

SPECIAL INVITED PAPER

CRITICAL PHENOMENA AND UNIVERSAL EXPONENTS IN STATISTICAL PHYSICS. ON DYSON'S HIERARCHICAL MODEL

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Dedicated to Professor Ja. G. Sinai, who introduced us to this subject

We are interested in the behavior of Gibbs states at or near the critical temperature. From the point of view of classical probability theory this is a problem about the limit distribution of partial sums of strongly dependent random variables. The problem is very hard in the general case, and almost no rigorous results are known, so we will discuss a special case, Dyson's hierarchical model, in detail. This model can be rigorously investigated, and it may help us to understand the behavior of the more general models. We present the most important results with a sketch of their proofs. Vector-valued models are also discussed, since in this case some new interesting phenomena appear. The last section deals with the translation invariant case. Some recent results are presented and some conjectures and open problems are formulated. The last section can be read independently of the previous sections, but the conjectures formulated there are strongly motivated by the previous results.

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0. Introduction. One of the most important problems in probability theory is the investigation of the limit distribution of appropriately normalized sums of

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random variables. The case of independent random variables is fairly well understood, but much less is known about the dependent case. On the other hand, such questions arise in a natural way when we investigate equilibrium states in statistical physics. In the case of dependent random variables it is more natural to pose the question in a slightly different way. A set of random variables X_n , $n \in \mathbb{Z}^d$, where \mathbb{Z}^d denotes the lattice in the d -dimensional Euclidean space R^d , is called a d -dimensional random field. Let us define the nonintersecting cubes $D_n^N \subset \mathbb{Z}^d$, $n \in \mathbb{Z}^d$, $N = 1, 2, \dots$,

$$D_n^N = \{k = (k_1, \dots, k_d) \in \mathbb{Z}^d, Nn_j \leq k_j < N(n_j + 1), j = 1, \dots, d\}$$

and the random fields Z_n^N , $n \in \mathbb{Z}^d$, $N = 1, 2, \dots$, as

$$(0.1) \quad Z_n^N = \frac{1}{A_N} \sum_{k \in D_n^N} (X_k - EX_k), \quad n \in \mathbb{Z}^d, \quad N = 1, 2, \dots,$$

where A_N is an appropriate norming constant. The random fields Z_n^N are called the renormalizations of the original field X_n . If the finite-dimensional distributions of the fields Z_n^N tend to those of a random field Z_n^* as $N \rightarrow \infty$, then the random field Z_n^* is called the large-scale limit of the random field X_n . We are interested in which random fields have a large-scale limit, which random fields can appear as a limit, and how the norming constant A_N must be chosen. We restrict ourselves to the case of stationary random fields X_n , i.e., to the case when the random vectors $(X_{n_1}, \dots, X_{n_k})$ and $(X_{n_1+p}, \dots, X_{n_k+p})$ have the same distribution for all $k = 1, 2, \dots$, and $n_1, \dots, n_k, p \in \mathbb{Z}^d$. Moreover, we assume that the random variables have sufficiently many moments.

If the random field X_n , $n \in \mathbb{Z}^d$, consists of independent—or in some sense weakly dependent—random variables, then its large-scale limit exists with $A_N = N^{d/2}$, and the limit is an independent Gaussian field, i.e., a set of independent Gaussian random variables. In statistical mechanics, however, there arise more sophisticated situations. Equilibrium states in statistical physics are probability measures on $R^{\mathbb{Z}^d}$ which depend on a physical parameter, for example, the temperature T . We shall give the definition of equilibrium states in the next section. Let $\mu(T)$ be an equilibrium state at temperature T , and let X_n , $n \in \mathbb{Z}^d$, be a $\mu(T)$ distributed random field. We are interested in the large-scale limit of the random field X_n . In the next few paragraphs we will describe a picture which is believed by physicists to hold for a large class of models but has been rigorously demonstrated in only a few cases.

There is a special parameter $T = T_{cr}$, the so-called critical temperature, where the equilibrium state has a very special behavior. For all $T \neq T_{cr}$ the large-scale limit of the $\mu(T)$ distributed fields X_n with the normalization $A_N = N^{d/2}$ is an independent Gaussian field. In the case $T = T_{cr}$, however, a normalization $A_N = (N^d)^\alpha$, $\alpha > \frac{1}{2}$, must be chosen, and the limit field consists of dependent random variables, which are not necessarily Gaussian. The particular behavior of the equilibrium state at $T = T_{cr}$ is closely connected with the dependence structure of the $\mu(T)$ distributed random fields X_n , $n \in \mathbb{Z}^d$. For $T < T_{cr}$, $EX_k X_{k+n} \rightarrow M = M(T)$ with some $M(T) \neq 0$ as $|n| \rightarrow \infty$, and for $T > T_{cr}$,

$EX_k X_{k+n} \rightarrow 0$ as $|n| \rightarrow \infty$. For both $T > T_{cr}$ and $T < T_{cr}$ the correlation function tends to zero exponentially fast, i.e., $E(X_k - EX_k)(X_{n+k} - EX_{n+k}) = O(\exp(-c|n|))$ with some $c > 0$. The only exceptional case is $T = T_{cr}$ when $EX_k X_{k+n} \sim \text{const.}|n|^{-c}$ with some $0 < c < d$. It is natural to choose the normalizing constant A_N in (0.1) as

$$A_N^2 = \text{Var}\left(\sum_{j \in D_n^N} X_j\right).$$

Hence the behavior of the correlation function suggests that the right choice of A_N is $A_N = N^{d/2}$ for $T \neq T_{cr}$ and $A_N = N^{(d-c)/2}$ for $T = T_{cr}$. In the latter case a new type of limit theorem can be expected. We are interested in the behavior of equilibrium states at or near the critical temperature. Here another interesting phenomenon appears, which is called universality. For $T \neq T_{cr}$ the random variable $N^{-d/2} \sum_{j \in D_n^N} (X_j - EX_j)$ is asymptotically normal with expectation zero and variance $\sigma(T)$. For T in the vicinity of T_{cr} the asymptotic relation $\sigma(T) \sim \text{const}|T - T_{cr}|^{-\gamma}$ holds. Moreover, the parameter γ depends only on some global characteristics of the model, like the dimension d , but it does not depend on the finer structure of the model at all. Constants like the above-defined γ are called universal exponents. Similar universal exponents appear in the normalization at the critical temperature, where $A_N = N^\alpha$, or the magnetization $EX_n \sim \text{const.}|T - T_{cr}|^\beta$ for $T < T_{cr}$.

It is widely believed among physicists that such results hold true. But, unfortunately, most of these results have no rigorous proofs. Hence a simplified model, where all the above statements can rigorously be proved, and which helps us to understand the general situation, is very interesting for us. This is the reason why we shall discuss a special model, Dyson's hierarchical model, in detail. We present the most important results and discuss what kind of results they suggest in the general case. We shall also consider the vector-valued model, where limit theorems similar to that at the critical temperature hold for all low temperatures.

Let us finally remark that the above-discussed critical behavior and universality is not a peculiarity of models in statistical physics.

There are several infinite systems where similar results are expected. Unfortunately, the investigation of most of these models seems to be extremely hard and very few rigorous proofs are known.

1. The definition of equilibrium states. In this section we define the equilibrium states. Let a lattice S and a metric space K be given. In this paper we choose either $S = \mathbb{Z}^d$ or $\mathbb{Z} = \{1, 2, \dots\}$ and either $K \subset \mathbb{R}^1$, in which case we speak of scalar valued or $K \subset \mathbb{R}^p$, $p \geq 2$, in which case we speak of vector-valued models. A function $\sigma: S \rightarrow K$ is called a configuration, and its value $\sigma(i)$, $i \in S$, the spin at the point i . Equilibrium states are implicitly defined probability measures on the space of all configurations K^S . They depend on the energy of configurations (or Hamiltonian function), a "free" measure ν on K and the temperature T . We restrict ourselves to the case when the Hamiltonian function

of a configuration $\sigma = \{\sigma(i), i \in S\}$ can be given by the formal series

$$(1.1) \quad H(\sigma) = - \sum_{i, j \in S} U(i, j)\sigma(i)\sigma(j) - h \sum_{i \in S} \sigma(i).$$

Here $h \in R^1$ in the scalar case, $h \in R^p$ in the vector case, and $h\sigma(i), \sigma(i)\sigma(j)$ denote scalar products if $K \subset R^p, p > 1$. We shall impose the restriction $U(i, j) \geq 0$. Models satisfying this condition are called ferromagnetic models. Let us remark that models where this condition is violated are no less interesting. In such cases some new effects may appear, but the behavior of such models is much less understood.

The precise meaning of formula (1.1) is the following: The energy of a configuration $\sigma = \{\sigma(i), i \in V\}$ in a finite set $V \subset S$ is

$$H_V(\sigma) = - \sum_{i, j \in V} U(i, j)\sigma(i)\sigma(j) - h \sum_{i \in V} \sigma(i),$$

and its conditional energy under some external configuration $\sigma' = \{\sigma(i), i \in S - V\}$ is

$$H_V(\sigma|\sigma') = H_V(\sigma) - \sum_{i \in V} \sum_{j \in S - V} U(i, j)\sigma(i)\sigma'(j).$$

We shall assume that we are given an even measure ν over K , i.e., $\nu(A) = \nu(-A)$, which we call the free measure, and it is invariant with respect to rotations of R^p in the vector-valued case. The measure ν will be a probability measure if it is not stated otherwise. Moreover, we assume that it tends to zero sufficiently fast at infinity, so that all integrals in the sequel will be convergent. Given a Hamiltonian function H , a free measure ν and a finite set $V \subset S$, we define the Gibbs distribution at temperature T by the formulas

$$(1.2) \quad \mu_V(d\sigma|T, \sigma', \nu) = \frac{1}{\Xi(T, \sigma', \nu)} \exp\left\{-\frac{1}{T}H_V(\sigma|\sigma')\right\} \prod_{i \in V} \nu(d\sigma(i)),$$

$$(1.2') \quad \Xi(T, \sigma', \nu) = \int \exp\left\{-\frac{1}{T}H_V(\sigma|\sigma')\right\} \prod_{i \in V} \nu(d\sigma(i)).$$

A probability measure $\mu = \mu(T)$ on K^S is called an equilibrium state with the above-defined Hamiltonian H at temperature T if the following property holds: For all finite sets $V \subset S$ and almost all configurations $\sigma' = \{\sigma'(j), j \in S - V\}$ (with respect to the measure μ), the conditional distribution of the configurations σ in V under the condition that the configuration is σ' in $S - V$ is given by formulas (1.2) and (1.2'). The first natural question which arises in connection with the above definition is whether there exist equilibrium states and if they exist whether they are unique or not. It has been proved under fairly general conditions that equilibrium states exist and they are unique for all high temperatures ([33], [35]). On the other hand, in many cases it has been proved that the equilibrium state is not unique at sufficiently low temperatures. In order to describe a more detailed picture let us first consider scalar-valued models. For the sake of simplicity let us restrict ourselves to the case when the free measure ν is such that $\nu(\{-1\}) = \nu(\{1\}) = \frac{1}{2}$. We shall also assume that $U(i, j) = U(j, i) = U(i - j) \geq 0, U(0) = 0$ and $0 < \sum_{j \in \mathbf{Z}^d} U(j) < \infty$.

The situation is different for $d = 1$ and $d \geq 2$. For $d = 1$ the equilibrium state is unique for all temperatures $T > 0$ if $\sum_{j=1}^{\infty} jU(j) < \infty$ (see [34]). On the other hand, as was proved by Dyson in [19], in the case $U(j) = j^{-a}$, $1 < a < 2$, there are two different ergodic equilibrium states $\mu^+(T) = \mu^+(d\sigma|T)$ and $\mu^-(T) = \mu^-(d\sigma|T)$ for all sufficiently small $T > 0$ and only one equilibrium state $\mu(T) = \mu(d\sigma|T)$ for large T . Fröhlich and Spencer [24] proved the same result in the very delicate border case $a = 2$. The equilibrium state μ^+ (resp. μ^-) can be obtained as the limit $\lim_{V \rightarrow \mathbf{Z}} \mu_V(d\sigma|T, \sigma', \nu)$ with the boundary condition $\sigma(i) = 1, i \in V$ [resp. $\sigma(i) = -1, i \notin V$].

Define the quantity $M(T) = E^+ \sigma(i) = \int \sigma(i) \mu^+(d\sigma|T)$, where we write μ instead of μ^+ if there is only one equilibrium state. [It can be proved that $M(T)$ does not depend on $i \in \mathbf{Z}$.] The function $M(T)$ is called the spontaneous magnetization. It is proved that $M(T) = 0$ for large T , and $M(T) > 0$ for sufficiently small T if $U(j) = j^{-a}$, $1 < a \leq 2$. In the latter case $\mu^+ \neq \mu^-$. It is also proved that $M(T)$ is a monotone decreasing function in T . Thus there exists a critical temperature T_{cr} such that $M(T) = 0$ for $T > T_{cr}$, and $M(T) > 0$ for $T < T_{cr}$.

There are several interesting conjectures about the behavior of equilibrium states near the critical temperature. We consider equilibrium states with the potential $U(j) = j^{-a}$, $1 < a < 2$. Let us define the random variables $\sigma_N = \sum_{i=1}^N \sigma(i)$, where $\sigma(i)$ are $\mu^+(T)$ [resp. $\mu(T)$ for $T \geq T_{cr}$] distributed random variables. It is believed that for $N \rightarrow \infty$

$$(1.3) \quad E\sigma_N^2 \sim \text{const.}(T - T_{cr})^{-\gamma}N, \quad \text{if } T > T_{cr},$$

$$(1.4) \quad E\sigma_N^2 \sim \text{const.} N^a, \quad \text{if } T = T_{cr},$$

$$(1.5) \quad E(\sigma_N - E\sigma_N)^2 \sim \text{const.}(T_{cr} - T)^{-\gamma}N, \quad \text{if } T < T_{cr},$$

$$(1.6) \quad E\sigma_N = NM(T), \quad M(T) \sim \text{const.}(T_{cr} - T)^{-\beta}, \quad \text{if } T < T_{cr},$$

with some positive parameters β and γ , which are called critical exponents. Moreover, we expect that the random variables $\sigma(i)$ are weakly dependent if $T > T_{cr}$ or $T < T_{cr}$, hence they satisfy the central limit theorem, i.e., $(1/\sqrt{N})\sigma_N$ for $T > T_{cr}$ and $(1/\sqrt{N})(\sigma_N - E\sigma_N)$ for $T < T_{cr}$ tend in distribution to the normal law. On the other hand, for $T = T_{cr}$ the $\sigma(i)$ are strongly dependent, and in that case we expect that $N^{-a/2}\sigma_N$ has a limit in distribution as $N \rightarrow \infty$. It is conjectured that the limit is Gaussian for $1 < a \leq \frac{3}{2}$ and non-Gaussian for $\frac{3}{2} < a < 2$.

For $d \geq 2$ it has been proved that for any potential U ,

$$U(j) \geq 0, \quad \sum_{j \in \mathbf{Z}^d} U(j) < \infty \quad \text{and} \quad U(j) > 0$$

for two linearly independent j , the equilibrium state is not unique for sufficiently low temperatures. Let us restrict ourselves to the case of a finite range potential, i.e., to the case when there exists some $R > 0$ such that $U(j) = 0$ if $|j| \geq R$. It is conjectured that the situation in this case is similar to the one-dimensional case with the potential $U(j) = j^{-a}$, $1 < a < 2$. Namely, if we

define the volumes $V_N = \{j = (j_1, \dots, j_d): 1 \leq j_k \leq N, k = 1, \dots, d\}$ and the random variables $\sigma_N = \sum_{j \in V_N} \sigma(j)$, $N = 1, 2, \dots$, then the following relations are expected to hold: There exists some T_{cr} such that

$$(1.7) \quad E\sigma_N^2 \sim \text{const.}(T - T_{\text{cr}})^{-\gamma}N^d, \quad \text{if } T > T_{\text{cr}},$$

$$(1.8) \quad E\sigma_N^2 \sim \text{const. } N^{d+2-\eta}, \quad \text{if } T = T_{\text{cr}},$$

$$(1.9) \quad E(\sigma_N - E\sigma_N)^2 \sim \text{const.}(T_{\text{cr}} - T)^{-\gamma}N^d, \quad \text{if } T < T_{\text{cr}},$$

$$(1.10) \quad E\sigma_N = N^d M(T), \quad M(T) \sim \text{const.}(T_{\text{cr}} - T)^\beta, \quad \text{if } T < T_{\text{cr}},$$

where γ, η, β are critical exponents and a logarithmic multiplier is possible. [The exponent in (1.8) is written in a strange form, because in such a formulation $\eta = 0$ is expected in the so-called classical cases. We are not going to discuss the details.] It is conjectured that $N^{-d/2}\sigma_N$ in the case $T > T_{\text{cr}}$ and $N^{-d/2}(\sigma_N - E\sigma_N)$ in the case $T < T_{\text{cr}}$ satisfy the central limit theorem. For $T = T_{\text{cr}}$ it is expected that $N^{-(d+2-\eta)/2}\sigma_N$ has a limit in distribution as $N \rightarrow \infty$. Moreover, it is believed that the critical exponents β, γ and η are universal in the following sense. They do not depend on the finite range potential $U(j)$. Let us remark that the critical temperature T_{cr} is not universal in the above sense.

The behavior of vector-valued models is similar to that of scalar-valued models, but some new phenomena appear. Here we want to discuss only some of them. We restrict ourselves to the case when the free measure ν is the uniform distribution on the p -dimensional unit sphere.

The problem about the existence of different equilibrium states in vector-valued models is only partly solved. One has to impose stronger conditions on the potential U in order to guarantee the existence of different equilibrium states at low temperatures. Some results [17] about the so-called absence of breakdown of continuous symmetry indicate that for dimension $d = 1, 2$, models with finite range potential must have only one translation invariant equilibrium state for all $T > 0$. On the other hand, it has been proved in certain cases, e.g., for models with potential $U(j) = |j|^{-a}$, $1 < a < 2$, $d \geq 1$, or $U(j) = 1$ if $|j| = 1$, $U(j) = 0$ if $|j| > 1$, $d \geq 3$, that for sufficiently low temperatures and arbitrary $e \in R^p$ there exists an ergodic equilibrium state $\mu^e(T) = \mu^e(d\sigma|T)$ such that $E^e\sigma(i) = \int \sigma(i)\mu^e(d\sigma|T) = M(T)e$ with some $M(T) > 0$ (see [23] and [22]). It is believed that the correlation function of scalar-valued models $r(i-j) = E^+[(\sigma(i) - M(T))(\sigma(j) - M(T))]$ and that of vector-valued models $r(i-j) = E^e\{(\sigma(i) - M(T)e)(\sigma(j) - M(T)e)\}$ behave differently. Namely, it has been proved that the correlation function of scalar-valued models is rapidly decreasing (in case of finite range potential exponentially fast), and, on the other hand, it is conjectured that in the vector-valued case $r(j)$ decreases only as a relatively small power of $|j|$. Some estimates given for certain models support this conjecture (see [13] and [38]). The slow decrease of the correlation function indicates strong dependence between the random variables $\sigma(i)$. Because of this strong dependence we conjecture that for all low temperatures the partial sums

of the $\mu^\epsilon(d\sigma|T)$ distributed random variables satisfy a limit theorem with an unusual normalization and a possibly non-Gaussian limit. Let us emphasize that this conjecture indicates an essential difference between scalar- and vector-valued models. In scalar models there appears a nonclassical limit theorem with an unusual normalization only at a very special parameter, at the critical temperature. On the other hand, in vector-valued models we expect a nonclassical limit theorem with some unusual normalization for all low temperatures.

The above-mentioned conjectures seem to be very hard to prove in the general case. Hence we shall consider a special case, Dyson's hierarchical model, which we are able to investigate in detail. The results obtained for this model are in full accordance with the above-mentioned conjectures. Moreover, our investigations can give us a better insight about what to expect in the case of general models.

2. An overview of the main results. In this section we briefly describe our program in this paper and make some comments. In Section 3 we define Dyson's hierarchical model and introduce the most important notions. This model is similar to the one-dimensional models with potential $U(j) = j^{-a}$, $1 < a < 2$, but it is not translation invariant. Instead of translation invariance it has another symmetry which makes it more tractable. In Sections 4 and 5 we discuss the Gibbs states without boundary conditions. Here the main problem is the investigation of an integral operator S_n defined in (4.2). This operator enables us to calculate the density function of the average of 2^n spins from the density function of the average of 2^{n-1} spins. In the case of independent random variables one has to apply the convolution operator to solve the analogous problem. Let us remark that the operator S_n is very similar to the convolution operator. Moreover, just as the convolution does, it turns a Gaussian density function into a Gaussian density function again. The formal difference between S_n and the convolution operator is the existence of a kernel function $\exp\{c^{n-1}/T(x^2 - u^2)\}$ in the definition of S_n . This kernel is extremely important for us. Actually, it is this kernel which reflects the dependence structure in our model and which is responsible for the fact that our model has much more complex behavior than that of independent random variables.

In the investigation of general statistical physical models some transformations appear which we expect to have a behavior similar to that of S_n . But these transformations are much more complicated in the general case, and it is the relative simplicity of S_n which makes Dyson's hierarchical model tractable.

In Section 4 we prove a limit theorem for the sum of spins in a large volume at the critical temperature (Theorems 4.1 and 4.3). The normalization in these theorems is unusual, and the limit is in certain cases Gaussian and in certain cases non-Gaussian. The limit which appears must be the fixed point of a rescaled version of the operator S_n . (This rescaled version does not depend on n .) This rescaled operator always has a Gaussian fixed point. But only fixed points with some stability property can appear as a limit, and the Gaussian fixed point is not always stable. In such cases a stable fixed point must be found, and this is done in Theorem 4.2 with the help of bifurcation theory. In Section 5 a large

deviation result is presented and the critical exponents are calculated. The investigation of critical exponents and large deviation results are closely related.

In Section 6 the equilibrium states are investigated. This investigation is based on the results of the previous paragraphs and the investigation of the operator T_m defined in Lemma 6.4. The main point is that T_m is strongly localized, hence we can give sufficiently accurate and simple asymptotic formulas (given in Lemma 6.5) for the Radon–Nikodym derivatives we are interested in.

In Section 7 we deal with vector-valued models. The starting formulas in the investigation of vector-valued models are the natural adaptations of those given in the scalar-valued case. The methods of proof in these two cases are also very similar. We need a good asymptotic formula for the distribution of the average spin for Gibbs states without boundary conditions and the Radon–Nikodym derivative of the equilibrium state with respect to the Gibbs state without boundary condition. However, when the starting formulas are approximated by simpler expressions during the proof, essentially different expressions appear in the scalar- and vector-valued case. We get some results for vector-valued models which have no analogues in the scalar case.

The main results of Section 7 are Theorems 7.2 and 7.4 and Lemma 7.3. Here we get limit theorems with an unusual normalization in the direction orthogonal to the spontaneous magnetization for all low temperatures. Let us emphasize that, unlike the scalar-valued case, here we get a nonclassical limit theorem with an unusual normalization not only at a special critical parameter value but for all low temperatures. In Theorem 7.4 a non-Gaussian limit distribution appears. It is given with the help of an integral equation defined in formula (7.1). We know of no analogous result in the previous literature.

In Section 8 general translation invariant models are discussed. This section can be read independently of the previous ones. However, the introduction of several notions and the formulation of several conjectures are strongly influenced by the results obtained for Dyson's hierarchical model. In the translation invariant case very few rigorous results are known. Hence we put emphasis on the formulation of some important conjectures and try to explain why it is natural to expect them.

3. Dyson's hierarchical model. Posing of the problems. Dyson's hierarchical model is a special case of the equilibrium states defined in Section 2. In this case we choose $S = \mathbb{Z} = \{1, 2, \dots\}$. To define the Hamiltonian function first we have to define the so-called hierarchical distance on \mathbb{Z} . Put $V_{k,n} = \{j, j \in \mathbb{Z}, (k-1)2^n < j \leq k \cdot 2^n\}$, $k = 1, 2, \dots$; $n = 1, 2, \dots$, and define $n(i, j) = \min\{n: \text{there is a } k \text{ such that } i \in V_{k,n}, j \in V_{k,n}\}$. We shall also write V_n instead of $V_{1,n}$. The hierarchical distance $d(i, j)$, $i, j \in \mathbb{Z}$, is defined by the formula

$$d(i, j) = \begin{cases} 0, & \text{if } i = j, \\ 2^{n(i,j)-1}, & \text{if } i \neq j. \end{cases}$$

Now we define the Hamiltonian function $H(\sigma)$, $\sigma = \{\sigma(i), i \in \mathbb{Z}\}$, of Dyson's

hierarchical model by formula (1.1) with the choice $h = 0$ and $U(i, j) = d^{-\alpha}(i, j)$, $1 < \alpha < 2$. The constant α is an important parameter of the model. It measures the order of interaction between distant spins. We shall often use the parameter $c = 2^{2-\alpha}$ instead of α . First we discuss scalar models. We shall mainly consider the so-called ϕ^4 model, where the free measure ν is defined by the formula

$$(3.1) \quad \frac{d\nu}{dx} = p_0(x) = C(u) \exp\left(-\frac{x^2}{2} - \frac{u}{4}x^4\right), \quad 0 < u < u_0.$$

Here u_0 is a sufficiently small fixed constant, and the norming factor $C(u)$ is chosen in such a way that ν is a probability measure. The function $p_0(x)$ has two important properties which we need during the proofs. It is sufficiently close to a Gaussian density and tends to zero at infinity faster than any Gaussian density function. The parameters α, β, γ mentioned in the introduction can be defined for these models and do not depend on the parameter u . This fact indicates universality. Mainly we shall be interested in the following two problems:

3.1. *The behavior of Gibbs states without boundary conditions.* Let $H_n(\sigma)$ denote $H_{V_n}(\sigma)$ for $\sigma = \{\sigma(i), i \in V_n\}$, and define the measures

$$(3.2) \quad \mu_n(d\sigma|T, \nu) = \frac{1}{\Xi_n(T, \nu)} \exp\left\{-\frac{1}{T}H_n(\sigma)\right\} \prod_{i \in V_n} \nu(d\sigma(i)),$$

$$(3.2') \quad \Xi_n(T, \nu) = \int \exp\left\{-\frac{1}{T}H_n(\sigma)\right\} \prod_{i \in V_n} \nu(d\sigma(i)).$$

Let $p_n(x) = p_n(x, T)$ denote the density function of the random variable $2^{-n} \sum_{i \in V_n} \sigma(i)$, where the joint distribution of the random variables $\sigma(i), i \in V_n$, is given by formulas (3.2) and (3.2'). (We assume that the measure ν has a density function.) We are interested in the asymptotic behavior of the density function $p_n(x, T)$ for large n . We want to prove that there is some $T = T_{cr}$ for which $c^{-n/2} p_n(c^{-n/2}x, T_{cr})$ has a limit as $n \rightarrow \infty$. This means an unusual rescaling. Whether the limit is Gaussian or not depends on the parameter c of the model. For $T > T_{cr}$, $p_n(x, T)$ is asymptotically Gaussian with variance $\text{const.} \cdot 2^{-n}$, and for $T < T_{cr}$ it is asymptotically the mixture of two Gaussian density functions with variance $\text{const.} \cdot 2^{-n}$ and expectations $M(T)$ and $-M(T)$, respectively. Moreover, we want to show that the function $p_n(x, T)$ has the universal exponents α, β, γ in the vicinity of T_{cr} to be defined later, which are the natural modifications of those mentioned in the introduction.

3.2. *The behavior of equilibrium states.* In the above problem we have investigated the average of μ_n distributed random variables. This measure μ_n is a good approximation of the equilibrium states of Dyson's hierarchical model. However, it is more natural to investigate the equilibrium states themselves. The first problem is to show that the equilibrium states really exist. Then if we know that the equilibrium states $\mu(T)$ exist let us consider a $\mu(T)$ distributed random

field $\sigma(i)$, $i \in \mathbb{Z}$, and its renormalization

$$(3.3) \quad Y_n(k) = \frac{1}{A_n} \sum_{i \in V_{k,n}} (\sigma(i) - E\sigma(i)), \quad k \in \mathbb{Z}.$$

We want to show that the joint distributions of the random variables $Y_n(k)$ tend with the normalization $A_n = 2^{n/2}$ to those of an independent Gaussian field if $T \neq T_{cr}$, and they tend with the normalization $A_n = 2^n c^{-n/2}$ to the joint distributions of a not necessarily Gaussian field if $T = T_{cr}$. The equilibrium state $\mu(T)$ is unique for $T \geq T_{cr}$, and $E\sigma(i) = 0$ in this case. For $T < T_{cr}$ the equilibrium state is not unique, and we have to explain which equilibrium state we consider. We construct an equilibrium state $\mu^+(T)$ for $T < T_{cr}$ which is the natural analogue of the measure μ^+ defined for equilibrium states in \mathbb{Z}^d . We consider a random field $\sigma(i)$ with this distribution $\mu^+(T)$ in (3.3). It turns out that $E\sigma(i) > 0$ for $T < T_{cr}$. Finally, we want to prove that the critical exponents exist and we want to calculate them.

We remark that the values T_{cr} agree in the two problems. The proofs of the theorems about the equilibrium states heavily depend on the results about the behavior of $p_n(x, T)$. This is the reason why we have to investigate the first problem in detail.

We shall also consider the vector-valued version of Dyson's hierarchical model when $\sigma(i) \in R^p$. We shall consider the equilibrium states at low temperatures. We construct an equilibrium state $\mu_e(T)$ such that for a $\mu_e(T)$ distributed random field $\sigma(k) = (\sigma^{(1)}(k), \dots, \sigma^{(p)}(k))$, $k \in \mathbb{Z}$, $E\sigma^{(1)}(k) > 0$, $E\sigma^{(j)}(k) = 0$ for $j \geq 2$. We shall show that its renormalizations $Y_n(k) = (Y_n^{(1)}(k), \dots, Y_n^{(p)}(k))$,

$$(3.4) \quad \begin{aligned} Y_n^{(1)}(k) &= \frac{1}{A_n} \sum_{i \in V_{k,n}} [\sigma^{(1)}(i) - E\sigma^{(1)}(i)], \\ Y_n^{(j)}(k) &= \frac{1}{B_n} \sum_{i \in V_{k,n}} \sigma^{(j)}(i), \quad j = 2, \dots, p, \end{aligned} \quad k \in \mathbb{Z},$$

have a limit as $n \rightarrow \infty$, but an unusual normalization is needed for all small T .

4. Gibbs states without boundary conditions. The Hamiltonian function $H_n(\sigma)$, $\sigma = \{\sigma(i), i \in V_n\}$, satisfies the relation

$$(4.1) \quad H_n(\sigma) = H_{n-1}(\sigma_1) + H_{n-1}(\sigma_2) - 2^{-(n-1)\alpha} \sum_{i \in V_{n-1}} \sigma(i) \sum_{j \in V_n - V_{n-1}} \sigma(j),$$

where $\sigma_1 = \{\sigma(i), i \in V_{n-1}\}$, $\sigma_2 = \{\sigma(i), i \in V_n - V_{n-1}\}$.

We assume that the free measure ν has a density function $p_0(x)$. We claim that the density function $p_n(x, T)$ of the average of the $\mu_n(d\sigma|T, \nu)$ distributed spins satisfies the recursive relation

$$(4.2) \quad \begin{aligned} \hat{p}_n^*(x, T) &= S_n p_{n-1}(x, T) \\ &= C_n(T) \int \exp\left\{ \frac{c^{n-1}}{T} (x^2 - u^2) \right\} p_{n-1}(x - u, T) p_{n-1}(x + u, T) du, \end{aligned}$$

where the norming constant $C_n(T)$ is defined in such a way that $p_n(x, T)$ is a density function. On the other hand,

$$(4.2') \quad p_0(x, T) = p_0(x) = \frac{d\nu}{dx}.$$

In order to prove (4.2) write

$$p_n(x, T) = C_n \int \delta\left(2^{-n} \sum_{i \in V_n} \sigma(i) - x\right) \exp\left\{-\frac{1}{T} H_n(\sigma)\right\} \prod_{i \in V_n} p_0(\sigma(i)) d\sigma(i),$$

where $\delta(2^{-n} \sum_{i \in V_n} \sigma(i) - x)$ denotes integration on the hyperplane

$$2^{-n} \sum_{i \in V_n} \sigma(i) = x$$

with respect to Lebesgue measure. Then formula (4.1) implies that

$$\begin{aligned} p_n(x, T) &= C_n \iint \delta(u + v - 2x) \exp\left(\frac{1}{T} 2^{(n-1)(2-a)} uv\right) \\ &\quad \times \left[\int \delta\left(2^{-(n-1)} \sum_{i \in V_{n-1}} \sigma(i) - u\right) \right. \\ &\quad \quad \times \exp\left(-\frac{1}{T} H_{n-1}(\sigma_1)\right) \prod_{i \in V_{n-1}} p_0(\sigma(i)) d\sigma(i) \left. \right] \\ &\quad \times \left[\int \delta\left(2^{-(n-1)} \sum_{i \in V_n - V_{n-1}} \sigma(i) - v\right) \right. \\ &\quad \quad \times \exp\left(-\frac{1}{T} H_{n-1}(\sigma_2)\right) \prod_{i \in V_n - V_{n-1}} p_0(\sigma(i)) d\sigma(i) \left. \right] du dv \\ &= C'_n \iint \delta(u + v - 2x) \exp\left(\frac{c^{n-1}}{T} uv\right) p_{n-1}(u, T) p_{n-1}(v, T) du dv. \end{aligned}$$

This relation implies (4.2). Thus the investigation of the distribution of the average of the $\mu_n(d\sigma|T, \nu)$ spins leads to the question about the behavior of the function p_n defined by (4.2) and (4.2'). As it will be seen the cases $\sqrt{2} < c < 2$ and $1 < c < \sqrt{2}$ are essentially different. First, we formulate the following result (see [10] and [37]):

THEOREM 4.1. *For $\sqrt{2} < c < 2$ there exists a $T_{cr} = T_{cr}(u)$, $T_{cr}(u) = 2/(2 - c) + O(u)$ as $u \rightarrow 0$, in Dyson's hierarchical ϕ^4 model [i.e., in the model with the free measure defined in (3.1)] such that $c^{-n/2} p_n(c^{-n/2} x, T_{cr})$ tends to the Gaussian density function*

$$p(x) = p(x, T_{cr}) = \frac{1}{\sqrt{(2 - c)\pi T_{cr}}} \exp\left(-\frac{x^2}{(2 - c)T_{cr}}\right), \quad \text{as } n \rightarrow \infty.$$

We remark that in [10] a slightly different result is proved. In that paper such a model is considered where the spins take the values ± 1 , the Hamiltonian function satisfies relation (4.1) for $n \geq n_0$ with some n_0 , the free measure ν is $\nu\{+1\} = \nu\{-1\} = \frac{1}{2}$ and H_{n_0} is defined in such a way that $p_{n_0}(x, T)$ is very close to a Gaussian density. The same proof works in both cases. What we really need is that the recursive relation (4.2) holds and the starting function is near to a Gaussian density. The question of what happens in the case of a general starting function is an open question.

We explain the main idea of the proof. First, we want to get rid of the dependence on n in the operator S_n . To this end we make the rescaling

$$P_n(x, T) = p_n\left(\sqrt{\frac{2-c}{4-c}} T c^{-n/2} x, T\right)$$

and rewrite (4.2) for the function P_n . We get

$$P_n(x, T) = C_n \int \exp\left\{\frac{2-c}{4-c} \left(\frac{x^2}{c} - u^2\right)\right\} P_{n-1}\left(\frac{x}{\sqrt{c}} - u, T\right) P_{n-1}\left(\frac{x}{\sqrt{c}} + u, T\right) du.$$

The function $\text{const} \exp(-x^2/(4-c))$ is a fixed point of the above iteration. Let us introduce the function $Q_n(x, T) = \exp(x^2/(4-c)) P_n(x, T)$. We get

$$(4.3) \quad Q_n(x, T) = \text{const. } S Q_{n-1}(x, T),$$

where the operator S is defined by the formula

$$(4.4) \quad S f(x) = S_c f(x) = \frac{1}{\sqrt{\pi}} \int e^{-u^2} f\left(\frac{x}{\sqrt{c}} - u\right) f\left(\frac{x}{\sqrt{c}} + u\right) du.$$

On the other hand, (3.1) and (4.2') and the definition of the functions P and Q yield

$$(4.5) \quad Q_0(x, T) = \exp\left\{\left(1 - \frac{2-c}{2} T\right) \frac{x^2}{4-c} - \frac{u}{4} \left(\frac{2-c}{4-c}\right)^2 T^2 x^4\right\}, \quad T > 0,$$

and

$$(4.6) \quad p_n(c^{-n/2} x, T) = \exp\left(-\frac{x^2}{(2-c)T}\right) Q_n\left(\sqrt{\frac{4-c}{(2-c)T}} x, T\right).$$

The constant in (4.3) is chosen in such a way that $p_n(x)$ is a density function. Observe that the function $t(x) \equiv 1$ is a fixed point of the operator S , and $Q_0(x, T)$ is close to 1 if $T = 2/(2-c)$. Now to prove Theorem 4.1 one has to show that $C_n Q_n(x, T) = C_n S^n Q_0(x, T)$ tends to 1 with an appropriate choice of C_n for some $T = T_{cr} = 2/(2-c) + O(u)$. To prove the above statement we have to investigate the stability of the operator S around its fixed point $t(x) \equiv 1$. To this end we consider the differential $D_1 S$, $D_1 S f(x) = (\partial/\partial \epsilon) S(1 + \epsilon f(x))|_{\epsilon=0}$, at this fixed point. We get

$$D_1 S f(x) = \frac{2}{\sqrt{\pi}} \int e^{-u^2} f\left(\frac{x}{\sqrt{c}} + u\right) du.$$

This is a Gaussian integral operator (see [2]). It is known that D_1S is a self-adjoint operator in the Hilbert space $L_2(R^1, \exp(-[(c - 1)/c]x^2) dx)$, and the Hermite polynomials $h_k(x) = h_k(x, (c - 1)/c)$ with weight function $\exp(-[(c - 1)/c]x^2)$ are its eigenvectors with eigenvalues $2c^{-k/2}$. The functions h_k form a complete orthogonal system in this Hilbert space. Since all functions $Q_n(x, T)$ are even functions in their first coordinate we can work in the subspace $L_{2, \text{ev}}(R^1, \exp(-[(c - 1)/c]x^2) dx)$. The eigenvectors $h_0 \equiv 1, h_2, h_4, \dots$ have eigenvalues $2, 2/c, 2/c^2, \dots$. It is natural to expect that the operator S behaves similarly to its linearization D_1S in a small neighborhood of the function $t(x) \equiv 1$. If $\sqrt{2} < c < 2$, then the operator D_1S has only one eigenvector, the function h_2 , whose eigenvalue $2/c$ is larger than 1. [The function $h_0(x) \equiv 1$ with the eigenvalue 2 is not interesting to us, because, as it turns out from a more detailed investigation, it influences only the norming constant in (4.3).] Since D_1S has an unstable direction we cannot expect that $C_n S^n Q_0(x, T)$ is convergent for a general T . But we may expect that there is a parameter $T = T_{\text{cr}}$ such that $C_n S^n Q_0(x, T_{\text{cr}})$ tends to 1, because, roughly speaking, the effects in the unstable direction h_2 balance each other for $T = T_{\text{cr}}$. The proof is actually a justification of this conjecture. Put

$$(4.7) \quad C_n Q_n(x, T) - 1 = \sum_{k=1}^{\infty} c_k^n(T) h_{2k}(x).$$

Then

$$D_1S[C_n Q_n(x, T) - 1] = \sum_{k=1}^{\infty} \frac{2}{c^k} c_k^{(n)}(T) h_{2k}(x)$$

and

$$\begin{aligned} C_n Q_{n+1}(x, T) &= SC_n Q_n(x, T) \sim 1 + D_1S[C_n Q_n(x, T) - 1] \\ &= 1 + \sum_{k=1}^{\infty} \frac{2}{c^k} c_k^{(n)}(T) h_{2k}(x). \end{aligned}$$

Hence

$$C_{n+1} Q_{n+1}(x, T) = 1 + \sum_{k=1}^{\infty} c_k^{(n+1)}(T) h_{2k}(x), \quad c_k^{(n+1)}(T) \sim \frac{2}{c^k} c_k^{(n)}(T).$$

The above argument can be carried out if the coefficients $c_k^{(n)}(T)$ are sufficiently small, and the substitution of S by its linearization D_1S causes a negligible error. In this case the coefficients $c_k^{(n)}, k \neq 1$, are exponentially decreasing. If $u > 0$ is sufficiently small, then (4.7) holds for $n = 0$ in a small interval $[T_0^{(1)}, T_0^{(2)}]$, $T_0^{(1)} < T_0 < T_0^{(2)}, T_0 = 2/(2 - c)$ and $c_1^{(0)}(T_0^{(1)}) < 0, c_1^{(0)}(T_0^{(2)}) > 0$. Then a sequence of decreasing intervals $[T_n^{(1)}, T_n^{(2)}]$ can be found in such a way that for all n the relation (4.7) holds if $T \in [T_n^{(1)}, T_n^{(2)}]$. Moreover, it can be guaranteed that $T_n^{(2)} - T_n^{(1)} \rightarrow 0, c_1^{(n)}(T_n^{(1)}) < 0, c_1^{(n)}(T_n^{(2)}) > 0$ and $\sup_{T \in [T_n^{(1)}, T_n^{(2)}]} |c_1^{(n)}(T)|$ tends to zero exponentially fast. Then we get, by defining $T = T_{\text{cr}}$ as the intersection of the intervals $[T_n^{(1)}, T_n^{(2)}]$, that $C_n S^n Q_0(x, T_{\text{cr}}) \rightarrow 1$. The proof of Theorem 4.1 can be obtained by working out the details of the above argument. In the proof we

have to overcome some technical difficulties which are connected with the following two facts: (a) We have introduced the auxiliary Hilbert space $L_2(R^1, \exp(-[(c - 1)/c]x^2) dx)$, and investigated the contraction properties of the operator S in it. But actually we need some estimates about the operator S in the supremum norm. (b) In order to carry out the above estimations we need some control about the behavior of the function $Q_n(x, T)$ at infinity; which guarantees that its effect is negligible. These difficulties can be overcome by some arguments from the elementary analysis, but we omit the details.

In the case $c < \sqrt{2}$ both the eigenfunctions h_2 and h_4 have an eigenvalue larger than 1; hence the fixed point $t(x) \equiv 1$ of the operator S is unstable, and we cannot expect that Theorem 1 holds in this case. Hence we need to look for another fixed point g of the operator S defined in (4.4) and consider the differential $D_g S$ of the operator S at this point g . We have

$$(4.8) \quad D_g S f(x) = \frac{2}{\sqrt{\pi}} \int e^{-u^2} f\left(\frac{x}{\sqrt{c}} + u\right) g\left(\frac{x}{\sqrt{c}} - u\right) du.$$

The following result holds true (see [11], [14] and [26]):

THEOREM 4.2. *There is some $\varepsilon > 0$ such that for $\sqrt{2} - \varepsilon < c < \sqrt{2}$ the operator S_c defined in (4.4) has a fixed point $g(x) = g(x, c)$, $g(x) > 0$, for all $x \in R^1$, which is different from the function $t(x) = 1$. The operator $D_g S$ has only one eigenvector with an eigenvalue larger than 1 (besides the eigenvector g with the eigenvalue 2 which is not interesting to us).*

Let us introduce the function

$$p(x, T) = K \exp\left(-\frac{x^2}{(2 - c)T}\right) g\left(\sqrt{\frac{2 - c}{4 - c}} T x, c\right),$$

where the constant K is chosen in such a way that p is a density function. Now we formulate:

THEOREM 4.3. *If $\sqrt{2} - \varepsilon < c < \sqrt{2}$ and the density of the free measure ν is near to the function $p(x, T_0)$ with some $T_0 > 0$, then there exists some critical temperature T_{cr} in the vicinity of T_0 such that $c^{-n/2} p_n(c^{-n/2} x, T_{cr})$ tends to $p(x, T_{cr})$ as $n \rightarrow \infty$.*

Theorem 4.3 can be proved with the help of Theorem 4.2 similarly to Theorem 4.1. We omit the details.

In order to prove Theorem 4.2 first we observe that for $c = \sqrt{2}$ the function $h_4(x, 1 - 1/\sqrt{2})$ is an eigenvector of the operator $D_1 S$ with the eigenvalue 1. Hence the finite-dimensional bifurcation theory suggests that for $c = \sqrt{2} - \varepsilon$, $0 < \varepsilon < \varepsilon_0$, with some small $\varepsilon_0 > 0$, the operator S_c has a new fixed point $g(\tilde{x}) = 1 - C\varepsilon h_4(x, 1 - 1/\sqrt{2}) + O(\varepsilon^2)$ in addition to the old one $t(x) \equiv 1$. We want to find this new fixed point, and to show that it satisfies Theorem 4.2. Since we have to work in an infinite-dimensional space, the results of the finite-dimensional theory cannot directly be applied. First, we apply the so-called ε -expansion

sion, i.e., we look for the fixed point g in the form

$$g(x) = \sum_{k=0}^{\infty} a_k(\varepsilon) h_{2k}(x),$$

$$a_j(\varepsilon) = \sum_{k=2}^{\infty} a_{j,k} \varepsilon^k, \quad j \neq 0, 2,$$

$$a_0(\varepsilon) = 1 + \sum_{k=2}^{\infty} a_{0,k} \varepsilon^k, \quad a_2(\varepsilon) = \sum_{k=1}^{\infty} a_{1,k} \varepsilon^k,$$

where h_j denotes the j th Hermite polynomial with weight function $\exp(-[(c-1)/c]x^2)$ and leading coefficient 1. Then the equation $S_\varepsilon g = g$ leads to the system of equations

$$a_k(\varepsilon) = \frac{1}{(h_{2k}; h_{2k})} (g; h_{2k}) = \frac{1}{(h_{2k}; h_{2k})} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} C_{k,m,p}(\varepsilon) a_m(\varepsilon) a_p(\varepsilon),$$

with

$$C_{k,m,p}(\varepsilon) = \int e^{-[(c-1)/c]u^2} h_{2k}(u) \frac{1}{\sqrt{\pi}} \int e^{-v^2} h_{2m}\left(\frac{u}{\sqrt{c}} + v\right) h_{2p}\left(\frac{u}{\sqrt{c}} - v\right) du dv,$$

where $(\cdot; \cdot)$ denotes scalar product in the space $L_2(R^1, \exp(-[(c-1)/c]x^2) dx)$. Since the terms $C_{k,m,p}(\varepsilon)$ can be calculated explicitly, the above relation makes it possible to calculate the coefficients $a_{j,k}$ recursively. In this way we can represent the function $g(x)$ in the form of a formal power series. Unfortunately, this series is divergent. But by preserving only the terms $a_{j,k} \varepsilon^k$, $k \leq k_0$, with some k_0 , we get an almost fixed point of S_ε , i.e., a function $\bar{g}(x) = \bar{g}_{k_0}(x, c)$ such that $\bar{g}(x) - S\bar{g}(x) = O(\varepsilon^{k_0})$ for $|x| < \text{const.}(\ln(1/\varepsilon))^{1/2}$. The power k_0 can be chosen arbitrarily large. Having this almost fixed point \bar{g} we can find a real fixed point g by applying a procedure similar to the proof of Theorem 4.1. First, we need some information about the spectrum of the operator $D_{\bar{g}}S$. Since $\bar{g} \sim 1$ for small ε , the eigenvectors Φ_k and the eigenvalues λ_k of $D_{\bar{g}}S$ are close to those of D_1S , i.e., to h_{2k} and $2c^{-k}$. Moreover, by applying an ε -expansion for the eigenvector Φ_2 and the eigenvalue λ_2 we get that $\lambda_2 = -C\varepsilon + O(\varepsilon^2)$ with some $C > 0$, therefore $\lambda_2 < 1$. Since the operator $D_{\bar{g}}S$ has two eigenvectors $\Phi_0 = \bar{g}$ and Φ_1 with eigenvalues larger than 1 in the space $L_{2, \text{ev}}(R^1, \exp(-[(c-1)/c]x^2))$, we start our iteration from the class of functions $g_0(x, s, t) = \bar{g}(x) + s\Phi_0(x) + t\Phi_1(x)$ and consider the iteration $g_n(x, s, t) = S^n g_0(x, s, t)$. It can be proved by refining the argument of Theorem 4.1 that there is a pair (\bar{s}, \bar{t}) such that the limit $g(x) = \lim_{n \rightarrow \infty} g_n(x, \bar{s}, \bar{t})$ exists. Moreover, $D_{\bar{g}}S$ has beside the function g only one eigenvector with an eigenvalue larger than 1. The precise proof depends heavily on some proofs and ideas of perturbation theory. Some technical difficulties arise because we have to work also in the spaces $C(-\infty, \infty)$ and $C^1(-\infty, \infty)$ where the results of perturbation theory, worked out mainly for Hilbert spaces, cannot be applied. We omit the details.

The proof of Theorem 4.2 works only for small ε . On the other hand, some computer calculations [11] suggest that it must hold for all $0 < \varepsilon < \sqrt{2} - 1$. No

rigorous proof is known. It seems very likely that the proof of Theorem 4.2 for all $0 < \varepsilon < \sqrt{2} - 1$ demands some computer work. First, one looks for an approximate solution of the equation $S_c g = g$ and investigates the spectrum of $D_g S$. Then an argument similar to that given above may supply the proof.

The case $c = \sqrt{2}$ deserves special attention. In this case the differential operator $D_1 S$ has an eigenvector h_4 with the eigenvalue 1. Hence in this case the linearized operator $D_1 S$ is not a sufficiently good approximation of the operator S , and we have to consider also a second-term approximation. It can be decided only with the help of this second-term approximation whether the Gaussian solution is stable or not. A refined analysis shows that it is stable, and the following result holds true (see [4] and [25]):

THEOREM 4.4. *For $c = \sqrt{2}$ in Dyson's hierarchical ϕ^4 model there is a $T_{cr} = T_{cr}(u)$, $T_{cr}(u) = 2/(2 - c) + O(u)$ as $u \rightarrow 0$, such that $c^{-n/2} p_n(c^{-n/2} x, T_{cr})$ tends to the Gaussian density function $p(x, T_{cr})$ defined in Theorem 4.1.*

In this case the speed of convergence in the relation $p_n \rightarrow p$ is very slow. Dyson's hierarchical model with the parameter $c = \sqrt{2}$ is especially important, because, as we shall see, it corresponds in some sense to the four-dimensional translation invariant equilibrium states.

5. Critical exponents, large deviation results. In this section we consider the critical exponents for Gibbs states without boundary conditions. Let $p_n(x, T)$ denote the density function of the average of the $\mu_n(d\sigma|T, \nu)$ distributed spins. The critical exponent α is defined as the number for which $2^{-n\alpha} \sum_{i \in V_n} \sigma(i)$ has a nontrivial limit for $n \rightarrow \infty$ if the spins $\sigma(i)$ are $\mu_n(d\sigma|T_{cr}, \nu)$ distributed. In Section 4 it was shown that such an α exists, and $\alpha = 1 - \frac{1}{2} \log_2 c$. The critical exponents β and γ are defined in the following way: It can be shown that $2^{-n/2} p_n(2^{-n/2} x, T)$ is asymptotically Gaussian distributed with some variance $\sigma(T) > 0$ if $T > T_{cr}$. Moreover, $\sigma(T) \sim \text{const.} |T - T_{cr}|^{-\gamma}$ where γ does not depend on the parameter u . For $T < T_{cr}$ the density function $p_n(x, T)$ is asymptotically the mixture of two Gaussian density functions with some variance $2^{-n}\sigma(T)$, $\sigma(T) > 0$, and expectations $M_n(T)$ and $-M_n(T)$, respectively. We claim that $\lim_{n \rightarrow \infty} M_n(T) = M(T) > 0$, and $M(T) \sim \text{const.} |T - T_{cr}|^\beta$ as $T \rightarrow T_{cr}$ where β like the exponent γ does not depend on the parameter u . Moreover, $\sigma(T) \sim \text{const.} |T - T_{cr}|^{-\gamma}$ for small $T_{cr} - T$ also in the case $T < T_{cr}$.

First, we consider the case $\sqrt{2} < c < 2$. We discuss the ϕ^4 model where the measure ν is defined in (3.1). We formulate an asymptotic formula for the function $p_n(x, T)$ for all x and T . Such a result is called a large deviation result. We show that the critical exponents can be determined by its help. To formulate this result we first introduce some notation. The function

$$(5.1) \quad \Phi(x, T) = \frac{u}{4} x^4 + \frac{T - a_0}{2T} x^2 + R(x, T),$$

$$x \in R^1, \quad T > 0, \quad a_0 = \frac{2}{2 - c}$$

belongs to the class of functions $S_u = S_u(c)$ if the function $R(x, T)$ satisfies the following properties:

- (i) $R(x, T)$ is an even function in its first coordinate;
- (ii) the derivatives $\partial^{i+j}R/\partial x^i \partial T^j$ exist and are continuous in $x \in \mathbb{R}^1, T > 0$, if $i + 2j \leq 4, j \leq 1$;
- (iii) the estimates

$$\left| \frac{\partial^4 R}{\partial x^4} \right| \leq Cu^2, \quad \left| \frac{\partial^3 R}{\partial x^3} \right| \leq Cu^2|x|, \quad \left| \frac{\partial^3 R}{\partial x^2 \partial T} \right| \leq Cu \frac{1}{T^2},$$

$$\left| \frac{\partial^2 R}{\partial x^2} \right| \leq Cu, \quad \left| \frac{\partial^2 R}{\partial x \partial T} \right| \leq Cu|x| \frac{1}{T^2}, \quad \left| \frac{\partial R}{\partial x} \right| \leq Cu|x|,$$

$R(0, T) = 0$ hold in the half-plane $(x, T) \in \mathbb{R}^1 \times (0, \infty)$.

The class S_u consists of such functions which are small and smooth perturbations of the function $\Phi_0(x, T) = (u/4)x^4 + [(T - a_0)/2T]x^2$. Given a function $\Phi \in S_u$ we define its critical parameter T_{cr} as the (unique) solution of the equation $\partial^2 \Phi(0, T_{cr})/\partial x^2 = 0$. This equation has a unique solution, which can be seen from the relation $\partial^3 \Phi(0, T)/\partial x^2 \partial T > 0$ for all $T > 0$. Some more analysis shows that $T_{cr} = a_0 + O(u)$ as $u \rightarrow 0$. For $T < T_{cr}$ the spontaneous magnetization corresponding to $\Phi \in S_u$ is defined as the (unique) positive solution (in x) of the equation $\partial \Phi(x, T)/\partial x = 0$. This equation has a unique positive solution for $T < T_{cr}$ and has no solution except the trivial one $x = 0$ for $T \geq T_{cr}$. Indeed, $\partial^2 \Phi(0, T)/\partial x^2 > 0$ for $T > T_{cr}$, and $\partial^2 \Phi(0, T)/\partial x^2 < 0$ for $T < T_{cr}$. Since $\partial^3 \Phi(x, T)/\partial x^3 > 0$, and it tends to infinity as $x \rightarrow \infty$ for all fixed T , $\partial \Phi(x, T)/\partial x$ is a strictly convex function tending to infinity as $x \rightarrow \infty$. This relation together with the fact that $\partial \Phi(0, T)/\partial x = 0$ imply that $\partial \Phi(x, T)/\partial x = 0$ has no positive solution in x if $\partial^2 \Phi(0, T)/\partial x^2 > 0$, and it has exactly one positive solution if $\partial^2 \Phi(0, T)/\partial x^2 < 0$. Some refinement of the above argument shows that $M(T) \rightarrow 0$ as $T \nearrow T_{cr}$. Now we formulate the following:

THEOREM 5.1. *For all $\epsilon > 0$ and $\sqrt{2} < c < 2$ there is a $u_0 > 0$ such that in Dyson's hierarchical ϕ^4 model with $0 < u < u_0$ the density function $p_n(x, T)$ of the average spin in the volume V_n satisfies the following relation: There is a function $\Phi \in S_u(c)$ such that in the domain*

$$U_n = \left\{ (x, T), |x| \geq \left(1 - \frac{1}{100} \left(\frac{c}{2} \right)^{n/2} \right) M(T) \text{ if } T < T_{cr} \right\} \cup \{ (x, T), T \geq T_{cr} \},$$

the relation

$$(5.2) \quad p_n(x, T) = C_n(T) \Psi_n(x, T) \times \exp \left\{ -2^n \Phi(x, T) - \frac{a_0 c^n}{2T} x^2 + O(2^{n(a-3/2+\epsilon)}) \right\}$$

holds with

$$\Psi_n(x, T) = \prod_{j=0}^{\infty} \left[\frac{a_1}{T} + \left(\frac{2}{c} \right)^{n+j} \frac{\partial^2 \Phi(x, T)}{\partial x^2} \right]^{2^{-j-2}},$$

with $a_0 = 2/(2 - c)$ and $a_1 = a_0 + 1$. (The parameter a was defined by the relation $2^{2-a} = c$.)

Moreover,

$$(5.3) \quad p_n(x, T) \leq (1 + O(\xi^n)) p_n \left(1 - 0.01 \left(\frac{c}{2} \right)^{n/2} M(T), T \right),$$

with some $0 < \xi < 1$ if $(x, T) \in R \times (0, \infty) \setminus U_n$.

The proof of Theorem 5.1 is similar to that of Theorem 4.1, but a much more refined analysis is needed. A local limit theorem can be proved for the function $p_n(x, T)$ in a small neighborhood of all points $(x, T) \in U_n$ [i.e., a local expansion of the function $p_n(x, T)$ can be given around (x, T)], and not only in a neighborhood of $(x, T) = (0, T_{cr})$ as in Theorem 4.1. Then Theorem 5.1 can be proved by the help of these formulas, “by sticking these local expansions together.” The starting function $p_0(x, T)$ can be written in the form $p_0(x, T) = \exp\{-\Phi_0(x, T) - (a_0/2T)x^2\}$. Then an appropriate linearization of the operator S_n defined in (4.2) suggests that formula (5.2) holds with some function Φ_n which is a small perturbation of Φ_0 , but a pre-exponential term Ψ_n also appears in this formula. (The linearization of S_n does not change the function Φ_0 , but a modification of Φ_0 appears when we work with the real operator S_n instead of its linearization.) Then a careful analysis shows that Φ_n tends to some $\Phi \in S_u$ as $n \rightarrow \infty$, and the convergence is sufficiently fast to write formula (5.2) with this function $\Phi \in S_u$. In this way Theorem 5.1 can be proved, but we cannot give an explicit formula for $\Phi \in S_u$ in (5.2). The proof of Theorem 5.1 is given in [8], and Section 3 therein contains the linearization procedure we have mentioned above.

In the domain $(x, T) \in R \times (0, \infty) \setminus U_n$ we gave only an upper bound for $p_n(x, T)$ instead of a good asymptotic formula. But even this upper bound is sufficient for our purposes. Actually formula (5.2) is not valid if $(x, T) \in R \times (0, \infty) \setminus U_n$. The reason for it is the following: We can prove formula (5.2) only in the domain of such points (x, T) where the following localization property holds: The integral defining $p_n(x, T)$ which is given on the right-hand side of formula (4.1) is essentially localized in a small neighborhood of the point x . This localization property holds if $(x, T) \in U_n$, but it is violated if $(x, T) \notin U_n$. An asymptotic formula for $p_n(x, T)$, $(x, T) \notin U_n$, can also be proved, by determining which domain gives the main contribution in the integral (4.2). This formula is completely different from (5.2). We shall not discuss this question here, although it is an interesting problem. It is closely related to the phase separation phenomenon (see [7]).

Theorem 5.1 implies Theorem 4.1, and what is more important for us, the critical exponents can be calculated with its help. This can be done by making a local expansion in formula (5.2). In the next section we shall see that Theorem

5.1 is needed also in the investigation of equilibrium states. Some analysis shows that the pre-exponential term $\Psi_n(x, T)$ changes very little in a small neighborhood of a point (x, T) , therefore it can be neglected in the following expansions. For $T = T_{cr}$ we get, by taking a Taylor expansion inside the exponent of formula (5.2),

$$p_n(x, T_{cr}) = p_n(0, T_{cr}) \exp \left\{ - \frac{\alpha_0}{2T_{cr}} c^n x^2 + O(\xi^n) \right\},$$

for $|x| < c^{-n/2} \xi^{-n}$ with some $\xi < 1$. This formula together with the fact that the probability measure with the density function $p_n(x, T)$ is essentially concentrated in the domain $|x| < c^{-n/2} \xi^{-n}$ imply Theorem 4.1. The latter statement holds since $\Phi(x, T_{cr})$ is an even function decreasing for $x \geq 0$. This follows from the relations

$$\begin{aligned} \frac{\partial \Phi(0, T_{cr})}{\partial x} &= \frac{\partial^2 \Phi(0, T_{cr})}{\partial x^2} = 0, \\ \frac{\partial^3 \Phi(x, T_{cr})}{\partial x^3} &> 0, \quad \text{for } x > 0. \end{aligned}$$

If $T > T_{cr}$, then we can write

$$2^n \Phi(x, T) + \frac{\alpha_0}{2T} c^n x^2 = 2^{n-1} (Ax^2 + O(\xi^n)),$$

with $A = \partial^2 \Phi(0, T) / \partial x^2 > 0$ for $|x| < 2^{-n/2} \xi^{-n}$. Some calculation shows that the probability measure with density $p_n(x, T)$ is essentially concentrated in the domain $|x| < 2^{-n/2} \xi^{-n}$. Hence the above expansion together with formula (5.2) imply that the densities $2^{-n/2} p_n(2^{-n/2} x, T)$ tend to a Gaussian density function with expectation zero and variance $\sigma(T) = (\partial^2 \Phi(0, T) / \partial x^2)^{-1} > 0$. Moreover, since $\sigma^2(T) \sim (\partial^3 \Phi(0, T_{cr}) / \partial x^2 \partial T)^{-1} (T - T_{cr})^{-1}$, $\partial^3 \Phi(0, T_{cr}) / \partial x^2 \partial T > 0$, the critical exponent γ exists, and $\gamma = 1$. To investigate the case $T < T_{cr}$ first we need some information about the behavior of the function $M(T)$ in the vicinity of T_{cr} . Since $M(T) \rightarrow 0$ as $T \rightarrow T_{cr}$, a Taylor expansion yields

$$\begin{aligned} 0 &= \frac{\partial \Phi(M(T), T)}{\partial x} - \frac{\partial \Phi(0, T_{cr})}{\partial x} \\ (5.4) \quad &= \frac{1}{6} \frac{\partial^4 \Phi(0, T_{cr})}{\partial x^4} M(T)^3 + \frac{\partial^3 \Phi(0, T_{cr})}{\partial x^2 \partial T} (T - T_{cr}) M(T) \\ &\quad + o(M(T)^3) + o((T - T_{cr}) M(T)). \end{aligned}$$

Here we have exploited that besides $\partial^2 \Phi(0, T_{cr}) / \partial x^2 = 0$, the relations

$$\frac{\partial \Phi(0, T_{cr})}{\partial x} = \frac{\partial^3 \Phi(0, T_{cr})}{\partial x^2} = \frac{\partial^2 \Phi(0, T_{cr})}{\partial x \partial T} = 0$$

also hold by the evenness of the function $\Phi(x, T)$ in the variable x . Relation (5.4)

implies that $M(T) \sim A(u)(T_{cr} - T)^{1/2}$ with

$$A(u) = \left(6 \frac{\partial^3 \Phi(0, T_{cr})}{\partial x^2 \partial T} \right) \left(\frac{\partial^4 \Phi(0, T_{cr})}{\partial x^4} \right)^{-1/2} = (a_0 u (1 + O(u^{1/2})))^{-1/2}.$$

Let $M_n(T)$ denote the positive place of maximum the function $\Phi(x, T) + a_0/2T(c/2)^n x^2$, i.e., let it be the positive solution (in the variable x) of the equation $\partial \Phi(x, T)/\partial x = -(c/2)^n a_0 x/T$. Then $M_n(T) \rightarrow M(T)$ as $n \rightarrow \infty$. Now it can be seen similarly to the case $T > T_{cr}$ that $p_n(x, T)$ is asymptotically the mixture of two Gaussian densities with expectations $M_n(T)$ and $-M_n(T)$, respectively and variance $2^{-n}(\partial^2 \Phi(M(T), T)/\partial x^2)^{-1}$, i.e., the measure with the density function $p_n(x, T)$ is concentrated in the domain $|x \pm M_n(T)| < 2^{-n/2} \xi^{-n}$, and the functions $2^{-n/2} p_n(2^{-n/2}(x \pm M_n(T)), T)$ tend to $\frac{1}{2} \times$ the Gaussian density with expectation zero and variance $[\partial^2 \Phi(M(T), T)/\partial x^2]^{-1}$. Now since $M(T) \sim \text{const.}(T_{cr} - T)^{1/2}$ the critical exponent β exists and $\beta = \frac{1}{2}$. On the other hand, we get from the Taylor expansion and the relation (5.3)

$$\begin{aligned} \frac{\partial^2 \Phi(M(T), T)}{\partial x^2} &\sim \frac{\partial^4 \Phi(0, T_{cr})}{\partial x^4} \frac{M(T)^2}{2} + \frac{\partial^3 \Phi(0, T_{cr})}{\partial x^2 \partial T} (T - T_{cr}) \\ &\sim 2 \frac{\partial^3 \Phi(0, T_{cr})}{\partial x^2 \partial T} (T_{cr} - T). \end{aligned}$$

Hence $\sigma(T) \sim \text{const.}|T - T_{cr}|^{-1}$ also in the case $T < T_{cr}$ as we claimed.

Now we turn to the case $\sqrt{2} - \epsilon < c < \sqrt{2}$. A large deviation result for this case similar to Theorem 5.1 would enable us to determine the critical exponents in this case too. Unfortunately, we are unable to prove such a result; we can describe the behavior of $p_n(x, T)$ only in a relatively small neighborhood of the point $(0, T_{cr})$. But even this weaker result is sufficient for us to determine the critical exponents. Before giving the value of the critical exponents we recall that the operator $D_g S$ defined in (4.8) with the function g constructed in Theorem 4.2 has two eigenvalues larger than 1, $\lambda_0 = 2$, and $\lambda_1 = \sqrt{2} + O(\epsilon)$. Now we formulate the following:

THEOREM 5.2. *Under the conditions of Theorem 4.3 for $T > T_{cr}$, $2^{-n/2} p_n(2^{-n/2} x, T)$ is asymptotically Gaussian with some positive variance $\sigma(T)$. For $T < T_{cr}$, $p_n(x, T)$ is asymptotically the mixture of two Gaussian densities with variance $2^{-n} \sigma(T)$, $\sigma(T) > 0$, and expectations $M(T)$ and $-M(T)$, respectively. The critical exponents β and γ exist, and $\beta = \frac{1}{2} \log_{\lambda_1} c$, $\gamma = \log_{\lambda_1}(2/c)$. Here λ_1 denotes the second largest eigenvalue of the operator $D_g S$ defined in (4.8) with the function g appearing in Theorem 4.2.*

We do not prove Theorem 5.2, but present only a rough heuristic argument, which may, however, explain why such critical exponents β and γ appear in this model. The formal proof is given in [11] and [14].

We need a good asymptotic formula for $p_n(x, T)$ in the case when $T - T_{cr}$ is small. Let us define the functions $P_n(x, T)$ and $Q_n(x, T)$ in the same way as in

the proof of Theorem 4.1. Then $Q_n(x, T) = \text{const. } S^n Q_0(x, T)$ and $\text{const. } Q_n(x, T_{\text{cr}}) \rightarrow g(x)$, where the operator S is defined in (4.4), and the function g appeared in Theorem 4.2. The function p_n can be expressed by Q_n with the help of relation (4.6). Let Φ_0, Φ_1, \dots denote the eigenvectors of the operator $D_g S$ with the corresponding eigenvalues $\lambda_0, \lambda_1, \dots$, where Φ_k is a small perturbation of the Hermite polynomial $H_{2k}(x)$. For $n = 0$ and small $T - T_{\text{cr}}$ we can write

$$\frac{Q_0(x, T)}{Q_0(x, T_{\text{cr}})} = \frac{Q_0\left(x \frac{\sqrt{T}}{\sqrt{T_{\text{cr}}}}, T_{\text{cr}}\right)}{Q_0(x, T_{\text{cr}})} \sim 1 + (T - T_{\text{cr}})F(x),$$

with some function $F(x)$, and

$$(5.5) \quad Q_0(x, T) \sim (1 + c_0(T - T_{\text{cr}}))Q_0(x, T_{\text{cr}}) - (T - T_{\text{cr}}) \sum_{k=1}^{\infty} c_k \Phi_k(x).$$

In this expression we may achieve by appropriately choosing the coefficient of Q_0 that the term $c_0 \Phi_0(x)$ is absent in the last sum. Some analysis shows that the coefficient c_1 of $\Phi_1(x)$ is positive. In this analysis we have to use $Q_0(x, T) \sim g(x) \sim \exp(-\varepsilon H_4(x))$ and $\Phi_1(x) \sim H_2(x)$.

First, let us consider the case $T > T_{\text{cr}}$. If $T - T_{\text{cr}}$ is small and n is not too large, namely $(T - T_{\text{cr}})\lambda_1^n \ll 1$, then S can be replaced by its linearization, i.e., $SQ_{n-1}(x, T) \sim Q_n(x, T_{\text{cr}}) + D_g S[Q_{n-1}(x, T) - Q_{n-1}(x, T_{\text{cr}})]$. Hence we have in this case

$$(5.6) \quad Q_n(x, T) \sim C_n \left[Q_n(x, T_{\text{cr}}) - (T - T_{\text{cr}}) \sum_{k=1}^{\infty} c_k \lambda_k^n \Phi_k(x) \right].$$

Moreover, since $\lambda_1 > 1$ and $\lambda_k < 1$ for $k \geq 2$, only the first term is important in the sum of the right-hand side of (5.6). Relation (5.6) is valid if $\lambda_1^n (T - T_{\text{cr}}) < \delta$, where $\delta > 0$ is a small but fixed constant independent of T . Let $n_0 = n_0(T)$ be the largest integer such that $\lambda_1^{n_0} (T - T_{\text{cr}}) < \delta$. Let us observe that the coefficient $c_k \lambda_1^{n_0} (T - T_{\text{cr}})$ of $\Phi_1(x)$ in the expression (5.6) for $Q_{n_0}(x, T)$ is positive. Let $n_1 = n_0 + K$, where K is a fixed large constant. Then $Q_{n_1}(x, T)$ will be such that $Q_{n_1}(x, T) < \exp(-\lambda(n_1)x^2)Q_{n_1}(0, T)$ with some $\lambda(n_1)$ and $\lambda(n_1) > 0$ is large. Hence we make a small error if we omit the multiplicative term $\exp(-u^2)$ in the integral defining $SQ_{n_1}(x, T)$. This means that for $n = n_1$ we can replace the operator S by convolution. If we do this for all $n > n_1$, then the central limit theorem suggests that for $n \gg n_0$,

$$2^{-(n-n_0)/2} c^{-n_0/2} p_n(2^{-(n-n_0)/2} c^{-n_0/2} x, T)$$

tends to a Gaussian density with zero expectation and $\text{const. } \lambda_1^{n_0} (T - T_{\text{cr}}) = \text{constant variance}$. Hence $2^{-n/2} p_n(2^{-n/2} x, T)$ is normal with expectation zero and variance $\sigma(T) \sim \text{const.}(2/c)^{n_0}$. Since $(T - T_{\text{cr}})\lambda_1^{n_0} \sim \delta = \text{const.}$, therefore $\sigma(T) \sim \text{const.}(T - T_{\text{cr}})^{-\gamma}$ with $\gamma = \log_{\lambda_1}(2/c)$.

The case $T < T_{cr}$ can be similarly discussed. For not too large n , when $\lambda_1^n(T_{cr} - T) \ll 1$, relation (5.6) holds true. Then there is some $n_2 = n_2(T)$, $\lambda_1^{n_2}(T_{cr} - T) \sim \text{const.}$ such that the function $Q_{n_2}(x, T)$ has two maxima M_2 and $-M_2$ which are separated both from zero and infinity. It is natural to expect such a result, because $Q_{n_2}(x, T) \sim g(x) + C\Phi_1(x)$ with some $C > 0$, $g(x)$ is monotone decreasing and $\Phi_1(x) \sim H_2(x)$ is monotone increasing for $x > 0$. Hence if C is sufficiently large then Q_{n_2} has a nonnegative maximum at a point M . By the evenness of the function Q_{n_2} , $-M$ is also a place of maximum. After finitely many steps these two maxima become localized, and after it they develop independently. Hence we can expect that for large n , $p_n(c^{-n_2/2}x, T)$ is asymptotically the mixture of two Gaussian densities with expectations M and $-M$ and variance $\text{const.} \cdot 2^{-(N-n_2)}$. This means that $p_n(x, T)$ is asymptotically the mixture of the two Gaussian densities with expectations $M(T)$ and $-M(T)$, $M(T) \sim \text{const.} \cdot c^{-n_2/2} = \text{const.} \cdot (T_{cr} - T)^\beta$, $\beta = \frac{1}{2} \log_{\lambda_1} c$ and $\sigma_n(T) \sim \text{const.} \cdot 2^{-n} (T_{cr} - T)^{-\gamma}$.

6. Equilibrium states. In this section we discuss the existence and uniqueness properties of equilibrium states of Dyson’s hierarchical model, their large-scale limit and critical exponents. First, we formulate the following:

THEOREM 6.1. *Under the conditions of Theorem 4.1 or 4.3 the equilibrium states of Dyson’s hierarchical model exist for all $T > 0$. For $T \geq T_{cr}$ the equilibrium state $\mu = \mu(T)$ is unique. For $T < T_{cr}$ there is an equilibrium state $\mu^+ = \mu^+(T)$ so that the μ^+ distributed random field $\sigma(i)$, $i \in \mathbb{Z}$, satisfies the inequality $E\sigma(i) > 0$, the distribution $\mu^- = \mu^-(T)$ of the random field $-\sigma(i)$ is also an equilibrium state, and any equilibrium state is a convex linear combination of μ^+ and μ^- .*

Let us emphasize that the critical temperature T_{cr} in Theorem 6.1 agrees with that in Theorems 4.1 or 4.3. In the next theorem we consider the large-scale limit of $\mu(T)$ resp. $\mu^+(T)$ distributed random fields.

THEOREM 6.2. *Let us consider the random fields*

$$Y_n(k) = \frac{1}{A_n} \sum_{i \in V_{k,n}} [\sigma(i) - E\sigma(i)], \quad k \in \mathbb{Z},$$

for all $n = 1, 2, \dots$, where the random field $\sigma(i)$, $i \in \mathbb{Z}$, is $\mu(T)$ distributed for $T \geq T_{cr}$, is $\mu^+(T)$ distributed for $T < T_{cr}$ and A_n is an appropriate norming constant. Put $A_n = 2^{n/2}$ for $T \neq T_{cr}$ and $A_n = 2^n c^{-n/2}$ for $T = T_{cr}$. Then under the conditions of Theorem 4.1 or 4.3 the finite-dimensional distributions of the random fields $Y_n(k)$, $k \in \mathbb{Z}$, tend to those of a random field $Y^*(k)$, $k \in \mathbb{Z}$, as $n \rightarrow \infty$. For $T \neq T_{cr}$, Y_k^* , $k \in \mathbb{Z}$, are independent Gaussian random variables. For $T = T_{cr}$ their joint distribution is the (unique) equilibrium state with the

Hamiltonian function of Dyson's hierarchical model at $T = T_{cr}$ with the free measure $\nu(dx) = p(x, T_{cr}) dx$, where p is the same as in Theorem 4.1 or 4.3.

We say that the critical exponents β and γ of the equilibrium states exist if the $\mu(T)$ resp. $\mu^+(T)$ distributed random fields $\sigma(k)$, $k \in \mathbb{Z}$, satisfy the relation

$$\lim_{n \rightarrow \infty} E \left[2^{-n/2} \sum_{i \in V_n} \sigma(i) - E\sigma(i) \right]^2 \sim \text{const.} |T - T_{cr}|^{-\gamma}, \text{ if } |T - T_{cr}| \rightarrow 0$$

and $E\sigma(k) \sim \text{const.} |T - T_{cr}|^\beta$ if $T < T_{cr}$ and $T \rightarrow T_{cr}$.

THEOREM 6.3. *Under the conditions of Theorem 4.1 or 4.3 the critical exponents β and γ of the equilibrium states of Dyson's hierarchical model exist. For $\sqrt{2} < c < 2$ they are $\beta = \frac{1}{2}$, $\gamma = 1$ and for $\sqrt{2} - \varepsilon < c < \sqrt{2}$ the exponents β and γ are the same as in Theorem 5.2.*

The proof of the above results can be found in [3], [11] and [14]. We briefly discuss their main ideas. We concentrate mainly on the case $\sqrt{2} < c < 2$. The proofs we are sketching are considerably simpler than the original ones.

For all positive integers n let us fix some configurations $\sigma' = \sigma'_n = \{\sigma'_n(j), j \in \mathbb{Z} - V_n\}$ such that $h(\sigma'_n) = \sum_{j \in \mathbb{Z} - V_n} d(1, j)^{-\alpha} \sigma'_n(j) \rightarrow 0$. Let us consider the Gibbs states $\mu_n(d\sigma|T, \sigma') = \mu_{V_n}(d\sigma|T, \sigma', \nu)$ on R^{V_n} , where $\mu_{V_n}(d\sigma|T, \sigma', \nu)$ is defined by formulas (2.2) and (2.2') with the choice $\sigma' = \sigma'_n$ and the Hamiltonian H_{V_n} and the free measure ν defined in Dyson's hierarchical model. It is known from the general theory of phase transitions that any measure μ on $R^{\mathbb{Z}}$, which can be obtained as $\mu = \lim_{n_k \rightarrow \infty} \mu_{n_k}(d\sigma|T, \sigma'_{n_k})$ with some sequence $n_k \rightarrow \infty$ and boundary condition $\sigma'_{n_k}, h(\sigma'_{n_k}) \rightarrow 0$, is an equilibrium state. Here the lim is meant as the convergence of the finite-dimensional distributions, i.e., we demand that for all finite sets $V, V \subset \mathbb{Z}$, the projection of the measures to R^V is convergent. Moreover, all equilibrium states can be obtained as the convex linear combination of such measures μ which are constructed in the above way.

It follows from the definition of Dyson's hierarchical model that $\mu_n(d\sigma|T, \sigma') = \mu_n^{h_n}(d\sigma|T)$ with $h_n = \sum_{j \in \mathbb{Z} - V_n} d(1, j)^{-\alpha} \sigma'(j)$ where $\mu_n^{h_n}(d\sigma|T, \nu)$ is also defined by (2.2) and (2.2'), but $H_{V_n}(\sigma|\sigma')$ is substituted by $H_{V_n}^h(\sigma) = H_{V_n}(\sigma) - h \sum_{i \in V_n} \sigma(i)$ in it. Let us fix some integers $n, N, N \geq n \geq 0$, and define the measure $\mu_{n, N}^{h_n}(d\sigma|T)$ as the projection of the measure $\mu_N^{h_n}(d\sigma|T)$ to R^{V_n} . In order to prove our results we need a good asymptotic formula for the measures $\mu_{n, N}^{h_n}(d\sigma|T)$. It can be obtained with the help of the following lemma in which we express the Radon-Nikodym derivative $d\mu_{n, N}^{h_n}/d\mu_n$. Introduce the notation $\xi_m = 2^{-m} \sum_{i \in V_m} \sigma(i)$ and let γ_m denote the distribution of ξ_m if the vector $(\sigma(1), \dots, \sigma(2^m))$ is $\mu_m(d\sigma|T)$ distributed.

LEMMA 6.4.

$$d\mu_{n, N}^{h_n}(\sigma|T) = Lf_{n, N}^h(\xi_n) d\mu_n(\sigma|T),$$

where the function $f_{n,N}^h(x)$ is defined by the recursive relations

$$(6.1) \quad f_{N,N}^h(x) = \exp\left(\frac{2^N h}{T} x\right),$$

$$f_{m,N}^h(x) = T_m f_{m+1,N}^h(x) = \int \exp\left(\frac{c^m}{T} xy\right) f_{m+1,N}^h\left(\frac{x+y}{2}\right) \gamma_m(dy),$$

for $m = N - 1, \dots, 1$, and L is an appropriate norming constant.

Lemma 6.4 together with its proof can be found in [6] under the name “main formula.” The proof is relatively simple, but we omit it. Let us remark that the Radon–Nikodym derivative $d\mu_{n,N}^h/d\mu_n$ in a point $(\sigma(1), \dots, \sigma(2^n))$ depends only on the average $2^{-n} \sum_{i \in V_n} \sigma(i)$.

In the case $\sqrt{2} < c < 2$ Theorem 5.1 yields a good asymptotic formula for the density function $p_n(x, T)$ of the probability measure $\gamma_n(dx)$. Hence we can get a sufficiently good asymptotic formula for $f_{n,N}^h(x)$ with the help of formula (6.1). Although we have to estimate rather complicated integrals, we are able to do this because the integrands we have to work with are strongly localized around their maximum. In the next lemma we give a good asymptotic formula for $f_{n,N}^h(x)$. Actually we need a good estimate only in a so-called typical region where the average $2^{-n} \sum_{j=1}^{2^n} \sigma(j)$ of the μ_n^h distributed spins is concentrated. This average has the density function $p_n(x, T) f_{n,N}^h(x)$. Hence, what we really need, is a good estimate of this product around its maximum.

Before formulating Lemma 6.5 we make some comments about its content and introduce some notations. Given some numbers $h \geq 0$, $T > 0$ and $N > 1$, let $M = M(N, h, T)$ be the place of minimum (in the variable x) of the function $\Phi(x, T) + (c/2)^N a_0 x^2/2T - hx/T$, where the function Φ and the number a_0 are the same as in Theorem 5.1. The number M is the solution of the equation

$$(6.2) \quad \frac{\partial \Phi(x, T)}{\partial x} + \left(\frac{c}{2}\right)^N \frac{a_0}{T} x = \frac{h}{T}.$$

If $T \geq T_{cr}$ or $h > 0$, then M is the unique nonnegative solution of (6.2). If $h = 0$ and $T < T_{cr}$, then (6.2) has two nonnegative solutions $x = 0$ and $x > 0$. In this case we define M as the latter solution of (6.2). We also introduce the notation $m_2 = m_2(T, h) = \partial^2 \Phi(x, T)/\partial x^2|_{x=M}$. In Lemma 6.5 we actually make a Taylor expansion of $\log f_{n,N}^h(x)$ in the typical region. We want to make this expansion with such an accuracy that the error term tends to zero exponentially fast in n , i.e., it is $O(\xi^n)$ with some ξ , $0 < \xi < 1$. The cases $T > T_{cr}$, $T = T_{cr}$ and $T < T_{cr}$ are different. If $T > T_{cr}$, then the typical region is around the point $M(N, h)$, where the function $p_n(x, T) f_{n,N}^h(x)$ takes its maximum, and its size is a bit larger than $2^{-n/2}$. If $T = T_{cr}$, then the typical region is again around $M(N, h)$, but if h is small, then the typical region is of size $c^{-n/2}$. [This is connected with the fact that the second derivative of $p_n(x, T_{cr})$ is almost zero for small x .] As a consequence, in this larger region we have to make the Taylor expansion up to the second term. Hence the formulas become more complicated.

For $T < T_{cr}$ we have two different cases. If h is large [this is case (c1)], then the situation is similar to the case $T > T_{cr}$. If h is small [case (c2)], then the magnetic field cannot select out a single pure state. In this case the average spin is essentially concentrated in two disjoint intervals around the points M and \bar{M} (the latter will be defined in Lemma 6.5), and we need a good formula in both intervals. We remark that $M \sim M(T)$ and $\bar{M} \sim -M(T)$, where $M(T)$, the spontaneous magnetization, is defined in Theorem 5.1. Now we formulate:

LEMMA 6.5. *Under the conditions of Theorem 4.1:*

(a) For $T > T_{cr}$

$$f_{n,N}^h(x) = L \exp\{g_n(x - M) + O(\xi^n)\}, \quad \text{if } |x - M| < 2^{-n/2}\xi^{-n},$$

with

$$(6.3) \quad g_n = g_{n,N}^h(T) = \frac{2^n h}{T} + 2c^n \frac{M}{T} \frac{1 - \left(\frac{1}{2}c\right)^{N-n}}{2 - c}.$$

(b) For $T = T_{cr}$

$$f_{n,N}^h(x) = L \exp\{g_n(x - M) + A_n(x - M)^2 + O(\xi^n)\},$$

if $|x - M| \leq (c^n + 2^n m_2)^{-1/2} \xi^{-n}$ where g_n is defined in (6.3) and $A_n = A_{n,N}^h(T)$ by the recursive relations

$$(6.4) \quad A_N = 0, \quad A_n = \frac{A_{n+1}}{4} + \left(\frac{A_{n+1}}{4} + \frac{c^n}{2T} \right) \frac{2c^n + A_{n+1}T}{2^{n+1}m_2 + 2a_0c^n - A_{n+1}T},$$

for $n = N - 1, \dots, 1$.

(c) For $T < T_{cr}$:

(c1) If $h \geq 2^{-N}\xi^{-N}$, then

$$f_{n,N}^h(x) = L \exp\{g_n(x - M) + O(\xi^n)\}, \quad \text{for } |x - M| < 2^{-n/2}\xi^{-n},$$

where g_n is defined in (6.3).

(c2) If $0 \leq h < 2^{-N}\xi^{-N}$, then

$$f_{n,N}^h(x) = (1 - p)L \exp\{g_n(x - M) + O(\xi^n)\}, \quad \text{for } |x - M| < 2^{-n/2}\xi^{-n}$$

and

$$f_{n,N}^h(x) = (p + O(\xi^n))L_n \exp\{-\bar{g}_n(x - \bar{M}) + O(\xi^n)\},$$

for $|x - \bar{M}| < 2^{-n/2}\xi^{-n}$.

Here \bar{M} is the unique solution of the equation (6.2) in the vicinity of $-M(T)$, where $M(T)$ is defined before Theorem 5.1. [This equation has a unique solution in the interval $-M(T) \leq x \leq -M(T) + ((c/2)^N + h)\xi^{-N}$.] The constants g_n and \bar{g}_n are defined by formula (6.3), but in the latter case M must be replaced by \bar{M} , and $p = 1/[1 + \exp(2^{N+1}Mh)]$.

In the above formulas $L = L(n, N, h)$ is an appropriate norming constant, $0 < \xi < 1$ (which is chosen sufficiently close to 1) is independent of N, h and x , and $O(\cdot)$ is uniform in these variables.

There is a flaw in the formulation and proof of Lemma 6.5. In the applications (and also in the proof) we need some bound on $f_{n,N}^h(x)$ for all x and not only in the typical region. This bound must guarantee that the main contribution to the integrals appearing in applications of Lemma 6.5 (and also in its proof) is in the typical region. A rather rough estimate would suffice for this aim. We could give such an estimate, but we have omitted it deliberately. It would make the results and proofs even more complicated. On the other hand, we did not find it instructive enough; hence we preferred an incomplete proof.

PROOF OF LEMMA 6.5. Part (a). The relation holds for $n = N$; hence it is enough to prove it by induction from $n + 1$ to n . Introduce the function

$$H_n(x, y) = \frac{c^n}{T}xy + g_{n+1}\left(\frac{x + y}{2} - M\right) - 2^n\Phi(y, T) - \frac{\alpha_0 c^n}{2T}y^2.$$

It follows from Theorem 5.1, Lemma 6.4 and the induction hypothesis for $n + 1$ that

$$(6.5) \quad f_{n,N}^h(x) = L \int \exp\{H_n(x, y) + O(\xi^n)\} dy.$$

We estimate the above integral by first determining the place of maximum $\bar{y}_n = \bar{y}_n(x)$ of $H_n(x, y)$ and taking an expansion of $H_n(x, y)$ around this point. We have to solve the equation

$$(6.6) \quad \frac{\partial H_n(x, y)}{\partial y} = \frac{c^n}{T}x + \frac{g_{n+1}}{2} - 2^n \frac{\partial \Phi(y, T)}{\partial x} - \frac{\alpha_0 c^n}{T}y = 0.$$

A simple calculation shows that

$$(6.7) \quad 2^{n+1} \frac{\partial \Phi(M, T)}{\partial x} + \frac{\alpha_0 c^{n+1}}{T}M - g_{n+1} = 0.$$

[This identity has the following deeper content: The place of the maximum of the function $p_n(x, T)f_{n,N}^h(x)$ is M , i.e., it does not depend on n . Moreover, M is asymptotically the expectation of the $\mu_{n,N}^h$ distributed spins $E2^{-n}\sum_{i \in V_n} \sigma(i) = E\sigma(i)$, which is independent of n .] By using the expansion

$$\frac{\partial \Phi(y, T)}{\partial x} = \frac{\partial \Phi(M, T)}{\partial x} + m_2(y - M) + O((y - M)^2),$$

we get by expressing $\partial \Phi(M, T)/\partial x$ with the help of (6.7) and then using the identity

$$\frac{1}{T}c^n M + \frac{\alpha_0 c^{n+1}}{2T}M - \frac{\alpha_0 c^n}{T}M = 0$$

that

$$\begin{aligned} \frac{\partial H_n}{\partial y} &= \frac{c^n}{T}x + \frac{\alpha_0 c^{n+1}}{2T}M - 2^n m_2(y - M) - \frac{\alpha_0 c^n}{T}y + O(2^n(y - M)^2) \\ &= \frac{1}{T}c^n(x - M) - \alpha_0 \frac{c^n}{T}(y - M) - 2^n m_2(y - M) + O(2^n(y - M)^2). \end{aligned}$$

Hence we choose \bar{y} as

$$\bar{y} - M = \frac{c^n}{2^n T m_2 + \alpha_0 c^n}(x - M).$$

Now we calculate $H_n(x, \bar{y})$ with an error of the order $O(\xi^n)$. In this calculation we exploit $\bar{y} - M = O((c/2)^n(x - m))$ and $x - M = O(2^{-n/2}\xi^{-n})$. By taking the Taylor expansion of the function $\Phi(x, T)$ around $x = M$ and using (6.7) we get

$$\begin{aligned} H_n(x, \bar{y}) &= \frac{c^n}{T}x\bar{y} + g_{n+1}\left(\frac{x + \bar{y}}{2} - M\right) - 2^n\Phi(M, T) \\ &\quad - \left(\frac{g_{n+1}}{2} - \frac{\alpha_0 c^{n+1}}{2T}M\right)(\bar{y} - M) - \frac{\alpha_0}{2T}c^n\bar{y}^2 + O(2^n(\bar{y} - M)^2) \\ &= K(N) + \frac{c^n}{T}M(x - M) + \frac{c^n}{T}M(\bar{y} - M) + \frac{g_{n+1}}{2}(x - M) \\ &\quad + \frac{\alpha_0 c^{n+1}}{2T}M(\bar{y} - M) - \frac{\alpha_0 c^n}{T}M(\bar{y} - M) + O(\xi^n). \end{aligned}$$

Hence

$$H_n(x, \bar{y}) = K(N) + g_n(x - M) + O(\xi^n).$$

On the other hand,

$$\begin{aligned} \frac{\partial H_n(x, \bar{y})}{\partial y} &= O(2^n(\bar{y} - M)), \\ \frac{\partial^2 H_n(x, \bar{y})}{\partial y^2} &= -2^n m_2 - \frac{\alpha_0 c^n}{T} + O(2^n(\bar{y} - M)). \end{aligned}$$

Now we apply (6.5) for calculating $f_{n,N}^h(x)$, but it is enough to integrate in the domain $|y - \bar{y}| \leq 2^{-n/2}\xi^{-n/2}$, because the contribution of the remaining domain is negligible. [We also have to observe that $|(x + y)/2 - M| \leq 2^{-(n+1)/2}\xi^{-(n+1)/2}$ in this domain; hence the induction hypothesis for $n + 1$ holds in this domain.] This fact is needed in the proof of (6.5). In the domain $|y - \bar{y}| \leq 2^{-n/2}\xi^{-n/2}$ the error terms in the expressions for $H_n(x, y)$, $\partial H_n/\partial y(y - \bar{y})$, $\partial^2 H_n/\partial y^2(y - \bar{y})^2$ and $\partial^3 H_n(x, y)/\partial y^3(y - \bar{y})^3 = O(2^{-n/2}\xi^{-3n/2})$ are negligible; hence the Taylor

expansion of $H_n(x, y)$ around \bar{y} yields

$$f_{n,N}^h(x) = L' \int \exp \left\{ g_n(x - M) - \left(2^{n-1}m_2 + \frac{\alpha_0 c^n}{2T} \right) (y - \bar{y})^2 + O(\xi^n) \right\} dy$$

$$= L \exp \{ g_n(x - M) + O(\xi^n) \}$$

as we claimed.

Part (b). In the proof we need the estimate $0 \leq A_n \leq [(2 - c)/cT]c^n$ for the sequence A_n defined in (6.4). In order to prove it we introduce the sequences \bar{A}_n and \tilde{A}_n , $n = N, \dots, 1$, defined by the relations $\bar{A}_n = c^{-n}TA_n$ and $\tilde{A}_N = 0$, $\tilde{A}_n = (c/4)\tilde{A}_{n+1} + ((c/4)\tilde{A}_{n+1} + \frac{1}{2})(2 + c\tilde{A}_{n+1})/(2\alpha_0 - c\tilde{A}_{n+1})$. Then we have $0 \leq \bar{A}_n \leq \tilde{A}_n$. (We have $\bar{A}_n = \tilde{A}_n$ if $m_2 = 0$.) Let us consider the equation $z = (c/4)z + (2 + cz)^2/(8\alpha_0 - 4cz)$. This equation has two solutions $z_1 = (2 - c)/c$ and $z_2 = 1/(2 - c)$, $z_1 < z_2$. Some elementary analysis shows that $0 \leq \tilde{A}_n \leq z_1$ for all $n = N, \dots, 1$. Hence $0 \leq A_n \leq z_1 c^n T$ as we have claimed.

The remaining part of the proof is very similar to part (a), the only difference is that we have to use expansions up to second order. Relation (6.5) remains valid if we substitute $H_n(x, y)$ by $\bar{H}_n(x, y) = H_n(x, y) + A_{n+1}((x + y)/2 - M)^2$. Some calculation shows that the place of maximum of $\bar{H}_n(x, y)$ is asymptotically given by the formula

$$\bar{y} - M = \frac{2c^n + A_{n+1}T}{2^{n+1}m_2T + 2\alpha_0c^n - A_{n+1}T}(x - M).$$

Some further calculation shows that

$$\bar{H}_n(x, \bar{y}) = K(n) + g_n(x - M) + A_n(x - M)^2 + O(2^n m_3 (\bar{y} - M)^3),$$

$$\frac{\partial \bar{H}_n(x, \bar{y})}{\partial y} = O(2^n m_3 (\bar{y} - M)^2),$$

$$\frac{\partial^2 \bar{H}_n(x, \bar{y})}{\partial y^2} = - \left(2^n m_2 + \frac{\alpha_0 c^n}{T} - \frac{A_{n+1}}{2} \right) + O(2^n m_3 (\bar{y} - M)),$$

with some $m_3 = \partial^3 \Phi(\eta, T)/\partial x^3$, $\eta \in [M, \bar{y}]$. We claim that in the interval $|y - \bar{y}| \leq (2^n m_2 + c^n)^{-1/2} \xi^{-n/2}$,

$$(6.8) \quad \bar{H}_n(x, y) = K(n) + g_n(x - M) + A_n(x - M)^2 + \left(2^{n-1}m_2 + \frac{\alpha_0 c^n}{2T} - \frac{A_{n+1}}{2} \right) (y - \bar{y})^2 + O(\xi^n).$$

This relation follows from the estimates given for \bar{H}_n , $\partial \bar{H}_n/\partial y$, $\partial^2 \bar{H}_n/\partial y^2$ together with the inequalities $|\partial^3 \Phi(y, T)/\partial x^3| \leq \text{const.}|y|$ and $m_2 \geq \text{const.}M^2$ which enable us to bound the error terms. These inequalities follow from the properties of the functions $\Phi \in S_u$. All error terms can be bounded similarly to

the following one: If $\eta \in [y, \bar{y}]$, then

$$\begin{aligned} \left| \frac{\partial^3 \bar{H}_n(x, \eta)}{\partial y^3} (y - \bar{y})^3 \right| &\leq 2^n |\eta| (2^n m_2 + c^n)^{-3/2} \xi^{-3n/2} \\ &\leq 2^n (M + (2^n m_2 + c^n)^{-1/2}) (2^n m_2 + c^n)^{-3/2} \xi^{-2n} \\ &\leq \text{const.} [2^n \sqrt{m_2} (2^n m_2)^{-1/2} c^{-n} + 2^n c^{-2n}] \xi^{-2n} = O(\xi^n). \end{aligned}$$

Observe that because of the inequality $A_{n+1} \leq [(2 - c)/cT]c^{n+1}$, we have $2^{n-1}m_2 + a_0c^n/2T - A_{n+1}/2 \geq K(2^n m_2 + c^n)$ with some $K > 0$. Hence in this case the essential contribution to the integral (6.5) is given by the domain $|y - \bar{y}| \leq (2^n m_2 + c^n)^{-1/2} \xi^{-n/2}$, and formula (6.8) yields

$$\begin{aligned} f_{n,N}^h(x) &= L' \int \exp \left\{ g_n(x - M) + A_n(x - M)^2 \right. \\ &\quad \left. + \left(2^{n-1}m_2 + \frac{a_0c^n}{2T} - \frac{A_{n+1}}{2} \right) (y - \bar{y})^2 + O(\xi^n) \right\} dy \\ &= L \exp \left\{ g_n(x - M) + A_n(x - M)^2 + O(\xi^n) \right\}, \end{aligned}$$

as we claimed.

Part (c). We have to calculate $f_{n,N}^h(x)$ in an appropriate neighborhood of equation (6.2), but in the case (c2) we have to consider also the solution \bar{M} . [It can be seen that \bar{M} belongs to the domain where Theorem 5.1 yields a good asymptotic formula for $p_n(x, T)$.] We calculate $f_{n,N}^h(x)$ by induction with the help of Lemma 6.4. Since the main contribution of the integral (6.1) for an $x \sim M$ (or $x \sim \bar{M}$) is given by a neighborhood of M (or \bar{M}) almost the same calculations supply the proof as in part (a). We omit the details. \square

A simple calculation similar to that given at the beginning of the proof of part (b) shows that if $h_N \rightarrow 0$, then for all $n \geq 0$ there exists some $N_0 = N_0(n)$ such that for $N > N_0$, $g_{n,N}^h = gc^n + O(2^{-n})$, $A_{n,N}^h = ATc^n + O(2^{-n})$, $M = M(N, h, T) = M(T) + O(2^{-n})$ with $g = (a_0/T)M(T)$, $A = (2 - c)/cT$, $M(T)$ is defined in Section 5 for $T < T_{cr}$, and $M(T) = 0$ for $T \geq T_{cr}$. The above relations imply that

$$(6.9) \quad f_{n,N}^h(x) = f_n(x)(1 + O(\xi^n)), \quad \text{if } h_N \rightarrow 0 \text{ and } N > N(n),$$

in the domain $|x| < 2^{-n/2}\xi^{-n}$ if $T > T_{cr}$, $|x| < c^{-n/2}\xi^{-n}$ if $T = T_{cr}$, $|x - M(T)| < 2^{-n/2}\xi^{-n}$ if $T < T_{cr}$, where $f_n(x) = L \exp(gc^n(x - M(T)) + Ac^n x^2)$ (with $A = 0$ if $T \neq T_{cr}$). Formula (6.9) enables us to investigate $\lim \mu_{N,N}^{h_N}$ as $h_N \rightarrow 0$ with $N \rightarrow \infty$, and thus to prove Theorem 6.1. We must be careful when carrying out the limiting procedure, because formula (6.9) is useful only for large n when $O(\xi^n)$ is small, and we want to prove that $\lim \mu_{k,N}^{h_N}(A)$ exists for an arbitrary measurable set $A \subset R^{V_k}$ where k is fixed. To overcome this difficulty we consider the cylindrical set $\tilde{A} = \tilde{A}(n) = A \times R^{V_n - V_k}$, $n \gg k$, and exploit the identity $\mu_{k,N}^{h_N}(A) = \mu_{n,N}^{h_N}(\tilde{A})$. Then it can be proved by tending first with N then with n

to infinity that for fixed k the measures $\mu_{k,N}^h$ are convergent in variational norm as $k \rightarrow \infty$. We omit the details of the proof. Let us remark that a very similar limiting procedure is worked out in detail in [9].

Besides the proof of Theorem 6.1 formula (6.9) yields the following corollary. Let $\bar{\mu}_n = \bar{\mu}_n(T)$ denote the projection of the equilibrium state μ [for $T < T_{cr}$ of $\mu^+(T)$] to V_n .

COROLLARY 6.6. *The Radon–Nikodym derivative $d\bar{\mu}_n/d\mu_n$ satisfies the relation*

$$d\bar{\mu}_n(\sigma|T) = Lf_n(\eta_n) d\mu_n(\sigma|T),$$

with $\sigma = (\sigma(1), \dots, \sigma(2^n))$, $\eta_n = 2^{-n} \sum_{i \in V_n} \sigma(i)$, where

$$f_n(x) = \begin{cases} \exp(O(\xi^n)), & \text{if } |x| < 2^{-n/2}\xi^{-n} \text{ for } T > T_{cr}, \\ \exp(Ac^n x^2 + O(\xi^n)), & \text{if } |x| < c^{-n/2}\xi^{-n} \text{ for } T = T_{cr}, \\ \exp(gc^n(x - M(T))) + O(\xi^n), & \text{if } |x - M(T)| < 2^{-n/2}\xi^{-n} \\ & \text{for } T < T_{cr}, \end{cases}$$

with $g = 2/(2 - c)$, $A = (2 - c)/c$. Here L is an appropriate norming constant, $0 < \xi < 1$, it is chosen independently of x and n , and $O(\cdot)$ is uniform in these variables.

We remark that in Corollary 6.6 we defined $d\bar{\mu}_n/d\mu_n$ in the typical region where the measure $\bar{\mu}_n$ is concentrated.

Now we explain the proof of Theorem 6.2. Let us define the renormalization transformation $\mathcal{R}_n = \mathcal{R}_n(k, c)$ which maps the probability measures on $R^{V_{n+k}}$ to the probability measures on R^{V_k} in the following way: Given a probability measure μ on $R^{V_{n+k}}$ let $\sigma = (\sigma(1), \dots, \sigma(2^{n+k}))$ be a μ distributed random vector. Then $\mathcal{R}_n\mu$ is the distribution of the vector

$$\mathcal{R}_n\sigma = (\bar{\sigma}(1), \dots, \bar{\sigma}(2^k)), \quad \bar{\sigma}(j) = \left(\frac{\sqrt{c}}{2}\right)^n \sum_{l \in V_{j,n}} \sigma(l), \quad j = 1, \dots, 2^k.$$

A simple algebraic calculation proves the following:

LEMMA 6.7. $\mathcal{R}_n\mu_{nk}(d\sigma|T, \nu) = \mu_k(d\sigma|T, \bar{\nu})$, where μ_n is the Gibbs state of Dyson’s hierarchical model without boundary conditions defined in (3.2) and (3.2’) and $\bar{\nu}$ is the distribution of the normalized sum $(\sqrt{c}/2)^n \sum_{i \in V_n} \sigma(i)$ of the $\mu_n(d\sigma|T, \nu)$ distributed vector $(\sigma(1), \dots, \sigma(2^n))$.

Lemma 6.7 agrees with Theorem 1 of [6]. We omit the proof.

If $\mu^{(1)}$ and $\mu^{(2)}$ are two probability measures on $R^{V_{n+k}}$ such that the Radon–Nikodym derivative $d\mu^{(1)}/d\mu^{(2)}$ has the form

$$\mu^{(1)}(d\sigma) = f\left(\sum_{i \in V_{n+k}} \sigma(i)\right)\mu^{(2)}(d\sigma),$$

i.e., it depends at a point $\sigma = (\sigma(1), \dots, \sigma(2^{n+k}))$ only on $\sum_{i \in V_{n+k}} \sigma(i)$, then $\mathcal{R}_n\mu^{(1)}(d\sigma) = f(\sum_{i \in V_k} \sigma(i))\mathcal{R}_n\mu^{(2)}(d\sigma)$.

Since the above-mentioned property holds with the choice $\mu^{(1)} = \bar{\mu}_{kn}$, $\mu^{(2)} = \mu_{kn}(d\sigma|T, \nu)$ and Theorem 5.1 gives a good asymptotic formula for the density function of the measure $\bar{\nu}$ defined in Lemma 6.7 [we need an expansion of $p_n(x, T)$ around $x = M(T)$] hence Theorem 6.2 can be deduced from Corollary 6.6 and Lemma 6.7 with the help of a simple calculation. Theorem 6.3 also follows from these calculations.

The proof in the case $\sqrt{2} - \varepsilon < c < \sqrt{2}$ is similar but technically more difficult. The most difficult case is when $T = T_{cr}$ (see [6]). In this case the Radon-Nikodym derivative $d\mu_{n,N}^h/d\mu_n$ has an essentially different form. We shall not discuss the problem here.

7. Vector-valued models. In this section we consider Dyson's vector-valued hierarchical ϕ^4 model. Its Hamiltonian function is defined by (2.1) with $h = 0$ and the same function $U(i, j)$ as in the scalar case, its free measure by (3.1), only in the present case $\sigma(x) \in R^p$, $x \in R^p$, $p \geq 2$, $\sigma(i)\sigma(j)$ denotes scalar product and x^2 resp. x^4 denotes $|x|^2$ and $|x|^4$. The results formulated for scalar-valued models remain valid, with some natural modifications, also in this case, although their proof is more difficult. We shall not consider this question. We only discuss the renormalization of vector-valued models at low temperatures, since here some new phenomena appear. The cases $1 < c < \sqrt{2}$ and $\sqrt{2} < c < 2$ are essentially different. (In the scalar case such a difference appears only at the critical temperature.)

First, we discuss the case $\sqrt{2} < c < 2$. We have to construct the pure phase we want to renormalize. We do it by first considering an equilibrium state with a small external field h and then by letting h go to zero. Let $\mu_N^h = \mu_N^h(T)$ denote the Gibbs measure in V_N with the Hamiltonian function given in (2.1) with $\mathbf{h} = he$, $e = (1, 0, \dots, 0)$, $h > 0$, at the temperature T . We have:

THEOREM 7.1. *The relations*

$$\lim_{N \rightarrow \infty} \mu_N^h = \mu^h, \quad \lim_{h \rightarrow 0} \mu^h = \mu_e,$$

hold true, where \lim means the convergence of the finite-dimensional distributions in the variational metric.

After establishing the above relations it is not difficult to see that $\mu^h = \mu^h(T)$ is an equilibrium state with external field h and $\mu_e = \mu_e(T)$ is an equilibrium state without external field. The double limiting procedure enables us to construct a pure state, i.e., an equilibrium state which cannot be represented as the mixture of other equilibrium states. Now we formulate:

THEOREM 7.2. *Let $\sigma(k) = (\sigma^{(1)}(k), \dots, \sigma^{(p)}(k))$, $k \in \mathbb{Z}$, be a $\mu_e = \mu_e(T)$ distributed random field where μ_e is defined in Theorem 7.1. For $T < T_{cr}$ the finite-dimensional distributions of the random fields $Y_n(k)$, $k \in \mathbb{Z}$, $n = 1, 2, \dots$, defined by formula (3.4) with $A_n = 2^{n/2}$, $B_n = 2^{nc^{-n/2}}$, tend to those of a Gaussian random field $Y(k) = (Y^{(1)}(k), \dots, Y^{(p)}(k))$, $k \in \mathbb{Z}$. The random fields*

$\{Y^{(j)}(k), k \in \mathbb{Z}\}, j = 1, \dots, p$, are independent for different j , $Y^{(1)}(k), k \in \mathbb{Z}$, is a sequence of independent Gaussian random variables with zero mean, and for $2 \leq j \leq p$ the density of the random vector $(Y^{(j)}(1), \dots, Y^{(j)}(2^k))$ is given by the formula

$$\text{const. exp} \left\{ \frac{1}{T} \left[\frac{(2-c)c^k}{2^{2k}} \left(\sum_{j=1}^{2^k} x_j \right)^2 - \sum_{j=1}^{2^k} \frac{2}{2-c} \frac{x_j^2}{2} + \sum_{j=1}^{2^k} \sum_{l=1}^{j-1} d(j, l)^{-a} x_j x_l \right] \right\}.$$

We remark that the last formula actually defines the finite-dimensional distributions of the equilibrium state of the scalar-valued hierarchical model at temperature T with the free measure $1/[(2-c)\pi T]^{1/2} \exp(-x^2/(2-c)T)$. Theorem 7.2 means that for $T < T_{cr}$ vector-valued models behave in the direction of the magnetization like scalar models at $T < T_{cr}$, but in the orthogonal direction they behave like scalar models for $T = T_{cr}$. An interesting feature of this result is that such behavior appears for a whole interval of parameters.

The proof of Theorems 7.1 and 7.2 is given in [9]. It is very similar to the arguments of Section 6. One has to prove the multidimensional version of Theorem 5.1, to calculate the Radon–Nikodym derivative $d\mu_{n,N}^h/d\mu_n$ and to carry out the limiting procedures $N \rightarrow \infty$ and $h \rightarrow 0$. We omit the details of the proof; we only explain why the normalization $B_n = 2^n c^{-n/2}$ must be chosen in the direction orthogonal to the magnetization. For the sake of simplicity we consider the case $p = 2$ and the limit of the one-dimensional distributions $A_n^{-1} \sum_{i \in V_n} \sigma^{(1)}(i), B_n^{-1} \sum_{i \in V_n} \sigma^{(2)}(i)$.

By carrying out the calculations mentioned above we find that the density of the average $2^{-n} \sum_{i \in V_n} \sigma(i)$ of the μ_e distributed spins in the volume V_n is of the form $f_n(x) p_n(|x|, T)$ with $p_n(|x|, T) = \exp\{-a_0 c^n / T(x_1^2 + x_2^2) - 2^n \Phi(|x|, T) + O(\xi^n)\}$ with some $\Phi \in S_u$, and

$$f_n(x) = \exp \left\{ \frac{2c^n M}{(2-c)T} x_1 + \frac{c^n(2-c)}{cT} x_2^2 + O(\xi^n) \right\}, \quad x = (x_1, x_2),$$

where $M = M(T)$ is the (unique) positive solution of the equation $\partial \Phi / \partial x = 0$, $a_0 = 2/(2-c)$. Here p_n is the density function of the average of the μ_n distributed spins in V_n , and $f_n(2^{-n} \sum_{j \in V_n} \sigma(j))$ is the Radon–Nikodym derivative of the projection of the equilibrium state μ_e with respect to μ_n . The above relations for $f_n(x) p_n(|x|, T)$ hold in the domain $|x_1 - M| < 2^{-n/2} \xi^{-n}$ and $x_2^2 < c^{-n} \xi^{-n}$, but some upper-bound estimates show that the random variable we are interested in falls into this domain with probability almost 1. Hence the distribution of the average spin we are investigating is $\text{const. exp}\{-H_n(x_1, x_2) + O(\xi^n)\}$ with

$$H_n(x_1, x_2) = 2^n \Phi(\sqrt{x_1^2 + x_2^2}, T) + \frac{a_0 c^n}{2T} (x_1^2 + x_2^2) - \frac{a_0 c^n M}{T} x_1 - \frac{(2-c)c^n}{cT} x_2^2.$$

We make the Taylor expansion of the function H_n up to the second order around

the point $(M, 0)$. We have

$$\begin{aligned} \frac{\partial \Phi(M, 0)}{\partial x_1} &= 0, & \frac{\partial \Phi(M, 0)}{\partial x_2} &= 0, \\ \frac{\partial^2 \Phi(M, 0)}{\partial x_1 \partial x_2} &= 0 & \text{and} & \frac{\partial^2 \Phi(M, 0)}{\partial x_2^2} = m, \end{aligned}$$

with some $m > 0$. It is very important for us that $\partial \Phi(M, 0)/\partial x_1 = 0$ together with the rotation invariance of the function Φ imply that $\partial^2 \Phi(M, 0)/\partial x_2^2 = 0$. Hence

$$\begin{aligned} H_n(x_1, x_2) &= H_n(M, 0) + \left(2^{n-1}m + \frac{\alpha_0 c^n}{2T} \right) (x_1 - M)^2 \\ &\quad + \frac{-c^2 + 5c - 4}{c(2 - c)T} c^n x_2^2 + O(\xi^n), \end{aligned}$$

i.e., the density of the average spin in V_n is asymptotically

$$\text{const. exp} \left\{ -2^n A (x_1 - M)^2 - B c^n x_2^2 + O(\xi^n) \right\}$$

with some $A > 0$ and $B > 0$. This means that $2^{-n/2} \sum_{i \in V_n} \sigma^{(1)}(i) - M$ and $2^{-n} c^{n/2} \sum_{i \in V_n} \sigma^{(2)}(i)$ are asymptotically independent Gaussian variables with positive variances as we claimed.

Now we present the result corresponding to Theorem 7.2 in the case $1 < c < \sqrt{2}$, together with a heuristic explanation for its validity. In the detailed proofs, one has to overcome several serious technical difficulties and we shall do so in a subsequent paper. For the sake of simpler notations we restrict ourselves to the case $p = 2$. First, we need an asymptotic formula for the density function $p_n(x) = p_n(x, T)$ of the average $2^{-n} \sum_{i \in V_n} \sigma(i)$ of $\mu_n(d\sigma|T)$ distributed spins if $T > 0$ is sufficiently small. [Here again μ_n is defined by (3.2) and (3.2').] The probability measure with density function $p_n(x, T)$ is concentrated around some $|x| = M = M(T) > 0$, but, and this is the essential difference between the cases $c < \sqrt{2}$ and $c > \sqrt{2}$, it is concentrated in the domain $||x| - M| < c^{-n}$, and not in the domain $||x| - M| < 2^{-n/2}$ as in the case $c > \sqrt{2}$. More precisely, we have:

LEMMA 7.3. *For $1 < c < \sqrt{2}$ and sufficiently small $T > 0$ in Dyson's vector-valued hierarchical ϕ^4 model the density function $p_n(x) = p_n(x, T)$ of the average $2^{-n} \sum_{i \in V_n} \sigma(i)$ of the $\mu_n(d\sigma|T)$ distributed spins has the form*

$$p_n(x, T) = \text{const. exp} \left(- \frac{\alpha_0 M (|x| - M)}{T} c^n \right) f(c^n (|x| - M), T) (1 + O(\xi^n)),$$

for $||x| - M| < c^{-n} \xi^{-n}$ with some $0 < \xi < 1$, where $M = M_0(T) + O(u)$, $M_0(T) = (\alpha_0 - T/uT)^{1/2}$ [$M_0(T)$ is the place of maximum of the function $\Phi_0(x) = (u/4)x^4 + [(T - \alpha_0)/2T]x^2$], $\alpha_0 = 2/(2 - c)$ and f is the unique

solution of the equation

$$(7.1) \quad f(x) = \iint \exp\left(-\frac{\alpha_1 v^2}{T}\right) f\left(\frac{x}{c} + u + \frac{v^2}{2M}\right) f\left(\frac{x}{c} - u + \frac{v^2}{2M}\right) du dv,$$

$$\alpha_1 = \alpha_0 + 1.$$

For $||x| - M| > c^{-n}\xi^{-n}$ the function p_n is negligibly small.

Lemma 7.3 means that for small $T > 0$, $c^{-n}p_n(c^{-n}(|x| - M), 0, T) \rightarrow p^*(x, T)$ with some function p^* as $n \rightarrow \infty$, i.e., an unusual normalization must be used. We remark that only vector-valued models have such an exceptional behavior, and it is closely related to the rotational invariance of the function p_n . We explain the main ideas of the proof.

We have to investigate the functions $p_n(x) = p_n(x, T)$ defined by formula (4.3), or after the substitution $p_n(x) = \text{const. exp}(-(\alpha_0 c^n / 2T)x^2)q_n(x)$ the functions $q_n(x)$ defined by the relations

$$q_n(x) = \int \exp\left(-\frac{\alpha_1 c^{n-1}}{T}u^2\right) q_{n-1}(x - u)q_n(x + u) du, \quad n = 1, 2, \dots,$$

and

$$q_0(x) = \exp\left(-\frac{u}{4}x^4 - \frac{\alpha_0 - T}{2T}x^2\right).$$

We show the rotational invariance of the function q_n . By introducing $u = (u, v)$, $q_n(x, 0) = Q_n(x)$, $x, u, v \in R^1$, we can rewrite the relation defining q_n as

$$(7.2) \quad Q_n(x) = \iint F_n(u, v, x) du dv,$$

with

$$(7.2') \quad F_n(u, v, x) = \exp\left(-\frac{\alpha_1 c^{n-1}}{T}(u^2 + v^2)\right) Q_{n-1}\left(\sqrt{(x + u)^2 + v^2}\right) \\ \times Q_{n-1}\left(\sqrt{(x - u)^2 + v^2}\right).$$

We want to get a good asymptotic formula for $Q_n(x)$ only for $x \sim M$, where M is the place of maximum of the function Q_n . The function Q_{n-1} and hence the integral (7.2) are strongly localized, and in the proof we must know the right size of localization. In the case $c > \sqrt{2}$, $Q_n(x)$ is negligibly small if $|x - M| > 2^{-n/2}\xi^{-n}$, the integral in (7.2) is concentrated in the domain $D_n = \{(u, v), |u| < 2^{-n/2}\xi^{-n}, |v| < c^{-n/2}\xi^{-n}\}$ and we make only a negligible error by substituting the integrand $F(u, v, x)$ by $Q_{n-1}(x + u)Q_{n-1}(x - u)$ in (7.2). But for $c < \sqrt{2}$, i.e., in the case we are now investigating, the typical domain D_n must be chosen otherwise. Indeed, let us consider the case $x = M$. A point (u, v) must be in D_n if $F_n(u, v, M) \geq \text{const. } F_n(0, 0, M) = \text{const. } Q_{n-1}^2(M)$. Hence, since we expect that $Q_{n-1}(M - \alpha_n) \geq \text{const. } Q_{n-1}(M)$ if and only if $Q_{n-1}(M + \alpha_n) \geq \text{const. } Q_{n-1}(M)$, the relation $(0, c^{-n/2}) \in D_n$ implies that $(c^{-n}, 0) \in D_n$. Since in

our case $c^{-n} > 2^{-n/2}\xi^{-n}$ this means that D_n must be chosen otherwise, and hence $Q_n(x)$ is also negligible in a different domain. Some consideration would suggest that in the case $c < \sqrt{2}$ the typical domain is $D_n = \{(u, v), |u| < c^{-n}\xi^{-n}, |v| < c^{-n/2}\xi^{-n}\}$ and the function $Q_n(x)$ is negligibly small if $|x - M| > c^{-n}\xi^{-n}$. Finally, the detailed proof of Lemma 7.3 shows that this conjecture is correct. It suggests a rescaling of the function Q_n , i.e., the introduction of the function $\bar{Q}_n(x) = c^{-3(n+1)/2}Q_n(M + c^{-n}x)$. Then formulas (7.2) and (7.2') together with the relations

$$\sqrt{(x \pm u)^2 + v^2} \sim x \pm u + \frac{v^2}{2M}, \quad \frac{a_1 c^{n-1}}{T} u^2 \ll 1,$$

for $|x - M| < c^{-n}\xi^{-n}$ and $(u, v) \in D_n$, imply that

$$(7.3) \quad \bar{Q}_{n+1}(x) = \bar{R}\bar{Q}_n(x) + \varepsilon_n(x),$$

with

$$(7.4) \quad \bar{R}f(x) = \iint \exp\left(-\frac{a_1}{T}v^2\right) f\left(\frac{x}{c} + u + \frac{v^2}{2M}\right) f\left(\frac{x}{c} - u + \frac{v^2}{2M}\right) du dv,$$

where $\varepsilon_n(x)$ is a small error term which we can control during the proof. Because of formula (7.3) it is natural to expect that \bar{Q}_n tends to the fixed point of the operator \bar{R} , i.e., to the solution of the integral equation (7.1). Now we solve this equation. Let us introduce the Fourier transform $\tilde{f}(s) = \int e^{isx}f(x) dx$. Then we get from (7.1) with the substitution $x/c + u + v^2/2M = z$, $x/c - u + v^2/2M = w$,

$$\begin{aligned} \tilde{f}(s) &= \int \exp\left(isx - \frac{a_1 v^2}{T}\right) f\left(\frac{x}{c} + u + \frac{v^2}{2M}\right) f\left(\frac{x}{c} - u + \frac{v^2}{2M}\right) du dv dx \\ &= c \int f(w) f(z) \exp\left\{is \frac{c}{2} \left(z + w - \frac{v^2}{M}\right) - \frac{a_1 v^2}{T}\right\} dw dz dv \\ &= \int \exp\left(i \frac{c}{2} sz\right) f(z) dz \int \exp\left(i \frac{c}{2} sw\right) f(w) dw \int c \exp\left(-v^2 \left(\frac{a_1}{T} + i \frac{cs}{2M}\right)\right) dv \\ &= \tilde{f}\left(\frac{c}{2}s\right)^2 \frac{c\sqrt{TM}\pi}{\sqrt{2Ma_1 + iscT}}. \end{aligned}$$

Let us introduce the function ϕ defined by the relation $\tilde{f}(s) = \exp(-\phi(s))$. Then we have

$$(7.5) \quad \phi(s) - 2\phi\left(\frac{c}{2}s\right) = \frac{1}{2} \log \frac{2Ma_1 + iscT}{c\pi TM}.$$

If $\phi(s) = \sum \alpha_k s^k$, then the left-hand side of (7.5) is $\sum (1 - 2(c/2)^k) \alpha_k s^k$. The coefficients α_k can be calculated by expanding the right-hand side of (7.5) into Taylor series. Then because of the special form of equation (7.5) we can continue this solution to the whole real line and determine the function f . A more careful analysis proves the contraction properties of the operator R which together with

(7.3) enables us to establish the convergence $\bar{Q}_n(x) \rightarrow f(x)$. Since $p_n(x) = \text{const. exp}(-(\alpha_0 c^n / 2T)x^2) \bar{Q}_n(c^n(|x| - M))$ this relation implies Lemma 7.3.

Let $\mu_N^h(d\sigma|T)$ denote the Gibbs distribution in V_N with the Hamiltonian of Dyson's hierarchical model with the external field he , $e = (1, 0)$, and let $\mu_{n,N}^h(d\sigma|T)$ denote its projection to V_n . In order to investigate the equilibrium states we need an asymptotic formula for the Radon-Nikodym derivative $d\mu_{n,N}^h/d\mu_n$. By the multidimensional version of Lemma 6.4 (see [6])

$$\mu_{n,N}^h(d\sigma|T) = Lf_{n,N}^h(\xi_n)\mu_n(d\sigma|T),$$

with

$$\begin{aligned} \xi_n &= 2^{-n} \sum_{i \in V_n} \sigma(i), & f_{N,N}^h(x) &= \exp\left(\frac{2^N h}{T} x_1\right), & x &= (x_1, x_2), \\ (7.6) \quad f_{n,N}^h(x) &= \int \exp\left(\frac{c^n}{T} xy\right) f_{n+1,N}^h\left(\frac{x+y}{2}\right) p_n(y) dy, \end{aligned}$$

$x, y \in R^2$, where p_n is the density function appearing in Lemma 7.3. We claim that

$$(7.7) \quad f_{n,N}^h(x) = C_n \{ \exp g_n(x_1 - M) + A_n x_2^2 + O(\xi^n) \},$$

with some constants $g_n = g_n(h, N, T)$, $A_n = A_n(h, N, T)$ which we shall define explicitly. Actually we claim this relation only for $x \sim \mathbf{M} = (M, 0)$ where $M = M(h, T)$ is the place of maximum of the function $\exp(hx/T)p_N(X)$. For other values of x it is enough to prove a rough estimate on $f_{n,N}^h(x)$ which guarantees that the average of the $\mu_{n,N}^h(d\sigma|T)$ distributed spins is concentrated around the point \mathbf{M} . We prove (7.7) with the help of Lemma 7.3 and relation (7.6) by induction from $n + 1$ to n . Write $x = (x_1 + M, x_2)$ and $y = (y_1 + M, y_2)$. We shall use that for $x \sim M$ the integral in (7.6) is concentrated in the domain $y \sim M$. Hence we can write $xy \sim \text{const.} + M(x_1 + y_1) + x_2 y_2$, $|x| - M \sim x_1 + x_2^2/2M$, $|y| - M \sim y_1 + y_2^2/2M$ and we get from (7.6) by omitting some error terms (which, as a more detailed analysis shows, is legitimate)

$$\begin{aligned} f_{n,N}^h(x) &= C'_n \int \exp\left\{ \frac{c^n}{T} M(x_1 + y_1) + \frac{c^n}{T} x_2 y_2 + g_{n+1} \frac{x_1 + y_1}{2} \right. \\ &\quad \left. + A_{n+1} \left(\frac{x_2 + y_2}{2}\right)^2 - \frac{\alpha_0 c^n}{T} (2My_1 + y_2^2) \right\} f\left(y_1 + \frac{y_2^2}{2M}\right) dy_1 dy_2, \end{aligned}$$

where the function f is the solution of equation (7.1). With the help of the substitution $y_1 + (y_2^2/2M) = z$, $y_2 = v$, the last formula can be rewritten as

$$\begin{aligned} f_{n,N}^h(x) &= C'_n \int f(z) \exp\left\{ -\frac{\alpha_0 c^n M}{T} z + \frac{g_{n+1}}{2} z + \frac{c^n}{T} Mz \right\} dz \\ &\quad \times \int \exp\left\{ \frac{c^n}{T} M\left(x_1 - \frac{v^2}{2M}\right) + \frac{c^n}{T} x_2 v + g_{n+1} \left(\frac{x_1}{2} - \frac{v^2}{4M}\right) \right. \\ &\quad \left. + A_{n+1} \left(\frac{x_2 + v}{2}\right)^2 \right\} dv. \end{aligned}$$

Here the first integral is some constant not depending on x , the second integral can be calculated explicitly, and it yields relation (7.7) for n with

$$g_n = \frac{g_{n+1}}{2} + \frac{c^n}{T}M$$

and

$$A_n = \frac{A_{n+1}}{4} + \frac{\left(\frac{c^n}{T} + \frac{A_{n+1}}{2}\right)^2}{2\frac{c^n}{T} + \frac{g_{n+1}}{M} - A_{n+1}}.$$

Let us choose a sequence $h_N \rightarrow 0$ as $N \rightarrow \infty$ and investigate the sequences $g_n = g_n(h_N, N, T)$ and $A_n = A_n(h_N, N, T)$ as $N \rightarrow \infty$. Introduce the quantities $\bar{g}_n = (g_n/c^n)T$ and $\bar{A}_n = (A_n/c^n)T$. Some analysis shows that $\bar{g}_n \rightarrow \bar{g} = 2M/(2 - c)$ where $M = M(T)$ is the same as in Lemma 7.3 and $\bar{A}_n \rightarrow \bar{A} = (2 - c)/c$. [The quantity \bar{g}_n can be calculated explicitly and it implies $\bar{g}_n \rightarrow \bar{g}$; the quantity \bar{A} appears as the smaller root of the equation

$$\bar{A} = \frac{c}{4}\bar{A} + \frac{\left(1 + \frac{c}{2}\bar{A}\right)^2}{2 + \frac{c\bar{g}}{M} - c\bar{A}}.]$$

The convergence relations $\bar{g}_n \rightarrow \bar{g}$ and $\bar{A}_n \rightarrow \bar{A}$ enable us to prove that the measures $\mu_{n,N}^{h_N}(d\sigma|T)$ tend to some measure $\bar{\mu}_n$ as $N \rightarrow \infty$, and these measures $\bar{\mu}_n$ are the finite-dimensional distributions of an equilibrium state $\mu_e = \mu_e(T)$ of Dyson's vector-valued hierarchical ϕ^4 model. Moreover, we get the following formula for the Radon-Nikodym derivative $d\bar{\mu}_n/d\mu_n$:

$$\bar{\mu}_n(d\sigma|T) = f_n\left(2^{-n} \sum_{i \in V_n} \sigma(i)\right)\mu_n(d\sigma|T),$$

with

$$f_n(x) = L_n \exp\left\{\frac{c^n}{T}\bar{g}(x_1 - M) + \frac{c^n}{T}\bar{A}x_2^2 + O(\xi^n)\right\}.$$

This relation together with the multidimensional version of Lemma 6.7 enables us to prove the following:

THEOREM 7.4. *Let $\sigma(k) = (\sigma^{(1)}(k), \sigma^{(2)}(k)) \in R^2$, $k \in Z$, be a $\mu_e = \mu_e(T)$ distributed equilibrium state in Dyson's vector-valued hierarchical ϕ^4 model with $1 < c < \sqrt{2}$. Let us consider the random fields $Y_n(k) = (Y_n^{(1)}(k), Y_n^{(2)}(k)) \in R^2$, $k \in Z$, $n = 1, 2, \dots$, defined by formula (3.4) with the normalizing constants $A_n = (2/c)^n$ and $B_n = (2/\sqrt{c})^n$ and the μ_e distributed random field $\sigma(k)$. Then the finite-dimensional distributions of the fields $Y_n(k)$ tend to those of a random field $Y(k) = (Y^{(1)}(k), Y^{(2)}(k))$, $k \in Z$, as $n \rightarrow \infty$. The probability density of the*

random vector $(Y(1), \dots, Y(2^k))$ is given by the formula

$$\text{const.} \prod_{j=1}^{2^k} f\left(x_j + \frac{y_j^2}{2M}\right) \exp\left\{\frac{1}{T} \left[\frac{2-c}{c \cdot 2^{2k}} c^k \left(\sum_{j=1}^{2^k} y_j\right)^2 - \sum_{j=1}^{2^k} \frac{y_j^2}{2-c} + \sum_{j=1}^{2^k} \sum_{l=1}^{j-1} d(j, l)^{-\alpha} y_l y_j \right]\right\},$$

where the function f is the solution of equation (7.1), and the constant $M = M(T)$ is the same as in Lemma 7.3.

Here the random field $Y^{(2)}(k)$, $k \in \mathbb{Z}$, is Gaussian, and similar random fields have appeared in Theorem 6.2 for $c > \sqrt{2}$ and $T = T_{\text{cr}}$ and in Theorem 7.1 in the direction orthogonal to the magnetization. The random field $Y^{(1)}(k) + Y^{(2)}(k)^2/2M$ is independent of the random field $Y^{(1)}(k)$, and it consists of independent random variables with the probability density $f(x)$.

8. On translation invariant equilibrium states. In this section we deal with the renormalization of translation invariant states. We present some results, formulate some conjectures and give heuristic arguments which may justify them. We are mainly interested in the following problem: Let us consider a translation invariant Hamiltonian H on the integer lattice \mathbb{Z}^d in the Euclidean space R^d , a free measure ν and the equilibrium states $\mu(T)$ corresponding to this H and ν . Prove that there exists a critical temperature $T = T_{\text{cr}}$ such that a $\mu(T_{\text{cr}})$ distributed random field X_n , $n \in \mathbb{Z}^d$, has a large-scale limit with an unusual normalization, i.e., the random fields defined by formula (0.1) with this field X_n have a limit as $N \rightarrow \infty$, and $\lim A_N N^{-d/2} = \infty$. How must the norming constants A_N be chosen, and which fields appear as a large-scale limit? The description of the possible fields leads to the following:

DEFINITION 8. The translation invariant random field X_n , $n \in \mathbb{Z}^d$, is self similar with self-similarity parameter α if the random fields defined in terms of this field by formula (0.1) with the choice $A_N = N^{\alpha d}$ have the same distribution as the original field X_n , $n \in \mathbb{Z}^d$.

Heuristically it is clear that only self-similar fields can appear as a large-scale limit. Hence the description of self-similar fields is an important problem, but it is only partially solved. The Gaussian self-similar fields are known and we describe them in the next theorem. Given a homogeneous function $q(p)$ on R^d introduce the function

$$(8.1) \quad q^D(p) = \sum_{k \in \mathbb{Z}^d} q(p + 2k\pi) |\tilde{\chi}_0(p + 2k\pi)|^2,$$

where

$$(8.2) \quad \tilde{\chi}_0(p) = \prod_{k=1}^d \frac{e^{ip_k} - 1}{ip_k}, \quad p = (p^{(1)}, \dots, p^{(d)})$$

is the Fourier transform of the uniform distribution on the unit cube $[0, 1]^d$.

THEOREM 8.1 (see, e.g., [16], [36]). *Let $q(p)$ be a homogeneous function of homogeneity order $d - a$, i.e., let $q(\lambda p) = \lambda^{d-a}q(p)$ for all $\lambda > 0$ and $p \in \mathbb{R}^d$, such that the function $q^D(p)$ defined in (8.1) is positive and integrable on the torus $T^d = \{-\pi \leq p^{(k)} \leq \pi, k = 1, \dots, d\}$. Then the stationary Gaussian random field $\sigma(j)$, $j \in \mathbb{Z}^d$, with $E\sigma(j) = 0$ and spectral density $q^D(p)$ is self-similar with self-similarity parameter $\alpha = a/2d$.*

Moreover, we have the following:

THEOREM 8.2. *If the series*

$$\sum_{j \in \mathbb{Z}^d} U(j), \quad U(j) = (2\pi)^{-d} \int_{[-\pi, \pi]^d} e^{ijp} \frac{1}{q^D(p)} dp$$

is absolutely convergent then the distribution of the Gaussian field defined in Theorem 8.1 is the (unique) Gibbs field with the Hamiltonian

$$H_0(\sigma) = \frac{1}{2} \sum U(i - j)\sigma(i)\sigma(j)$$

and the Lebesgue measure as free measure at temperature $T = 1$.

The Gibbs measure with Lebesgue measure as free measure can be defined in the usual way. In this section we choose the Lebesgue measure for free measure if it is not otherwise stated. Moreover, we consider more general Hamiltonians $H(\sigma) = \sum_k \sum U(j_1, \dots, j_k)\sigma(j_1) \dots \sigma(j_k)$. The modification of the definitions to this more general case causes no problem.

After the description of Gaussian self-similar fields our aim is to study their stability with respect to the transformation (0.1) with $A_n = N^{\alpha d}$. The two most important Gaussian self-similar fields are those with $q(p) \equiv 1$ for which $\alpha = \frac{1}{2}$, and $q(p) = 1/|p|^2$ for which $\alpha = \frac{1}{2} + 1/d$. The first field consists of independent identically distributed random variables and it is called the white noise. The second one is called the massless free field. Its correlation function decreases at infinity asymptotically as $\text{const}|n|^{-d+2}$, $n \in \mathbb{Z}^d$. The white noise is a Gibbsian field with the Hamiltonian function $H = H^0 = C \sum_{j \in \mathbb{Z}^d} \sigma^2(j)$, and the massless free field is a Gibbsian field with a Hamiltonian

$$H = H^1 = C \sum_{i, j \in \mathbb{Z}^d} U(i - j)\sigma(i)\sigma(j)$$

such that $U(j)$ decreases exponentially fast as $j \rightarrow \infty$. The last statement can be seen by observing that $U(j)$ can be calculated with the help of Theorem 8.2, and it yields in the present case that $U(j)$ is the Fourier transform of an analytic function.

Let us introduce the space \mathcal{H} of Hamiltonians

$$(8.3) \quad H(\sigma) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j_1, \dots, j_k \in \mathbb{Z}^d} U_k(j_1, \dots, j_k) \sigma(j_1) \dots \sigma(j_k),$$

where the potentials $U(j_1, \dots, j_k)$ are translation invariant, i.e.,

$$(8.4) \quad U_k(j_1 + a, \dots, j_k + a) = U_k(j_1, \dots, j_k), \quad \forall a \in \mathbb{Z}^d,$$

permutation invariant, i.e., $U_k(j_{\tau(1)}, \dots, j_{\tau(k)}) = U_k(j_1, \dots, j_k)$ for all permutations $\tau = (\tau(1), \dots, \tau(k))$ and exponentially decreasing at infinity, i.e.,

$$(8.5) \quad |U_k(j_1, \dots, j_k)| \leq C_k \exp\left\{-\alpha_k \max_{1 \leq p < r \leq k} |j_p - j_r|\right\},$$

with some $C_k > 0$ and $\alpha_k > 0$.

A Hamiltonian \mathcal{H} is called even if $U_k = 0$ whenever k is odd. We denote by \mathcal{H}_{ev} the space of even Hamiltonians $\mathcal{H}(\sigma) \in \mathcal{H}$. Clearly, $H^0, H^1 \in \mathcal{H}_{\text{ev}}$.

We begin the study of the stability of Gaussian self-similar fields with the investigation of the spectrum of the linearized renormalization transformation $\mathcal{D}_n^{(\alpha)}$. This transformation acts in the space of Hamiltonians H , and it is defined in the following way:

$$(8.6) \quad \mathcal{D}_n^\alpha H(\sigma) = \lim_{L \rightarrow \infty} E(H_V(\bar{\sigma}) | R_n^{(\alpha)} \bar{\sigma} = \sigma),$$

where V is the cube $\{0, 1, \dots, Ln - 1\}^d$, $R_n^{(\alpha)} \bar{\sigma} = \{n^{-\alpha d} \sum_{i \in D_j^n} \bar{\sigma}(i), j \in \mathbb{Z}^d\}$, $H_V(\sigma)$ is just as $H(\sigma)$ in (8.3) with the difference that now the summation is taken only for $j_1, \dots, j_k \in V$, and the conditional expectation is taken with respect to the Gaussian self-similar field whose stability is investigated. The convergence in (8.6) is meant in the following sense: We can write

$$E(H_V(\bar{\sigma}) | R_n^{(\alpha)} \bar{\sigma} = \sigma) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j_1, \dots, j_k \in \mathbb{Z}^d} U_{k,V}(j_1, \dots, j_k) \sigma(j_1) \dots \sigma(j_k)$$

and we want $U_{k,V}(j_1, \dots, j_k) \rightarrow U_k(j_1, \dots, j_k)$. The motivation behind the definition of the linearized renormalization group can be explained by the following heuristic argument: Consider a Gibbs state $\mu_V(d\sigma) = \Xi^{-1} \exp(-H_{0,V}(\sigma) - \varepsilon H_V(\sigma)) \prod_{i \in V} d\sigma(i)$ in a cube with edges of length nL , where $H_0(\sigma)$ is the Hamiltonian of the underlying Gaussian field. Let us write the renormalization $R_n^{(\alpha)} \mu_V$ of this measure in the form of a Gibbs state

$$\Xi'^{-1} \exp\left(-\sum_{k=0}^{\infty} \varepsilon^k H_{k,V',n}(\sigma)\right) \prod_{i \in V'} d\sigma(i)$$

over a cube V' with edges of length L . Then we expect that the first two coefficients in the last series satisfy the relations $\lim_{L \rightarrow \infty} H_{0,V',n} = H_0$ and $\lim_{L \rightarrow \infty} H_{1,V',n} = \mathcal{D}_n^\alpha H$. Thus, given a Gibbs state with the Hamiltonian $H_0 + \varepsilon H$, its renormalization defined in (0.1) with $A_n = n^{\alpha d}$ is a new Hamiltonian $H_0 + H(\varepsilon, n)$ (although even this statement is not proved), and \mathcal{D}_n^α is the linearization of the transformation $\varepsilon H \rightarrow H(\varepsilon, n)$.

The semigroup of the linear operators $\{\mathcal{D}_n^\alpha, n = 1, 2, \dots\}$ is called the linearized renormalization group. The following theorem describes the eigen-Hamiltonians and eigenvalues of the operators \mathcal{D}_n^α (see [36] and [5]). Let μ_0 be a Gaussian self-similar field with self-similarity parameter α . Let $q^D(p)$ denote its

spectral density. In the most important applications μ_0 is either white noise or the massless free field.

THEOREM 8.3. *Let $h(p_1, \dots, p_m)$, $p_1 \in R^d, \dots, p_m \in R^d$, be an analytic homogeneous function with homogeneity order $\gamma = 0, 2, 4, \dots$, which is isotropic, i.e., $h(tp_1, \dots, tp_m) = t^\gamma h(p_1, \dots, p_m)$, $t > 0$. Then*

$$(8.7) \quad H(\sigma) = \tilde{H}(\tilde{\sigma}) = \int_{T^{md}} \delta((p_1 + \dots + p_m) \bmod 2\pi) \times h^D(p_1, \dots, p_m) : \tilde{\sigma}(p_1) \cdots \tilde{\sigma}(p_m) : dp_1 \cdots dp_m$$

is an eigen-Hamiltonian of the linearized renormalization transformation \mathcal{D}_n^α with the eigenvalue n^λ ,

$$(8.8) \quad \lambda = -\gamma + m(\alpha - 1)d + d.$$

Here T^{md} is the md -dimensional torus $\{|p_{i,k}| \leq \pi, i = 1, \dots, m, k = 1, \dots, d\}$, $\delta(\cdot)$ is the δ -function

$$h^D(p_1, \dots, p_m) = \sum_{p_1 + 2\pi k_1 + \dots + p_m + 2\pi k_m = 0} h(p_1 + 2\pi k_1, \dots, p_m + 2\pi k_m) \times \alpha(p_1 + 2\pi k_1) \cdots \alpha(p_m + 2\pi k_m),$$

$$\alpha(p) = \tilde{\chi}_0(p)q(p) \frac{1}{q^D(p)},$$

where $q(p)$ and $q^D(p)$ are the same as in Theorem 8.2, $\tilde{\sigma}(p) = \sum_{j \in \mathbb{Z}^d} \exp(ijp) \sigma(j)$ and $: \tilde{\sigma}(p_1) \cdots \tilde{\sigma}(p_m) :$ denotes the Wick polynomial with respect to the underlying Gaussian self-similar field with the distribution μ_0 . Formula (8.7) gives all eigen-Hamiltonians $H(\sigma) \in \mathcal{H}$.

(The definition of Wick polynomials, which are a multidimensional generalization of Hermite polynomials, can be found, e.g., in [31].)

The eigen-Hamiltonian with the eigenvalue n^λ is called stable if $\lambda < 0$, unstable if $\lambda > 0$ and neutral if $\lambda = 0$. In physics literature they are called irrelevant, relevant and marginal, respectively. The existence of unstable eigen-Hamiltonians leads to the instability of the self-similar random field when the large-scale limit is taken. We list now the unstable and neutral eigen-Hamiltonians of the white noise and massless field. We are only interested in eigen-Hamiltonians even in σ , i.e., $H(\sigma) \in \mathcal{H}_{ev}$.

- (1) White noise $q(p) \equiv 1$, $\alpha = \frac{1}{2}$

$$m = 2, \quad h \equiv 1, \quad \gamma = 0, \quad \lambda = 0;$$
- (2) Massless free field $q(p) = |p|^{-2}$, $\alpha = \frac{1}{2} + 1/d$

$$m = 2, \quad h \equiv 1, \quad \gamma = 0, \quad \lambda = 2,$$

$$m = 4, \quad h \equiv 1, \quad \gamma = 0, \quad \lambda = 4 - d,$$

$$m = 6, \quad h \equiv 1, \quad \gamma = 0, \quad \lambda = 6 - 2d$$

.....

$$m = 2, \quad h = |p|^2, \quad \gamma = 2, \quad \lambda = 0$$

⋮

Due to this list we get the following picture about the stability of Gaussian self-similar random fields. Note that the Hamiltonian H_0 of the Gaussian self-similar field is always a neutral eigen-Hamiltonian. Indeed, $\{(1 + \varepsilon)^{-1/2}\sigma(j), j \in \mathbb{Z}^d\}$ is a Gaussian self-similar field with the Hamiltonian $(1 + \varepsilon)H_0$, hence the renormalization transformation maps εH_0 to εH_0 , and H_0 is a neutral eigen-Hamiltonian. We shall call H_0 the trivial neutral eigen-Hamiltonian. Now we consider the stability properties of the white noise and massless free field.

8.1. *White noise, or independent Gaussian random variables.* There is no unstable eigen-Hamiltonian and only one trivial neutral eigen-Hamiltonian. One can expect that this fixed point is stable with respect to small perturbations in the large-scale limit. This can be proved, and this fact is called in the physical literature the stability of the high-temperature fixed point. In the classical probability theory this expresses the central limit theorem for a small perturbation of the field of independent Gaussian random variables. We formulate a theorem which expresses the above-mentioned stability.

THEOREM 8.4. *Let $H^0 = \sum_{j \in \mathbb{Z}^d} \sigma^2(j)$, and*

$$H'(\sigma) = \sum_{k=1}^N \sum_{j_1, \dots, j_k} U_k(j_1, \dots, j_k) \sigma(j_1) \dots \sigma(j_k)$$

be a translation invariant Hamiltonian, even in σ , and such that the coefficients $U_k(j_1, \dots, j_k)$ decrease exponentially as $\text{diam}\{j_1, \dots, j_k\} \rightarrow \infty$ and $H'(\sigma) \geq 0$. Then there exists some $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the Gibbs state μ with the Hamiltonian $H = H^0 + \varepsilon H'$ exists, it is unique and its large-scale limit consists of independent Gaussian random variables.

In other words, Theorem 8.4 states that if the free measure is Gaussian then the large-scale limit of a field with Hamiltonian H' at high temperature is a field of independent Gaussian random variables.

The proof of Theorem 8.4 is based on the so-called cluster expansion estimates of the semi-invariants of Gibbs states. The conditions both on $H(\sigma)$ and $H'(\sigma)$ can be weakened.

8.2. *The massless free field.* Let us first consider the case $d > 4$. There is one unstable and one trivial neutral eigen-Hamiltonian. The following result is expected to hold (see a discussion in [27]):

CONJECTURE 1. *Let H^1 be the Hamiltonian of the massless free random field, $H^0 = \sum_j \sigma^2(j)$ and let H' be the same as in Theorem 8.4. Then there exists an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, a parameter $\mu_c = \mu_c(\varepsilon, H')$ exists such that there is a unique translation invariant Gibbs state with the Hamiltonian $H = H^1 + \mu_c H^0 + \varepsilon H'$. Its large-scale limit with the renormalizing constant $A_N = N^{(d+2)/2}$ is the massless free field. For $\mu > \mu_c$ there exists a unique Gibbs state, and its large-scale limit is white noise. For $\mu < \mu_c$ there are two extremal Gibbs states, and both have white noise as the large-scale limit.*

In connection with the above conjecture we mention a result of [1] and [20] (see also [21]). It states, in accordance with the above conjecture, that if in the so-called Φ^4 or Ising model for dimension $d \geq 4$, the large-scale limit of the correlation function exists, then the large-scale limit is a Gaussian random field. (For $d = 4$ some additional conditions are needed.)

Recently in a series of papers [27]–[29] Gawędzki and Kupiainen made substantial progress toward the proof of Conjecture 1 (see below).

For $d < 4$ the linearized renormalization of the massless free field has at least two unstable eigen-Hamiltonians. Hence we cannot expect that in the general case a one-parameter family of Hamiltonians contains a critical parameter where the large-scale limit is the massless free field. We remark that if H_0 has m unstable eigen-Hamiltonians H_1, \dots, H_m , we may expect the existence of a critical point $\mu_1^c(\epsilon), \dots, \mu_m^c(\epsilon)$ such that the equilibrium state with the Hamiltonian $H = H_0 + \mu_1^c(\epsilon)H_1 + \dots + \mu_m^c(\epsilon)H_m + \epsilon H'$ has the large-scale limit with the Hamiltonian H_0 .

CONJECTURE 2. *For $d = 2, 3$ there exists a (unique up to a factor) rotationally invariant self-similar random field with an exponent $\alpha > \frac{1}{2}$, which is an equilibrium state with a Hamiltonian $H^2 \in \mathcal{H}_{ev}$ such that the corresponding linearized renormalization transformation has exactly one unstable eigen-Hamiltonian. For the Hamiltonian $H = H_0 + \mu H_1 + \epsilon H'$ all statements of Conjecture 1 are valid.*

This self-similar random field cannot be Gaussian. Conjecture 2 states that for $d = 2, 3$ a non-Gaussian self-similar random field exists, which plays the same role as the massless free field for $d \geq 4$. An exact formula for the correlation function of the two-dimensional Ising model at the critical temperature suggests that for $d = 2$, $\alpha = \frac{15}{16}$. There is a conjecture about what the two-dimensional self-similar field appearing in Conjecture 2 should look like.

CONJECTURE 3. *For $d = 2$ the non-Gaussian self-similar random field in Conjecture 2 is the discretization*

$$\sigma(j) = \int_0^1 \int_0^1 \xi(x+j) dx_1 dx_2, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad j \in \mathbb{Z}^2,$$

of the generalized random field $\xi(x)$, $x \in \mathbb{R}^2$, with the correlations

$$E\xi(x_1) \dots \xi(x_{2k}) = \sqrt{\sum_{\eta} \exp\left(-\frac{1}{2} \sum_{i \geq j} \eta_i \eta_j \ln|x_i - x_j|\right)},$$

$$E\xi(x_1) \dots \xi(x_{2k+1}) = 0,$$

where the summation is taken for such $\eta = (\eta_1, \dots, \eta_{2k})$ for which $\eta_i = \pm 1$ and $\sum_{i=1}^{2k} \eta_i = 0$.

This conjecture has a long history, and it is connected with several rigorous and nonrigorous papers devoted to the exact solution of the two-dimensional

Ising model. The formula presented above was obtained in a nonrigorous way in physics (see [41] and others). Actually it has not even been proved that the field defined in Conjecture 3 really exists.

In the case $d = 4$ the massless free field has one unstable and two neutral eigen-Hamiltonians. It is expected that Conjecture 1 holds again for $d = 4$ (compare with the case $c = \sqrt{2}$ in the hierarchical model), only the rate of convergence is slow.

In the above conjectures the cases $d > 4$, $d < 4$ and $d = 4$ are similar to those $c > \sqrt{2}$, $c < \sqrt{2}$, $c = \sqrt{2}$ in Dyson's hierarchical model. The spectrum of the linearized renormalization transformation in the translation invariant case has a behavior similar to the differential operator DS . The conjectures actually state that this similarity is preserved for the original renormalization operators.

As in the case $c = \sqrt{2}$ in Dyson's hierarchical model a bifurcation of non-Gaussian random fields is expected for "dimension" $d = 4 - \epsilon$ starting from the four-dimensional massless free field. Wilson ([39], [40]) investigated the ϵ -expansion for these fields. It is a rather tricky thing to define "random fields" for noninteger dimensions, but for rotationally invariant random fields it can be done with the help of analytic continuation in dimensionality. This is similar to the analytic continuation of $\Gamma(n + 1) = n!$ to noninteger n , and it uses an integral representation of correlation functions. Wilson and later other authors investigated the first terms of the ϵ -expansion of the correlation functions and other characteristics of these self-similar fields.

There are some interesting models where the convergence of the large-scale limit to the massless free field is proved. Such models are considered in [32] and [28] (see also [27]). They proved that the large-scale limit of the Gibbs states with the Hamiltonian $H = H^1 + \epsilon H'$ is the massless free field for all small $\epsilon > 0$ if H' has a special structure. Namely H^1 is the Hamiltonian of the massless free field, $H' = \sum_p \Phi'(T_p \sigma)$, $\Phi(\sigma) = \phi(\nabla^2 \sigma)$ in [32] $\Phi(\sigma) = \phi(\nabla \sigma)$ in [28], where ϕ is a sufficiently smooth function,

$$\nabla \sigma = \{ \nabla_i \sigma, i = 1, \dots, d \}, \quad \nabla^2 \sigma = \{ \nabla_{ik} \sigma, i, k = 1, \dots, d \}$$

and

$$\begin{aligned} \nabla_i \sigma(j) &= \sigma(j + e_i) - \sigma(j), & e_i &= (0, \dots, 0, \underset{i}{1}, 0, \dots, 0), \\ \nabla_{ik} \sigma(j) &= \sigma(j + e_i + e_k) - \sigma(j + e_i) - \sigma(j + e_k) + \sigma(j). \end{aligned}$$

In the proof the cluster-expansion technique was used. The applicability of this technique is connected with the convergence of the series $\sum_{p \in \mathbf{Z}^d} E |\nabla_{ik} \sigma(0) \nabla_{lm} \sigma(p)|$, where E denotes expectation with respect to the massless free field with Hamiltonian H^1 . Since the series $\sum_{p \in \mathbf{Z}^d} E |\nabla_i \sigma(0) \nabla_k \sigma(p)|$ is logarithmically divergent, in [28] the cluster-expansion technique could not be applied directly, it must be refined. This refinement uses a combination of the cluster-expansion technique with a renormalization procedure and some estimations of the analytic continuation of the random field to the complex space. Let us emphasize that in all models of [32] and [28] the large-scale limit of the equilibrium states is the massless free field. This is in contrast to Conjecture 1,

where we expect that only one element of a one-parameter family of equilibrium states has the massless free field as the large-scale limit. The reason for this difference is the very special form of the Hamiltonian H' in [32] and [28]. Recently with the help of their technique Gawędzki and Kupiainen [29] proved the existence of the critical point μ_c for the one-parameter family of four-dimensional Gibbs states with the Hamiltonian $H = H^1 + \mu H^0 + \varepsilon H'$, where $H^0 = \sum_j \sigma^2(j)$, $H^1 = \sum_j \sigma^4(j)$, $0 < \varepsilon \ll 1$, i.e., they proved that for $\mu = \mu_c = \mu_c(\varepsilon)$ the large-scale limit of the Gibbs state with Hamiltonian H is the massless free field.

It is known that the large-scale limit of some infinite particle systems like the voter model, the critical branching diffusion etc. is again the massless free field (see [12], [30] and [15]). It is a surprising fact that in these models an unstable fixed point of the renormalization group appears as the large-scale limit. This may be connected with some hidden symmetries of these models, but the real reason is not completely understood.

Very little is rigorously known about the large-scale limit of vector-valued translation invariant Gibbs states at critical and low temperatures.

As a first step we would like to describe the decrease of the correlation function at infinity. In certain cases this problem has been solved for two-component systems (see [13]).

Let us consider pure states at low temperature and decompose the spin variables $\sigma(n)$, $n \in \mathbb{Z}^d$, as $\sigma(n) = \sigma^\perp(n) + \sigma^\parallel(n)$, where $\sigma^\perp(n)$ denotes the component of $\sigma(n)$ orthogonal and $\sigma^\parallel(n)$ the component parallel to the spontaneous magnetization $E\sigma(n) = M$. It is known that $E\sigma^\perp(0)\sigma^\perp(n)$ is of order $|n|^{-d+2}$ ([13] and [38]). On the other hand, $|E\sigma^\parallel(0)\sigma^\parallel(n) - |M|^2| \geq (E\sigma^\perp(0)\sigma^\perp(n))^2$ also holds (see [18]). These results together with the results obtained for Dyson's hierarchical vector model suggest the following picture:

For all $d > 2$ and all low temperatures in the direction orthogonal to the spontaneous magnetization one has to divide by $A_N = N^{(d+2)/2}$ in (0.1), and the large-scale limit is the massless free field. (For $d = 2$ there is no phase transition.) For $d > 4$ one has to divide by $A_N = N^{d/2}$ in the direction of the spontaneous magnetization, and the limit is a field of independent Gaussian variables, which are also independent of the large-scale limit of the component orthogonal to the spontaneous magnetization. The case $d = 4$ is similar, only some logarithmic term appears in the renormalizing constant A_N . For $d = 3$ the large-scale limit in the direction of the spontaneous magnetization is a non-Gaussian self-similar random field, and $A_N = N^2$. We expect that the large-scale limit in this case has a structure similar to the limit field appearing in Theorem 7.4, but this question demands further investigation.

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