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Central Limit Theorems for Non-Linear Functionals of Gaussian Fields

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In the present paper it is shown that the central limit theorem holds for some non-linear functionals of stationary Gaussian fields if the correlation function of the underlying field tends fast enough to zero. The results are formulated in terms of the Hermite rank of the functional and of the rate of the correlation function. Then we show an example when the limit field is self-similar and Gaussian but not necessarily consisting of independent elements.

I. INTRODUCTION

Recently several papers have dealt with the asymtotical distribution of non-linear functionals of Gaussian fields (see, e.g., [1–3, 6, 7]). It has been shown that if the correlation function of the underlying Gaussian field tends sufficiently slowly to zero, i.e., the dependence between distant terms is large, then a new type of limit theorem appears. In this paper we aim at giving some results in a different situation. Roughly speaking, we state that if the correlation function tends to zero faster than in the case investigated in the above papers then the central limit theorem holds again.

In order to formulate our results we introduce some notations. Let Z^ν denote the integer lattice in the ν-dimensional Euclidean space $\mathbb{R}^\nu$. Let X_n, $n \in Z^\nu$, be a ν-dimensional stationary Gaussian field with zero mean and unit

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variance. Put $r(n) = EX_m X_{n+m}$, n, $m \in Z^\nu$. Let $H(x)$ be a real-valued function such that

$$\int H(x) \exp(-x^2/2)\, dx = 0,$$
$$\int H^2(x) \exp(-x^2/2)\, dx < \infty. \tag{1.1}$$

Then $H(x)$ can be expanded in the form

$$H(x) = \sum_{j=1}^{\infty} c_j H_j(x), \tag{1.2}$$

$$\sum_{j=1}^{\infty} c_j^2\, j! < \infty, \tag{1.2'}$$

where H_j is the jth Hermite polynomial with leading coefficient 1. We say that $H(x)$ has Hermite rank k if in the expansion (1.2) $c_1 = \cdots = c_{k-1} = 0$ and $c_k \neq 0$. (The setup (1.1) and (1.2) and the notion of Hermite rank was introduced in [5].) We define the sets

$$B(n, N) \subset Z^\nu, \qquad n \in Z^\nu, \quad N = 1, 2, \ldots,$$

$B(n, N) = \{s = (s^{(1)}, \ldots, s^{(\nu)}) \in Z^\nu,\ n^{(t)} N < s^{(t)} \leqslant (n^{(t)} + 1)N,\ t = 1, \ldots, \nu\}$ and the random fields

$$Z_n^N, \qquad n \in Z^\nu, \quad N = 1, 2, \ldots,$$
$$Z_n^N = Z_n^N(H) = A_N^{-1} \sum_{j \in B(n,N)} H(X_j), \tag{1.3}$$

where A_N are appropriate norming constants. Now we formulate the following

THEOREM 1. *Suppose that the function H has Hermite rank k and the correlation function of the stationary Gaussian field X_n satisfies the condition*

$$\sum_{n \in Z^\nu} |r(n)|^k < \infty. \tag{1.4}$$

Put $A_N = N^{\nu/2}$. Then the limits

$$\lim_{N \to \infty} E(Z_0^N(H_l))^2 = \lim_{N \to \infty} A_N^{-2}\, l! \sum_{i \in B(0,N)} \sum_{j \in B(0,N)} r^l(i-j) = \sigma_l^2\, l!$$

exist for all $l \geqslant k$, and the infinite sum

$$\sigma^2 = \sum_{l=k}^{\infty} c_l^2\, l!\, \sigma_l^2 < \infty.$$

The finite dimensional distributions of the fields $Z_n^N(H)$ defined in formula (1.3) tend, as $N \to \infty$, to the finite dimensional distributions of the field σZ_n^*, where Z_n^*, $n \in Z^\nu$ are independent standard normal random variables.

Some earlier results on L_2-functionals of stationary Gaussian sequence appeared in [4]. In that paper Sun considered Gaussian sequences with square-integrable spectral density function (i.e., $\sum r^2(n) < \infty$) and suppose in addition that

$$\lim_{N\to\infty} E(Z_0^N(H_1))^2 = \lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} r(i-j)$$

exists and finite. Then he showed that a proposition similar to our theorem holds. Sun's proof relies upon some calculations on the spectral density function while our approach seems to be more direct.

Condition (1.4) can be slightly weakened. Theorem 1 remains valid with a possibly different normalization if the sum in (1.4) slowly tends to infinity. This fact is formulated in the following

THEOREM 1′. *Let the function H be the same as in Theorem 1. Assume that the correlation function of the Gaussian field X_n satisfies the conditions*

$$\sum_{n\in \bar{B}(0,N)} |r(n)|^k = L(N) \tag{1.4′}$$

and

$$\lim_{N\to\infty} (L(N))^{-1} \sum_{j\in \bar{B}(0,N)} r^l(j) \tag{1.4″}$$

exists for all $l \geqslant k$, where $L(N)$ is a slowly varying function and

$$\bar{B}(0,N) = \{n = (n^{(1)},\dots, n^{(\nu)}) \in Z^\nu, -N \leqslant n^{(j)} \leqslant N, j = 1,\dots, \nu\}.$$

Put $A_N = N^{\nu/2} L(N)^{1/2}$. Then the limits

$$\lim_{N\to\infty} E(Z_0^N(H_l))^2 = \sigma_l^2\, l! = \lim_{N\to\infty} (N^\nu L(N))^{-1} l! \sum_{i\in B(0,N)} \sum_{j\in B(0,N)} r^l(i-j)$$

exist for all $l \geqslant k$, and the infinite sum

$$\sigma^2 = \sum_{l=k} c_l^2\, l!\, \sigma_l^2 < \infty.$$

The finite dimensional distributions of the fields $Z_n^N(H)$ (defined with the new norming constants A_N) tend to those of the fields σZ_n^*, as $N \to \infty$.

In [1] the limit behaviour of the fields was investigated in the case when the correlation function of the underlying Gaussian field satisfies the relation

$$r(n) \sim n^{-\alpha} L'(|n|)\, a\left(\frac{n}{|n|}\right), \qquad k\alpha < \nu, \tag{1.4a}$$

where $L'(.)$ is a slowly varying function on Z_+ and $a(.)$ is a continuous function on the unit sphere. It was shown that in this case a new type of limit theorem holds. On the other hand, Theorems 1 and 1′ imply that in the case $k\alpha \geqslant \nu$ the central limit theorem holds again. In the case $k\alpha = \nu$ it may happen that the norming constant A_N must be chosen as $A_N = N^{\nu/2} L(N)$, $L(N) \to \infty$. Thus, e.g., if in the formula (1.4a) $L'(N) \equiv 1$, $a(x) \equiv 1$ then we have to choose $A_N = N^{\nu/2} (\log N)^{1/2}$ in Theorem 1′. Of course there is a gap between the conditions (1.4) and (1.4a). In the case when the sequence X_n has a spectral density function the condition (1.4) for $k = 2$ is equivalent with the square-integrability of the spectral density function which in turn does not imply (1.4a).

Let us emphasize that it may occur that $\sigma = 0$ in Theorem 1 or 1′. We show such an example. In this case the fields Z_n^N tend to zero. Hence it is natural to look for a different normalization.

EXAMPLE (see also [5, Remark, page 298]). Let $\nu = 1$, $H(x) = x$, X_n be a stationary Gaussian sequence with spectral density $g(x)$, $-\pi \leqslant x < \pi$, which is sufficiently smooth outside zero. Moreover, let $g(x) \sim |x|^{\alpha}$, $1 > \alpha > 0$, in a neighbourhood of zero. Then some calculation shows that

$$E(Z_0^N(H))^2 = \frac{1}{N} \int_{-\pi}^{\pi} \frac{1 - \cos Nx}{1 - \cos x}\, g(x)\, dx$$

$$= \int_{|x| \leqslant 1/N} + \int_{1/N < |x| \leqslant \pi} = I_1 + I_2.$$

Since $(1 - \cos Nx)/(1 - \cos x) \leqslant N^2$ hence $I_1 \leqslant \text{const} \cdot N^{-\alpha}$ and since $(1 - \cos Nx)/(1 - \cos x) \leqslant \text{const}\, |x|^{-2}$ hence $I_2 \leqslant \text{const} \cdot N^{-\alpha}$. Therefore $\sigma_1^2 = \sigma^2 = 0$. We remark that in our example the correlation function of X_n satisfies $r(n) \sim cn^{-1-\alpha}$. It is not difficult to make such examples where $\sigma_k^2 = 0$, $k \geqslant 2$. This can be done if the spectral measure of the field $H_k(X_n)$ is similar to the spectral measure of the previous example. On the other hand, the spectral measure of $H_k(X_n)$ is a multiple of the k-fold convolution of the spectral measure of the sequence X_n. (The convolution is taken on the unit circle.)

Theorems 1 and 1′ can be generalized. The condition on stationarity can be dropped, e.g., but in this case a uniform estimate is needed on the correlation function. Thus condition (1.4) can be substituted by

$$\sum_{n \in Z^\nu} |EX_m X_{n+m}|^k < C \tag{1.4b}$$

with some $C > 0$ for all $m \in Z^\nu$.

One can also consider more general functionals of the Gaussian fields. Let us choose some function $H(x_1, ..., x_s)$, $H: \mathbb{R}^s \to \mathbb{R}$ and some lattice points $d_1, ..., d_s \in Z^\nu$ such that $EH(X_{d_1}, ..., X_{d_s}) = 0$, $EH^2(X_{d_1}, ..., X_{d_s}) < \infty$. We define $T^m H$, $m \in Z^\nu$, as $T^m H(X_{d_1}, ..., X_{d_s}) = H(X_{d_1+m}, ..., X_{d_s+m})$. Then if the field X_n satisfies the conditions of Theorem 1 then the multidimensional distributions of the fields

$$Z_n^N(H) = N^{-\nu/2} \sum_{m \in B(n,N)} T^m H, \qquad n \in Z^\nu, \quad N = 1, 2, ...$$

tend to those of the field σZ_n^*. The Hermite rank of the function H is defined as the biggest k such that $H(X_{d_1}, ..., X_{d_s})$ is orthogonal to all polynomials of the random variables X_n, $n \in Z^\nu$, of order less than k.

The above statement can be proved similarly to Theorem 2 with the help of some ideas from the last section of [1]. We remark that in our theorems not only the behaviour of the underlying Gaussian field but also the Hermite rank of the function H plays an important role. The next example shows that these results do not follow from the customary central limit theorems for weakly dependent random variables.

Given a Gaussian sequence X_n, $n \in \mathbb{Z}$, and a real Borel-measurable function $G(x)$ define the σ-fields

$$\mathfrak{F}_{a,b}(G) = \sigma\{G(X_j), a < j < b\}, \qquad -\infty \leqslant a < b \leqslant \infty.$$

We can find two functions $G_2(x)$ and $G_3(x)$ with the following properties:

(i) With $G_1(x) = x$ the σ-fields $\mathfrak{F}_{a,b}(G_i)$, $i = 1, 2, 3$, are identical for all a and b.

(ii) G_i has Hermite rank i.

This has the following consequence: If the correlation function of the underlying Gaussian sequence X_n is

$$r(n) \sim n^{-\alpha}, \qquad \tfrac{1}{3} < \alpha < \tfrac{1}{2},$$

then the central limit theorem holds for the sequence $G_3(X_n)$, $n \in Z$. On the other hand, a non-central limit theorem holds for the sequence $G_2(X_n)$, $n \in Z$. Since these sequences generate the same σ-fields our central limit theorem cannot follow from any mixing-type conditions.

For example, $G_3(x)$ may be

$$\begin{array}{ll} x, & \text{if } |x| \geqslant c, \\ -x, & \text{if } |x| < c, \end{array}$$

where c is the only positive solution of the equality

$$1/\sqrt{2\pi} \int_0^c x^2 \exp(-x^2/2)\, dx = \tfrac{1}{4}.$$

The construction of $G_2(x)$ is also very simple but more tedious, therefore we omit the details.

This paper consists of three sections. In Section 2 we prove Theorems 1 and 1′. In Section 3 we discuss a model where the limit is a self-similar but not necessarily independent Gaussian field. The investigation of this model was motivated by [2].

II. Proofs

We shall prove only Theorem 1 in detail. We apply the following formula: if ξ and η are jointly Gaussian random variables, $E\xi = E\eta = 0$, $E\xi^2 = E\eta^2 = 1$, $E\xi\eta = r$, then

$$EH_k(\xi)\, H_l(\eta) = \delta(k, l)\, r^k k!, \tag{2.1}$$

where δ denotes the Kronecker delta. (Later we shall apply a more general formula.) It follows from (2.1) that

$$E(Z_n^N(H))^2 = \sum_{l=1}^{\infty} c_l^2 E(Z_n^N(H_l))^2, \tag{2.2}$$

$$E(Z_n^N(H_l))^2 \leqslant l! \sum_{n \in B(0,N)} |r(n)|^l \leqslant l! \sum_{n \in Z^\nu} |r(n)|^k \tag{2.3}$$

for $l \geqslant k$ and

$$|EZ_0^N(H_l)^2 - EZ_0^M(H_l)^2| \leqslant \varepsilon \sum_{n \in Z^\nu} |r(n)|^l + \sum_{n:\, \min |n^{(j)}| > K} |r(n)|^l \tag{2.4}$$

for arbitrary $K > 0$ and $\varepsilon > 0$ if $N > N(\varepsilon, K)$ and $M > M(\varepsilon, K)$. It follows from (1.4) and (2.4) that the limit σ_l^2 defined in (1.5) exists. Then (2.3), (1.4) and (1.2′) imply that $\sigma^2 < \infty$. Moreover, because of the orthogonality relations in (2.1),

$$E\left\{Z_n^N\left(\sum_{l=T}^{\infty} c_l H_l(x)\right)\right\}^2 = \sum_{l=T}^{\infty} c_l^2 E(Z_n^N(H_l))^2 < \varepsilon \tag{2.5}$$

for all $\varepsilon > 0$ if $T > T(\varepsilon)$. Because of relation (2.5) we can restrict ourselves to the special case when H is a polynomial, i.e., when the sum in (1.2) is finite. So we shall assume that H is a polynomial or order T. We remark that Sun [4] also shows that $H(x)$ can be replaced by polynomials.

Let us fix some lattice points $n_1,..., n_d \in Z^v$ and real numbers $b_1,..., b_d$.

We shall apply the method of moments, i.e., we shall show that the moments of the random variables $\sum_N = \sum_{j=1}^d b_j Z_{n_j}^N$ tend to the moments of an appropriate normal random variable. More precisely,

$$\begin{aligned} \lim_{N\to\infty} E\left(\sum\nolimits_N\right)^p &= (p-1)!!\left(\sum_{j=1}^d b_j^2\sigma^2\right)^{p/2}, \qquad \text{if } p \text{ is even}, \\ &= 0, \qquad \text{if } p \text{ is odd}. \end{aligned} \tag{2.6}$$

Theorem 1 follows from relation (2.6).

To prove relation (2.6) we need a so-called diagram formula about the expectation of a product of Hermite polynomials of standard Gaussian random variables. It can be found, e.g., in Lemma 3.2 of [5], although with a different notation.

We call an undirected graph G with $u_1 + \cdots + u_p$ vertices a diagram of order $(u_1,..., u_p)$ if:

(i) The set of vertices V of the graph G has the form

$$V = \bigcup_{j=1}^p L_j,$$

where

$$L_j = \{(j, l)\colon 1 \leqslant l \leqslant u_j\}, \qquad j = 1,..., p$$

(for $u_j = 0$ define $L_j = \varnothing$). We call L_j the jth level of the graph G.

(ii) Each vertex is of degree 1.

(iii) Edges may pass only between different levels, i.e., for $((j_1, l_1), (j_2, l_2)) \in G$ we have $j_1 \neq j_2$.

Let $\Gamma = \Gamma(u_1,..., u_p)$ denote the set of diagrams with properties (i)–(iii). Given a graph $G \in \Gamma$ let $G(V)$ denote the set of the edges of G. For a

$w \in G(V)$ $w = ((j_1, l_1), (j_2, l_2))$, $j_1 < j_2$, we define the functions $d_1(w) = j_1$ and $d_2(w) = j_2$. Now we formulate the

LEMMA (DIAGRAM FORMULA). *Let* $(X_1, ..., X_p)$, $p \geq 2$, *be a Gaussian vector,* $EX_j = 0$, $EX_j^2 = 1$, $EX_jX_k = r(j, k)$, $j, k = 1, ..., p$. *Then for the Hermite polynomials* $H_{l_1}(x), ..., H_{l_p}(x)$ *we have*

$$E\left\{\prod_{i=1}^{p} H_{l_i}(X_i)\right\} = \sum_{G \in \Gamma} I_G,$$

where $\Gamma = \Gamma(l_1, ..., l_p)$ *and* $I_G = \prod_{w \in G(V)} r(d_1(w), d_2(w))$.

We may observe that formula (2.1) is a special case of the diagram formula: when $p = 2$, there are $k!$ diagrams of order (k, k). In this context it is clear that in (j, l) the index j corresponds to that of the variables and l to the order of the Hermite polynomials.

We shall call a diagram regular if its levels can be paired in such a way that no edge passes between levels in different pairs. Now we turn to the proof of relation (2.6). We have

$$\left(\sum\nolimits_N\right)^p = \left(\sum_{j=1}^{d} \sum_{l=1}^{T} b_j c_l A_N^{-1} \sum_{m \in B(n_j, N)} H_l(X_m)\right)^p$$
$$= \sum_{(\mathbf{j}, \mathbf{l}) \in \mathscr{T}^{(p)}} \prod_{i=1}^{p} S^N(j_i, l_i), \tag{2.7}$$

where

$$\mathscr{T}^{(p)} = \{(\mathbf{j}, \mathbf{l}) = ((j_1, ..., j_p), (l_1, ..., l_p)),\ 1 \leq j_i \leq d,$$
$$1 \leq l_i \leq T,\ j_i \text{ and } l_i \text{ are integers; } i = 1, ..., p\} \tag{2.8}$$

and $S^N(j, l) = A_N^{-1} b_j c_l \sum_{m \in B(n_j, N)} H_l(X_m)$.

We shall prove formula (2.6) by means of the diagram formula and relation (2.7). The main idea of the proof is that in our case only the regular diagrams count, and the contribution of the terms I_G is negligible for nonregular diagrams G.

Let us fix some $(\mathbf{j}, \mathbf{l}) = \{(j_1, ..., j_p), (l_1, ..., l_p)\} \in \mathscr{T}^{(p)}$. We shall investigate the expression

$$E\left\{\prod_{i=1}^{p} S^N(j_i, l_i)\right\}$$
$$= E\left\{\prod_{i=1}^{p} A_N^{-1} b_{j_i} c_{l_i} \sum_{m_i \in B(n_{j_i}, N)} H_{l_i}(X_{m_i})\right\}$$
$$= K(\mathbf{j}, \mathbf{l}) A_N^{-p} \sum_{\mathbf{m} \in M} \sum_{G \in \Gamma} \prod_{w \in G(V)} r(m_{d_1(w)} - m_{d_2(w)}), \tag{2.9}$$

where $K(\mathbf{j}, \mathbf{l}) = \prod_{i=1}^{p} b_{j_i} c_{l_i}$, $\Gamma = \Gamma(l_1, ..., l_p)$ and $M = M(\mathbf{j}, N) = \{\mathbf{m} = (m_1, ..., m_p): m_i \in B(n_{j_i}, N),\ i = 1, ..., p\}$. For a fixed $(\mathbf{j}, \mathbf{l}) = ((j_1, ..., j_p), (l_1, ..., l_p)) \in \mathscr{T}^{(p)}$ and $G \in \Gamma(l_1, ..., l_p)$ define

$$T_G(\mathbf{j}, \mathbf{l}, N) = T_G(N) = A_N^{-p} \sum_{\mathbf{m} \in M(\mathbf{j}, N)} \prod_{w \in G(V)} r(m_{d_1(w)} - m_{d_2(w)}). \quad (2.10)$$

We shall prove the following

PROPOSITION. *If $G = G(\mathbf{l}) = G(l_1, ..., l_p)$ is not a regular diagram then, for all $\mathbf{j} = (j_1, ..., j_p)$,*

$$\lim_{N \to \infty} T_G(\mathbf{j}, \mathbf{l}, N) = 0.$$

First we prove relation (2.6) with the help of the Proposition. By relations (2.7), (2.9) and (2.10) we have

$$E\left(\sum\nolimits_N\right)^p = \sum_{(\mathbf{j}, \mathbf{l}) \in \mathscr{T}^{(p)}} K(\mathbf{j}, \mathbf{l}) \sum_{G \in \Gamma(\mathbf{l})} T_G(\mathbf{j}, \mathbf{l}, N). \quad (2.11)$$

Let $\Gamma^*(l_1, ..., l_p)$ denote the set of regular diagrams in $\Gamma(l_1, ..., l_p)$. If p is an odd number then $\Gamma^*(l_1, ..., l_p)$ is empty. Hence the Proposition and relation (2.11) imply that

$$\lim_{N \to \infty} E\left(\sum\nolimits_N\right)^p = 0 \qquad \text{if } p \text{ is an odd number.} \quad (2.12)$$

If p is an even number then write $p = 2q$. Let us fix a diagram $G \in \Gamma^*(l_1, ..., l_p)$ and define the pairs $(i(1), i(2)), (i(3), i(4)), ..., (i(p-1), i(p))$, where $(i(1), ..., i(p))$ is such a permutation of the set $\{1, 2, ..., p\}$ that edges go only between the levels $i(2m-1)$ and $i(2m)$, $m = 1, ..., q$. Let the $i(2m-1)$th and $i(2m)$th levels of G have cardinality $t(m)$, $m = 1, ..., q$. We can write

$$T_G(\mathbf{j}, \mathbf{l}, N) = \prod_{m=1}^{q} \left\{ A_N^{-2} \sum_{u \in B_1(m)} \sum_{v \in B_2(m)} r^{t(m)}(u - v) \right\},$$

where $B_1(m) = B(n_{j_{i(2m-1)}}, N)$, $B_2(m) = B(n_{j_{i(2m)}}, N)$. Since

$$\lim_{N \to \infty} A_N^{-2} \sum_{u \in B(n, N)} \sum_{v \in B(m, N)} r^l(u - v) = \delta(n, m)\, \sigma_l^2$$

the last relation implies that

$$\begin{aligned} &\lim_{N \to \infty} T_G(\mathbf{j}, \mathbf{l}, N) \\ &\quad = \sigma_{t(1)}^2 \cdot \cdots \cdot \sigma_{t(q)}^2 \qquad \text{if} \quad j_{i(2m-1)} = j_{i(2m)},\ m = 1, ..., q, \\ &\quad = 0 \qquad \text{otherwise.} \end{aligned} \quad (2.13)$$

Define

$$\sum_N(G) = \sum_{\mathbf{j}=(j_1,\ldots,j_p)} K(\mathbf{j}, \mathbf{l})\, T_G(\mathbf{j}, \mathbf{l}, N),$$

where $\mathbf{l} = (l_1, \ldots, l_p)$. It follows from (2.13) that

$$I(G) = \lim_{N\to\infty} \sum_N(G) = \left(\sum_{j=1}^{d} b_j^2\right)^q c_{t(1)}^2 \sigma_{t(1)}^2 \cdots c_{t(q)}^2 \sigma_{t(q)}^2.$$

Because of relation (2.11), the Proposition and the definition of $I(G)$,

$$\lim_{N\to\infty} E\left(\sum_N\right)^p = \sum{}' I(G), \tag{2.14}$$

where the summation in $\sum'$ goes over the regular diagrams with p levels. The number of regular diagrams which contain $2m_j$ levels of cardinality k_j, $j = 1, \ldots, s$, with some integer s such that $\sum_{j=1}^{s} m_j = q$ and with all k_j being different, is

$$\frac{(2q)!}{(2m_1)! \cdots (2m_s)!} \prod_{j=1}^{s} (2m_j - 1)(2m_j - 3) \cdots 1 \cdot \prod_{i=1}^{s} (k_i!)^{m_i}$$
$$= (2q-1)(2q-3) \cdots 1 \cdot \frac{q!}{m_1! \cdots m_s!} (k_1!)^{m_1} \cdots (k_s!)^{m_s}.$$

For such regular diagrams

$$I(G) = \left(\sum_{j=1}^{d} b_j^2\right)^q (c_{k_1}^2 \sigma_{k_1}^2)^{m_1} \cdots (c_{k_s}^2 \sigma_{k_s}^2)^{m_s}.$$

Hence relation (2.14) implies that

$$\lim_{N\to\infty} E\left(\sum_N\right)^p$$
$$\times \sum_{s=1}^{\infty} \sum_{m_1+\cdots+m_s=q} \sum_{\substack{k_j=1\\ j=1,\ldots,s}}^{T} \frac{q!}{m_1! \cdots m_s!} \prod_{j=1}^{s} (k_j!\, \sigma_{k_j}^2 c_{k_j}^2)^{m_j}$$
$$= (2q-1)(2q-3) \cdots \cdot 1 \cdot \left(\sum_{j=1}^{d} b_j^2\right)^q \left(\sum_{l=k}^{T} l!\, c_l^2 \sigma_l^2\right)^q.$$

This relation, togather with (2.12), implies (2.6).

Now we turn to the

Proof of the Proposition. Given a permutation π of the set $\{1,\dots,p\}$ and a diagram $G \in \Gamma(l_1,\dots,l_p)$ we define the diagram πG in the following way: The $\pi(j)$th level of πG has cardinality l_j, $j=1,\dots,p$, and $w=\{(j_1,k_1),(j_2,k_2)\}\in G(V)$ if and only if $\pi(w)=\{(\pi(j_1),k_1),(\pi(j_2),k_2)\}\in \pi G(V)$. Given a diagram $G\in\Gamma(l_1,\dots,l_p)$ we define the integer-valued function k_G on the set $\{1,\dots,p\}$ in the following way: $k_G(j)$ is the cardinality of the edges $w\in G(V)$ such that $d_1(w)=j$.

Observe that for $G\in\Gamma(l_1,\dots,l_p)$, $\mathbf{j}=(j_1,\dots,j_p)$, $\mathbf{l}=(l_1,\dots,l_p)$,

$$T_G(\mathbf{j},\mathbf{l},N)=T_{\pi G}(\pi(\mathbf{j}),\pi(\mathbf{l}),N), \tag{2.15}$$

where $\pi(\mathbf{j})=(\pi(j_1),\dots,\pi(j_p))$ and $\pi(\mathbf{l})=(\pi(l_1),\dots,\pi(l_p))$. For all diagrams G there exists a permutation π such that $G'=\pi G$ has the following property: $G'\in\Gamma(l'_1,\dots,l'_p)$ with some integers $l'_1,\dots,l'_p$ and

$$l'_1\leqslant l'_2\leqslant\cdots\leqslant l'_p. \tag{$*$}$$

Because of relation (2.15) it is enough to prove the Proposition only for such diagrams $G\in\Gamma(l_1,\dots,l_p)$ which have the property $(*)$. We can write

$$|T_G(\mathbf{j},\mathbf{l},N)|\leqslant A_N^{-p}\sum_{\mathbf{m}\in M(\mathbf{j},N)}\prod_{i=1}^{p}\prod_{\substack{w\in G(V)\\ d_1(w)=i}}|r(m_i-m_{d_2(w)})|. \tag{2.16}$$

In the inner product of the expression in the right-hand side of (2.16) there are $k_G(i)$ terms and the inequality

$$\prod_{\substack{w\in G(V)\\ d_1(w)=i}}|r(m_i-m_{d_2(w)})|\leqslant\frac{1}{k_G(i)}\sum_{\substack{w\in G(V)\\ d_1(w)=i}}|r(m_i-m_{d_2(w)})|^{k_G(i)}$$

holds. Hence we get, first setting $m_2,\dots,m_p$ fixed and summing up for m_1 in (2.16), that

$$|T_G(\mathbf{j},\mathbf{l},N)|\leqslant A_N^{-p}\sup_{v\in B(0,CN)}\left(\sum_{m\in B(n_{j_1},N)}|r(m-v)|^{k_G(1)}\right)$$
$$\times\sum_{\substack{m_i\in B(n_{j_i},N)\\ i=2,\dots,p}}\prod_{i=2}^{p}\prod_{\substack{w\in G(v)\\ d_1(w)=i}}|r(m_i-m_{d_2(w)})|,$$

where $C>0$ is chosen so that $B(n_j,N)\subset B(0,CN)$ for $j=1,\dots,p$. Then iterating the above procedure for $m_2,\dots,m_p$ and exploiting that $m-v\in\bar{B}(0,2CN)$ if $m\in B(n_{j_s},N)$, $s=1,\dots,p$, and $v\in B(0,CN)$ we get that

$$|T_G(\mathbf{j},\mathbf{l},N)|\leqslant A_N^{-p}\prod_{i=1}^{s}\sum_{m\in\bar{B}(0,2CN)}|r(m)|^{k_G(i)}. \tag{2.17}$$

(For the definition of $\bar{B}$ we refer to Theorem 1′.) Obviously, since $l_i \geqslant k$, we have

$$\sum_{m \in \bar{B}(0,2CN)} |r(m)|^{k_G(i)} \leqslant \text{const} \cdot N^{(1-g(i))\nu} \tag{2.18}$$

if $k_G(i) = 0$ or $k_G(i) = l_i$, where $g(i) = k_G(i)/l_i$. On the other hand, we claim that

$$\sum_{m \in \bar{B}(0,2CN)} |r(m)|^{k_G(i)} = o(N^{(1-g(i))\nu}) \tag{2.18′}$$

if $0 < k_G(i) < l_i$. Indeed, because of (1.4) there is a finite set $B = B(\varepsilon) \subset Z^\nu$ such that $\sum_{m \in Z^\nu - B} |r(m)|^{l_i} < \varepsilon$. Hence by Hölder's inequality we get

$$\sum_{m \in \bar{B}(0,2CN)} |r(m)|^{k_G(i)} \leqslant C(\varepsilon) + \varepsilon^{g(i)} \cdot (4CN)^{(1-g(i))\nu}.$$

Since ε is arbitrary small, relation (2.18′) holds. Relations (2.18) and (2.18′) imply that

$$|T_G(\mathbf{j}, \mathbf{l}, N)| \leqslant \mathcal{O}(N^{(p/2 - \sum_{i=1}^{p} g(i))\nu}) \tag{2.19}$$

and (2.19) holds with $o(\cdot)$ if $0 < k_G(i) < l_i$ for some i. In a non-regular diagram at least one of the following properties hold: either $0 < k_G(i) < l_i$ for some i or G contains an edge between levels of different cardinality. Now the Proposition follows from the inequality

$$\sum_{i=1}^{p} g(i) \geqslant p/2, \tag{2.20}$$

where there is strict inequality if G contains an edge connnecting levels of different cardinality.

Given an edge $w \in G(V)$ we define the numbers $p_1(w)$ and $p_2(w)$ as the cardinalities of the $d_1(w)$th and the $d_2(w)$th levels, respectively. Because of property (∗) we have $p_1(w) \geqslant p_2(w)$ for all $w \in G(V)$. Hence

$$2 \sum_{i=1}^{p} \frac{k_G(i)}{l_i} = 2 \sum_{w \in G(v)} \frac{1}{p_1(w)} \geqslant \sum_{w \in G(v)} \left(\frac{1}{p_1(w)} + \frac{1}{p_2(w)} \right) = p$$

because the term $1/l_j$ appears exactly l_j times among the summands $1/p_1(w)$ and $1/p_2(w)$. Relation (2.20) is thus proved, for there are exactly l_j edges arriving at the jth level from levels of either lower or higher indices.

The proof of Theorem 1′ is almost the same, therefore we remark only the most important changes. In relations (2.18) and (2.18′) we have to multiply the right-hand side by $L(N)^{g(i)}$. Then we get a non-positive power of $N/L(N)$

in the right-hand side of (2.19). The proof of (2.18′) must be slightly changed. We have to split up the investigated sum as

$$\sum_{m\in\bar{B}(0,2CN)} |r(m)|^{k_G(i)} = \sum_{m\in\bar{B}(0,\varepsilon N)} + \sum_{m\in\bar{B}(0,2CN)-\bar{B}(0,\varepsilon N)} = I_1 + I_2,$$

where $\varepsilon > 0$ is sufficiently small. Both I_1 and I_2 can be estimated by Hölder's inequality. In the estimation of I_1 we can exploit the fact that it contains only $(2\varepsilon N)^\nu$ summands and in the estimation of I_2 we use the relation

$$\sum_{m\in\bar{B}(0,2CN)-B(0,\varepsilon N)} |r(m)|^{l_i} = o(L(N)),$$

where $L(N)$ is a slowly varying function. The remaining modification in the proof of Theorem 1′ is the following: At the reduction of $H(x)$ to a polynomial we have to make more careful estimations, and we have to observe that $Z_n^N(H_l)$ and $Z_m^N(H_l)$ remain almost uncorrelated for $n \neq m$, n, $m \in Z^\nu$.

Finally, we remark that the following Theorem 1″ can be proved similarly to Theorems 1 and 1′. (The only difference is that one has to supply the auxiliary sequence b with two indices.)

THEOREM 1″. *Under the condition* (1.4) *or* (1.4′) *the joint distribution of the fields* $Z_n^N(H_k)$, $Z_n^N(H_{k+1})$,... *tends to those of the fields* $\sigma_k \sqrt{k!}\, Z_n^*(k)$, $\sigma_{k+1}\sqrt{(k+1)!}\, Z_n^*$,..., *where* Z_n^{*k}, $Z_n^{*(k+1)}$,..., $n \in Z^\nu$, *are independent random fields consisting of independent standard normal variables.*

III. A DIFFERENT MODEL

In this section we prove a limit theorem of a different type for non-linear functionals of stationary Gaussian sequences. In this example the limit will be Gaussian but the elements of the limit sequence are generally not independent. Since the proofs follow the ideas of the previous section together with some calculations from [2] we shall outline them very briefly.

Let X_n, $n = \cdots -1, 0, 1,\ldots$, be a stationary Gaussian sequence, $EX_n = 0, EX_n^2 = 1$ and denote $r(n) = EX_0X_n$. Let the real function $H(x)$ satisfy (1.1) and suppose that $H(x)$ has Hermite rank k. Set $Y_n = H(X_n)$ and define

$$U_m = U_m(H) = \sum_{n=-\infty}^{\infty} a_n Y_{n+m} \tag{3.1}$$

for $m = \cdots -1, 0, 1, \ldots$, where a_n, $n = \cdots -1, 0, 1, \ldots$, is a real sequence satisfying the following conditions: For some $-\frac{1}{2} < \beta < \frac{1}{2}$

$$\begin{aligned} a_n = a_n(\beta) &= C(1)\, n^{-\beta-1} + o(n^{-\beta-1}) && \text{for} \quad n > 0, \\ a_n = a_n(\beta) &= C(2)\, |n|^{-\beta-1} + o(|n|^{-\beta-1}) && \text{for} \quad n < 0. \end{aligned} \tag{3.2}$$

Set $b_n = \frac{1}{2}(a_n + a_{-n})$ and assume that

$$\sum_{n=-\infty}^{\infty} a_n = 0, \qquad \text{if} \quad 0 < \beta < \tfrac{1}{2}, \tag{3.3}$$

$$C(1) = -C(2) \qquad \text{and} \qquad \sum |b_n| < \infty, \qquad \text{if} \quad \beta = 0.$$

Furthermore, suppose that

$$\sum_{n=-\infty}^{\infty} |r(n)|^k < \infty. \tag{3.4}$$

Now we can formulate our

THEOREM 2. *Define*

$$Z_n^N = Z_n^N(H) = A_N^{-1} \sum_{m=nN}^{(n+1)N-1} U_m \tag{3.5}$$

for $n = \cdots -1, 0, 1, \ldots$, $N = 1, 2, \ldots$, *where* U_m *is defined in* (3.1) *and the norming constants are* $A_N = N^{1/2-\beta}$ *with the* β *defined in formula* (3.2).

Under the assumptions (3.2), (3.3) *and* (3.4) *the infinite sum in* (3.1) *is convergent in the* L_2*-sense, hence the random variables* $Z_n^N(H)$ *exist. Then there also exist the limits*

$$\lim_{N\to\infty} E(Z_0^N(H_l))^2 = l!\, \sigma_l^2 \qquad \textit{for all} \quad l \geqslant k,$$

and

$$\sigma^2 = \sum_{j=k}^{\infty} c_j^2\, j!\, \sigma_j^2 < \infty, \tag{3.6}$$

and the sequence $Z_n^N(H)$ *tends to a stationary Gaussian sequence* σZ_n^*, *where* $EZ_n^* = 0$, $EZ_n^{*2} = 1$, *and* Z_n^* *is self-similar with self-similarity parameter* $\frac{1}{2} - \beta$.

We remark that the above properties determine the distribution of the limit sequence Z_n^*.

Proof of Theorem 2. It can be proved similarly to [2] that the random variables U_m and Z_n^N are meaningful. The next step of the proof is the investigation of the covariance structure of the sequences Z_n^N. We can write

$$Z_n^N(H) = N^{\beta - 1/2} \sum_{j=-\infty}^{\infty} \gamma_{j+nN}(N)\, H(X_{-j}), \tag{3.7}$$

where

$$\gamma_j(N) = \sum_{k=-j}^{N-j-1} a_k.$$

Hence

$$\begin{aligned} r(N, n, l) &= EZ_0^N(H_l)\, Z_n^N(H_l) \\ &= N^{2\beta - 1} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \gamma_j(N)\, \gamma_{k+nN}(N)\, r^l(k-j) \\ &= \sum_{p=-\infty}^{\infty} t_N(p+nN)\, r^l(p) \end{aligned} \tag{3.8}$$

with

$$t_N(p) = N^{2\beta - 1} \sum_{j=-\infty}^{\infty} \gamma_j(N)\, \gamma_{p-j}(N).$$

It follows from the estimations of Section 3 of [2] that $t_N(p)$ is bounded for both variables N and p, and there exists a function $t^*(x)$ on $\mathbb{R}$ such that

$$\lim_{N\to\infty} t_N\left(\frac{p_N}{N}\right) \to t^*(x) \qquad \text{if} \quad \frac{p_N}{N} \to x.$$

Hence relations (3.8) and (3.5) imply that

$$R_l(n) = \lim_{N\to\infty} EZ_0^N(H_l)\, Z_n^N(H_l) = t^*(n) \sum_{p=-\infty}^{\infty} r^l(p) \tag{3.9}$$

for $l \geqslant k$. It follows from (3.9) that the limits

$$R(n) = \lim_{N\to\infty} EZ_0^N(H)\, EZ_n^N(H)$$

exist, and we can reduce the proof of Theorem 2, just as in Theorem 1, to the case when H is a polynomial. It follows from the definition of the sequences Z_n^N that for all positive integers N, n, k,

$$EZ_0^{kN} Z_n^{kN} = \frac{1}{k^{1-2\beta}} \sum_{j=0}^{k-1} \sum_{l=kn}^{(k+1)n-1} EZ_j^N Z_l^N.$$

Letting N tend to infinity we get that a stationary Gaussian sequence with mean zero and correlation function $R(n)$ is self-similar with self-similarity parameter $\frac{1}{2}-\beta$. Hence to prove Theorem 2 it is enough to show that the pth moment of the linear combinations $\sum_N = \sum b_j Z^N_{n_j}$ tend to the pth moment of a normal random variable with expectation zero and variance $\sum b_j b_l R(n_j - n_l)$. This can be proved similarly to Theorem 1. We can write

$$E\left(\sum\nolimits_N\right)^p = \sum_{(\mathbf{j},\mathbf{l})\in\mathscr{T}^{(p)}} E\prod_{i=1}^{p} S^N(j_i, l_i), \tag{3.10}$$

where $\mathscr{T}^{(p)}$ is defined in (2.8), and

$$S^N(j,l) = A_N^{-1} b_j c_l Z^N_{n_j}(H_l). \tag{3.11}$$

For a fixed $(\mathbf{j},\mathbf{l})\in\mathscr{T}^{(p)}$ we have, by the diagram formula and (3.7),

$$\begin{aligned} E&\left\{\prod_{i=1}^{p} S^N(j_i,l_i)\right\} \\ &= E\left\{\prod_{i=1}^{p} b_{j_i} c_{l_i} A_N^{-1} \sum_{u_i=-\infty}^{\infty} \gamma_{n_{j_i}N+u_i}(N)\, H_{l_i}(X_{-u_i})\right\} \\ &= K(\mathbf{j},\mathbf{l})\, A_N^{-p} \sum_{\mathbf{u}\in\mathscr{U}} \prod_{i=1}^{p} \gamma_{n_{j_i}N+u_i}(N) \sum_{G\in\Gamma} \prod_{w\in G(V)} r(u_{d_1(w)} - u_{d_2(w)}), \end{aligned} \tag{3.12}$$

where $\mathscr{U} = \{\mathbf{u} = (u_1,\dots,u_p)\colon u_i \in Z,\ i=1,2,\dots,p\}$, $\Gamma = \Gamma(l_1,\dots,l_p)$ and $K(\mathbf{j},\mathbf{l})$ is the same as in Section 2. Define for a fixed $\mathbf{j} = (j_1,\dots,j_p)$ and $\mathbf{l} = (l_1,\dots,l_p)$ and $G \in \Gamma(l_1,\dots,l_p)$ the quantity

$$T_G(\mathbf{j},\mathbf{l},N) = A_N^{-p} \sum_{\mathbf{u}\in\mathscr{U}} \prod_{i=1}^{p} \gamma_{n_{j_i}}N + u_i(N) \prod_{w\in G(v)} r(u_{d_1(w)} - u_{d_2(w)}).$$

Let G be a regular diagram whose levels are paired as $(i(1), i(2)),\dots,$ $(i(p-1), i(p))$, and the edges go only between the levels $i(2m-1)$ and $i(2m)$, $m=1,\dots,q$, $p=2q$. Let $t(m)$ be the cardinality of the $i(2m-1)$th and $i(2m)$th levels $m=1,\dots,q$, $p=2q$. Then

$$\lim_{N\to\infty} T_G(\mathbf{j},\mathbf{l},N) = R_{t(1)}(j_{i(1)} - j_{i(2)}) \cdots R_{t(q)}(j_{i(p-1)} - j_{i(p)}). \tag{3.13}$$

By relations (3.10) and (3.12) formula (2.11) remains valid in this setting with the newly defined functions T_G. Because of relation (3.13) a calculation similar to that in Section 2 shows that

$$\lim_{N\to\infty} \sum_{(\mathbf{j},\mathbf{l})\in\mathcal{T}(p)} K(\mathbf{j},\mathbf{l}) \sum_{G\in\Gamma^*(l_1,\ldots,l_p)} T_G(\mathbf{j},\mathbf{l},N)$$

$$= (p-1)!! \left[\sum b_j b_l R(n_j - n_l)\right]^{p/2} \qquad \text{if } p \text{ is even,}$$

$$= 0 \qquad \text{if } p \text{ is odd.}$$

Hence in order to complete the proof of Theorem 2 it is enough to show that

$$\lim_{N\to\infty} T_G(\mathbf{j},\mathbf{l},N) = 0$$

if G is not a regular diagram. This relation can be proved just as the proposition, therefore we omit the proof.

References

[1] Dobrushin, R. L., and Major, P. (1979). Non-central limit theorems for non-linear functionals of gaussian fields. *Z. Wahrsch. Verw. Gebiete* **50** 27–52.

[2] Major, P. (1981). Limit theorems for non-linear functionals of Gaussian sequences. *Z. Wahrsch. Verw. Gebiete* **57** 129–158.

[3] Rosenblatt, M. (1981). Limit theorems for Fourier transforms of Gaussian sequences. *Z. Wahrsch. Verw. Gebite* **55** 123–132.

[4] Sun, T. C. (1965). Some further results on central limit theorems for non-linear functions of normal stationary process. *J. Math. and Mech.* **14** 71–85.

[5] Taqqu, M. S. (1975). Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrsch. Verw. Gebiete* **31** 287–302.

[6] Taqqu, M. S. (1977). Law of the iterated logarithm for sums of non-linear functions of Gaussian variables. *Z. Wahrsch. Verw. Gebiete* **40** 203–238.

[7] Taqqu, M. S. (1979). Convergence of iterated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete* **50** 53–83.