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On the Asymptotic Behavior of Some Self-Similar Random Fields

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We consider some classes of self-similar non-Gaussian generalized fields over the ν-dimensional space, which were constructed in [1] as functionals over Gaussian fields. The definition of these fields is then only meaningful if their self-similarity parameter κ is bigger than $\nu/2$, which is the self-similarity parameter of the white noise field. In this paper we show that if κ tends to $\nu/2$, then these self-similar fields tend, after an appropriate normalization, to the white noise field. A discrete version of this result will also be proved.

1. Introduction

In [1] a new class of self-similar fields was constructed by means of multiple Wiener-Itô integrals with respect to the random spectral measure of a Gaussian self-similar field. This construction is meaningful only if an integrability condition is satisfied. Now we are interested in the case when a sequence of well-defined self-similar fields is given, but their formal limit is meaningless. We show for a large class of such fields that their limit exists and that this limit is the white noise field. In terms of the topological structure of self-similar fields, which has been worked out by Sinai [9] at the level of a formal asymptotic expansion, this means that in the space of self-similar fields the self-similar fields considered above constitute a branch starting from the Gaussian branch at the point corresponding to the white noise field.

In order to formulate the above results more precisely we have to recall some well-known results and definitions from the literature (see, e.g., [1], [5], [2]). Let $\mathbb{S} = \mathbb{S}(R^\nu)$ denote the Schwartz space of the infinitely differentiable rapidly decreasing functions over the ν-dimensional Euclidean space R^ν. A general random field over $\mathbb{S}$ is a set of random variables $X(\varphi)$,

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$\varphi \in \mathbb{S}$, such that

$$X(c_1\varphi_1 + c_2\varphi_2) = c_1X(\varphi_1) + c_2X(\varphi_2) \tag{a}$$

with probability 1 for all $\varphi \in \mathbb{S}$ and real numbers c_1, c_2 and

$$X(\varphi_n) \to X(\varphi) \tag{b}$$

in probability, if $\varphi_n \to \varphi$ in the topology of $\mathbb{S}$. A generalized random field is called a stationary Gaussian random field with zero mean if $X(\varphi)$ is a Gaussian random variable with expectation $\mathrm{E}X(\varphi) = 0$ for all $\varphi \in \mathbb{S}$, and $\mathrm{E}X(\varphi)X(\psi) = \mathrm{E}X(\varphi')X(\psi')$ for all $\varphi, \psi \in \mathbb{S}$ and $t \in R^\nu$, where $\varphi'(x) = \varphi(x+t)$ and $\psi'(x) = \psi(x+t)$ (see [5]). The finite dimensional distributions of such a field are determined by its correlation function $\mathfrak{R}(\varphi,\psi) = \mathrm{E}X(\varphi)X(\psi), \varphi,\psi \in \mathbb{S}$, or by its spectral measure G given by the formula

$$\mathfrak{R}(\varphi,\psi) = \int_{R^\nu} \tilde{\varphi}(x)\bar{\tilde{\psi}}(x)G(dx). \tag{1.1}$$

Here and in the following $\tilde{\varphi}$ denotes the Fourier transform of the function φ. The measure G is even, i.e., $G(A) = G(-A)$ for all Borel sets $A \subset \mathrm{R}^\nu$. If $G(A) = \int_A g(x)dx$ for all bounded Borel sets A, where g is a measurable function, then g is called the spectral density function of the spectral measure G. The stationary Gaussian field with spectral density $g(x) \equiv 1$ is called the white noise field. The random spectral measure of a stationary Gaussian field can be defined (see, e.g., [1] or [5] for the definition). If the spectral measure is nonatomic, i.e., $G(\{x\}) = 0$ for all $x \in R^\nu$, then the n-fold Wiener-Itô integral[1]

$$I_G^n(f) = \frac{1}{n!}\int f(x_1,\ldots,x_n)Z_G(dx_1)\ldots Z_G(dx_2)$$

can be defined. This integral is defined for the class of functions $f \in H_G^n$. A complex-valued function $f = f(x_1,\ldots,x_n)$ of n variables (the variables x_j are points of R^ν) belongs to the class H_G^n if the following conditions are satisfied:

a) f is a symmetrical function; i.e., it is invariant under all permutations of its variables.

b) $f(x_1,\ldots,x_n) = \overline{f(-x_1,\ldots,-x_n)}$.

c) $\int_{R^\nu}\ldots\int_{R^\nu}|f(x_1,\ldots,x_n)|^2G(dx_1)\ldots G(dx_n) < \infty$.

If the spectral measure $\hat{G}$ is absolutely continuous with respect to the spectral measure $G_1|\gamma(x)|^2 = (d\hat{G}/dG)(x)\gamma(-x) = \gamma(x)$, then

$$I_G^n(f) \stackrel{\Delta}{=} I_{\hat{G}}^n(\hat{f}), \tag{1.2}$$

where $\hat{f}(x_1,\ldots,x_n) = f(x_1,\ldots,x_n)\gamma(x_1)\ldots\gamma(x_n)$. Here $\stackrel{\Delta}{=}$ denotes equality in distribution.

[1] In the physics literature similar objects are called Wick polynomials.

An important class of generalized random fields is the class of self-similar fields. A generalized field is called a self-similar field with self-similarity parameter κ (see [2]) if the distributions of the random variables $X(\varphi)$ and $X(\varphi_\lambda)$ agree for all $\varphi \in \mathbb{S}$ and $\lambda > 0$, where $\varphi_\lambda(x) = \lambda^{\kappa - \nu}\varphi(\lambda^{-1}x)$. It is known (see [1]) that a Gaussian stationary field is self-similar with self-similarity parameter $\kappa > 0$ if and only if its spectral measure G is such that $G(\{0\}) = 0$, and there exists a finite measure μ defined on the Borel σ-algebra $\mathfrak{B}^{\nu-1}$ of the unit sphere $S^{\nu-1}$ in the ν-dimensional Euclidean space such that $G(B) = G(-B)$ for all $B \in \mathfrak{B}^{\nu-1}$, and

$$G(B_r) = r^{2\kappa}\mu(B) \tag{1.3}$$

for $B_r = \{x \in R_1^\nu 0 < |x| < r_1(x/|x|) \in B\}$, where $r > 0$, and $B \in \mathfrak{B}^{\nu-1}$. In particular, the white noise field whose correlation function is given by the formula

$$\mathfrak{R}(\varphi, \psi) = \int_{R^\nu} \tilde{\varphi}(x)\bar{\tilde{\psi}}(x)\,dx = \int_{R^\nu} \varphi(x)\psi(x)\,dx$$

is a Gaussian self-similar field with self-similarity parameter $\kappa = \nu/2$.

In the case when the spectral measure G of the Gaussian field with self-similarity parameter κ has a spectral density g, condition (1.3) is equivalent to the relation

$$g(x) = |x|^{-\alpha} a\left(\frac{x}{|x|}\right), \qquad x \neq 0, \tag{1.4}$$

where $\alpha = \nu - 2\kappa$ and $a(v)$, $v \in S^{\nu-1}$ is an even integrable function with respect to the Lebesgue measure over $S^{\nu-1}$. If, moreover, $0 < \alpha < \nu$, and the function a has $\nu - 1 - [\alpha]$ derivatives ([ ] here and in the following denotes the integral part) then the correlation function of the self-similar field can be written in the form

$$\mathfrak{R}(\varphi, \psi) = \int_{R^\nu}\int_{R^\nu} \varphi(s)\psi(t)|s - t|_{b_\alpha}^{-\alpha}\left(\frac{s-t}{|s-t|}\right) ds\,dt \tag{1.5}$$

with an appropriate even real function $b_\alpha(v)$, $v \in S^{\nu-1}$, and the function b_α can be expressed by means of an integral transformation of the function a (see Remark 1, Section 2).

Much less is known about non-Gaussian self-similar fields. In [1] (Theorem 6.2), the following class of self-similar fields is constructed. For $p = 1, 2, \ldots,$ the field

$$X(\varphi) = \sum_{n=1}^{p} \frac{1}{n!} \int \tilde{\varphi}(x_1 + \cdots + x_n) h_n(x_1, \ldots, x_n) Z_G(dx_1) \ldots Z_G(dx_n) \tag{1.6}$$

is self-similar with self-similarity parameter κ_0 if the spectral measure G satisfies condition (1.3) and the function $h_n(x_1, \ldots, x_n)$ is a symmetric even function $h_n(-x_1, \ldots, -x_n) = \overline{h_n(x_1, \ldots, x_n)}$ that satisfies the condition

$$h_n(\lambda x_1, \ldots, \lambda x_n) = \lambda^{-n\kappa_0+\kappa} h_n(x_1, \ldots, x_n)$$

for all $\lambda > 0$ and

$$\sum_{n=1}^{p} \frac{1}{n!} \int_{R^{\nu n}} |h_n(x_1, \ldots, x_n)|^2 |\varphi(x_1 + \cdots + x_n)|^2 G(dx_1) \ldots G(dx_n) < \infty$$

for all $\varphi \in \mathcal{S}$.

We shall consider the self-similar fields $H_{k,\epsilon,a}$ given by the formula

$$H_{k,\epsilon,a}(\varphi) = \frac{1}{k!} \int \tilde{\varphi}(x_1 + \cdots + x_k) Z_{G_{k,\epsilon,a}}(dx_1) \ldots Z_{G_{k,\epsilon,a}} dx_k,$$

where $k \geqslant 2$ is an integer, $0 < \epsilon < \nu$, $\varphi \in \mathcal{S}$, the spectral measure $G_{k,\epsilon,a}$ has a spectral density

$$g_{k,\epsilon,a}(x) = |x|^{-\alpha(k,\epsilon)} a\left(\frac{x}{|x|}\right), \qquad \alpha(k,\epsilon) = \nu\left(1 - \frac{1}{k}\right) + \frac{\epsilon}{k}, \tag{1.7}$$

and $a(v)$, $v \in S^{\nu-1}$, is an even function having continuous derivatives up to order $]\nu/k[-1$, where $]u[$ is the smallest integer which is not smaller than u. We assume that α is not identically zero. We shall write $H_{k,\epsilon}$ instead of $H_{k,\epsilon,a}$, and $g_{k,\epsilon}$, $G_{k,\epsilon}$ instead of $g_{k,\epsilon,a}$, $G_{k,\epsilon,a}$ where it causes no ambiguity. The field $H_{k,\epsilon}$ exists if for all $\varphi \in \mathcal{S}$ the variance

$$D(H_{k,\epsilon}(\varphi)) = \int_{R^{k\nu}} |\tilde{\varphi}(x_1 + \cdots + x_k)|^2 g_{k,\epsilon}(x_1) \ldots g_{k,\epsilon}(x_k)\, dx_1 \ldots dx_k. \tag{1.8}$$

The integral (1.8) converges for $0 < \epsilon < \nu$, and diverges for $\epsilon \geqslant \nu$ or $\epsilon \leqslant 0$. This can be proved by applying the well-known power counting theorem of statistical physics (see, e.g., [1]). We shall, however, prove some more general facts in this paper (see Remark 2, Section 2) that imply the convergence of the integral (1.8). Formula (1.2), the formula "for change of variables," shows that a large class of fields given by a formula of the form (1.6) with $h_n \equiv 0$, $n \neq k$, has the same distribution as $H_{k,\epsilon}$.

We are interested in the case $\epsilon \to 0$. In this case $\alpha(k,\epsilon) \to \alpha_0 = \nu(1 - 1/k)$, and (see Remark 2, Section 2) the variance satisfies the relation

$$D(H_{k,\epsilon}(\varphi)) \sim \epsilon^{-1} B_k \int |\varphi(x)|^2\, dx, \tag{1.9}$$

where

$$B_k = \frac{(2\pi)^{-\nu}}{k!} \int_{S^{\nu-1}} (b_{\alpha_0}(v))^k\, dv, \tag{1.10}$$

and b_{α_0} is the function whose substitution into formula (1.5) defines the self-similar Gaussian field with spectral density $g_{k,0}$. Here we shall investigate the limiting behavior of the normalized fields $(\epsilon/B_k)^{1/2} H_{k,\epsilon}(\varphi)$ as

$\epsilon \to 0$. We say that the generalized fields X_ϵ tend in distribution to a generalized field X as $\epsilon \to 0$ if $X_\epsilon(\varphi)$ tends to $X(\varphi)$ in distribution for all $\varphi \in \mathcal{S}$ as $\epsilon \to 0$.

Our main result is the following:

Theorem 1. *For all $k = 2, 3, \ldots$ the generalized fields*

$$\mathcal{T}_\epsilon(\varphi) = \epsilon^{1/2} H_{k,\epsilon}(\varphi), \qquad \varphi \in \mathcal{S}, \tag{1.11}$$

tend in distribution to the white noise field as $\epsilon \to 0$.

The self-similar fields $\overline{H}_{k,\epsilon,a}$ defined below are the natural discrete field counterparts of the fields $H_{k,\epsilon,a}$. Set

$$\overline{H}_{k,\epsilon,a}(n) = \frac{1}{k!} \int \tilde{\varphi}_n(x_1 + \cdots + x_k) Z_{G_{k,\epsilon}}(dx_1) \ldots Z_{G_{k,\epsilon}}(dx_k), \qquad n \in Z^\nu \tag{1.12}$$

where Z^ν denotes the ν-dimensional integer lattice, i.e., the set of points with integer coordinates from R^ν, $G_{k,\epsilon}$ is as defined in (1.7), and φ_n is the indicator function of the rectangle

$$\times_{j=1}^{\nu} \left[n^{(j)}, n^{(j)} + 1 \right),$$

where $n = (n^{(1)}, \ldots, n^{(\nu)})$. These random fields are again self-similar (see [1], [9]). The discrete analogue of the ν-dimensional white noise field is the set of independent standard normal random variables X_n, $n \in Z^\nu$. The next result is the discrete counterpart of Theorem 1.

Theorem 1′. *For all $k = 2, 3, \ldots$ the finite-dimensional joint distributions of the random fields $\epsilon^{1/2} B_k^{-1/2} \overline{H}_{k,\epsilon}(n)$ tend to the joint distributions of a set of independent standard normal random variables indexed by Z^ν.*

Theorem 1′ can be interpreted in the language of the theory of formal power series expansions worked out in Sinai's paper [9]. Informally, we can say that for all $k = 2, 3, \ldots$ and functions a, the random fields constitute a continuous branch in the space of self-similar fields with discrete parameters, which is parametrized by the self-similarity parameter $\nu/2 - \epsilon/2$ and starts at $\epsilon = 0$ from the Gaussian branch at the discrete white noise field.

It may be interesting to investigate a more general class of self-similar fields defined by (1.6). We shall consider the class of self-similar fields of the form

$$\hat{H}_{\epsilon, a_1, \ldots, a_p, c_1, \ldots, c_p} = \sum_{k=2}^{p} c_k \hat{H}_{k,\epsilon}, \tag{1.13}$$

$$\hat{H}_{k,\epsilon}(\varphi) = \frac{1}{k!} \int \tilde{\varphi}(x_1 + \cdots + x_k) \left[g_{k,\epsilon,a_k}(x_1) \ldots g_{k,\epsilon,a_k}(x_k) \right]^{1/2} \times W(dx_1) \ldots W(dx_k), \tag{1.14}$$

where W denotes the random spectral measure of the white noise field, the function g_{k,ϵ,a_k} is as defined by formula (1.7), a_k, $k = 1, \ldots, p$, are nonnegative integrable even functions over the sphere $S^{\nu-1}$ and $c_1, \ldots, c_p$ are certain constants. With continuous derivatives up to order $]\nu/k[-1$, the "formula for change of variables" (1.2) shows that the kth term in (1.13) has the same distribution as the field $H_{k,\epsilon}$ defined before.

We say that the generalized fields $X^{(1)}, \ldots, X^{(p)}$ defined on the same probability space are independent if the random variables $X^{(1)}(\varphi_1), \ldots, X^{(p)}(\varphi_p)$ are independent for all $\varphi_1, \ldots, \varphi_p \in \mathcal{S}$. A sequence of p-tuples $(X_\epsilon^{(1)}, \ldots, X_\epsilon^{(p)})$ of generalized fields tends to a p-tuple of generalized fields $(X^{(1)}, \ldots, X^{(p)})$ in distribution as $\epsilon \to 0$, if the random vectors

$$\left(X_\epsilon^{(1)}(\varphi_1), \ldots, X_\epsilon^{(p)}(\varphi_p)\right)$$

tend to the random vector $(X^{(1)}(\varphi_1), \ldots, X^{(p)}(\varphi_p))$ as $\epsilon \to 0$ for all $\varphi_1, \ldots, \varphi_p \in \mathcal{S}$.

Theorem 2. *Let the functions $a_1, \ldots, a_p$ be such that for all $j, k = 1, \ldots, p$ the functions $d(v) = [a_j(v)a_k(v)]^{1/2}$ have continuous derivatives up to order $](\nu/2)(1/k + 1/j)[-1$. Then the p-tuples of generalized random fields*

$$\mathcal{F}_\epsilon^{(k)}(\varphi) = (B_k)^{-1/2}\epsilon^{1/2}\hat{H}_{k,\epsilon}(\varphi), \qquad k = 1, \ldots, p \tag{1.15}$$

tend to p independent white noise fields. Hence, in particular, the random fields

$$\mathcal{F}_\epsilon(\varphi) = \left(\sum_{k=1}^{p} c_k^2 B_k\right)^{-1/2} \epsilon^{1/2} H_{\epsilon, a_1, \ldots, a_p, c_1, \ldots, c_p}(\varphi) \tag{1.16}$$

tend to the white noise field as $\epsilon \to 0$.

Theorem 1′ has a similar generalization.

We remark that it was not important in the above results that $g_{k,\epsilon,a}$ is the spectral density of a self-similar field. If $g_{k,\epsilon,a}$ is replaced by a spectral density $g^*_{k,\epsilon,a}$ which for small ϵ is near the function $g_{k,\epsilon,a}$ in a natural sense, then Theorems 1 and 2 remain valid, with a possibly different normalization.

We shall present two essentially different proofs of the above results. The first is based on a direct estimation of the moments of the random variables under investigation. It depends on the diagram formula for the moments of products of Wiener-Itô integrals. The crucial point of this proof is to show that the contribution of most diagrams is negligible. This method is quite frequently encountered in the physics literature, but in probability theory it is not so well known. It may also be useful in other problems; hence it may be interesting for probabilists. The second method is the method of "cutting off" introduced by Bernstein, which relates our problem to the central limit theorem for independent random variables. This method is well known in the theory of limit theorems for weakly dependent random variables, but has not been used in statistical physics.

This "cutting off" is carried out in the space of the spectral measure, and not in the time space as is typically done in probability theory. The authors think it useful to present both methods by proving Theorems 1 and 2 in both ways.

Section 2 contains some lemmas necessary for both proofs. In Section 3, Theorems 1, 1′, and 2 are proved by the method of moments. In Section 4, Theorems 3 and 4 are proved by the method of "cutting off." Theorem 1′ could also be proved by this method, but it would demand some more complicated estimates; hence we have omitted it.

2. Some preparatory lemmas

We shall use the Fourier method for estimating some integrals which appear in the formulas expressing the moments of the random variables we are interested in.

Let us consider the function

$$f_\alpha(x) = |x|^{-\alpha} a\left(\frac{x}{|x|}\right), \qquad x \in R^\nu - \{0\}, \tag{2.1}$$

where $a(v)$ is an integrable real function on the unit sphere $S^{\nu-1}$ in R^ν, and let $0 < \alpha < \nu$. The function f_α can also be interpreted as a generalized function in $\mathcal{S}'$, i.e., as a continuous linear functional over $\mathcal{S} = \mathcal{S}(R^\nu)$, by defining

$$f_\alpha(\varphi) = \int_{R^\nu} \varphi(x) f_\alpha(x)\,dx, \qquad \varphi \in \mathcal{S}(R^\nu).$$

Lemma 1. *Let the real even function $a(v)$ have continuous derivatives up to order $l(l = 0, 1, \dots, \nu - 1)$. If $\nu - l - 1 < \alpha < \nu$, then the Fourier transform of the generalized function $f_\alpha(\varphi)$ is given by the formula*

$$\tilde{f}_\alpha(\varphi) = \int \tilde{f}_\alpha(u)\varphi(u)\,du, \qquad \varphi \in \mathcal{S}, \tag{2.2}$$

where $\tilde{f}_\alpha(u)$ is given by the relation

$$\tilde{f}_\alpha(u) = |u|^{\alpha-\nu} b_\alpha\left(\frac{u}{|u|}\right), \qquad u \in R^\nu - \{0\}, \tag{2.3}$$

with an appropriate real even function $b_\alpha(v)$, $v \in S^{\nu-1}$ which is continuous in both variables v and α on the set $S^{\nu-1}x(\nu - 1 - l_1\nu)$.

Let

$$f_\alpha^A(x) = f_\alpha(x)\exp\left\{-\frac{|x|^2}{2A^2}\right\}, \qquad A \in (0, \infty). \tag{2.4}$$

The Fourier transform $\tilde{f}_\alpha^A(x)$ of $f_\alpha^A(x)$ has the following properties: The relation

$$\lim_{A\to\infty} \tilde{f}_\alpha^A(u) = \tilde{f}_\alpha(u) \tag{2.5}$$

holds for arbitrary $u \in R^\nu - \{0\}$, and there is a constant C_α such that

$$|\tilde{f}_\alpha^A(u)| \leqq C_\alpha |u|^{\alpha-\nu}. \tag{2.6}$$

The main result of the lemma is a consequence of some well-known results in the theory of homogeneous generalized functions (see [4]), and is due to Gårding [3]. For the sake of completeness, and since [3] is hard to read, we present another proof using some simple facts and ideas from [4]. This proof also provides explicit formulas for b_α.

Proof. First we restrict ourselves to the case $\nu - 1 < \alpha < \nu$.
Set

$$f_{\alpha,\tau}(x) = f_\alpha(x) e^{-\tau|x|}, \qquad \tau > 0.$$

By calculating in polar coordinates we see that [2])

$$\begin{aligned}
\tilde{f}_{\alpha,\tau}(u) &= \int_{R^\nu} f_{\alpha,\tau}(x) e^{i(u,x)}\, dx \\
&= \int_{S^{\nu-1}} ds\, a(s) \int dr\, |r|^{\nu-1-\alpha} \exp\{-\tau r + i(u,s)\} \\
&= i \exp\left\{ i \frac{\pi}{2}(\nu - 1 - \alpha) \right\} \Gamma(\nu - \alpha) \int_{S^{\nu-1}} a(s) \left[(u,s) + i\tau \right]^{\alpha-\nu} ds.
\end{aligned}$$

(In the last line we applied a result of [4], Chapter 2, Section 2.3.) Now taking the limit $\tau \to 0$ and exploiting the continuity of the Fourier transformation in $\mathbb{S}'$, we get that

$$\begin{aligned}
\tilde{f}_\alpha(u) = \lim_{\tau\to 0} \tilde{f}_{\alpha,\tau}(u) &= i \exp\left\{ i \frac{\pi}{2}(\nu - 1 - \alpha) \right\} \Gamma(\nu - \alpha) |u|^{\alpha-\nu} \\
&\quad \times \int_{S^{\nu-1}} a(s)\, ds \left(\left(\frac{u}{|u|}, s \right) + i0 \right)^{\alpha-\nu}
\end{aligned}$$

(see [4], Chapter 1, Section 3.6).

Exploiting the evenness of the function $a(v)$, we find that relation (2.3) holds with

$$b_\alpha(v) = -\Gamma(\nu - \alpha) \sin\left(\frac{\pi}{2}(\nu - \alpha - 1) \right) \int_{S^{\nu-1}} ds\, a(s) |(v,s)|^{\alpha-\nu}, \quad v \in S^{\nu-1}. \tag{2.7}$$

Fix $v \in S^{\nu-1}$. To prove relation (2.3) for all $0 < \alpha < \nu$, we first show that the function

$$G(\alpha) = \int a(s) |(v,s)|^{\alpha-\nu}\, ds, \qquad \nu - 1 < \operatorname{Re}\alpha < \nu$$

has an analytic continuation to the region $\nu > \operatorname{Re}\alpha > \nu - l - 1$ with poles only at the points $\nu - 1, \nu - 3, \ldots, \nu - 2[(l-1)/2] + 1$, and that these

[2]Here we consider integrals with respect to the nonnormalized Lebesgue measure over $S^{\nu-1}$.

poles have order 1. Put

$$g(t)=\int_{\{s\in S^{\nu-1},\,(v,s)=t\}} a(s)\,ds,$$

where ds denotes the Lebesgue measure on $S^{\nu-1}\cap\{(v,s)=t\}$. Then $g(t)$ is an l-times differentiable function, and

$$\begin{aligned} G(\alpha) &= \int_{-1}^{1} g(t)|t|^{\alpha-\nu}\,dt \\ &= \int_{-1}^{1}|t|^{\alpha-\nu}\left[g(t)-g(0)-\dots-\frac{t^{l-1}}{(l-1)!}g^{(l-1)}(0)\right]dt \\ &\quad+\sum_{j=0}^{[(l-1)/2]}\frac{g^{(2j)}(0)}{(2j)!}\,\frac{2}{\alpha-\nu+2j+1}, \end{aligned}$$

$$\nu-1<\operatorname{Re}\alpha<\nu,\qquad v\in S^{\nu-1}. \tag{2.8}$$

Formula (2.8) implies that $G(\alpha)$ has an analytic continuation with the desired properties. Let us consider $b_\alpha(v)$ as a function of α. By comparing formula (2.7) with the definition of $G(\alpha)$, and by exploiting the analyticity property of $G(\alpha)$, we see that $b_\alpha(v)$ can be extended to an analytic function in the whole region $\nu>\operatorname{Re}\alpha>\nu-l-1$, since $\sin((\pi/2)(\nu-\alpha-1))$ vanishes at the poles of $G(\alpha)$. The function $\tilde f_\alpha(u)$ defined by formulas (2.3) and (2.7) is an analytic function of α in the same region as b_α. Relations (2.8) and (2.7) imply that for all $\epsilon>0$ the function $b_\alpha(v)$ is bounded and continuous in both variables on the set

$$[\nu-l-1+\epsilon,\nu]xS^{\nu-1}.$$

Hence for all $\varphi\in\mathcal{S}$ the function

$$\tilde f_\alpha(\tilde\varphi)=\int_{R^\nu}\tilde\varphi(x)\tilde f_\alpha(x)\,dx$$

is analytic for $\nu>\alpha>\nu-l-1$. The same statement holds for $f_\alpha(\varphi)$. Relations (2.2) and (2.3) show that

$$f_\alpha(\varphi)=\tilde f_\alpha(\tilde\varphi).$$

We already know this for $\nu-1<\alpha<\nu$. Since both sides of this identity are analytic functions the same relation holds for $\nu-1-l<\alpha<\nu$.

To obtain formulas (2.5) and (2.6), it is enough to remark that, as is well known—see, e.g., [7], Theorem IX.4—the Fourier transform of a product of a generalized function in $\mathcal{S}'(R^\nu)$ with a test function in $\mathcal{S}(R^\nu)$ can be given by a convolution formula. In our case

$$\tilde f_\alpha^A(u)=\int\tilde f_\alpha(t)(2\pi)^{-\nu/2}A^\nu\exp\left\{-\frac{A^2|u-t|^2}{2}\right\}dt.$$

This relation implies (2.5) immediately. By making the substitution $t=$

$A^{-1}w$ in the last integral, one can see that

$$|u|^{\nu-\alpha}|\tilde{f}_\alpha^A(u)| \leqq B(2\pi)^{-\nu/2}|Au|^{\nu-\alpha}\int |w|^{\alpha-\nu}\exp\left\{-\frac{|w-Au|^2}{2}\right\}dw$$

with $B = \max_{v \in S^{\nu-1}} b_\alpha(v)$. The right side in the last formula is a continuous function of Au for $Au \in R^\nu - \{0\}$. It tends to zero as $Au \to 0$, and its upper limit is bounded as $Au \to \infty$. These relations imply (2.6).

Remark 1. Relation (1.1) means that

$$\mathfrak{R}(\varphi, \psi) = \tilde{G}(\varphi * \psi'),$$

where $\tilde{G}$ is the Fourier transform of G, considering it as an element from $\mathcal{S}'(R^\nu)$, $\varphi * \psi'$ is the convolution of φ, and $\psi'(t) = \bar{\psi}(-t)$. Hence formula (1.5) follows immediately from Lemma 1.

Lemma 2. *Let a function $\varphi \in \mathcal{S}$ be given. For all α such that $\nu > \alpha > \alpha_0 = ((k-1)/k)\nu$ and for all $k \geqslant 2$, consider the integral*

$$K(\varphi, \alpha) = \int_{R^{k\nu}} |\tilde{\varphi}(x_1 + \cdots + x_k)|^2 f_\alpha(x_1) \ldots f_\alpha(x_k)\, dx_1 \ldots dx_k, \tag{2.9}$$

where f_α is as defined in (2.1) and the function a has the same properties as in Lemma 1. Then the integral $K(\varphi, \alpha)$ is convergent, and

$$\lim_{\alpha \to \alpha_0} (\alpha - \alpha_0) K(\varphi, \alpha) = \frac{1}{k(2\pi)^\nu} \int_{S^{\nu-1}} [b_{\alpha_0}(v)]^k\, dv \int_{R^\nu} |\varphi(u)|^2\, du. \tag{2.10}$$

If the function a is nonnegative and is not identically zero, then the inequality

$$\int_{S^{\nu-1}} [b_{\alpha_0}(v)]^k\, dv > 0 \tag{2.11}$$

holds.

Proof. Let $\psi(u)$ be the inverse Fourier transform of $|\tilde{\varphi}(t)|^2$. Then

$$|\tilde{\varphi}(t)|^2 = \frac{1}{(2\pi)^\nu} \int \exp[i(u, t)]\psi(u)\, du.$$

By using this relation together with the definition (2.4), we get that

$$\begin{aligned}
&\int_{R^{k\nu}} |\tilde{\varphi}(x_1 + \cdots + x_k)|^2 f_\alpha^A(x_1) \ldots f_\alpha^A(x_k)\, dx_1 \ldots dx_k \\
&= \int_{R^{k\nu} x R^\nu} (2\pi)^{-\nu} \exp\{i(x_1 + \cdots + x_k, u)\} \\
&\quad \times \psi(u) f_\alpha^A(x_1) \ldots f_\alpha^A(x_k)\, dx_1 \ldots dx_k\, du \\
&= (2\pi)^{-\nu} \int_{R^\nu} \psi(u) [\tilde{f}_\alpha^A(u)]^k\, du.
\end{aligned}$$

Taking the limit $A \to \infty$ we get, using relation (2.5), that

$$K(\varphi, \alpha) = (2\pi)^{-\nu} \int_{R^\nu} \psi(u) [\tilde{f}_\alpha(u)]^k\, du. \tag{2.12}$$

This limiting procedure is legitimate because of (2.6), the convergence of the integral

$$\int |\psi(u)||u|^{k(\alpha-\nu)}\,du, \tag{2.13}$$

and the dominated convergence theorem. From the convergence of the integral (2.13) the convergence of the integrals (2.12) and (2.9) follows.

Exploiting formula (2.3) we find that for any $\gamma \to 0$ and $\alpha \to \alpha_0$,

$$\int_{\{|x|<\gamma\}} \psi(x)(\tilde{f}_\alpha(x))^k\,dx = \int_0^\gamma \int_{S^{\nu-1}} \psi(rv) r^{k(\alpha-\nu)+\nu-1}(b_\alpha(v))^k\,dr\,dv$$

$$\sim \psi(0) \int_{S^{\nu-1}} (b_\alpha(v))^k\,dv \int_0^\gamma r^{k(\alpha-\nu)+\nu-1}\,dr. \tag{2.14}$$

The analogous integral in the domain $\{|x| > \gamma\}$ is uniformly bounded in α as $\alpha \to \alpha_0$, and since $\psi(0) = \int_{R^\nu} |\tilde{\varphi}(t)|^2\,dt$, these relations imply (2.10). The simplest way to prove inequality (2.11) is to estimate (2.9) directly. It follows from relation (2.10) that it is enough to show that

$$\liminf_{\alpha\to\alpha_0} (\alpha - \alpha_0)^{-1} K(\varphi, \alpha) > 0 \tag{2.15}$$

with an appropriately chosen $\varphi \in \mathcal{S}$. We may assume that $\tilde{\varphi}(x) \geqslant 1$ for $|x| \leqslant 1$. Because of the continuity of the function a, $a(x) \geqslant \eta > 0$ with some $\eta > 0$ on an open subset of $S^{\nu-1}$. Because of spherical symmetry, we may assume that this set is

$$A = \left\{x = (x^{(1)}, \ldots, x^{(\nu)}) \in R^\nu, |x| = 1, |x^{(l)}| < \epsilon, l = 2, \ldots, k\right\}.$$

Let us consider the sets

$$D_n = \Bigg\{ x = (x_1, \ldots, x_k) \in R^{k\nu}, -n - \frac{1}{3} < x_1^{(1)} < -n;$$

$$\frac{n}{2k} < x_j^{(1)} < \frac{n}{k}, j = 2, \ldots, k-1;$$

$$n - (x_2^{(1)} + \cdots + x_{k-1}^{(1)}) < x_k^{(1)} < \left(n + \frac{1}{3}\right) - (x_2^{(1)} + \cdots + x_{k-1}^{(1)});$$

$$|x_j^{(l)}| < \frac{\epsilon}{4k^2} |x_j^{(1)}|, j = 1, \ldots, k-1, l = 2, \ldots \nu;$$

$$-\frac{\epsilon}{2} - (x_1^{(l)} + \cdots + x_{k-1}^{(l)}) < x_k^{(l)}$$

$$< \frac{\epsilon}{2} - (x_1^{(l)} + \cdots + x_{k-1}^{(l)}), l = 2, \ldots, \nu \Bigg\}$$

$n = 1, 2, \ldots.$

The sets D_n are pairwise disjoint, since $D_n C \{-n - \frac{1}{3} < x_1^{(1)} < -n\}$. The

Lebesgue measure of the set D_n satisfies the inequality $\lambda(D_n) > Cn^{\nu(k-1)-1}$ with a $C>0$ which is independent of n. To verify this inequality it is enough to remark that if $(x_1, \dots, x_{k-1}) \in R^{\nu(k-1)}$ is such that the intersection of D_n with the hyperplane consisting of the points $(x_1, \dots, x_{k-1}, y)$, $y \in R^\nu$ is non-empty, then this intersection has Lebesgue measure at least $\frac{1}{3}\epsilon^{\nu-1}$, and we have to calculate the Lebesgue measure of the set of such points $(x_1, \dots, x_{k-1})$. Assuming that $\epsilon < \nu^{-1}$, it is not difficult to check, for all $x = (x_1, \dots, x_k) \in D$, the relations

$$|x_1 + \cdots + x_k| < 1, \qquad |x_j| < 2n, \qquad j = 1, \dots, k$$

and $x_j/|x_j| \in A$, $j = 1, \dots, k$. Hence

$$|\tilde{\varphi}(x_1 + \cdots + x_k)|^2 |x_1|^{-\alpha} \dots |x_k|^{-\alpha} a\left(\frac{x_1}{|x_1|}\right) \dots a\left(\frac{x_k}{|x_k|}\right) > C' n^{-k\alpha} \tag{2.16}$$

with a constant $C' > 0$ independent of n, and

$$\begin{aligned} K(\varphi, \alpha) &\geqslant \sum_{n=1}^{\infty} \int_{D_n} |\tilde{\varphi}_n(x_1 + \cdots + x_k)|^2 \\ &\quad \times |x_1|^{-\alpha} \dots |x_k|^{-\alpha} a\left(\frac{x_1}{|x_1|}\right) \dots a\left(\frac{x_k}{|x_k|}\right) dx_1 \dots dx_k \\ &\geqslant CC' \sum_{n=1}^{\infty} n^{-k\alpha + \nu(k-1) - 1}. \end{aligned} \tag{2.17}$$

Relation (2.15) follows from these estimates.

Remark 2. Comparing relations (1.8) and (2.9), we see that the variance

$$D(H_{k,\epsilon,a}(\varphi)) = \frac{1}{k!} K(\varphi, \alpha(k, \epsilon)). \tag{2.18}$$

Hence it follows from Lemma 2 that the integral (1.8) is convergent and the asymptotic formulas (1.9) and (1.10) for the variance are valid.

Lemma 3. *Let there be given points $x_1, \dots, x_k$, $k \geqslant 2$, in R^ν and a function $\varphi \in \mathcal{S}$. For all numbers $\alpha_1, \dots, \alpha_k$ satisfying the relations $\alpha_i \geqslant \delta \geqslant 0$, $i = 1, \dots, k$, $\sum_{j=1}^k \alpha_j \leqslant \beta$, and $\beta - \delta < \nu$ there exists some constant $C(\varphi, \beta - \delta)$ depending only on φ, the difference $\beta - \delta$, and the dimension ν, such that the relation*

$$\begin{aligned} &\left| \int_{R^\nu} |x_1 - u|^{-\alpha_1} \dots |x_k - u|^{-\alpha_k} \varphi(u)\, du \right| \\ &\qquad \leqq k 2^\delta C(\varphi, \beta - \delta) \sum_{\substack{j,l = 1, \dots, k \\ j \neq l}} |x_j - x_l|^{-\delta} \end{aligned} \tag{2.19}$$

is true.

Proof. Let us remark that

$$\max_{i=1,\dots,k} |x_i - u| \geqslant \tfrac{1}{2} \max_{i \neq j} |x_i - x_j|$$

for arbitrary $u \in R^\nu$. Let the index $i_0(u)$ be chosen in such a way that

$$|x_{i_0(u)} - u| = \max_{i=1,\ldots,k} |x_i - u|.$$

Then

$$\prod_{i=1}^{k} |x_i - u|^{-\alpha_i} \leqslant |x_{i_0(u)} - u|^{-\delta} \prod_{i=i_0(u)} |x_i - u|^{-\alpha_i}$$
$$\leqslant 2^\delta \Big(\max_{i \neq j} |x_i - x_j| \Big)^{-\delta} \min_{i \neq i_0} |x_i - u|^{-\sum_{i \neq i_0(u)} \alpha_i}$$
$$\leqslant 2^\delta \Big(\sum_{i \neq j} |x_i - x_j|^{-\delta} \Big) \Big(\sum_{i=1}^{k} |x_i - u|^{-(\beta-\delta)} \Big).$$

Multiplying both sides of this inequality by $\varphi(u)$ and integrating with respect to u, we get the estimate (2.19) with

$$C(\varphi, \beta - \delta) = \max_x \int_{R^\nu} |x - u|^{-(\beta-\delta)} |\varphi(u)|\, du.$$

3. Proof of the theorems by the method of moments

To prove Theorem 1 it is enough to show that the moments of the random variable $\mathcal{F}_\epsilon(\varphi)$ tend to the moments of the normal random variable with mean zero and variance $\int |\varphi(x)|^2 dx$. We shall apply the so-called diagram formula (see, e.g., [1], Proposition 4.1) to express the moments of Wiener-Itô integrals. Unfortunately there is no uniform terminology for these notions in the mathematical literature; hence we formulate it in detail. We follow the terminology of [1].

Let some positive integers $n_1, \ldots, n_m$, $m \geqslant 2$, be given. We shall use the term *diagram* for an undirected graph of $N = n_1 + \cdots + n_m$ vertices such that its vertices are indexed by the pairs of integers (j, l), $j = 1, \ldots, n_l$; $l = 1, \ldots, m$, such that exactly one branch enters each vertex and such that branches can connect only pairs of vertices (j_1, l_1), (j_2, l_2) for which $l_1 \neq l_2$. The set of all vertices with second index l will be called the lth row of the diagram, and the number n_l the length of this row. The set of all diagrams of order $(n_1, \ldots, n_l)$ will be denoted by $\Gamma(n_1, \ldots, n_m)$. In the case $n_1 = \ldots = n_m = k$ we shall also write $\Gamma(n_1, \ldots, n_m) = \Gamma_k(m)$. We shall consider only the case when N is even because $\Gamma(n_1, \ldots, n_m)$ is empty for odd N.

Let there be given a set of functions $h_1 \in H_G^{n_1}, \ldots, h_m \in H_G^{n_m}$ (see Section 1). We introduce the function $\hat{h}$ of N variables $x_{j,l} \in R^\nu$ corresponding to the vertices of the diagram by the formula

$$\hat{h}(x_{j,l};\, l = 1, \ldots, m;\, j = 1, \ldots, n_l) = \prod_{l=1}^{m} h_l(x_{j,l};\, j = 1, \ldots, n_l). \quad (3.1)$$

Fixing the diagram $\gamma \in \Gamma(n_1, \ldots, n_m)$ we label the branches of this diagram with the numbers $1, \ldots, N/2$ in an arbitrary way; the vertices connected by the pth branch (and also the variables corresponding to them) will be indexed by p and $p + N/2$. Put

$$h_\gamma = \int_{R^\nu} \cdots \int_{R^\nu} \hat{h}(x_1, \ldots, x_{N/2}, -x_1, \ldots, -x_{N/2}) G(dx_1) \ldots G(dx_{N/2}). \tag{3.1'}$$

It is not difficult to see by exploiting the definition of the class H and the evenness of the spectral measure G that the number h_γ does not depend on the enumeration of the branches and vertices.

The following result holds true: All integrals in (3.1′) are absolutely convergent and

$$E\left(I_G^{(n_1)}(h_1) \ldots I_G^{(n_m)}(h_m)\right) = \begin{cases} n_1! \ldots n_m! \sum_{\gamma \in \Gamma(n_1, \ldots, n_m)} h_\gamma \text{ if } N \text{ is even,} \\ 0 \text{ if } N \text{ is odd.} \end{cases} \tag{3.2}$$

For an even number m we call a diagram $\Gamma(n_1, \ldots, n_m)$ paired if its rows can be put into pairs in such a way that branches connect only vertices of those rows which are paired. The set of all paired diagrams of $\Gamma(n_1, \ldots, n_m)$ will be denoted by $\Gamma^0(n_1, \ldots, n_m)$, and that of the nonpaired diagrams by $\bar{\Gamma}^0(n_1, \ldots, n_m)$. The notation $\Gamma_k^0(m)$, $\bar{\Gamma}_k^0(m)$ will be introduced analogously.

As is well known, the mth moment of a standard normal random variable equals $(m-1)(m-3) \ldots 1$ for an even number m and zero for an odd m. Hence to prove Theorem 1 it is enough to show that

$$\lim_{\epsilon \to 0} \epsilon^{m/2} (B_k)^{-m/2} E\left(H_{k,\epsilon}(\varphi)^m\right) = \begin{cases} \left(\int_{R^\nu} |\varphi(x)|^2 dx\right)^{m/2} (m-1)(m-3) \ldots 1 \text{ if } m \text{ is even} \\ 0 \text{ if } m \text{ is odd} \end{cases} \tag{3.3}$$

for all $\varphi \in \mathcal{S}$.

We can apply the diagram formula (3.2) with $n_1 = \cdots = n_m = k$, $N = mk$, $f_1(x_1, \ldots, x_k) = \cdots = f_m(x_1, \ldots, x_k) = \tilde{\varphi}(x_1 \cdots + x_k)$, $G = G_{k,\epsilon}$ for computing the moments of $E(H_{k,\epsilon}(\varphi)^m)$. (For $m = 2$ the relation (3.3) has already been proved: see (1.9) and Remark 1 in Section 2).

The relation $f_i \in H_G^k$ follows from inequality (1.8) (see Remark 2 in Section 2). Hence

$$E\left(H_{k,\epsilon}(\varphi)^m\right) = \begin{cases} (k!)^{-m} \sum_{\gamma \in \Gamma_k(m)} h_\gamma(\epsilon, \varphi) \text{ if } N \text{ is even,} \\ 0 \text{ if } N \text{ is odd,} \end{cases} \tag{3.4}$$

where $h_\gamma(\epsilon, \varphi)$ is as defined by formula (3.1′) and

$$\hat{h}(x_{j,l}; j = 1, \ldots, k; l = 1, \ldots, m) = \prod_{l=1}^{m} \tilde{\varphi}(x_{1,l} + \cdots + x_{k,l}). \tag{3.5}$$

Let us define (for a fixed diagram γ with a prescribed enumeration of their branches and vertices) $\kappa(q) = l_1$ and $\lambda(q) = l_2$ if the qth branch of the diagram connects vertices from the l_1th and l_2th row and $1 \leqslant l_1 \leqslant N/2$, $N/2 + 1 \leqslant l_2 \leqslant N$, i.e., if the vertex labelled by q had an original index (j, l_1) and the vertex labelled by $q + N/2$ had the index $(\bar{j}, l_2)$ with some j and $\bar{j}$. Set

$$\sigma_l(q) = \begin{cases} 1 & \text{if } \kappa(q) = 1, \\ -1 & \text{if } \lambda(q) = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$h_\gamma(\epsilon, \varphi) = \int_{R^\nu} \cdots \int_{R^\nu} \prod_{l=1}^{m} \tilde{\varphi}\left(\sum_{q=1}^{N/2} \sigma_l(q) x_q \right) q_{k,\epsilon}(x_q)\, dx_1 \ldots dx_q. \tag{3.6}$$

Comparing relations (3.6) and (1.8) we can see that for a paired diagram $\gamma \in \Gamma_k^0(m)$,

$$h_\gamma(\epsilon, \varphi) = (k!)^{m/2}(D(H_{k,\epsilon}(\varphi))^{m/2}. \tag{3.7}$$

The number of paired diagrams is $(k!)^m(m-1)\ldots 1$ if m is even, and zero if m is odd. Hence the asymptotic formula (1.9) implies that

$$\sum_{\gamma \in \Gamma_k^0(m)} h_\gamma(\epsilon, \varphi)\epsilon^{m/2} B_k^{-m/2}$$

$$= \begin{cases} (m-1)(m-3)\ldots 1\left[\int_{R^\nu} |\varphi(x)|^2 dx \right]^{m/2} & \text{if } m \text{ is even}, \\ 0 \text{ if } m \text{ is odd}. \end{cases} \tag{3.8}$$

A comparison of formulas (3.3) and (3.4) shows that to prove Theorem 1 it is enough to verify that

$$\lim_{\epsilon \to 0} \sum_{\gamma \in \bar{\Gamma}_k^0(m)} h_\gamma(\epsilon, \varphi) = 0.$$

As the number of nonpaired diagrams does not depend on ϵ, it is enough to show that for all $\gamma \in \bar{\Gamma}_k^0(m)$

$$\lim_{\epsilon \to 0} \epsilon^{m/2} h_\gamma(\epsilon, \varphi) = 0. \tag{3.9}$$

Further transformations are introduced to prove this relation.

We shall rewrite the integral (3.6) by means of the Fourier transform. Let us observe that

$$\bar{\varphi}\left(\sum_{q=1}^{N/2} \sigma_l(q) x_q \right) = \int_{R^\nu} \exp\left\{ i\left(u_l, \sum_{q=1}^{N/2} \sigma_l(q) x_q \right) \right\} \varphi(u_l)\, du_l.$$

Therefore, introducing the function

$$g^A_{k,\epsilon}(x) = g_{k,\epsilon}(x)\exp\left(-\frac{|x|^2}{2A^2}\right)$$

(see formula (2.4)), we can write

$$\int_{R^\nu}\cdots\int_{R^\nu}\prod_{l=1}^{m}\tilde\varphi\left(\sum_{q=1}^{N/2}\sigma_l(q)x_q\right)\prod_{q=1}^{N/2}g^A_{k,\epsilon}(x_q)\,dx_1\ldots dx_{N/2}$$

$$=\int_{R^\nu}\cdots\int_{R^\nu}\varphi(u_1)\ldots\varphi(u_m)\prod_{l=1}^{m}\exp\left\{i\left(u_l,\sum_{q=1}^{N/2}\sigma_l(q)x_q\right)\right\}$$

$$\times\prod_{q=1}^{N/2}g^A_{k,\epsilon}(x_q)\,dx_1\ldots dx_{N/2}\,du_1\ldots du_m$$

$$=\int_{R^\nu}\cdots\int_{R^\nu}\varphi(u_1)\ldots\varphi(u_m)\prod_{q=1}^{N/2}\tilde g^A_{k,\epsilon}(u_{\kappa(q)}-u_{\lambda(q)})\,du_1\ldots du_m. \tag{3.10}$$

Taking the limit $A\to\infty$ in this relation, we get, by applying formulas (1.7), (2.5) and (2.3), the identity

$$h_\gamma(\epsilon,\varphi)=\int_{R^\nu}\cdots\int_{R^\nu}\varphi(u_1)\ldots\varphi(u_m)$$

$$\times\prod_{q=1}^{N/2}|u_{\kappa(q)}-u_{\lambda(q)}|^{-(\nu-\epsilon)/k}b_\alpha\left(\frac{u_{\kappa(q)}-u_{\lambda(q)}}{|u_{\kappa(q)}-u_{\lambda(q)}|}\right)du_1\ldots du_m. \tag{3.11}$$

The limiting procedure taken under the integral sign in formula (3.10) is legitimate, as follows from formula (2.6), the dominated convergence theorem, and the finiteness of the integral

$$\bar h_\gamma(\epsilon,\varphi)=\int_{R^\nu}\cdots\int_{R^\nu}|\varphi(u_1)|\ldots|\varphi(u_m)|$$

$$\times\prod_{q=1}^{N/2}|u_{\kappa(q)}-u_{\lambda(q)}|^{-(\nu-\epsilon)/k}\,du_1\ldots du_m, \tag{3.12}$$

to be proved later. This finiteness will be proved for all diagrams $\gamma\in\Gamma(n_1,\ldots,n_m)$ such that $n_1\leqslant k,\ldots,n_m\leqslant k$. The set of all such diagrams will be denoted by $\Gamma_{\leqslant k}(m)$, and the subsets of paired and nonpaired diagrams from $\Gamma_{\leqslant k}(m)$ will be denoted by $\Gamma^0_{\leqslant k}(m)$ and $\bar\Gamma^0_{\leqslant k}(m)$.

First we show that for all paired diagrams $\gamma\in\Gamma^0_{\leqslant k}(m)$ the integral (3.12) is convergent, and moreover

$$\bar h_\gamma(\epsilon,\varphi)=O(\epsilon^{-m/2})\qquad\text{for}\quad\epsilon\to 0,\quad\gamma\in\Gamma^0_{\leqslant k}(m). \tag{3.13}$$

Indeed, for $n\leqslant k$ we can make, with constants C_1, C_2, C_3, depending only on φ, the estimate

$$\begin{aligned}\int_{R^\nu}\int_{R^\nu}&|u_1-u_2|^{-n(\nu-\epsilon)/k}|\varphi(u_1)||\varphi(u_2)|\,du_1du_2\\ &\leqslant \int_{R^\nu}|\varphi(u_2)|\Big(\int_{|u_1-u_2|\leqslant 1}C_1|u_1-u_2|^{-n(\nu-\epsilon)/k}\,du_1\Big)\,du_2\\ &\quad+\int\int_{|u_1-u_2|\geqslant 1}|\varphi(u_1)||\varphi(u_2)|\,du_1\,du_2\\ &\leqslant C_2\Big(\nu-\frac{n}{k}(\nu-\epsilon)\Big)^{-1}+C_3\leqslant C_2\cdot\frac{1}{\epsilon}+C_3.\end{aligned} \tag{3.14}$$

For all diagrams $\gamma\in\Gamma^0_{\leqslant k}(m)$ the integral (3.12) is a product of m such integrals, which were estimated in (3.14). Hence formula (3.14) implies (3.13). We shall prove by induction for $m\geqslant 2$ that

$$\bar h_\gamma(\epsilon,\varphi)=O(\epsilon^{-(m-1)/2})\qquad\text{for all}\quad \gamma\in\bar\Gamma^0_{\leqslant k}(m). \tag{3.15}$$

Relation (3.15) obviously holds for $m=2$, since all diagrams are paired for $m=2$. Let us assume that (3.15) holds for all $m'<m$. Fixing a nonpaired diagram $\gamma\in\bar\Gamma^0_{\leqslant k}(m)$ we denote by $r(j)$, $j=1,\dots,m-1$, the number of vertices in the jth row of this diagram connected to a vertex from the mth row by a branch. Let $R(\gamma)$ denote the set of indices j, $j=1,\dots,m-1$, such that $r(j)\neq 0$. We may assume without loss of generality that $R(\gamma)$ contains at least two points, since this always can be achieved in the case of a nonpaired diagram by renumbering the rows of the diagrams, if it is necessary. Let us rewrite the integral (3.12) as

$$\begin{aligned}\bar h_\gamma(\epsilon,\varphi)=\int_{R^\nu}\cdots\int_{R^\nu}&|\varphi(u_1)|\dots|\varphi(u_{m-1})|\prod_{\substack{q\,:\,\kappa(q)\neq m\\ \lambda(q)\neq m}}|u_{\kappa(q)}-u_{\lambda(q)}|^{-(\nu-\epsilon)/q}\\ &\times\Big(\int|\varphi(u_m)|\prod_{j\in R(\gamma)}|u_j-u_m|^{-r(j)(\nu-\epsilon)/k}\,du_m\Big)du_1\dots du_{m-1}.\end{aligned} \tag{3.16}$$

We can estimate the inner integral of (3.16) with the help of formula (2.19) in Lemma 3 by choosing $\beta=\nu$ and $\delta=(\nu-\epsilon)/k\geqslant\nu/2k$, since

$$\sum_{j\in R(\gamma)}r(j)\leqslant k\qquad\text{and}\qquad \epsilon\leqslant\frac{\nu}{2}.$$

We get that the inequality

$$\bar h_\gamma(\epsilon,\varphi)\leqslant C\sum_{\substack{j,l\in R(\gamma)\\ j\neq l}}h_{\gamma(j,l)}(\epsilon,\varphi) \tag{3.17}$$

holds true with a constant C independent of ϵ where the diagram $\gamma(j,l)$ is defined as follows: We delete from γ the mth row together with the vertices connected with the vertices of the mth row. Then a new vertex is attached to the jth and kth rows, which are connected by a new branch. Obviously

$\gamma(j,l) \in \Gamma_{\leqslant k}(m')$ with some $m' \leqslant m-1$. Hence

$$\bar{h}_{\gamma(j,l)}(\epsilon,\varphi) = O(\epsilon^{-(m-1)/2}). \tag{3.18}$$

If $\gamma(j,l) \in \Gamma^0_{\leqslant k}(m)$, $m' \leqslant m-1$, then (3.18) follows from (3.7), and if $\gamma(j,l) \in \bar{\Gamma}^0_{\leqslant k}(m')$, then it follows from the induction hypothesis. Since $h_\gamma(\epsilon,\varphi) \leqslant \max_{v \in S^{\nu-1}} b_{\alpha(k,\epsilon)}(v)\bar{h}_\gamma(\epsilon,\varphi)$ relation (3.15) implies (3.9). Theorem 1 is proved.

The proof of Theorem 1′ is analogous to that of Theorem 1. Since to prove weak convergence of random vectors it is enough to show that all the linear combinations of their components converge weakly, it is enough to show that for all positive integers M, real numbers $c_1, \ldots, c_M$, and points $n_1, \ldots, n_M$ from the lattice Z^ν, the random variables

$$\epsilon^{1/2}B^{-1/2}\sum_{j=1}^{M} c_j H_{k,\epsilon}(n_j) = \epsilon^{1/2}B^{-1/2}\int \tilde{\varphi}^*(x_1 + \cdots + x_k) \times Z_{G_{k,\epsilon}}(dx_1)\ldots Z_{G_{k,\epsilon}}(dx_k), \tag{3.19}$$

where

$$\varphi^*(u) = \sum_{j=1}^{M} c_j \varphi_{n_j}(u)$$

tends weakly to a normal random variable with expectation zero and variance $\sum c_j^2$. Let us observe that all the estimates of Section 2 and those of this section remain valid if the functions $\varphi \in \mathbb{S}$ are replaced by φ^*. The reason for this is that the only property of the function φ exploited during these proofs was that $\varphi(x)$ (in the proof of Lemma 2 also $\varphi(x) * \varphi(-x)$, which is the inverse Fourier transform of $|\tilde{\varphi}(x)|^2$) is bounded and integrable, and the function φ^* also has these properties. One also has to remark that

$$\int_{R^\nu} |\varphi^*(x)|^2\,dx = \sum_{j=1}^{M} c_j^2.$$

Let us now turn to the proof of Theorem 2. It is also analogous to the proof of Theorem 1, and we work out only those parts of the proof where some modification is needed. When using the method of moments it is enough to show that for all positive integers $m_2, \ldots, m_p$ and functions $\varphi_1, \ldots, \varphi_p \in \mathbb{S}$, we have, with $m = m_1 + \cdots + m_p$,

$$\lim_{\epsilon \to 0} E\epsilon^{m/2}(\hat{H}_{2,\epsilon}(\varphi_2)^{m_2} \ldots \hat{H}_{p,\epsilon}(\varphi_p)^{m_p}) = \begin{cases} \prod_{j=2}^{p}\left(B_j^{m_j}(m_j-1)(m_j-3)\ldots 1\left[\int |\varphi_j(x)|^2\,dx\right]^{m_j/2}\right) & \text{if } m \text{ is even}, \\ 0 \text{ if } m \text{ is odd}, \end{cases} \tag{3.20}$$

since the right side of these expressions agrees with the corresponding joint moment of the independent limiting normal random variables. To calculate these moments we apply again the diagram formula for the random variables $\hat{H}_{k,\epsilon}(\varphi)$, defined in (1.14). We denote by $\Gamma[m_1, \ldots, m_p]$ the set of all diagrams whose first m_2 rows contain two vertices, the following m_3 rows three vertices $\ldots$, the last m_p rows p vertices. The number of rows in such a diagram is $m = m_2 + \cdots + m_p$. We say that a row containing q vertices in $\Gamma[m_2, \ldots, m_p]$ has order q. Let $\Gamma^0[m_2, \ldots, m_p]$ and $\bar{\Gamma}^0[m_2, \ldots, m_p]$ denote the subsets of paired and nonpaired diagrams in $\Gamma[m_2, \ldots, m_p]$. The moment that concerns us equals the sum of the integrals $h_\gamma(\epsilon, \varphi_2, \ldots, \varphi_p)$, defined by formulas (3.1) and (3.1′) with

$$h_l(x_j, j = 1, \ldots, n_l) = \tilde{\varphi}_{n_l}(x_1 + \cdots + x_{n_l})\left[g_{n_l,\epsilon}(x_1) \cdots g_{n_l,\epsilon}(x_{n_l})\right]^{1/2}, \tag{3.21}$$

where $n_l = s$ if the lth row has order s and the summation is taken over all $\gamma \in \Gamma[m_2, \ldots, m_p]$. The sum of the integrals h_γ for paired diagrams is estimated in the same way as in Theorem 1. The main problem again is to prove that $h_\gamma \to 0$ for all nonpaired diagrams. It can be reduced to the statement that

$$\bar{h}_\gamma(\epsilon, \varphi_2, \ldots, \varphi_p) = O(\epsilon^{-(m-1)/2}), \tag{3.22}$$

where

$$\bar{h}_\gamma(\epsilon, \varphi_2, \ldots, \varphi_p) = \int_{R^\nu} \cdots \int_{R^\nu} \prod_{l=1}^{m} |\varphi_{n_l}(u)| \times \prod_{q=1}^{N/2} |u_{\kappa(q)} - u_{\lambda(q)}|^{-(1/2)[\mu(\kappa(q),\epsilon) + \mu(\lambda(q),\epsilon)]}\, du_1 \ldots du_m, \tag{3.23}$$

with

$$\mu(l, \epsilon) = \frac{\nu - \epsilon}{n_l}.$$

This reduction is based on the same construction as in the previous case, but in order to avoid some additional smoothness conditions on a_j, we have to make some modifications. We introduce the quantity $h'_\gamma(\epsilon, \varphi_2^0, \ldots, \varphi_m^0)$, defined by formulas (3.1), (3.1′), and (3.21), except that $g_{k,\epsilon}(x)$ is replaced by $g'_{k,\epsilon}(x) = (\max_s a_k(s))|x|^{-\alpha(k,\epsilon)}$ and φ_i by a $\varphi_0 \in \mathcal{S}$ with the property $\tilde{\varphi}_i^0(x) \geq |\tilde{\varphi}_i(x)|$ for all $x \in R^\nu$. (Actually we could also choose φ_i^0 as $\tilde{\varphi}_i^0(t) = C(1 + |t|)^{-2r}$ with sufficiently large positive integers r and C, although in this case $\varphi_i^0 \notin \mathcal{S}$. Then $h_\gamma(\epsilon, \varphi_1, \ldots, \varphi_r) \leq h'_\gamma(\epsilon, \varphi_1^0, \ldots, \varphi_r^0)$, as can be seen by rewriting both sides of this inequality like formulas (3.10) and (3.11); Theorem 2 can then be deduced from formula (3.23), except that φ_i must be replaced by φ_i^0.

To prove relation (3.23) we introduce a larger class of diagrams $\Gamma_{\leqslant}[m_2, \dots, m_p]$, which contains all those diagrams whose rows have at most as many vertices as are in the corresponding row in $\Gamma[m_2, \dots, m_p]$. The order of a row in $\Gamma_{\leqslant}[m_2, \dots, m_p]$ will be defined as the order of the corresponding row in $\Gamma[m_2, \dots, m_p]$. Let

$$\Gamma^0_{\leqslant}[m_2, \dots, m_p] \text{ and } \bar{\Gamma}^0_{\leqslant}[m_2, \dots, m_p]$$

denote the set of paired and nonpaired diagrams, respectively. The same argument as before shows that relation (3.13) remains valid for $\gamma \in \Gamma^0_{\leqq}[m_2, \dots, m_p]$. Now we show by induction on m that relation (3.15) holds for $\gamma \in \bar{\Gamma}^0_{\leqq}[m_2, \dots, m_p]$. There are two cases. In the first case the branches of the diagram γ connect only vertices from rows of the same order. In this case

$$\bar{h}_\gamma(\epsilon, \varphi_2, \dots, \varphi_p) = \bar{h}_{\gamma_2}(\epsilon, \varphi_2) \dots \bar{h}_{\gamma_p}(\epsilon, \varphi_p),$$

where $\gamma_k \in \bar{\Gamma}^0_{\leqslant k}(m)$ is obtained by deleting all rows with an order different from k from the diagram γ. One of the diagrams γ_k is nonpaired; therefore the estimate (3.15) for such diagrams can be verified by applying relations (3.13) and (3.15) for the case already proved. In the second case there are rows with vertices which are connected to vertices of rows of different order. Let us pick out one of these rows with minimal order. Let it be the jth row. Let $r(i)$ denote the number of branches connecting vertices from the jth and ith rows. We rewrite the integral (3.23) in the form

$$\begin{aligned}\bar{h}_\gamma(\epsilon, \varphi_2, \dots, \varphi_p) = \int_{R^\nu} \cdots \int_{R^\nu} \prod_{l\,:\,l\neq j} |\varphi_{n_l}(u)| \\ \times \prod_{q\,:\,\kappa(q)\neq j,\,\lambda(q)\neq j} |u_{\kappa(q)} - u_{\lambda(q)}|^{-(1/2)[\mu(\kappa(q),\epsilon)+\mu(\lambda(q),\epsilon)]} \\ \times \left[\int \varphi_{n_j}(u_j) \prod_{i\,:\,i\neq j} |u_i - u_j|^{-(1/2)[(\nu-\epsilon)/n_i + (\nu-\epsilon)/n_j]r(i)}\, du_j \right] \\ \times du_1 \dots du_{j-1}\, du_{j+1} \dots du_m. \qquad (3.24)\end{aligned}$$

Now we apply, to estimate the inner integral in (3.24), Lemma 3 with $\delta = 0$ and $\beta = (\nu/2)[(1/n_j) + \sum_{i\,:\,i\neq j} r(i)/n_i]$. The condition $\beta \leqslant \nu - c$ is satisfied with $c = \nu/(p-1) - \nu/p$, since $\sum_{i\neq j} r(i) = n_j$, $n_k \geqslant n_j$ if $r(i) \neq 0$, and there is some i with $r(i) \neq 0$ and $n_i > n_j$. Hence the inner integral in (3.24) is bounded by a constant C independent of ϵ, and

$$\bar{h}_\gamma(\epsilon, \varphi_2, \dots, \varphi_p) \leqslant C h_{\gamma'}(\epsilon, \varphi_2, \dots, \varphi_p),$$

where γ' is the diagram obtained from γ by deleting from γ the jth row and all vertices connected with vertices of the jth row, together with the connecting branches. As γ' has less than m rows, we obtain (3.15) by applying the induction hypothesis if γ' is a nonpaired and (3.13) if γ' is a paired diagram.

4. The proof of the theorems using the method of "cutting off"

We begin this section by proving Theorem 2, which is more general than Theorem 1. As the convergence of random vectors in distribution follows from the convergence of all linear combinations of their components, it suffices to show that the random variables

$$\xi(\epsilon) = \epsilon^{1/2} \sum_{k=2}^{p} B_k^{-1/2} \hat{H}_{k,\epsilon}(\varphi_k)$$

tend in distribution, as $\epsilon \to 0$, to a normal random variable with expectation zero and variance

$$\sigma^2 = \sum_{k=2}^{p} \int \varphi_k(x)^2\, dx$$

for arbitrary $\varphi_2, \ldots, \varphi_p \in \mathcal{S}$.

It follows from relation (1.9) and the orthogonality of Wiener-Itô integrals of different multiplicity that the variance of the random variables $\xi(\epsilon)$ tends to σ^2 as $\epsilon \to 0$.

Let us introduce the measures $\mu_{k,\epsilon,\varphi}\varphi \in \mathcal{S}$ on the Borel σ-algebra $\mathcal{B}^{\nu k}$ of $R^{\nu k}$, by means of the formula

$$\mu_{k,\epsilon,\varphi}(A) = \int_A |\tilde{\varphi}(x_1 + \cdots + x_k)|^2 G_{k,\epsilon}(dx_1) \ldots G_{k,\epsilon}(dx_k).$$

We shall construct a sequence of real numbers $u_0 = a_0(\epsilon)$, $a_1 = a_1(\epsilon), \ldots$; $0 = a_0 < a_1 < \ldots$, which has some nice properties. By means of the sequence we define the sets

$$D_j = D_j(\epsilon), \qquad D_j^k \subset R^{\nu k}, \qquad j = 1, 2, \ldots, \qquad k = 2, \ldots, p$$

in the following way.

$$D_j = \{x \in R^\nu, a_{j-1} < |x| < a_j\}, \qquad j = 1, 2, \ldots$$

$$D_j^k = \{x = (x_1, \ldots, x_k) \in R^{\nu k};\ x_l \in D_j,\ l = 1, \ldots, k\}.$$

Set

$$\overline{D}^k = R^{\nu k} \bigcup_{j=1}^{\infty} D_j^k.$$

We claim that the sequence $a_0(\epsilon)$, $a_1(\epsilon), \ldots$ can be constructed in such a way that

$$\lim_{\epsilon \to 0} \epsilon \sup_j \mu_{k,\epsilon,\varphi}(D_j^k) = 0, \qquad k = 2, \ldots, p \tag{4.1}$$

and

$$\lim_{\epsilon \to 0} \epsilon \mu_{k,\epsilon,\varphi}(\overline{D}^k) = 0. \tag{4.2}$$

First we show that the existence of sequences $a_j(\epsilon)$ satisfying (4.1) and (4.2) implies Theorem 2. Indeed, let us define the random variables

$$\xi_j(\epsilon) = \epsilon^{1/2} \sum_{p=2}^{k} B_k^{-1/2}(k!)^{-1} \int \tilde{\varphi}(x_1 + \cdots + x_k) \Gamma_{D_j^k}(x_1, \ldots, x_k) \times \left(\prod_{j=1}^{k} g_{k,\epsilon}(x_j) \right)^{1/2} W(dx_1) \ldots W(dx_k)$$

and

$$\eta(\epsilon) = \epsilon^{1/2} \sum_{k=2}^{p} B_k^{-1/2}(k!)^{-1} \int \tilde{\varphi}(x_1 + \cdots + x_k) I_{\bar{D}^k}(x_1, \ldots, x_k) \times \left[\prod_{j=1}^{k} g_{k,\epsilon}(x_j) \right]^{1/2} W(dx_1) \ldots W(dx_k),$$

where I_A denotes the indicator set of A.

$$\xi(\epsilon) = \sum_{j=1}^{\infty} \xi_j(\epsilon) + \eta(\epsilon) \tag{4.3}$$

because of definition (1.14).

By the definition of Wiener-Itô integrals (see, e.g., [1]) and the independence of the random spectral measure W on the disjoint sets D_j the random variables $\xi_j(\epsilon)$, $j = 1, 2, \ldots$ are independent.[3] Their expectations are $E\xi_j(\epsilon) = 0$, $j = 1, 2, \ldots$. Condition (4.1) means that $\sup_j E(\xi_j(\epsilon)^2) \to 0$ as $\epsilon \to 0$. It follows from Nelson's hypercontractive estimate (see, e.g., [8]) that there exists some constant C depending only on p, such that for all random variables that can be represented as sums of Wiener–Itô integrals with multiplicity not exceeding p, $E\xi^4 \leqslant (E\zeta^2)^2 C_p$. Hence for all $\epsilon > 0$ and $j = 1, 2, \ldots,$

$$E\left(\xi_j(\epsilon)^4\right) \leqslant C_p E\left(\xi_j(\epsilon)^2\right)^2.$$

Hence the central limit theorem can be applied for the sum of the variables $\xi_j(\epsilon)$ (see, e.g., [6], Section 49). Finally, relation (4.2) shows that $E\eta(\epsilon)^2 \to 0$ as $\epsilon \to 0$. Therefore $\xi(\epsilon)$ is asymptotically normal.

Now we turn to the construction of the sequences $a_j(\epsilon)$ satisfying (4.1) and (4.2). To carry out these constructions we need to show the following inequalities: For all

$$0 < \epsilon < \frac{\nu}{2}, \quad x = (x_1, \ldots, x_k) \in R^{\nu k}, \quad i = 1, \ldots, k, k \geqslant 2, \quad A > 0$$
$$I_1^\epsilon(A) = \mu_{k,\epsilon,\varphi}(x : |x_1 + \cdots + x_k| \geqslant A) \leqslant C_1 A \epsilon^{-1}. \tag{4.4}$$

[3] It may be interesting to remark that there seems to be no natural way to formulate this argument in terms of functional analytic concepts of Wick polynomials. Hence in such situations it seems to be inevitable to apply Wiener–Itô integrals, which are their probabilistic counterparts.

Furthermore for all $A > 0$, $0 < B_1 < B_2$,

$$I_2^\epsilon(A, B_1, B_2) = \mu_{k,\epsilon,\varphi}(x : |x_1 + \cdots + x_k| < A, B_1 < |x_1| < B_2)$$
$$\leqslant \begin{cases} CA^\nu \epsilon^{-1}\left[(B_1 - A)^{-\epsilon} - (B_2 + A)^{-\epsilon}\right] \text{ if } 3A \leqslant B_1, \\ CA^\nu\left\{\max\left[0, \epsilon^{-1}((2A)^{-\epsilon} - (B_2 + A)^{-\epsilon})\right] + A^{-\epsilon}\right\} \\ \text{if } 3A > B_1. \end{cases} \tag{4.5}$$

Finally for all $A \geqslant 1$, $M \geqslant 5$,

$$I_3^\epsilon(A, M) = \mu_{k,\epsilon,\varphi}(x : |x_1 + \cdots + x_k| < A, |x_1| > A, |x_1| > M|x_2|)$$
$$\leqslant CA^\nu M^{-\nu/k}\epsilon^{-1}. \tag{4.6}$$

Here and in the following the same letter C may denote different constants which do not depend on ϵ, A, B_1, B_2, and M, but may depend on k, φ and ν.

To prove these inequalities we make the following remarks. Put

$$F_\alpha^k(A) = \int_{\{|x_1 + \cdots + x_k| < A\}} |x_1|^{-\alpha} \ldots |x_k|^{-\alpha}\, dx_1 \ldots dx_k. \tag{4.7}$$

We can see, by making the substitution $x_i = Ax_i'$, that $F_\alpha^k(A)$ is a homogeneous function of the parameter A with order $k(\nu - \alpha)$. If the function F_α^k is finite for all $A > 0$ (or what is equivalent, for one $A > 0$), then because of the homogeneity property of F_α^k there exists a constant $C_\alpha^k > 0$ such that for all measurable nonnegative functions $\psi(u)$, $u \in [0, \infty)$,

$$\int_{R^{\nu k}} |x_1|^{-\alpha} \ldots |x_k|^{-\alpha}\psi(|x_1 + \cdots + x_k|)\, dx_1 \ldots dx_k$$
$$= C_\alpha^k \int_0^\infty \psi(u) u^{k(\nu-\alpha)-1}\, du. \tag{4.8}$$

Let us apply Lemma 2 with $a(v) = 1$ and $\tilde\varphi(x) \geqslant 1$ for $|x| \leqslant 1$. Then $F_\alpha^k(1) \leqslant k(\varphi, \alpha)$, hence Lemma 2 implies that (4.8) holds true for $\nu > \alpha > ((k-1)/k)\nu$, and

$$C_\alpha^k \leqslant C_k\left(\alpha - \frac{k-1}{k}\nu\right)^{-1}, \qquad \nu > \alpha > \frac{k-1}{k}\nu, \tag{4.9}$$

where $C_k < \infty$.

Let $K_\varphi < \infty$ be such that $|\tilde\varphi(x)|^2 < K_\varphi |x|^{-2\nu} x \in R^\nu$, and $\bar a = \max_{v \in S^{\nu-1}} a(v)$. Then it follows from (4.8) that

$$I_1^\epsilon(A) \leqslant K_\varphi \bar a^k \int_{\{|x_1 + \cdots + x_k| > A\}} |x_1|^{-\alpha(k,\epsilon)} \ldots |x_k|^{-\alpha(k,\alpha)}$$
$$\times |x_1 + \cdots + x_k|^{-2\nu}\, dx_1 \ldots dx_k$$
$$\leqslant K_\varphi \bar a^k C_{\alpha(k,\epsilon)}^k \int_A^\infty u^{k(\nu - \alpha(k,\epsilon)) - 2\nu - 1}\, du \leqslant C_k A^\nu \epsilon^{-1}, \tag{4.10}$$

which implies (4.4).

Now let $\bar{K}_\varphi < \infty$ be such that $|\tilde{\varphi}(x)|^2 \leqslant \bar{K}_\varphi, x \in R^\nu$.
Then

$$I_2^\epsilon(A, B_1, B_2) \leqslant \bar{a}^k \bar{K}_\varphi \int |x_2|^{-\alpha(k,\epsilon)} \dots |x_k|^{-\alpha(k,\epsilon)} \times \left[\int_{\mathcal{Q}(x_2 + \cdots + x_k)} |x_1|^{-\alpha(k,\epsilon)} \, dx_1 \right] dx_2 \dots dx_k, \tag{4.11}$$

where

$$\mathcal{Q}(x) = \{ x_1 \in R^\nu, B_1 < |x| < B_2, |x_1 + x| < A \}.$$

Let us first consider the case $B_1 > 3A$. Let us observe that the set $\mathcal{Q}(x)$ is empty if $|x| < B_1 - A$ or $|x| > 2A$. Moreover $\mathcal{Q}(x) \subseteq c\{x_1 \in R^\nu, |x_1| > |x|/2\}$ if $|x| > 2A$, which is always satisfied if $|x| > B_1 - A$. Hence relations (4.8) and (4.9) imply that

$$\begin{aligned} I_2^\epsilon(A, B_1, B_2) &\leqslant \int_{\{B_1 - A < |x_2 + \cdots + x_k| < B_2 + A\}} |x_2|^{-\alpha(k,\epsilon)} \dots |x_k|^{-\alpha(k,\epsilon)} \\ &\quad \times CA^\nu |x_2 + \cdots + x_k|^{-\alpha(k,\epsilon)} \, dx_2 \dots dx_k \\ &\leqslant CC_{\alpha(k,\epsilon)}^{k-1} A^\nu \int_{B_1 - A}^{B_2 + A} |u|^{(k-1)(\nu - \alpha(k,\epsilon)) - \alpha(k,\epsilon) - 1} \, du \\ &\leqslant C\epsilon^{-1} A^\nu \left[(B_1 - A)^{-\epsilon} - (B_2 + A)^{-\epsilon} \right]. \end{aligned} \tag{4.12}$$

Now we turn to the case $B_1 < 3A$. Since $\mathcal{Q}(x) \subset \{x_1 \in R^\nu, |x_1| \leqslant 3A\}$ for $|x| < 2A$,

$$\begin{aligned} &\int_{\{|x_2 + \cdots + x_k| < 2A\}} |x_2|^{-\alpha(k,\epsilon)} \dots |x_k|^{-\alpha(k,\epsilon)} \\ &\quad \times \left[\int_{\mathcal{Q}(x_2 + \cdots + x_k)} |x_1|^{-\alpha(k,\epsilon)} \, dx_1 \right] dx_2 \dots dx_k \\ &\leqslant CA^{\nu - \alpha(k,\epsilon)} F_{\alpha(k\epsilon)}^{k-1}(2A) = CA^{k(\nu - \alpha(k,\epsilon))} = CA^{\nu - \epsilon}. \end{aligned}$$

This integral can be estimated in the region $\{|x_2 + \cdots + x_k| > 2A\}$ in the same way as in (4.12), only $B_1 - A$ must be replaced by $2A$. These relations imply (4.5).

For $k > 2$ we have

$$I_3^\epsilon(A, M) \leqslant \bar{a} \bar{K}_\varphi \int_{R^{(k-2)\nu}} |x_3|^{-\alpha(k,\epsilon)} \dots |x_k|^{-\alpha(k,\epsilon)} \times \left[\int_{\mathcal{B}(x_3 + \cdots + x_k)} |x_1|^{-\alpha(k,\epsilon)} |x_2|^{-\alpha(k,\epsilon)} \, dx_1 \, dx_2 \right] dx_3 \dots dx_k, \tag{4.13}$$

where

$$\mathcal{B}(x) = \left\{ (x_1, x_2) \in R^{2\nu}, |x_1| > A, |x_2| > M|x_1|, |x_1 + x_2 + x| < A \right\}.$$

Put

$$\mathcal{B}_{x_1}(x) = \{x_2 \in R^\nu, (x_1, x_2) \in \mathcal{B}(x)\}.$$

Since $|x_1 + x_2| > |x_2| - |x_1| > (M-1)|x_1|$ for $|x_2| > M|x_1|$, the set $\mathcal{B}(x)$ is empty for $|x| < (M-2)A$. If $|x| > (M-2)A$, and $x_2 \in \mathcal{B}_{x_1}(x)$ with some x_2, then $M|x_1| < |x_2| < A + |x_1| + |x| < |x_1| + (1 + (1/(M-2))|x|$. This means that $\mathcal{B}_{x_1}(x)$ is empty if $|x_1| > (1/(M-2))|x|$. Finally, since $M \geqslant 5$ and $|x_2| \geqslant |x_1 + x_2| - |x_1|$, we have $\min_{x_2 \in \mathcal{B}_{x}1(x)} |x_2| \geqslant |x| - A - |x_1| > |x|/3$ if $|x| > (M-2)A$ and $|x| > (M-2)|x_1|$. The volume of the set $\mathcal{B}_{x_1}(x)$ is less then CA^ν. These facts together imply that

$$\begin{aligned}
&\int_{\mathcal{B}(x)} |x_1|^{-\alpha(k,\epsilon)} |x_2|^{-\alpha(k,\epsilon)}\, dx_1\, dx_2 \\
&\quad \leqslant \left(\frac{|x|}{3}\right)^{-\alpha(k\,\epsilon)} \int_{\{|x_1| < (M-2)^{-1}|x|\}} |x_1|^{-\alpha(k,\epsilon)} \left[\int_{\mathcal{B}_{x_1}(x_2)} dx_2\right] dx_1 \\
&\quad \leqslant C|x|^{-\alpha(k,\epsilon)} A^\nu (M^{-1}|x|)^{\nu - \alpha(k,\epsilon)} = CA^\nu M^{-(\nu-\epsilon)/k} |x|^{\nu - 2\alpha(k\,\epsilon)}.
\end{aligned}$$

Substituting this estimate into (4.13) and exploiting relations (4.8) and (4.9), we get that

$$\begin{aligned}
I_3(A, M) &\leqslant CA^\nu M^{-(\nu-\epsilon)/k} \int_{\{|x_3 + \cdots + x_k| > (M-2)A\}} |x_3|^{-\alpha(k,\epsilon)} \ldots |x_k|^{-\alpha(k,\epsilon)} \\
&\qquad \times |x_3 + \cdots + x_k|^{\nu - 2\alpha(k,\epsilon)}\, dx_3 \ldots dx_k \\
&\leqslant CC_{\alpha(k,\epsilon)}^{k-2} A^\nu M^{-(\nu-\epsilon)/k} \int_{(M-2)A}^{\infty} u^{k(\nu - \alpha(k,\epsilon)) - \nu - 1}\, du \\
&\leqslant CA^\nu M^{-(\nu-\epsilon)/k} ((M-2)A)^{-\epsilon} \epsilon^{-1}.
\end{aligned}$$

This inequality proves relation (4.6) for $k \geqslant 2$. The case $k = 2$ is trivial, since in this case the set $\{x = (x_1, x_2) \in R^{2\nu}, |x_1 + x_2| < A, |x_1| > A, |x_2| > M|x_1|\}$ is empty.

Let us now define the sequences $a_0 = 0$, $a_j = a_j(\epsilon) = \exp\{j\epsilon^{-1/2}\}$ $j = 1, 2, \ldots$. By applying estimates (4.4) and (4.5) with $A = \epsilon^{-(1/4\nu)}$, we get for sufficiently small $\epsilon > 0$, the estimate

$$\begin{aligned}
\epsilon\mu_{k,\epsilon,\varphi}(D_j^k) &\leqslant \epsilon\left[I_1^\epsilon(A) + I_2^\epsilon(A, a_j, a_{j+1})\right] \\
&\leqslant C\left[A^{-\nu} + A^\nu\left((a_j - A)^{-\epsilon} - (a_{j+1} + A)^{-\epsilon}\right)\right] \\
&\leqslant C\left[A^{-\nu} + A^\nu\left(\left(\exp\{(j - \tfrac{1}{2})\epsilon^{-1/2}\}\right)^{-\epsilon} - \left(\exp\{(j + \tfrac{3}{2})\epsilon^{-1/2}\}\right)^{-\epsilon}\right)\right] \\
&\leqslant C\left(\epsilon^{1/4} + \epsilon^{-1/4} \exp\{-(j - \tfrac{1}{2})\epsilon^{1/2}\}\left[1 - \exp\{-2\epsilon^{1/2}\}\right]\right) \to 0
\end{aligned}$$

uniformly for all $j = 1, 2, \ldots$. The case $j = 0$ is simpler. One has only to remark that $A^{-\epsilon} \to 1$, and $(2A)^{-\epsilon} - (A + a_2)^{-\epsilon} \leqslant 1 - a_3^\epsilon = O(\epsilon^{1/2})$. The estimate (4.1) is proved.

Let us observe that for all $j = 0, 1, 2, \ldots$

$$\bar{D}^k \cap \{a_j < |x_1| \leqslant a_{j+1}\}$$

$$\subset \bigcup_{l=2}^{k} \left[\{a_j < |x_1| < a_{j+1}, |x_l| \leqslant a_j\} \cup \{a_j < |x_1| < a_{j+1}, |x_l| > a_{j+1}\}\right]$$

$$\subset \bigcup_{l=2}^{k} \left[\{|x_l| < M^{-1}|x_1|\} \cup \{a_j \geqslant |x_l| > M^{-1}a_j\}\right.$$

$$\left. \cup \{|x_l| > M|x_1|\} \cup \{a_{j+1} \leqslant |x_l| \leqslant Ma_{j+1}\}\right].$$

Hence, exploiting the invariance with respect to the permutations of the indices $j = 1, \ldots, k$, we get

$$\mu_{k,\epsilon,\varphi}(\bar{D}_k) \leqslant 2(k-1)\,\mu_{k,\epsilon,\varphi}\{|x_2| > M|x_1|\}$$

$$+ \mu_{k,\epsilon,\varphi}\{|x_1| < a_1\} + k\mu_{k,\epsilon,\varphi}\left(\bigcup_{j=1}^{\infty}\{M^{-1}a_j < |x_1| < Ma_j\}\right)$$

$$\leqslant 2(k-1)I_3^{\epsilon}(A, M) + 2(k-1)I_2^{\epsilon}(A, 0, A) + I_2^{\epsilon}(A, 0, a_1)$$

$$+ [2(k-1) + k + 1]I_1^{\epsilon}(A) + k\sum_{j=1}^{\infty} I_2^{\epsilon}(A, M^{-1}a_j, Ma_j).$$

Put $A = \epsilon^{-1/6\nu}$, and $M = \exp\{\epsilon^{-1/6}\}$. It is not difficult to see by exploiting relations (4.4), (4.5), and (4.6) that all terms on the right side of (4.14) have order $o(\epsilon^{-1})$ as $\epsilon \to 0$. We prove this estimate only for the last summand.

$$\sum_{j=1}^{\infty} I_2^{\epsilon}(A, M^{-1}a_j, Ma_j) \leqslant \sum_{j=1}^{\infty} CA^{\nu}\epsilon^{-1}\left[(M^{-1}a_j)^{-\epsilon}(Ma_j)^{-\epsilon}\right]$$

$$= CA^{\nu}\epsilon^{-1}(M^{\epsilon}M^{-\epsilon})\sum_{j=1}^{\infty} a_j^{-\epsilon}$$

$$\leqslant C\epsilon^{-1/6}\epsilon^{-1}\epsilon^{5/6}\sum_{j=1}^{\infty}\exp(-j\epsilon^{1/2})$$

$$\leqslant C\epsilon^{-1}\epsilon^{2/3}\epsilon^{-1/2} = o(\epsilon^{-1}).$$

We have proved the estimate (4.2), and therefore the proof of Theorem 2 is complete.

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