_u U_n): \; = g(T_u U_1, \dots, T_u U_n)$$

with the same function g. (In the present case $g(x_1, \dots, x_n)$ is a polynomial of order n.) On the other hand, $T_u :U_1 \cdots U_n: \; = T_u g(U_1, \dots, U_n) = g(T_u U_1, \dots, T_u U_n)$ in this case. The above argument implies the desired identity.

Thus we can state that if Y is defined by formula (2.6) with a function

$$h(x_1, \dots, x_n) = \varphi_1(x_1) \cdots \varphi_n(x_n)$$

with the above properties, then

$$\begin{aligned} T_u Y &= :(T_u U_1) \cdots (T_u U_n): \\ &= \int e^{i(u, x_1 + \cdots + x_n)} h(x_1, \dots, x_n) Z_{G, j_1}(\, dx_1) \dots Z_{G, j_n}(\, dx_n) \end{aligned}$$

because of Theorem 2.2 and the relation $T_u U_s = \int e^{i(u,x)} \varphi_s(x) Z_{G,j_s}(\, dx)$ for all indices $1 \le s \le n$.

From the result in this case follows that relation (2.8) also holds if Y is defined by (2.6) with a function $h(x_1, \dots, x_n)$ of the form of a finite sum

$$h(x_1, \dots, x_n) = \sum_k \varphi_{1,k}(x_1) \varphi_{2,k}(x_2) \cdots \varphi_{n,k}(x_n)$$

with $\varphi_{s,k} \in \mathcal{K}_{1,j_s}$, $1 \le s \le n$.

Since functions of the above form are dense in $\mathcal{K}_{n,j_1,\dots,j_n}$, T_u is a unitary operator, and both (linear) transformations

$$h(x_1, \dots, x_n) \to e^{i(u, x_1 + \cdots + x_n)} h(x_1, \dots, x_n)$$

and $h \to I_n(h | j_1, \dots, j_n)$ from the space $\mathcal{K}_{n,j_1,\dots,j_n}$ to the spaces $\mathcal{K}_{n,j_1,\dots,j_n}$ and $\mathcal{H}_n$ are of bounded norms, it is not difficult to see that Proposition 2.4 holds in the general case. Proposition 2.4 is proved. □

3. On the proof of limit theorems for non-linear functionals of vector-valued stationary Gaussian random fields

First I recall the limit theorem problem we are interested in.

Let $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, be a d-dimensional vector-valued stationary Gaussian random field, and let a function $H(x_1, \dots, x_d)$, $H\colon \; \mathbb{R}^d \to \mathbb{R}^1$, of d variables be given. Let us define with their help the random variables $Y(p) = H(X_1(p), \dots, X_d(p))$ for all $p \in \mathbb{Z}^\nu$, and introduce for all $N = 1, 2, \dots$ the normalized random sum

$$S_N = A_N^{-1} \sum_{p \in B_N} Y(p) \tag{3.1}$$

with an appropriate norming constant $A_N > 0$, where

$$B_N = \{p = (p_1, \dots, p_\nu)\colon\ 0 \le p_k < N \text{ for all } 1 \le k \le \nu\}. \tag{3.2}$$

Let us also fix the vector-valued random spectral measure $(Z_{G,1}, \dots, Z_{G,d})$ on the torus $[-\pi, \pi)^\nu$ for which $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\,dx)$, $1 \le j \le d$, $p \in \mathbb{Z}^\nu$. We are interested in the question what kind of limit theorems may hold for the normalized sums S_N defined in (3.1) as $N \to \infty$ with appropriate norming constants A_N. Here we are interested in the case when the correlation functions $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $1 \le j, j' \le d$, tend to zero slowly as $|p| \to \infty$. This means strong dependence of the random variables in the stationary random fields. In such cases, we can get limit theorems with a non-Gaussian limit.

We have studied the above problem in [5] for scalar-valued stationary random fields, i.e., in the case $d = 1$, and we have proved some new kinds of limit theorems. Let me remark that at the same time M. Taqqu also proved similar results with the help of a different method, see [13]. I do not discuss Taqqu's work, because here I am interested in the question of how to generalize the method in [5] to prove limit theorems also for non-linear functionals of vector-valued stationary Gaussian random fields.

In paper [10] and Section 2 of this work I discussed the notions and results we have to adopt some important methods of [8] when we are working with multivariate models. In this section, I explain how to generalize those methods of [8] which lead to non-central limit theorems when we are working with non-linear functionals of vector-valued stationary Gaussian random fields. I shall give the proof of the limit theorems in paper [11] with the help of these results.

In the first step of this discussion I rewrite the limit problem we are interested in in a different form. Let us observe that we have $X_j(p) = T_p X_j(0)$ with the shift transform T_p for all $p \in \mathbb{Z}^\nu$ and $1 \le j \le d$, hence $Y(p) = T_p Y(0)$, and we can rewrite the sum in (3.1) in the form

$$S_N = A_N^{-1} \sum_{p \in B_N} T_p Y(0). \tag{3.3}$$

As it will turn out the crucial point in the investigation of our limit theorems is the study of limit theorems in the special case when $Y(0)$ is a Wick polynomial, and here we restrict our attention to this case.

Let us consider the case when $Y(0)$ is a Wick polynomial of order k which has the form

$$Y(0) = {:} \sum_{\substack{(k_1,\dots,k_d)\\ k_1+\dots+k_d=k}} a_{k_1,\dots,k_d} X_1(0)^{k_1} \cdots X_d(0)^{k_d} {:} \tag{3.4}$$

with some real coefficients $a_{k_1,\dots,k_d}$. Then by the corollary of Theorem 2.2 and the identities $X_j((0) = \int \mathbb{I}_1(x) Z_{G,j}(\,dx)$, $1 \le j \le d$, where $\mathbb{I}_1(\cdot)$ denotes the indicator function of the torus $[-\pi, \pi)^\nu$, the random variable $Y(0)$ can

be written in the form

$$Y(0)=\sum_{\substack{(k_1,\dots,k_d)\\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} a_{k_1,\dots,k_d}\int \mathbb{I}_1(x_1)\dots\mathbb{I}_1(x_k)\prod_{j=1}^d\left(\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dx_t)\right),$$

where for $j=1$ we define $\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dx_t)=\prod_{t=1}^{k_1} Z_{G,1}(\,dx_t)$, and if $k_j=0$ for some $1\le j\le d$, then the product $\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G_j}(\,dx_t)$ is omitted from this express By Proposition 2.4 we can write

$$T_pY(0)=\sum_{\substack{(k_1,\dots,k_d)\\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} a_{k_1,\dots,k_d}\int e^{i(p,x_1+\dots+x_k)}\prod_{j=1}^d\left(\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G.j}(\,dx_t)\right) \tag{3.5}$$

for all indices $p\in\mathbb{Z}^\nu$.

We get by summing up formula (3.5) for all $p\in B_N$ with our choice of $Y(0)$ that

$$S_N=A_N^{-1}\sum_{\substack{(k_1,\dots,k_d)\\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} a_{k_1,\dots,k_d}\int \quad \prod_{l=1}^\nu \frac{e^{iN(x_1^{(l)}+\dots+x_k^{(l)})}-1}{e^{i(x_1^{(l)}+\dots+x_k^{(l)})}-1}$$

$$\prod_{j=1}^d\left(\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dx_t)\right),$$

where we write $x=(x^{(1)},\dots,x^{(\nu)})$ for all $x\in[-\pi,\pi)^\nu$. (The set B_N was defined in (3.2).) I shall rewrite the above identity in a form more appropriate for us, First I apply the change of variables $y_l=Nx_l$, $1\le l\le n$. It yields that

$$S_N=\sum_{\substack{(k_1,\dots,k_d)\\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}}\int h^N_{k_1,\dots,k_d}(y_1,\dots,y_k)\prod_{j=1}^d\left(\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G^{(N)},j}(\,dy_t)\right), \tag{3.6}$$

where

$$h^N_{k_1,\dots,k_d}(y_1,\dots,y_k)=a_{k_1,\dots,k_d}\prod_{l=1}^\nu\frac{e^{i(y_1^{(l)}+\dots+y_k^{(l)})}-1}{N(e^{i(y_1^{(l)}+\dots+y_k^{(l)})/N}-1)}$$

is a function on $[-N\pi,N\pi)^\nu$, and $Z_{G^{(N)},j}(A)=N^{\nu/k}A_N^{-1/n}Z_{G,j}(\frac{A}{N})$ is defined for all measurable sets $A\subset[-N\pi,N\pi)^\nu$ and $j=1,\dots,d$. Here we

use the notation $y_s = (y_s^{(1)}, \dots, y_s^{(\nu)})$, $1 \le s \le k$. Let us observe that $(Z_{G^{(N)},1}, \dots, Z_{G^{(N)},d})$ is a vector-valued random spectral measure on the torus $[-N\pi, N\pi)^\nu$, corresponding to the matrix valued spectral measure $G^{(N)} = (G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-N\pi, N\pi)^\nu)$, defined by the formula

$$G^{(N)}_{j,j'}(A) = N^{2\nu/k} A_N^{-2/n} G_{j,j'}\left(\frac{A}{N}\right), \quad 1 \le j, j' \le d,$$

for all measurable sets $A \subset [-N\pi, N\pi)^\nu$, where $G = (G_{j,j'})$, $1 \le j, j' \le d$, is the matrix valued spectral measure of the original vector-valued stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$. On the other hand, $h^N_{k_1,\dots,k_d} \in \mathcal{K}_{k,j_1,\dots,j_k}(G^{(N)}_{j_1,j_1}, \dots, G^{(N)}_{j_k,j_k})$ with $j_p = s$ if $k_1 + \cdots + k_{s-1} < p \le k_1 + \cdots + k_s$, $1 \le s \le d$. (For $s = 1$ we define $k_1 + \cdots + k_{s-1} = 0$.)

In formula (3.6) we have taken summation for the series $(k_1, \dots, k_d) \in \mathcal{K}$, where $\mathcal{K} = \{(k_1, \dots, k_d)\colon k_s \ge 0, 1 \le s \le d,\ k_1 + \cdots + k_d = k\}$, and in each term of the sum at its right-hand side there was a product of the form $\prod_{s=1}^k Z_{G^{(N)},j_s}(du_t)$ with a sequence $(j_1, \dots, j_k) \in \mathcal{J}$, where $\mathcal{J} = \{(j_1, \dots, j_k)\colon 1 \le j_1 \le j_2 \le \cdots \le j_k \le d\}$, and it depended on the sequence $(k_1, \dots, k_d)$ which indices $j_1, \dots, j_k$ appeared in the above product

We want to rewrite expression (3.6) in a more pleasant form. For this goal, we make the following observation. There is a natural invertible map $T\colon \mathcal{J} \to \mathcal{K}$, defined as $T(j_1, \dots, j_k) = (k_1(j_1, \dots, j_k), \dots, k_d(j_1, \dots, j_k))$ for all $(j_1, \dots, j_k) \in \mathcal{J}$, where for all indices $1 \le s \le d$, $k_s(j_1, \dots, j_k)$ equals the number of those elements j_p in the sequence $(j_1, \dots, j_k)$ for which $j_p = s$. Its inverse is defined by the formula

$$T^{-1}(k_1, \dots, k_d) = (j_1(k_1, \dots, k_d), \dots, j_k(k_1, \dots, k_d))$$

for all $(k_1, \dots, k_d) \in \mathcal{K}$, where $j_s(k_1, \dots, k_d) = \min\{u\colon k_1 + \cdots + k_u \ge s\}$.

With the help of the above defined functions $k_s = k_s(j_1, \dots, j_k)$, $1 \le s \le d$, we can rewrite the identity in (3.6) in a form more appropriate for us. In this new formula we take summation for $(j_1, \dots, j_k) \in \mathcal{J}$ instead of $(k_1, \dots, k_d) \in \mathcal{K}$. We get that

$$S_N = \sum_{\substack{(j_1,\dots,j_k)\\ 1 \le j_1 \le \cdots \le j_k \le d}} \int h^N_{j_1,\dots j_k}(y_1, \dots, y_k) Z_{G^{(N)},j_1}(\,dy_1) \dots Z_{G^{(N)},j_k}(\,dy_k) \tag{3.7}$$

with

$$h^N_{j_1,\dots,j_k}(y_1, \dots, y_k) = a_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)} \prod_{l=1}^{\nu} \frac{e^{i(y_1^{(l)} + \cdots + y_k^{(l)})} - 1}{N(e^{i(y_1^{(l)} + \cdots + y_k^{(l)})/N} - 1)}.$$

Let us observe that

$$\lim_{N \to \infty} h^N_{j_1,\dots,j_k}(y_1, \dots, y_k) = h^0_{j_1,\dots,k}(y_1, \dots, y_k)$$

with the function

$$h^0_{j_1,\dots,j_k}(y_1,\dots,y_k)=a_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)}\prod_{l=1}^{\nu}\frac{e^{i(y_1^{(l)}+\dots+y_k^{(l)})}-1}{i(y_1^{(l)}+\dots+y_k^{(l)})}$$

defined on $\mathbb{R}^{k\nu}$, and this convergence is uniform in all bounded subsets of $\mathbb{R}^{k\nu}$.

It is natural to expect that if the matrix valued spectral measures $G^{(N)}=(G^{(N)}_{j,j'})$, $1\le j,j'\le d$, converge to a matrix valued spectral measure $G^{(0)}=(G^{(0)}_{j,j'})$, $1\le j,j'\le d$, defined on $\mathbb{R}^{\nu}$ in an appropriate way, then a limiting procedure in formula (3.7) supplies the limit theorem $S_N\to S_0$ in distribution with

$$S_0=\sum_{\substack{(j_1,\dots,j_k)\\1\le j_1\le\dots\le j_k\le d}}\int h^0_{j_1,\dots j_k}(y_1,\dots,y_k)Z_{G^{(0)},j_1}(\,dy_1)\dots Z_{G^{(0)},j_k}(\,dy_k)$$

as $N\to\infty$, where $(Z_{G^{(0)},1},\dots,Z_{G^{(0)},d})$ is a vector-valued random spectral measure on $\mathbb{R}^{\nu}$ corresponding to the matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1\le j,j'\le d$. On the other hand, the convergence of the spectral measures $G^{(N)}$ to the spectral measure $G^{(0)}$ is satisfied in many interesting models, for instance in the models investigated in paper [11].

Next, I explain how to work out a precise method to prove limit theorems on the basis of the above heuristic argument. In particular, we are interested in the question of when the above sketched heuristic argument can be carried out. In the scalar-valued case, this problem was solved in Lemma 8.3 of [8]. Here I prove the vector-valued variant of this result.

In the formulation of Lemma 8.3 of [8], we had to introduce a version of the notion of weak convergence of finite measures to a larger class of measures, to the class of so-called locally finite measures. They are measures, whose restrictions to any compact set are finite. Here I introduce a slight generalization of the notion called vague convergence in [8] to the case when we are working with complex measures of locally finite total variation. In [8] we have worked with (positive) measures. (The definition of complex measures on $\mathbb{R}^{\nu}$ with locally finite total variation was explained in Section 4 of the paper [10].)

Definition of vague convergence of complex measures on $\mathbb{R}^{\nu}$ with locally finite total variation. *Let G_N, $N=1,2,\dots$, be a sequence of complex measures on $\mathbb{R}^{\nu}$ with locally finite total variation. We say that the sequence G_N vaguely converges to a complex measure G_0 on $\mathbb{R}^{\nu}$ with locally finite total variation (in notation $G_N\xrightarrow{v}G_0$) if*

$$\lim_{N\to\infty}\int f(x)\,G_N(\,dx)=\int f(x)\,G_0(\,dx) \tag{3.8}$$

for all continuous functions f on $\mathbb{R}^{\nu}$ with a bounded support.

I shall take a sequence of sums of k-fold Wiener–Itô integrals, and then I formulate Proposition 3.1 which states that under some appropriate conditions these sums of random integrals have a limit that can be expressed in an explicit form. This result together with the representation of non-linear functionals of vector-valued stationary Gaussian random fields by means of multiple Wiener–Itô integrals enable us to prove limit theorems with a non-Gaussian limit for non-linear functionals of vector-valued stationary Gaussian random fields.

For all $N = 1, 2, \dots$ take a sequence of matrix valued non-atomic spectral measures $(G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-A_N\pi, A_N\pi)^\nu$ with parameter A_N such that $A_N \to \infty$ as $N \to \infty$. Let us also take some functions

$$h^N_{j_1,\dots,j_k}(x_1, \dots, x_k) \in \mathcal{K}_{k,j_1,\dots,j_k} = \mathcal{K}_{k,j_1,\dots,j_k}(G^{(N)}_{j_1,j_1}, \dots, G^{(N)}_{j_k,j_k})$$

on the torus $[-A_N\pi, A_N\pi)^\nu$ for all $(j_1, \dots, j_k)$ with $1 \le j_s \le d$, $1 \le s \le k$, and $N = 1, 2, \dots$. For all $N = 1, 2, \dots$ fix a vector-valued random spectral measure

$$(Z_{G^{(N)},1}, \dots, Z^{(N)}_{G^{(N)},d})$$

on the torus $[-A_N\pi, A_N\pi)^\nu$ corresponding to the matrix valued spectral measure $(G^{(N)}_{j,j'})$, $1 \le j, j' \le d$. Let us define with the help of these quantities the sums of n-fold Wiener–Itô integrals

$$Z_N = \sum_{\substack{(j_1,\dots,j_k)\\ 1\le j_s\le d \text{ for all } 1\le s\le k}} \int h^N_{j_1,\dots,j_k}(x_1, \dots, x_k) Z_{G^{(N)},j_1}(\,dx_1) \dots Z_{G^{(N)},j_k}(\,dx_k), \tag{3.9}$$

$N = 1, 2, \dots$. In the next result I show that under appropriate conditions these random variables Z_N converge in distribution to a random variable Z_0 expressed in the form of a sum of multiple Wiener–Itô integrals.

Proposition 3.1. *Let us consider for all $N = 1, 2, \dots$ the sums of k-fold Wiener–Itô integrals Z_N defined in formula (3.9) with the help of certain vector-valued random spectral measures $(Z_{G^{(N)},1}, \dots, Z_{G^{(N)},d})$ corresponding to some non-atomic matrix valued spectral measures $(G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, defined on tori $[-A_N, A_N)^\nu$ such that $A_N \to \infty$ as $N \to \infty$, and functions*

$$h^N_{j_1,\dots,j_k}(x_1, \dots, x_k) \in \mathcal{K}_{k,j_1,\dots,j_k}(G^{(N)}_{j_1,j_1}, \dots, G^{(N)}_{j_k,j_k}).$$

Let the coordinates $G^{(N)}_{j,j'}$, $1 \le j, j' \le d$, of the matrix valued spectral measures $(G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, converge vaguely to the coordinates $G^{(0)}_{j,j'}$ of a non-atomic matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$ for all $1 \le j, j' \le d$ as $N \to \infty$, and let $(Z_{G^{(0)},1}, \dots, Z_{G^{(0)},d})$ be a vector-valued random spectral measure on $\mathbb{R}^\nu$ corresponding to the matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$. Let us also have some functions $h^0_{j_1,\dots,j_k}$ for

all $(j_1, \dots, j_k)$ *with* $1 \le j_s \le d$, $1 \le s \le k$, *such that these functions and matrix valued spectral measures satisfy the following conditions.*

(a): *The functions* $h^0_{j_1,\dots,j_k}(x_1, \dots, x_k)$ *are continuous on* $\mathbb{R}^{k\nu}$ *for all* $1 \le j_s \le d$, $1 \le s \le k$, *and for all* $T > 0$ *and indices* $1 \le j_s \le d$, $1 \le s \le k$, *and the functions* $h^N_{j_1,\dots,j_k}(x_1, \dots, x_k)$ *converge uniformly to the function* $h^0_{j_1,\dots,j_k}(x_1, \dots, x_k)$ *on the cube* $[-T, T]^{k\nu}$ *as* $N \to \infty$.

(b): *For all* $\varepsilon > 0$ *there is some* $T_0 = T_0(\varepsilon) > 0$ *such that*

$$\int_{\mathbb{R}^{k\nu}\setminus[-T,T]^{k\nu}} |h^N_{j_1,\dots,j_k}(x_1, \dots, x_k)|^2 G^{(N)}_{j_1,j_1}(\,dx_1) \dots G^{(N)}_{j_k,j_k}(dx_k) < \varepsilon^2 \quad (3.10)$$

for all $1 \le j_s \le d$, $1 \le s \le k$, *and* $N = 1, 2 \dots$ *if* $T > T_0$.

Then inequality (3.10) holds also for $N = 0$,

$$h^0_{j_1,\dots,j_k} \in \mathcal{K}_{k,j_1,\dots,j_k} = \mathcal{K}_{k,j_1,\dots,j_k}(G^{(0)}_{j_1,j_1}, \dots G^{(0)}_{j_k,j_k}), \quad (3.11)$$

the sum of random integrals

$$Z_0 = \sum_{\substack{(j_1,\dots,j_k)\\ 1\le j_s\le d,\ \text{for all } 1\le s\le k}} \int h^0_{j_1,\dots,j_k}(x_1, \dots, x_k) Z_{G^{(0)},j_1}(\,dx_1) \dots Z_{G^{(0)},j_k}(\,dx_k) \quad (3.12)$$

exists, and the random variables Z_N *defined in (3.9) satisfy the relation* $Z_N \xrightarrow{\mathcal{D}} Z_0$ *as* $N \to \infty$, *where* $\xrightarrow{\mathcal{D}}$ *denotes convergence in distribution.*

Remark 1. A complex measure $G^{(N)}_{j,j'}$ with finite total variation defined on the torus $[-A_N\pi, A_N\pi)^\nu$ can be identified in a natural way with a complex measure on $\mathbb{R}^\nu$ which is concentrated on its subset $[-A_N\pi, A_N\pi)^\nu$. We take this identification of $G^{(N)}_{j,j'}$ with a complex measure on $\mathbb{R}^\nu$ when we give meaning to formula (3.8) with $G_N = G^{(N)}_{j,j'}$ and $G_0 = G^{(0)}_{j,j'}$ in the definition of the vague convergence of the complex measures $G^{(N)}_{j,j'}$ to $G^{(0)}_{j,j'}$ as $N \to \infty$.

Remark 2. In Proposition 3.1 we imposed two conditions for the convergence of the sums of multiple Wiener–Itô integrals Z_N defined in (3.9) to the sum of multiple Wiener–Itô integrals Z_0 defined in (3.12). First we demanded that the spectral measures and kernel functions appearing in the definition of the expressions Z_N should converge to the corresponding spectral measure and kernel functions appearing in the definition of the expression Z_0 in an appropriate way. We still imposed an additional condition in part (b) of Proposition 3.1. This is a compactness type condition which implies that the random integrals in the definition of the random variables Z_N are essentially concentrated in a compact set not depending on the parameter N. I shall show in the Appendix with the help of an example that without condition (b) Proposition 3.1 may not hold any longer. I shall also make some additional remarks about Proposition 3.1.

Proposition 3.1 is a multivariate version of Lemma 8.3 in [8]. I gave a simpler proof for this lemma in Lemma 6.3 of [9], and here I shall adopt this proof. We have to overcome some additional difficulties, which arose because we are working with vector-valued random fields. First I briefly explain the method of the proof.

In the first step we show that relation (3.11) holds, and the random integrals appearing in the definition of Z_0 really exist. In the proof of this statement condition (b) of Proposition 3.1 plays an important role.

In the next step, we reduce the proof of the limit theorem in Proposition 3.1 to the proof of a simpler statement formulated in relation (3.13). The main point in this reduction is that here we have to prove the limit theorem only for the sums of such Wiener–Itô integrals whose kernel functions do not depend on the parameter N. The proof of this reduction is based on some results on Wiener–Itô integrals proved in [10] and the characterization of convergence of distribution with the help of characteristic functions.

Then we make a further simplification of the statement we have to prove. We show that the statement in formula (3.18) implies relation (3.13), hence Proposition 3.1, too. Formula (3.18) states, similarly to relation (3.13), the convergence of some sums of Wiener–Itô integrals in distribution, but the kernel functions of the random integrals appearing here are simple functions, and this makes its proof simpler.

The reduction of relation (3.13) to (3.18) is done similarly to the reduction of Proposition 3.1 to relation (3.13). But here we need in addition some results which provide a good approximation of the kernel functions in (3.13) by simple functions. (See Section 5 of [10] for the definition of simple functions and of their properties needed in our proof.) The formulation of the precise statements we need for the reduction of (3.13) to (3.18) is given in formulas (3.16) and (3.17).

One must be careful in the proof of these formulas. The main difficulty arises because in (3.17) we demand a good approximation simultaneously for all sufficiently large indices N.

The results of Section 5 in [10] enable us to construct such simple functions which satisfy (3.16). Moreover, they make possible to construct these approximating simple functions with the following additional property. We define an appropriate measure μ_0 on $\mathbb{R}^\nu$ with some nice properties, and the simple functions we construct are adapted to such a regular system whose elements have boundaries with zero μ_0 measure. With the help of this extra property (and with a good definition of the measure μ_0) we can achieve that relation (3.17) also holds.

Then it remains to prove relation (3.18). Here the convergence of certain Gaussian polynomials in distribution has to be proved. The main step of the proof is to show that under the conditions of Proposition 3.1 $G^{(N)}_{j,j'}(\Delta) \to G^{(0)}_{j,j'}(\Delta)$ as $N \to \infty$ for all $1 \le j, j' \le d$ for those measurable sets $\Delta \subset \mathbb{R}^\nu$ whose boundaries behave nicely in a certain sense. At this point, some new

arguments are needed in the proof.The arguments applied in the study of scalar valued random fields are not sufficient here.

The proof of this convergence is fairly simple if $j = j'$. In this case, we can apply some results about weak convergence of measures. But the proof is more difficult for pairs (j, j') with $j \neq j'$. This difficulty arises, because if $j \neq j'$, then it may happen that $G^{(N)}_{j,j'}(\cdot)$ with a fixed parameter N is not a (real-valued, positive) measure. To get a proof in this case we apply a special argument, where we exploit that $G^{(N)}_{j,j'}$ is an element of a positive semidefinite matrix valued measure $G^{(N)}$.

If the limit behavior of the complex measures $G^{(N)}_{j,j'}$ with locally finite total variation is already known, then relation (3.18) can be proved in a standard way. For all $N = 0, 1, 2, \dots$ we consider the random vector $Z_N(\mathcal{D})$ we obtain by restricting the random spectral measure Z_{G^N} to the elements of the regular system $\mathcal{D}$ we are working with. We have to prove a limit theorem for a polynomial of these random vectors as $N \to \infty$. This can be done with the help of the results we proved about spectral and random spectral measures in [10].

Proof of Proposition 3.1. First I show that relation (3.10) holds also for $N = 0$. To see this let us first show that the measures $\mu^{(N)}_{j_1,\dots,j_k}$, $N = 1, 2, \dots$, defined as

$$\mu^{(N)}_{j_1,\dots,j_k}(A) = \int_A |h^N_{j_1,\dots,j_k}(x_1, \dots, x_k)|^2 G^{(N)}_{j_1,j_1}(\,dx_1) \dots G^{(N)}_{j_k,j_k}(dx_k),\ A \subset \mathbb{R}^{k\nu},$$

converge vaguely to the locally finite measure $\mu^{(0)}_{j_1,\dots,j_k}$ defined as

$$\mu^{(0)}_{j_1,\dots,j_k}(A) = \int_A |h^0_{j_1,\dots,j_k}(x_1, \dots, x_k)|^2 G^{(0)}_{j_1,j_1}(\,dx_1) \dots G^{(0)}_{j_k,j_k}(dx_k),\ A \subset \mathbb{R}^{k\nu},$$

if $N \to \infty$.

Indeed, it follows from the vague convergence of the measures $G^{(N)}_{j,j}$ to $G^{(0)}_{j,j}$ as $N \to \infty$ and the continuity of the function $h^{(0)}_{j_1,\dots,j_n}$ that this relation holds if we replace the kernel function $|h^N_{j_1,\dots,j_k}(x_1, \dots, x_k)|^2$ by the kernel function $|h^0_{j_1,\dots,j_k}(x_1, \dots, x_k)|^2$ in the definition of the measures $\mu^{(N)}_{j_1,\dots,j_k}$. Then condition (a) of Proposition 3.1 implies that this relation also holds with the original definition of the measures $\mu^{(N)}_{j_1,\dots,j_k}$.

Next I state that the measure $\mu^{(0)}_{j_1,\dots,j_k}$ is finite, and the measures $\mu^{(N)}_{j_1,\dots,j_k}$ converge to it not only vaguely but also weakly. Indeed, condition (b) implies that the sequence of measures $\mu^{(N)}_{j_1,\dots,j_k}$ is compact with respect to the topology defining the weak convergence of finite measures, hence any subsequence of it has a convergent sub-subsequence. But the limit of such a sub-subsequence can be only its limit with respect to the vague convergence,

i.e., it is $\mu^{(0)}_{j_1,\dots,j_k}$. This implies that $\mu^{(0)}_{j_1,\dots,j_k}$ is a finite measure, and the sequence of measures $\mu^{(N)}_{j_1,\dots,j_k}$ converges also weakly to it.

Finally the properties of the functions $h^N_{j_1,\dots,j_k}$, and their convergence to $h^0_{j_1,\dots,j_k}$ formulated in condition (a) imply that also the symmetry property $h^0_{j_1,\dots,j_k}(-x_1,\dots,-x_k) = \overline{h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)}$ holds, hence relation (3.11) is valid, and the random integral Z_0 defined in (3.12) is meaningful. Next I reduce the proof of the relation $Z_N \xrightarrow{\mathcal{D}} Z_0$ to the proof of the following statement:

Under the conditions of Proposition 3.1

$$\begin{aligned}\sum_{\substack{(j_1,\dots,j_k)\\ 1\le j_s\le d,\text{ for all } 1\le s\le k}} &\int h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)\chi_T(x_1,\dots,x_k) \\ &\qquad Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k) \\ \xrightarrow{\mathcal{D}} \sum_{\substack{(j_1,\dots,j_k)\\ 1\le j_s\le d,\text{ for all } 1\le s\le k}} &\int h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)\chi_T(x_1,\dots,x_k) \\ &\qquad Z_{G^{(0)},j_1}(\,dx_1)\dots Z_{G^{(0)},j_k}(\,dx_k),\end{aligned} \tag{3.13}$$

as $N\to\infty$, where $\chi_T(x_1,\dots,x_k)$ is the indicator function of the cube $[-T,T]^{k\nu}$. We make a small, not so important technical restriction in the choice of the number T in (3.13). Let me recall that for all all vector-valued spectral measures G_N there is a finite measure μ_N on $\mathbb{R}^\nu$ such that all coordinates $G^{(N)}_{j,j'}$ of G_N, (more precisely their restrictions to any compact sets) are such measures which are absolutely continuous measures with respect to μ_N. (See Section 4 of [10].) We fix such a measure μ_N for all $N=0,1,2,\dots$, and we shall call them dominating measures. We shall work with such measures μ_N in the proof of Proposition 3.1. We state formula (3.13) for all such $T>0$ for which the boundary of the cube $[-T,T]^{k\nu}$ has zero measure with respect to the measure $\underbrace{\mu_0\times\dots\times\mu_0}_{k\text{ times}}$.

To prove this reduction let us observe that by formulas (5.6) in [10] and (3.10)

$$\begin{aligned}E\Bigg[\int &[1-\chi_T(x_1,\dots,x_k)]h^N_{j_1,\dots,j_k}(x_1,\dots,x_k) \\ &\qquad Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k)\Bigg]^2 \\ \le k!\int_{\mathbb{R}^{k\nu}\setminus[-T,T]^{k\nu}} &|h^N_{j_1,\dots,j_k}(x_1,\dots,x_n)|^2 G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k) < k!\varepsilon^2\end{aligned}$$

for all sequences $(j_1,\dots,j_k)$, $1\le j_s\le d$, $1\le s\le k$, and $N=0,1,2,\dots$ if $T>T_0(\varepsilon)$. Hence

$$E\Bigg[\sum_{\substack{(j_1,\dots,j_k)\\ 1\le j_s\le d \text{ for all } 1\le s\le k}}\int [1-\chi_T(x_1,\dots,x_k)]h^N_{j_1,\dots,j_k}(x_1,\dots,x_k) \\ Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k)\Bigg]^2\le d^k k!\varepsilon^2 \tag{3.14}$$

for all $N=0,1,\dots$ if $T>T_0(\varepsilon)$.

Since $G^{(N)}_{j,j}\overset{v}{\to}G^{(0)}_{j,j}$ for all $1\le j\le d$ as $N\to\infty$, hence for all $T>0$ there is some number $C(T)$ such that $G^{(N)}_{j,j}([-T,T])\le C(T)$ for all $N=1,2,\dots$ and $1\le j\le d$. Because of this estimate and the uniform convergence $h^N_{j_1,\dots,j_k}\to h^0_{j_1,\dots,j_k}$ on any cube $[-T,T]^{k\nu}$ we have

$$E\Bigg[\int [h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)-h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)]\chi_T(x_1,\dots,x_k) \\ Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k)\Bigg]^2 \\ \le k!\int_{[-T,T]^{k\nu}}|h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)-h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)|^2 \\ G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k)<\varepsilon^2$$

for all $T>0$ and $(j_1,\dots,j_k)$, $1\le j_s\le d$, $1\le s\le k$, if $N>N_1$ with some $N_1=N_1(T,\varepsilon)$. Hence

$$E\Bigg[\sum_{1\le j_1,\dots,j_k\le d}\int [h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)-h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)] \\ \chi_T(x_1,\dots,x_k)Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k)\Bigg]^2\le d^k\varepsilon^2 \tag{3.15}$$

for all $T>0$ if $N>N_1$ with some $N_1=N_1(T,\varepsilon)$.

Let us define the quantities

$$U_N=U_N(T)=\sum_{1\le j_1,\dots,j_k\le d}\int h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)\chi_T(x_1,\dots,x_k) \\ Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k),$$

and

$$V_N=V_N(T)=\sum_{1\le j_1,\dots,j_k\le d}\int h^0_{j_1,\dots,j_n}(x_1,\dots,x_k)\chi_T(x_1,\dots,x_k) \\ Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k),$$

$N=1,2,\dots$. We introduce the definition of $V_N=V_N(T)$ also for $N=0$, where we replace the spectral measures $Z_{G^{(N)},j}$, $N\ge 1$, $1\le j\le d$, by

$Z_{G^{(0)},j}$, $1 \le j \le d$, in the definition of V_N. We can reduce the proof of the relation $Z_N \xrightarrow{\mathcal{D}} Z_0$ to formula (3.13) in the following way. By formula (3.14) we can state that

$$\begin{aligned} |E(e^{itZ_N} - e^{itU_N})| &\le E|(1 - e^{it(Z_N - U_N)})| \le E|(t(Z_N - U_N)| \\ &\le |t|(E(Z_N - U_N)^2)^{1/2} \le |t|(d^k k!)^{1/2}\varepsilon. \end{aligned}$$

for all $t \in \mathbb{R}^1$ with the random variable Z_N defined in (3.9) if $T > T_0$ and $N > N_0(\varepsilon)$. Similarly, $|E(e^{itU_N} - e^{itV_N})| \le |t|(E(U_N - V_N)^2)^{1/2} \le |t|d^{k/2}\varepsilon$ for all $t \in \mathbb{R}^1$ and $N > N_0$ by inequality (3.15). Besides, (3.14) with $N = 0$ implies that

$$E|e^{itV_0} - Ee^{itZ_0}| \le |t|(E(Z_0 - V_0)^2)^{1/2} \le |t|d^{n/2}\varepsilon$$

for all $t \in \mathbb{R}^1$ if $T > T_0(\varepsilon)$, where Z_0 is defined in (3.12) and V_0 after the definition of V_N for $N \ge 1$. Finally, $Ee^{itV_N} \to Ee^{itV_0}$ for all $t \in \mathbb{R}^1$ if relation (3.13) holds. These relations together imply that $|Ee^{itZ_N} - Ee^{itZ_0}| \le C(t)\varepsilon$ if $N > N_0(t, \varepsilon)$ with some numbers $C(t)$ and $N_0(t, \varepsilon)$. Since this inequality holds for all $\varepsilon > 0$, it implies that $Z_N \xrightarrow{\mathcal{D}} Z_0$. (In formula (3.13) we imposed a condition on the parameter $T > 0$. We demanded that the boundary of $[-T, T]^{k\nu}$ must have measure zero with respect to the product measure of μ_0. It causes no problem that we can apply the above argument only for parameters T with this property.)

We shall prove (3.13) with the help of some statements formulated below. To formulate them let us first fix a number $T > 0$ such that the boundary of the cube $[-T, T]^{k\nu}$ has zero measure with respect to the measure $\underbrace{\mu_0 \times \cdots \times \mu_0}_{n \text{ times}}$. Observe that

$$h^0_{j_1,\dots,j_k}(x_1, \dots, x_k)\chi_T(x_1, \dots, x_k) \in \mathcal{K}_{k,j_1,\dots,j_k}(G^{(N)}_{j_1,j_1}, \dots, G^{(N)}_{j_k,j_k})$$

for all $T > 0$ and $N = 0, 1, 2, \dots$. I claim that for all $\varepsilon > 0$ a regular system $\mathcal{D} = \mathcal{D}(\varepsilon) = \{\Delta_k,\ k = \pm 1, \dots, \pm M\}$ can be constructed for which all of its elements have zero measure with respect the a dominating measure μ_0, i.e. $\mu_0(\partial\Delta_k) = 0$ for all $1 \le |k| \le M$, $\Delta_k \subset [-T, T]^\nu$ for all $1 \le |k| \le M$, and such that there exist some simple functions

$$f^\varepsilon_{j_1,\dots,j_k} \in \hat{\mathcal{K}}_{n,j_1,\dots,j_k}(G^{(0)}_{j_1,j_1}, \dots, G^{(0)}_{j_n,j_k})$$

indexed by the parameters $(j_1, \dots, j_k)$, $1 \le j_s \le d$, $1 \le s \le k$ which are adapted to this regular system and satisfy the inequalities written down in the following two formulas (3.16 and (3.17):

$$\int |h^0_{j_1,\dots,j_k}(x_1, \dots, x_k)\chi_T(x_1, \dots, x_k) - f^\varepsilon_{j_1,\dots,j_k}(x_1, \dots, x_k)|^2 \\ G^{(0)}_{j_1,j_1}(\,dx_1) \dots G^{(0)}_{j_k,j_k}(\,dx_k) < \varepsilon^2 \tag{3.16}$$

for all $1 \le j_s \le d$, $1 \le s \le k$, and also

$$\int |h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)\chi_T(x_1,\dots,x_k) - f^\varepsilon_{j_1,\dots,j_k}(x_1,\dots,x_k)|^2 \\ G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k) < \varepsilon^2 \tag{3.17}$$

for all $1 \le j_s \le d$, $1 \le s \le k$, and $N \ge N_0$ with some $N_0 = N_0(\varepsilon, T)$.

I also claim that with such a choice of simple functions

$$Y_N \xrightarrow{\mathcal{D}} Y_0 \tag{3.18}$$

as $N \to \infty$, where

$$\begin{aligned} Y_N &= Y_N(\varepsilon, T) \\ &= \sum_{\substack{(j_1,\dots,j_k) \\ 1\le j_s\le d \text{ for all } 1\le s\le k}} \int f^\varepsilon_{j_1,\dots,j_k}(x_1,\dots,x_k) Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k) \end{aligned}$$

for $N = 0, 1, 2, \dots$.

Let us show that for all $\varepsilon > 0$ there exists a regular system $\mathcal{D}$ together with some simple functions $f^\varepsilon_{j_1,\dots,j_k}$ adapted to it which satisfy the desired properties.

Indeed, by Lemma (5.2) of [10] for all $\varepsilon > 0$ and parameters $(j_1, \dots, j_k)$, $1 \le j_s \le d$, $1 \le s \le k$, there exists such a simple function $f^\varepsilon_{j_1,\dots,j_k}$ adapted to a regular system $\mathcal{D}_{j_1,\dots,j_k}$ in such a way that this function $f^\varepsilon_{j_1,\dots,j_k}$ satisfies (3.16), and the elements of $\mathcal{D}_{j_1,\dots,j_k}$ have boundaries with zero μ_0 measure. Let us make such a construction for all parameters $(j_1, \dots, j_k)$. It can be seen that there is a refinement $\mathcal{D}$ of the regular systems $\mathcal{D}_{j_1,\dots,j_k}$ such that all simple functions $f^\varepsilon_{j_1,\dots,j_k}$ are adapted to it, and its elements have boundaries with zero μ_0 probability. (I omit the details of this construction.) This regular system together with the functions $f^\varepsilon_{j_1,\dots,j_k}$ adapted to it satisfy the desired requirements, because, as we shall see, their properties imply that these functions satisfy not only (3.16), but also (3.17).

Relation (3.13) can be proved with the help of relations (3.16), (3.17) and (3.18) similarly to the reduction of the relation $Z_N \xrightarrow{\mathcal{D}} Z_0$ to formula (3.13). Indeed, one gets from inequalities (3.16), (5.6) in [10] and the definition of the quantities V_N and Y_0, by applying an argument similar to the proof of relation (3.14) that

$$E(V_0 - Y_0)^2 \le k! k^d \varepsilon^2,$$

and also

$$E(V_N - Y_N)^2 \le k^d k! \varepsilon^2$$

if $N > N_0(\varepsilon, T)$ by (3.17) and (5.6) in [10].

Then we can show with the help of these relations similarly to the reduction of the relation $Z_N \xrightarrow{\mathcal{D}} Z_0$ to formula (3.13) that $|Ee^{itV_N} - Ee^{itY_N}| \le \varepsilon$, $|Ee^{itY_N} - Ee^{itY_0}| \le \varepsilon$, and $|Ee^{itY_0} - Ee^{itV_0}| \le \varepsilon$ if $N > N_0(\varepsilon, t, T)$ with some threshold index $N_0(\varepsilon, t, T)$. Here in the first and third inequality we apply the last two inequalities which were consequences of (3.16) and (3.17),

while the second inequality follows from (3.18). Since these relations hold for all $\varepsilon > 0$ they imply that $Ee^{itV_N} \to Ee^{itV_0}$ for all $t \in \mathbb{R}^1$ as $N \to \infty$, i.e. $V_N \xrightarrow{\mathcal{D}} V_0$ as $N \to \infty$, and this is formula (3.13) written with a different notation.

It remains to prove (3.16), (3.17) and (3.18). We made such a construction of a regular system and simple functions adapted to it with the help of Lemma 5.2 in [10] which satisfy (3.16). Then formula (3.17) follows from some classical results about vague (and weak) convergence of measures. Since we are working in the proof of (3.17) in a cube $[-T,T]^{k\nu}$ it is enough to know the results about weak convergence to carry out our arguments.

Let us first observe that since the restrictions of the measures $G^{(N)}_{j,j}$ to $[-T,T]^\nu$ tend weakly to the restriction of the measure $G^{(0)}_{j,j}$ to the cube $[-T,T]^\nu$ as $N \to \infty$, we can also say that the restrictions of the product measures $G^{(N)}_{j_1,j_1} \times \cdots \times G^{(N)}_{j_k,j_k}$ to the cube $[-T,T]^{k\nu}$ converge weakly to the restriction of the product measure $G^{(0)}_{j_1,j_1} \times \cdots \times G^{(0)}_{j_k,j_k}$ on the cube $[-T,T]^{k\nu}$, as $N \to \infty$. On the other hand, the function

$$H^0_{j_1,\dots j_k}(x_1,\dots,x_k) = |h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)\chi_T(x_1,\dots,x_k) - f^\varepsilon_{j_1,\dots,j_k}(x_1,\dots,x_k)|^2$$

is almost everywhere continuous with respect to the measure $G^{(0)}_{j_1,j_1} \times \cdots \times G^{(0)}_{j_k,j_k}$. By the general theory about convergence of measures these properties imply that

$$\int H^0_{j_1,\dots j_k}(x_1,\dots,x_k) G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k) \to \int H^0_{j_1,\dots j_k}(x_1,\dots,x_k) G^{(0)}_{j_1,j_1}(\,dx_1)\dots G^{(0)}_{j_k,j_k}(\,dx_k)$$

as $N \to \infty$. (Such a convergence is proved for probability measures for example in [2].) A careful analysis shows that this result remains valid for sequences of finite, but not necessarily probability measures. Let me remark that here we are working with (real, non-negative) measures. The last relation together with (3.16) imply (3.17).

To prove relation (3.18) first we show that $G^{(N)}_{j,j'}(\Delta_k) \to G^{(0)}_{j,j'}(\Delta_k)$ as $N \to \infty$ for all $1 \le j, j' \le d$ and $\Delta_k \in \mathcal{D}$ with the regular system $\mathcal{D}$ we are working with. (Let me recall that the boundary of all sets $\Delta_k \in \mathcal{D}$ has zero μ_0 measure and hence also zero $G^{(0)}_{j,j'}$ measure.)

If $j = j'$ then this relation follows immediately from the facts that $G^{(N)}_{j,j} \xrightarrow{v} G^{(0)}_{j,j}$, $G^{(0)}_{j,j}(\partial\Delta_k) = 0$ for all $1 \le |k| \le M$, and $G^{(N)}_{j,j}$ is a locally finite measure for all $N = 0, 1, 2, \dots$. If $j \ne j'$, then we have to apply a more refined argument, since in this case we only know that $G^{(N)}_{j,j'}$ is a complex measure with locally finite total variation. In this case we will exploit that the matrix

valued measures $(G_{j,j'}^{(N)})$, $1 \le j, j' \le d$, are positive semidefinite. This implies that the Radon–Nikodym derivatives $g_{j,j'}^{(N)}$ of the complex measures $G_{j,j'}^{(N)}$ with respect to the dominating measure μ_N have the following property. For all $N = 0, 1, 2, \dots$ and $1 \le j, j' \le d$ such that $j \ne j'$ the 2×2 matrices

$$g^{(N)}(x|j, j') = \begin{pmatrix} g_{j,j}^{(N)}(x), & g_{j,j'}^{(N)}(x) \\ g_{j',j}^{(N)}(x), & ng_{j',j'}^{(N)}(x) \end{pmatrix}$$

are positive semidefinite for μ_N almost all $x \in \mathbb{R}^\nu$. Let us define for all non-negative functions $v(x)$, $x \in \mathbb{R}^\nu$ the vector $S(x|v) = (\sqrt{v(x)}, \sqrt{v(x)})$. By exploiting that the matrices $g^{(N)}(x|j, j')$ are positive semidefinite we get that

$$\begin{aligned} &\int v(x)[G_{j,j}^{(N)}(\,dx) + G_{j,j'}^{(N)}(\,dx) + G_{j',j}^{(N)}(\,dx) + G_{j',j'}^{(N)}(\,dx)] \\ &\quad = \int S(x|v) g^{(N)}(x|j, j') S(x|v)^* \mu_N(\,dx) \ge 0 \end{aligned}$$

for all functions v such that $v(x) \ge 0$, $x \in \mathbb{R}^\nu$. Hence $H_{j,j'}^{(N)} = [G_{j,j}^{(N)} + G_{j,j'}^{(N)} + G_{j',j}^{(N)} + G_{j',j'}^{(N)}]$ is a locally finite measure on $\mathbb{R}^\nu$. Moreover $H_{j,j'}^{(N)} \xrightarrow{v} H_{j,j'}^{(0)}$ as $N \to \infty$. This implies that $H_{j,j'}^{(N)}(\Delta_k) \to H_{j,j'}^{(0)}(\Delta_k)$, therefore $G_{j,j'}^{(N)}(\Delta_k) + G_{j',j}^{(N)}(\Delta_k) \to G_{j,j'}^{(0)}(\Delta_k) + G_{j',j}^{(0)}(\Delta_k)$ as $N \to \infty$ for all $\Delta_k \in \mathcal{D}$.

We get similarly by working with the vectors $R(x|v) = (\sqrt{v(x)}, i\sqrt{v(x)})$ instead of the vectors $S(x|v) = (\sqrt{v(x)}, \sqrt{v(x)})$ for all functions $v(x) \ge 0$, $x \in \mathbb{R}^\nu$, that $K_{j,j'}^{(N)} = [G_{j,j}^{(N)} + iG_{j,j'}^{(N)} - iG_{j',j}^{(N)} + G_{j',j'}^{(N)}]$ is a a locally finite measure for all $N = 0, 1, 2, \dots$, and $K_{j,j}^{(N)} \xrightarrow{v} K_{j,j'}^{(0)}$ as $N \to \infty$. Thus $K_{j,j'}^{(N)}(\Delta_k) \to K_{j,j'}^{(0)}(\Delta_k)$, therefore $G_{j,j'}^{(N)}(\Delta_k) - G_{j',j}^{(N)}(\Delta_k) \to G_{j,j'}^{(0)}(\Delta_k) - G_{j',j}^{(0)}(\Delta_k)$ as $N \to \infty$ for all $\Delta_k \in \mathcal{D}$. These relations imply that $G_{j,j'}^{(N)}(\Delta_k) \to G_{j,j'}^{(0)}(\Delta_k)$ for all $\Delta_k \in \mathcal{D}$.

Let us define for all $N = 0, 1, 2, \dots$ and our regular system $\mathcal{D} = \{\Delta_k,\ 1 \le |k| \le M\}$ the Gaussian random vector

$$Z_N(\mathcal{D}) = \Big(\operatorname{Re} Z_{G^{(N)},j}(\Delta_k), \operatorname{Im} Z_{G^{(N)},j}(\Delta_k), \quad |k| \le M,\ \ 1 \le j \le d\Big)$$

I claim that the elements of the covariance matrices of the random vectors $Z_N(\mathcal{D})$ can be expressed by means of the numbers $G_{j,j'}^{(N)}(\Delta_k)$, $1 \le |k| \le M$ and $1 \le j, j' \le d$, and the covariance matrices of $Z_N(\mathcal{D})$ converge to the covariance matrix of $Z_0(\mathcal{D})$ as $N \to \infty$. (In the proof of this statement I repeat some arguments applied in the investigation of random spectral measures in Section 3 of [10].)

To prove these relations observe that

$$\mathrm{Re}\, Z_{G^{(N)},j}(\Delta_k) = \frac{Z_{G^{(N)},j}(\Delta_k) + \overline{Z_{G^{(N)},j}(\Delta_k)}}{2},$$
$$\mathrm{Im}\, Z_{G^{(N)},j}(\Delta_k) = \frac{Z_{G^{(N)},j}(\Delta_k) - \overline{Z_{G^{(N)},j}(\Delta_k)}}{2i},$$

and $\overline{Z_{G^{(N)},j}(\Delta_k)} = Z_{G^{(N)},j}(-\Delta_k) = Z_{G^{(N)},j}(\Delta_{-k})$. (In the last identity we also exploited the properties of the regular systems $\mathcal{D}$.) Also the properties of the regular systems imply that if $\Delta_k, \Delta_l \in \mathcal{D}$, then we have either $\Delta_k \cap \Delta_l = \Delta_k$ or $\Delta_k \cap \Delta_l = \emptyset$. The first identity holds if $l = k$ and the second one if $l \neq k$. Hence we have either $EZ_{G^{(N)},j}(\Delta_k)\overline{Z_{G^{(N)},j'}(\Delta_l)} = G^{(N)}_{j,j'}(\Delta_k)$ if $k = l$ or $EZ_{G^{(N)},j}(\Delta_k)\overline{Z_{G^{(N)},j'}(\Delta_l)} = 0$ if $k \neq l$. These relations imply that we can express all covariances

$$E\mathrm{Re}\, Z_{G^{(N)},j}(\Delta_k)\mathrm{Re}\, Z_{G^{(N)},j'}(\Delta_l), \quad E\mathrm{Re}\, Z_{G^{(N)},j}(\Delta_k)\mathrm{Im}\, Z_{G^{(N)},j'}(\Delta_l)$$
$$\text{and } E\mathrm{Im}\, Z_{G^{(N)},j}(\Delta_k)\mathrm{Im}\, Z_{G^{(N)},j'}(\Delta_l)$$

with the help of the quantities $G^{(N)}_{j,j'}(\Delta_k)$, $1 \le j, j' \le d$, $1 \le |k| \le M$. The convergence of the numbers $G^{(N)}_{j,j'}(\Delta_k)$ to $G^{(0)}_{j,j'}(\Delta_k)$ also implies that the covariance matrices of $Z_N(\mathcal{D})$ converge to the covariance matrix of $Z_0(\mathcal{D})$ as $N \to \infty$.

The convergence of the covariance matrices of the Gaussian random vectors $Z_N(\mathcal{D})$ with expectation zero also implies that the distributions of $Z_N(\mathcal{D})$ converge weakly to the distribution of $Z_0(\mathcal{D})$ as $N \to \infty$. But then the same can be told about any continuous functions of the coordinates of the random vectors $Z_N(\mathcal{D})$. Because of the definition of the multiple Wiener–Itô integrals of simple functions the random variables Y_N in formula (3.18) are polynomials, hence continuous functions of the coordinates of the random vectors $Z_N(\mathcal{D})$. Besides, these polynomials do not depend on the parameter N. Hence the previous results imply that formula (3.18) holds. Proposition 3.1 is proved. □

To simplify the application of Proposition 3.1 we also prove the following lemma.

Lemma 3.2. *Let us have a sequence of matrix valued spectral measures* $(G^{(N)}_{j,j'})$, $N = 1, 2, \dots$, $1 \le j, j' \le d$, *on the torus* $[-A_N\pi, A_N\pi]^\nu$ *such that* $A_N \to \infty$, *and* $G^{(N)}_{j,j'} \xrightarrow{v} G^{(0)}_{j,j'}$ *with some complex measure* $(G^{(0)}_{j,j'})$ *with locally finite total variation for all* $1 \le j, j' \le d$ *as* $N \to \infty$. *Then* $G^{(0)} = (G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, *is a positive semidefinite matrix valued even measure on* $\mathbb{R}^\nu$.

Remark. Lemma 3.2 helps to show that in many interesting cases the limit matrix $G^{(0)} = (G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, of the spectral measures $G^{(N)} = (G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, $N = 1, 2, \dots$, in Proposition 3.1 is the spectral measure of a generalized stationary Gaussian random field. Indeed, in Theorem 4.1

of [10] it was shown that a positive semidefinite matrix valued measure on $\mathbb{R}^\nu$ whose distribution is moderately increasing at infinity is the spectral measure of such a random field. (See Section 4 of [10] for the definition of these notions.) So by Lemma 3.2 to prove that $G^{(0)}$ is the spectral measure of a generalized stationary Gaussian random field it is enough to show that the distribution of $G^{(0)}$ is moderately increasing.

Proof of Lemma 3.2. We have to show that $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite matrix valued measure. To do this take a vector $v(x) = (v_1(x), \dots, v_d(x))$ whose coordinates $v_k(x)$, $1 \le k \le d$, are continuous functions with compact support. We have

$$\lim_{N\to\infty} \sum_{j=1}^d \sum_{j'=1}^d \int v_j(x) v_{j'}(x) G^{(N)}_{j,j'}(\,dx) = \sum_{j=1}^d \sum_{j'=1}^d \int v_j(x) v_{j'}(x) G^{(0)}_{j,j'}(\,dx) \ge 0. \tag{3.19}$$

The identity in (3.19) holds, since $G^{(N)}_{j,j'} \xrightarrow{v} G^{(0)}_{j,j'}$ for all $1 \le j, j' \le d$. The inequality at the end of (3.19) also holds, because $(G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite matrix valued measure for all $N = 1, 2, \dots$, and this implies that the left-hand side of (3.19) is non-negative for all $N = 1, 2, \dots$. Thus we got that if $g^{(0)}_{j,j'}(x)$ is the Radon–Nikodym derivative of $G^{(0)}_{j,j'}$ with respect to some dominating measure μ_0 in the point $x \in \mathbb{R}^\nu$ for all $1 \le j, j' \le d$, we take the $d \times d$ matrix $g^{(0)}(x) = (g^{(0)}_{j,j'}(x))$, $1 \le j, j' \le d$, and the coordinates of the vector $v(x) = (v_1(x), \dots, v_d(x))$ are continuous functions with compact support, then

$$\int v(x) g^{(0)}(x) v^*(x) \mu_0(\,dx) \ge 0.$$

In the proof of Theorem 2.2 of [10] we have shown that this relation implies that $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, is a positive semidefinite matrix valued measure.

We still have to show that the complex measure $G^{(0)}_{j,j'}$ with locally finite variation is even for all $1 \le j, j' \le d$. To do this fix a pair j, j' of indices, $1 \le j, j' \le d$, and define for all $N = 0, 1, 2, \dots$ the complex measure $(G')^{(N)}_{j,j'}$ by the relation $(G')^{(N)}_{j,j'}(A) = \overline{G^{(N)}_{j,j'}(-A)}$ for all bounded, measurable sets $A \subset \mathbb{R}^\nu$. It is not difficult to see that not only $G^{(N)}_{j,j'} \xrightarrow{v} G^{(0)}_{j,j'}$, but also $(G')^{(N)}_{j,j'} \xrightarrow{v} (G')^0_{j,j'}$ as $N \to \infty$. The evenness of the measures $G^{(N)}_{j,j'}$ for $N = 1, 2, \dots$ means that $G^{(N)}_{j,j'} = (G')^{(N)}_{j,j'}$ for all $N = 1, 2, \dots$. By taking the limit $N \to \infty$ we get that $G^{(0)}_{j,j'} = (G')^{(0)}_{j,j'}$. This means that $G^{(0)}_{j,j}$ is an even complex measure with locally finite variation. Lemma 3.2 is proved. □

Appendix A. On the results in Section 2.

The Appendix consists of two parts. In the first part some questions related to Section 2 while in the second part some questions related to Section 3 are discussed.

In the first part the idea of the proof of Itô's formula is explained together with the reason why to work with Wick polynomials in it. It is also shown how Itô's formula can be applied in the proof of the other results in Section 2.

The second part deals with the investigation of limit problems for non-linear functionals of a stationary Gaussian random field and the content of Proposition 3.1. Here I try to explain the role of the compactness type condition (b) in this proposition with the help of an example.

First I try to explain the idea behind the proof of Itô's formula. To do this first I consider its one-dimensional version. In that case we want to prove the following statement. Let $\varphi(x)$, $\varphi(-x) = \overline{\varphi(x)}$, be such a function for which $\int |\varphi(x)|^2 G(\,dx) = 1$ with a spectral measure G, and consider a random spectral measure Z_G corresponding to the spectral measure G. Then $Y = \int \varphi(x) Z_G(\,dx)$ is a standard normal random variable, and the identity

$$H_n\left(\int \varphi(x) Z_G(\,dx)\right) = \int \varphi(x_1)\cdots\varphi(x_n) Z_G(\,dx_1)\dots Z_G(\,dx_n). \quad \text{(A.1)}$$

holds for all $n \ge 1$. Actually, Itô's formula is a more general result, but here it will be enough to consider this special case.

It is proved in the general theory that the above-defined random variable Y has standard normal distribution. Formula (A.1) is proved by induction with respect to n. In this induction, we apply the recursion formula (2.2) for Hermite polynomials and the diagram formula for the product of multiple Wiener–Itô integrals. We exploit that they "fit to each other".

Formula (A.1) clearly holds for $n = 1$. To prove it for n if we know it for $m < n$ we rewrite the left-hand side of (A.1) with the help of the recursion formula (2.2). We rewrite the random integral

$$\int \varphi(x_1)\cdots\varphi(x_n) Z_G(\,dx_1)\dots Z_G(\,dx_n)$$

at the right hand of (A.1) by means of the identity that we get by applying the diagram formula for the product

$$\int \varphi(x_1)\cdots\varphi(x_{n-1}) Z_G(\,dx_1)\dots Z_G(\,dx_{n-1}) \int \varphi(x) Z_G(\,dx).$$

(I remark that Proposition 5.1 of [8] yields a generalization of the formula we get in such a way.) Then some calculations with the help of these formulas and the inductive hypothesis yield the proof of formula (A.1) for n.

Itô's formula for vector-valued stationary Gaussian random fields can be proved by an appropriate adaptation of the above argument. In the proof, we apply a useful special case of the diagram formula for vector-valued

stationary Gaussian random fields presented in [10]. It is formulated in the corollary of Theorem 6.1 in[10].

On the other hand, we need a new identity instead of formula (2.2) in the proof that we can formulate it with the help of Wick polynomials. This was the reason for the introduction of Wick polynomials in this paper. They are defined in Section 2 with the help of some results in [10]. First we have to understand that this definition is correct. Namely, we have to show that the Wick polynomial $:P(\xi_1, \dots, \xi_m):$ of a homogeneous polynomial $P(\xi_1, \dots, \xi_m)$ of order n depends only on the random variables $\xi_1, \dots, \xi_m$, although in its definition we applied a projection to a Hilbert space $\mathcal{H}_n$ which may depend on other random variables, too.

The result of Theorem 2B implies the correctness of this definition.It states that the value of the Wick polynomial does not change if we take projection to the Hilbert space $\mathcal{H}_n(\xi_1, \dots, \xi_m)$ introduced before the formulation of this result instead of the projection to $\mathcal{H}_n$. The definition of this new Hilbert space is similar to that of $\mathcal{H}_n$, the only difference is that here we work only with the random variables $\xi_1, \dots, \xi_m$.

The proof of Theorem 2B exploits the following property of Gaussian random vectors. If some coordinates of a Gaussian random vector are uncorrelated, then they are also independent. This implies that the elements of the underlying Gaussian random field can be decomposed as $X_t = \eta_{1,t} + \eta_{2,t}$, $t \in T$, in such a way that $\eta_{1,t}$ is a linear combination of the random variables $\xi_1, \dots, \xi_m$, while $\eta_{2,t}$ is uncorrelated, hence independent of them. The proof of Theorem 2B is based on this fact. I omit the details of the proof.

Theorem 2B implies in particular that $:\xi^n: \, = H_n(\xi)$ if ξ is a standard normal random variable. Corollary 2C describes a deeper relation between Hermite and Wick polynomials. This can be exploited. For instance, identity (2.3) formulated in Proposition 2.1 can be proved with its help and formula (2.2) about Hermite polynomials. This identity plays an important role in the proof of Itô's formula.

The proof of Itô's formula for vector-valued stationary Gaussian random fields is made with the help of the identity (2.3) for Wick polynomials and the Corollary of Theorem 6.1 in [10] which is a special case of the diagram formula for vector-valued stationary Gaussian random fields. It is a natural adaptation of the previously discussed proof in the scalar-valued case.

It is easy to deduce from Theorem 2.2 its Corollary. In this Corollary the Wick polynomials of such homogeneous polynomials are considered whose arguments are elements of one of the Hilbert spaces $\mathcal{H}_{1,j}$, $1 \le j \le d$. In the Corollary of Theorem 2.2 such expressions are expressed in the form of a sum of multiple Wiener–Itô integrals. The class of homogeneous polynomials considered in the Corollary of Theorem 2.2 is fairly large. This fact is exploited in the proof of Proposition 2.3, which states that the set of all finite sums of n-fold Wiener–Itô integrals constitute an everywhere dense class of functions in $\mathcal{H}_n$.

In the last result of Section 2, in Theorem 2.4 a random variable, defined in formula (2.6) in the form of a multiple Wiener–Itô integral is considered, and its shift transforms are calculated in formula (2.8). This is an important result, and its proof is based also on the Itô formula. The validity of formula (2.8) can be checked first for the shifts of one-fold Wiener–Itô integrals. Then it can be proved with the help of Itô's formula for such multiple Wiener–Itô integrals whose kernel functions have the special form

$$h(x_1, \dots, x_n) = \varphi_1(x_1) \cdots \varphi_n(x_n).$$

After this Proposition 2.4 can be proved in the general case by means of a standard method.

Appendix B. On the results in Section 3.

At the beginning of Section 3 I formulated a limit problem. I considered a vector-valued stationary Gaussian random field $X(p)$, $p \in \mathbb{Z}^\nu$, defined a Wick polynomial of order k of the coordinates of the vector $X(0) = (X_1(0), \dots, X_d(0))$, and I was interested in a limit problem for the expressions S_N defined in (3.3). These expressions are normalized partial sums whose elements are shift transforms $T_pY(0)$ of the random variable $Y(0)$ defined in (3.4).

This problem is a multivariate version of the problem studied in [5].

The expressions S_N can be rewritten in an interesting simple form. To do this first we rewrite the random variable $Y(0)$ in the form of a sum of multiple Wiener–Itô integrals and express its shift transforms $T_pY(0)$ with the help of Proposition 2.4. This enables us to express the appropriately rescaled versions of the random variables S_n as sums of multiple Wiener–Itô integrals with such kernel functions which have a limit as $N \to \infty$. This is done in (3.7). This formula suggests that if the matrix valued spectral measures $G^{(N)}$ defined in Section 3 have a limit, then the normalized versions of the random variables S_N are convergent in distribution, and we can get their limit by means of a natural limiting procedure. Maybe, this limiting procedure can be carried out only under some not too restrictive additional conditions. If this limiting procedure can be carried out then we get a limit theorem for the normalized versions of the random variables S_n.

Proposition 3.1 gives a useful sufficient condition for the application of such a limiting procedure. In its formulation some random variables Z_N, $N = 0, 1, 2, \dots$, defined in (3.9) and (3.12) are considered. First it is shown that under the conditions of Proposition 3.1 these random variables exist (the multiple integrals appearing in their definition are well-defined), and then also the convergence $Z_N \xrightarrow{\mathcal{D}} Z_0$ is proved.

The conditions of Proposition 3.1 formulated in part (a) are natural. They demand that the kernel functions $h^N_{j_1,\dots,j_k}(x_1, \dots, x_k)$ of the random integrals in Z_N and the matrix valued spectral measures $G^{(N)}$ converge to $h^0_{j_1,\dots,j_k}(x_1, \dots, x_k)$ and $G^{(0)}$ in an appropriate way. (Actually, there is also

a hidden condition here. The functions $h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)$ must be continuous. In the scalar-valued version of this result, in Proposition 8.3 of [8] a slightly weaker continuity condition is imposed. Here we do not discuss the question how the continuity property in Proposition 3.1 can be weakened.)

On the other hand, part (b) of Proposition 8.3 contains an additional condition that deserves special attention. We can understand its role better by considering the application of Proposition 3.1 in the proof of the main result in [11] or the application of its scalar-valued version in the proof of Theorem 8.2 in [8].

In Theorem 8.2 of [8] a scalar-valued stationary Gaussian random field $X(p)$, $p \in \mathbb{Z}^\nu$, is considered. The random variables $Y(p) = H_k(X(p)) = T_pY(0)$ are introduced, where $H_k(\cdot)$ is the Hermite polynomial of order k, and the normalized partial sums S_N are defined by formula (3.3) with these random variables $Y(p)$. A non-central limit theorem is proved with normalizing constants $A_N = N^{\nu-k\alpha/2}L(N)^{k/2}$ if the correlation function $r(p) = EX(0)X(p)$ satisfies the relation $r(p) = |p|^{-\alpha}a\left(\frac{p}{|p|}\right)L(|p|)$ with some $0 < \alpha < \frac{\nu}{k}$, where $L(\cdot)$ is a slowly varying function at infinity, and $a(\cdot)$ is a function on the unit sphere of $\mathbb{R}^\nu$. It shows the dependence of the correlation function $r(p)$ on the direction of the vector p.

Paper [11] contains a multivariate version of this result. Here a vector-valued stationary Gaussian random field $X(p) = (X_1(p),\dots,X_d(p))$, $p \in \mathbb{Z}^\nu$, is considered, and a limit theorem is proved for the random variables S_N, defined in (3.3) and (3.4) under appropriate conditions. A condition, similar to the condition of the correlation function in Theorem 8.2 of [8] is imposed on the correlation function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$ which is described in formula (1.3) of [10]. A non-central limit theorem with normalizing constants $A_N = N^{\nu-k\alpha/2}L(N)^{k/2}$ is proved if this condition holds for the correlation function with exponent $0 < \alpha < \frac{\nu}{k}$.

It is worth understanding why the condition $\alpha < \frac{\nu}{k}$ in the exponent of the formula expressing the decrease of the correlation function is needed in the proof of these results. In the first step of these proofs, we have to describe the asymptotic behavior of the spectral measure of the underlying stationary Gaussian random field. In the scalar-valued case this is done in Lemma 8.2 of [8]. It describes the limit behavior of the appropriately rescaled versions G_N of the spectral measure G of the stationary Gaussian random field we are working with. These measures are defined by the identity $G_N(A) = \frac{N^\alpha}{L(N)}G\left(\frac{A}{N}\right)$ for all measurable sets A. In this lemma, it is proved that these measures G_N have a vague limit G_0 as $N \to \infty$, and the limit measure has the homogeneity property $G_0(A) = t^{-\alpha}G_0(tA)$ for all measurable sets A and $t > 0$. There is a similar result also in the case of vector-valued stationary Gaussian fields.

The above results imply that the spectral measures and kernel functions in the representation of S_N in formula (3.7) satisfy the starting conditions of Proposition 3.1 if the correlation function of the underlying stationary

Gaussian field has such an asymptotic behavior at the infinity as it is demanded in the above-mentioned results. Moreover, they are satisfied if the correlation function satisfies the desired relation with arbitrary "homogeneity parameter" $\nu > \alpha > 0$.

One would like to understand where we exploited the condition $\alpha < \frac{\nu}{k}$, and what can be told if this condition is violated. (The number k was the order of the Wick polynomial we were working with.) This question is related to condition (b) in Proposition 3.1. It is proved that this condition is satisfied under the additional condition $\alpha < \frac{\nu}{k}$. (This is proved in [5] or [8] in the scalar and in [11] in the vector-valued case.) On the other hand, it can be proved that if the asymptotic formula prescribed for the correlation function satisfies the prescribed asymptotic relation with $\alpha \geq \frac{\nu}{k}$, then relation (3.11) does not hold, and as a consequence, the definition of the random variable Z_0 in (3.12) is incorrect.

We have a more detailed knowledge on the behavior of the random sum S_N defined in (3.3) if $\alpha > \frac{\nu}{k}$. It is known that it satisfies the central limit theorem with the standard normalization $N^{\nu/2}$. This follows from the central limit theorem proved in [3] in the scalar and in Theorem 4 of [1] in the vector-valued case. One only has to check that the conditions of these results are satisfied in this case, and this can be done by calculating the necessary covariances. A similar central limit theorem also holds if $\alpha = \frac{\nu}{k}$, but in this case, it may happen that the norming constant is $N^{\nu/2}L(N)$ with a slowly varying function $L(N)$ tending to infinity as $N \to \infty$.

The above-discussed results suggest the following heuristic picture about a generalized version of the results discussed in the Appendix.

Let us have a stationary Gaussian random field $X(p)$, $p \in \mathbb{Z}^\nu$ and a non-linear functional Y of this random field. Take the shifts $Y(p) = T_p Y$, $p \in \mathbb{Z}^\nu$, of this non-linear functional, and consider their normalized sums S_N defined in formula (3.1). We are interested in what kind of limit theorem holds for this sequence S_N as $N \to \infty$ with an appropriate norming constants A_N. In Section 3 we considered a special case of this problem and proved that it can be reformulated to the problem about the limit theorem for a sequence Z_N defined in (3.9). Such a sequence has a limit if the kernel functions and spectral measures in the definition of Z_N behave nicely. A similar reformulation of the above-mentioned limit problem is possible in a more general case, only different kernel functions appear in the definition of the random integrals in (3.9). Proposition 3.1 may help in the study of the limit behavior of the random variables Z_N defined in (3.9). It turned out that condition (b) of Proposition 3.1 is an important condition of this result. Let us understand its role better.

Condition (b) of Proposition 3.1 is a compactness type condition formulated in (3.10). Let us consider the integrals in it when we integrate on the whole space $\mathbb{R}^\nu$. If the values of these integrals tend to infinity as $N \to \infty$, then for large N the essential part of the random integrals in (3.9) comes

from a region that contains vectors in $\mathbb{R}^{k\nu}$ with very big norms. It is natural to expect that in such cases the random variables Z_N satisfy the central limit theorem with the classical norming constants $A_N = N^{\nu/2}$ under very general conditions. Such a result is proved besides the above-mentioned papers [1] and [3] also in the book [12]. On the other hand, Proposition 3.1 implies the existence of a non-Gaussian limit, expressed by means of a sum of multiple Wiener–Itô if the spectral measures and the kernel functions have a limit, and condition (b) of Proposition 3.1 holds.

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Alfréd Rényi Institute of Mathematics Budapest, P.O.B. 127 H–1364, Hungary
Email address: major@renyi.hu