Limit theorems on the direct product of a non-compact Lie group and a compact group

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Abstract: Let us consider a triangular array of random vectors $(X_j^{(n)}, Y_j^{(n)})$, $n = 1, 2, \dots$, $1 \le j \le k_n$, such that the first coordinates $X_j^{(n)}$ take their values in a non-compact Lie group and the second coordinates $Y_j^{(n)}$ in a compact group. Let the random vectors $(X_j^{(n)}, Y_j^{(n)})$ be independent for fixed n, but we do not assume any (independence type) condition about the relation between the components of these vectors. We show under fairly general conditions that if both random products $S_n = \prod\limits_{j=1}^{k_n} X_j^{(n)}$ and $T_n = \prod\limits_{j=1}^{k_n} Y_j^{(n)}$ have a limit distribution, then also the random vectors (S_n, T_n) converge in distribution as $n \to \infty$. Moreover, the non-compact and compact coordinates of a random vector with this limit distribution are independent.

1. Motivations for the investigation of the problem.

The problem investigated in this work appeared as a by-product of the investigation in paper [3]. In that paper the limit behaviour of the appropriate normalizations of k-order symmetric polynomials $S_n^{(k)} = \sum\limits_{1 \le j_1 < j_2 < \cdots < j_k \le n} \xi_{j_1} \cdots \xi_{j_k}$ of i.i.d. random variables $\xi_1, \dots, \xi_n$ with a fixed distribution function F was considered as $n \to \infty$ in the case when the order $k = k(n)$ of the symmetric polynomials strongly depends on the number n. Namely, it was assumed that the fractions $\alpha(n) = \dfrac{k(n)}{n}$ satisfy the relation $\lim\limits_{n \to \infty} \alpha(n) = \alpha$ with some number $0 < \alpha < 1$.

The investigation in paper [3] was based on the proof of a relation which shows that the description of the limit behaviour of the random variables $S_n^{(k)}$ can be reduced to the investigation of a non-linear functional of a random vector $(S(n, \alpha(n)), T(n, \alpha(n)))$, where $S(n, \alpha(n)) = \dfrac{1}{\sqrt{n}} \sum\limits_{j=1}^{n} \xi_j(\alpha(n))$ and $T(n, \alpha(n)) = \sum\limits_{j=1}^{n} \eta_j(\alpha(n)) \bmod 2\pi$ with a sequence $(\xi_j(\alpha(n)), \eta_j(\alpha(n)))$, $E\xi_j(\alpha(n)) = 0$, $j = 1, \dots, n$, of independent and identically distributed random vectors. The distribution of the random vector $(\xi_1(\alpha(n)), \eta_1(\alpha(n)))$ could be given explicitly, and it depended only on the distribution function F of the random variables ξ_j, $j = 1, \dots, n$, appearing in the definition of the symmetrical polynomials $S_n^{(k)}$ and the fraction $\alpha(n) = \dfrac{k(n)}{n}$. Beside this, the distribution of this random vector depended continuously on the parameter α, $0 \le \alpha \le 1$.

To solve the problem investigated in paper [3] a limit theorem had to be proved for the distribution of the random vectors $(S(n,\alpha(n)),T(n,\alpha(n)))$ as $n\to\infty$. The usual central limit theorem describes the limit distribution of the random variables $S(n,\alpha(n))$, the (known) limit theorems for products of independent and identically distributed random variables on the unit circle describe the limit behaviour of the random variables $T(n,\alpha(n))$ as $n\to\infty$. (The additive group modulo 2π is isomorph to the group on the unit circle with the usual multiplication.) But in the investigations of paper [3] also a limit theorem for the joint distribution of the random vectors $(S(n,\alpha(n)),T(n,\alpha(n)))$ was needed. It was shown in that work that these random vectors have a limit distribution, and the components of the limit are independent. This independence of the components in the limit distribution appeared not because of some uncorrelatedness property of the coordinates. It had a structural reason.

The proof of the limit theorem in paper [3] was based on the characteristic function technique, and it exploited the fact that the characters of the additive group of the real line are the functions $e_t(x)=e^{itx}$ where $t\in R^1$, and the characters of the additive group with addition modulo 2π on the interval $[0,2\pi]$ are the functions $e_m(x)=e^{imx}$ where the parameter m takes integer values. This means that the character group of the groups we have considered is a continuous group in the first case and a discrete group in the second case. The independence of the coordinates of the limit distribution was the consequence of these facts. It is natural to expect similar results in a more general case when the (appropriately normalized) products of random vectors are considered with first coordinates in a non-compact and with second coordinates in a compact group. But we have found only one work in the literature where such results were proved. It was the paper of A. Raugi [6]. Raugi's results were not sufficiently general for our purposes, because they do not cover the case needed in paper [3]. So our main goal was to prove an appropriate generalization of Raugi's results.

Let us explain what kind of generalization of Raugi's results we need. The main point is that for a fixed n the distribution of the random vectors $(\xi_j(\alpha(n)),\eta_j(\alpha(n)))$, $j=1,\dots,n$, we have considered depends on a parameter $\alpha(n)$ which may be different for different n. We only know that the parameters $\alpha(n)$ are convergent as $n\to\infty$. This means that we need a triangular array type generalization of limit theorems for products of independent vectors. This dependence of the distribution of the terms in the random product have a deep consequence in particular if the compact group coordinate is considered. Indeed, let us consider the following simple example: Let the group $\mathbf{G}$ be the interval $[0,2\pi]$, with addition modulo 2π as the group multiplication. If the product of such independent and identically distributed random variables are considered on this group which take only values $\frac{2k\pi}{3}$, $k=1,2,3$, then the limit distribution of the random products is the uniform distribution on the values $\frac{2k\pi}{3}$, $k=1,2,3$. But even a very small perturbation of the distributions may radically change the limit behaviour of the random product. Typically the products of the random variables with the perturbed distribution converge to the uniform distribution on the interval $[0,2\pi]$. On the other hand, we shall show that such counter examples do not appear if the limit distribution of the product of the unperturbed random variables is the Haar measure of the whole group. Our goal is to show that under very general conditions the (appropriately normalized) products of independent and identically distributed random vectors with the first coordinate in a non-compact and the second coordinate in a compact group have a limit distribution, and the compact and non-compact coordinates of a random vector with the limit distribution are independent. Beside this, we want to show that a similar result holds after a small perturbation of the distribution of the random vectors if the projection of the limit distribution (of the random products whose terms have the unperturbed distribution) to the compact group coordinate is the Haar measure of this group.

This paper consists of 4 sections. In Section 2 we prove independence type results which enable us to reduce the problem of limit theorems for products of independent random vectors with a non-compact and a compact group valued component to the separate investigation of these components. In Section 3 we prove with the help of some classical theorems the results we need about products of random variables on a compact group. Here we formulate and prove the main result of this paper Theorem 3.2. This theorem can be considered as a definite formulation of the heuristic statements formulated in this Section. In Section 4 we recall some limit theorems on product of Lie group valued random variables which are useful for us and also formulate some open problems.

To avoid some unpleasant measure theoretical problems we restrict our attention in this paper to the case when the groups we handle are complete separable metric spaces.

2. Independence type results.

In this section we prove two results. The first result, Proposition 2.1, gives a condition for the convergence of a sequence of probability measures on a product space to a product measure on this product space which is simpler and can be better checked than the original definition. The second result, Proposition 2.2, gives a sufficient condition for the asymptotic independence of a product of independent random vectors whose first coordinates are in a non-compact and the second coordinates in a compact group. This asymptotic independence of the compact and non-compact component of the product may also appear if no independence like relation holds between the compact and non-compact terms taking part in the product. These results enable us to reduce the proof of limit theorems for the distribution of these random vectors under some not too restrictive conditions to the investigation of the limit theorems for the products of the non-compact and compact components of these vectors separately.

Proposition 2.1. *Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be two complete separable metric spaces, and let P_n, $n=1,2,\dots$, be a sequence of probability measures on the product space $(X\times Y,\mathcal{A}\times\mathcal{B})$ such that there exist some probability measures μ on the space $(X,\mathcal{A})$ and ν on the space $(Y,\mathcal{B})$ which satisfy the relations*

$$\lim_{n\to\infty}\int f(u)g(v)P_n(\,du,\,dv)=\int f(u)g(v)\mu(\,du)\nu(\,dv) \tag{2.1}$$

for all continuous and bounded functions $f(\cdot)$ on the space $(X,\mathcal{A})$ and continuous and bounded functions $g(\cdot)$ on the space $(Y,\mathcal{B})$. Then the measures P_n converge weakly to the product measure $\mu\times\nu$.

Proof of Proposition 2.1. Proposition 2.1 can be proved as a relatively simple consequence of Theorem 3.1 in Chapter 1 of Billingsley's book [1]. By this result it is enough to show that if relation (2.1) holds, then for all such $A \in \mathcal{A}$ and $B \in \mathcal{B}$ for which the boundaries ∂A and ∂B of the sets A and B satisfy the relation $\mu(\partial A) = 0$ and $\nu(\partial B) = 0$ $\lim\limits_{n\to\infty} P_n(A \times B) = \mu(A)\nu(B)$.

Instead of this relation it is enough to prove the statements that $\limsup\limits_{n\to\infty} P_n(A\times B) \le \mu(A)\nu(B)$ if the sets A and B are closed and $\liminf\limits_{n\to\infty} P_n(A\times B) \ge \mu(A)\nu(B)$ if the sets A and B are open. Indeed, by replacing the sets A and B by their closure and interior the above statements imply the desired relation. To prove the first statement let us choose for all $\varepsilon > 0$ two open sets $G \supset A \cup \partial A$ and $H \supset B \cup \partial B$ such that $\mu(G) < \mu(A) + \varepsilon$ and $\nu(H) < \nu(B) + \varepsilon$. Then there exist two continuous functions $f(\cdot)$ on the space $(X, \mathcal{A})$ and $g(\cdot)$ on the space $(Y, \mathcal{B})$ such that $0 \le f(u) \le 1$ for all $u \in X$, $0 \le g(v) \le 1$ for all $v \in Y$, $f(u) = 1$ if $u \in A$, $f(u) = 0$ if $u \notin G$, and $g(v) = 1$ if $v \in B$, $g(v) = 0$ if $v \notin H$. Then

$$\limsup_{n\to\infty} P_n(A \times B) \le \lim_{n\to\infty} \int f(u)g(v)P_n(\,du,\,dv) \le \mu(G)\nu(H) \le (\mu(A)+\varepsilon)(\nu(B)+\varepsilon).$$

Since the above relation holds for all $\varepsilon > 0$ it implies the first statement. The proof of the second statement is similar. Only in this case we have to exploit that the measure of an open set can be approximated arbitrary well by the measure of a closed set contained in this open set.

Proposition 2.2, which together with Proposition 2.1 helps to prove results of the type indicated in Section 1 is a generalization of Lemma 1.5 in Raugi's paper [6]. The proof heavily exploits Raugi's ideas. Before the formulation of this result we recall some facts and notations from the theory of group representations on compact groups. The theory of group representations appears in a natural way if we want to apply the characteristic function technique in the case of general compact groups.

Let $\mathbf{K}$ be a compact group. A representation of the group $\mathbf{K}$ is a continuous homomorphism of the group $\mathbf{K}$ to the group of unitary transformations $\mathcal{U}(H)$ of a Hilbert space H. We call a representation $D\colon \mathbf{K} \to \mathcal{U}(H)$ irreducible if there is no non-trivial closed subspace of the Hilbert space H invariant with respect to all unitary transformations $D(g)$, $g \in \mathbf{K}$. Two representations D_1 and D_2 are called unitarily equivalent if there is a unitary transformation U of the Hilbert space H such that $UD_1(g)U^* = D_2(g)$ for all $g \in \mathbf{K}$. It follows from the general theory of group representations that all irreducible representations of a compact group $\mathbf{K}$ are finite dimensional, that is they map the group to the unitary matrices of a finite dimensional Euclidean space. Let $\mathrm{Irr}\,(\mathbf{K})$ denote the class of all irreducible not unitarily equivalent representations of the group $\mathbf{K}$.

Given a $D \in \mathrm{Irr}\,(\mathbf{K})$ of dimension $d = d(D)$, let $D(i,j)(g)$, $g \in \mathbf{G}$, $1 \le i, j \le d(D)$ denote the elements of the matrix we get if the transformations $D(g)$ are written in the form of a matrix with a fixed orthonormal basis of the d-dimensional space. By a most important result of the group representations, the Peter–Weyl theorem, the set of functions $\frac{1}{d(D)}D(i,j)(\cdot)$, $1 \le i,j \le d(D)$, $D \in \mathrm{Irr}\,(\mathbf{K})$, is a complete orthonormal basis in the space $L_2(\mathbf{K},\mathcal{K},\mu)$, where $\mathcal{K}$ denotes the Borel σ-algebra of the group $\mathbf{K}$ and μ is the Haar measure in this space. Beside this, the finite linear combinations of the functions $D(i,j)(\cdot)$ constitute an everywhere dense set in the space of continuous functions on the compact group $\mathbf{K}$ with respect to the supremum norm.

If X is a random variable taking values in a compact group $\mathbf{K}$ then let us define its Fourier transform $\mathcal{F}_X = \mathcal{F}_X(D)$, $D \in \mathrm{Irr}\,\mathbf{K}$, as

$$\mathcal{F}_X(D) = ED(X(\cdot)), \quad \text{that is} \quad \langle \mathcal{F}_X(D)(u), v\rangle = E\langle D(X(\cdot))(u), v\rangle, \quad D \in \mathrm{Irr}\,\mathbf{K}$$

if $u, v \in H(D)$, where $H(D)$ denotes the (finite dimensional) Hilbert space where the group representation D is acting. The Fourier transform of a compact group $\mathbf{K}$ valued random variable is the natural analog of the characteristic function of real valued random variables. In particular, the Fourier transform of the product of independent random variables on the group $\mathbf{K}$ equals the product of the Fourier transforms of these random variables.

Let us also recall that given a random variable X with probability distribution μ on a separable metric space M, there exists a smallest closed subset $F \subset M$ such that $\mu(F) = 1$. We shall call this set the support of the random variable X and denote it by $\mathrm{supp}\,(X)$.

The above facts help us to prove Proposition 2.2 formulated below. Before the proof we shall discuss the content of the conditions imposed in this result.

Proposition 2.2. *Let $\mathbf{N}$ be a locally compact and $\mathbf{K}$ a compact group. Let $\mathbf{G} = \mathbf{N} \times \mathbf{K}$ denote their direct product. For each $n = 1, 2, \dots$ let $(X_j^{(n)}, Y_j^{(n)})$, $j = 1, 2, \dots, k_n$, $k_n \to \infty$ if $n \to \infty$, be a sequence of independent random variables on $\mathbf{G}$. Let us define the random products*

$$U_n = \prod_{j=1}^{k_n} X_j^{(n)}, \quad V_n = \prod_{j=1}^{k_n} Y_j^{(n)}, \qquad n = 1, 2, \dots,$$

for all $n = 1, 2, \dots$. Let us assume that the random variables $X_j^{(n)}$ satisfy the following condition (i):

(i) The relation

$$\lim_{n\to\infty} E\left| f\left(\prod_{j=1}^{k_n} X_j^{(n)}\right) - f\left(\prod_{j=1}^{k_n-p} X_j^{(n)}\right)\right| = 0 \tag{2.2}$$

holds for all $p = 1, 2, \dots$ and continuous and bounded functions $f(\cdot)$ on the locally compact group $\mathbf{N}$.

Let $\mathbf{K}'$ be a closed subgroup of $\mathbf{K}$ such that $\mathrm{supp}\,(Y_j^{(n)})$ lies in one of its two–sided cosets, $a\mathbf{K}' = \mathbf{K}'a$ with some $a \in \mathbf{K}$ for all $n = 1, 2, \dots$ and $j = 1, 2, \dots, k_n$, and fix *some $y \in a\mathbf{K}'$. Suppose that the random variables $Y_j^{(n)}$ and their Fourier transforms satisfy the following condition (ii):*

(ii) If V is a random variable in $\mathbf{K}$ with uniform distribution on the subgroup $\mathbf{K}'$, then

$$\lim_{p\to\infty} \sup_{k_n \ge p} \left\| \mathcal{F}_{vy^{-p} \prod\limits_{j=k_n-p+1}^{k_n} Y_j^{(n)}}(D) - \mathcal{F}_V(D) \right\| = 0 \tag{2.3}$$

with the element $y \in a\mathbf{K}'$ we have fixed for all irreducible representations $D \in$ Irr $(\mathbf{K})$ and all $v \in \mathbf{K}'$.

Then the sequences U_n, $n = 1, 2, \dots$ and $y^{-k_n} V_n$, $n = 1, 2, \dots$, are asymptotically independent, i.e. for all continuous and bounded functions f on the group $\mathbf{N}$ and continuous and bounded functions g on the group $\mathbf{K}$

$$\lim_{n\to\infty} \left(E[f(U_n)\, g(y^{-k_n} V_n)] - E[f(U_n)]\, E[g(y^{-k_n} V_n)] \right) = 0 \tag{2.4}$$

with the element $y \in a\mathbf{K}'$ we have fixed.

If the conditions of Propositions 2.1 and 2.2 are satisfied then to prove a limit theorem for the random vectors $(U_n, y^{-k_n} V_n)$, $n = 1, 2, \dots$, it is enough to prove a limit theorem for the random variables U_n and $y^{-k_n} V_n$ separately. Then also the joint distributions of these random variables converge in distribution, and the components of the limit distributions are independent.

The condition (i) in Proposition 2.2 expresses the non-compact character of the group $\mathbf{N}$. Its heuristic content is that by omitting finitely many terms from the end of the product U_n we make a very small modification of this product. We shall return to the discussion of this property in the next Section in the formulation of Theorem 3.2.

Condition (ii) is slightly more general than the condition we need in the sequel. It also helps to consider the case when the random variables $y^{-k_n} V_n$ converge in distribution to the Haar measure of a proper subgroup of the group $\mathbf{K}$ with some appropriate "shift" y^{-k_n}. But we shall be interested mainly in the case when the products V_n converge to the Haar measure of the whole group $\mathbf{K}$, and the "shift" factors y^{-k_n} do not appear. In this case the factor y^{-k_n} does not appear in formula (2.3), i.e. y has to be chosen as the unit element of the group $\mathbf{K}$ in this formula. As we shall see in the next section condition (ii) of Proposition 2.1 holds in a very general case. Let us also remark that condition (2.3) is equivalent to the following formally weaker statement:

$$\lim_{p\to\infty} \sup_{k_n \ge p} \left\| \mathcal{F}_{y^{-p} \prod\limits_{j=k_n-p+1}^{k_n} Y_j^{(n)}}(D) - \mathcal{F}_V(D) \right\| = 0 \tag{2.3$'$}$$

with an element $y \in a\mathbf{K}'$ for all irreducible representations $D \in \mathrm{Irr}\,(\mathbf{K})$, i.e. $v \in \mathbf{K}'$ can be replaced by the identity of the group $\mathbf{K}$. Indeed, if $v \in \mathbf{K}$ then vV has the same distribution as V, $\mathcal{F}_V(D)=\mathcal{F}_{vV}(D)$,

$$\mathcal{F}_{vy^{-p}\prod\limits_{j=k_n-p+1}^{k_n} Y_j^{(n)}}(D)-\mathcal{F}_V(D)=D(v)\left(\mathcal{F}_{y^{-p}\prod\limits_{j=k_n-p+1}^{k_n} Y_j^{(n)}}(D)-\mathcal{F}_V(D)\right),$$

$D(v)$ is a unitary matrix, hence relation (2.3′) implies relation (2.3). We formulated our condition in the form (2.3) because this formula can be better applied in the proof.

Before the proof of Proposition 2.2 we briefly explain its main ideas. Because of the Peter–Weil theorem we can reduce the statement to be proved to a relation formulated in relation (2.5). Then we exploit Conditions (i) and (ii) of Proposition 2.2. In an informal way the content of Condition (i) is that a negligibly small error is committed if finitely many terms are omitted from the end of the products U_n on the locally compact group $\mathbf{N}$. Condition (ii) says that the behaviour of the random product V_n on the compact group $\mathbf{K}$ shows a different character. Here the product of sufficiently many terms at the end of the product V_n determines the distribution of the random variable V_n with a very good accuracy. To get a good approximation of this distribution we have to make sufficiently many terms but their number does not depend on the parameter n. Condition (ii) expresses this property in a rather hidden way. It says in the language of Fourier transforms that the product of finitely many terms at the end of the product U_n is close to the Haar measure of a subgroup of the group $\mathbf{K}$ or to some of its shift. Then by multiplying it with the independent product from the left we have to multiply with to get the product V_n we do not deteriorate this property.

The formal proof exploits these observations. First we show with the help of Property (i) that by omitting finitely many terms from the end of V_n a negligible error is committed and the proof of Propositon 2.2 can be reduced to a good bound on the expression $\gamma(n,p)$ introduced in formula (2.8). To exploit the available independence we make a conditioning of $\gamma(n,p)$ with respect to the condition $U_{n,p}=u$, $V_{n,p}=v$ where $U_{n,p}$ and $V_{n,p}$ are defined in formula (2.6). The conditional expectation we have to handle can be bounded well with the help of Condition (ii). In the exact proof we need uniform bounds on the conditional expectation we have to handle with respect to the conditions. They can be proved with the help of usual compactness arguments.

Proof of Proposition 2.2. Because of the Peter–Weyl theorem it is enough to prove instead of formula (2.4) that

$$\lim_{n\to\infty}(E[f(U_n)D(y^{-n}V_n)]-E[f(U_n)]E[D(y^{-k_n}V_n)])=0 \tag{2.5}$$

for all bounded and continuous functions f on $\mathbf{N}$ and irreducible representations $D\in\operatorname{Irr}(\mathbf{K})$. (Here $D(y^{-k_n}V_n)$ is understood as a random matrix, and formula (2.5) means that all coordinates of a matrix satisfies the corresponding relation.)

Let us define for all $p=1,2,\dots$ and n such that $k_n\ge p$ the random products

$$U_{n,p}=\prod_{j=1}^{k_n-p} X_j^{(n)} \quad\text{and}\quad V_{n,p}=\prod_{j=1}^{k_n-p} Y_j^{(n)}. \tag{2.6}$$

Then we have

$$\|E[f(U_n)D(y^{-k_n}V_n)] - E[f(U_n)]E[D(y^{-k_n}V_n)]\| \le \alpha(n,p) + \beta(n,p) + \gamma(n,p), \quad (2.7)$$

where

$$\alpha(n,p) = \|E[(f(U_n) - f(U_{n,p}))D(y^{-k_n}V_n)]\| \le E|f(U_n) - f(U_{n,p})|,$$
$$\beta(n,p) = \|E[f(U_n) - f(U_{n,p})]E[D(y^{-k_n}V_n)]\| \le E|f(U_n) - f(U_{n,p})|,$$

and

$$\gamma(n,p) = \|E[f(U_{n,p})D(y^{-k_n}V_n)] - E[f(U_{n,p})]E[D(y^{-k_n}V_n)]\|, \quad (2.8)$$

since $\|D(x)\| = 1$ for all $x \in \mathbf{K}$.

Condition (i) implies that

$$\lim_{n\to\infty} \alpha(n,p) = 0, \qquad \lim_{n\to\infty} \beta(n,p) = 0 \quad (2.9)$$

for all $p = 1, 2, \ldots$. On the other hand,

$$\begin{aligned}\gamma(n,p) &= \|E[f(U_{n,p})\{D(y^{-k_n}V_n) - ED(y^{-k_n}V_n)\}]\| \\ &= \|E(E\left[f(U_{n,p})\{D(y^{-k_n}V_n) - ED(y^{-k_n}V_n)\} \mid U_{n,p}, V_{n,p}\right])\| \\ &= \left\|\int H_{n,p}(u,v)\mu_{n,p}(\,du,\,dv)\right\| \\ &\le \|f\|_\infty \int \left\|E\left[D\left(y^{-k_n}v \prod_{j=k_n-p+1}^{k_n} Y_j^{(n)}\right) - ED(y^{-k_n}V_n)\right]\right\| \nu_{n,p}(\,dv),\end{aligned}$$

where $\mu_{n,p}(\cdot,\cdot)$ denotes the distribution of the vector $(U_{n,p}, V_{n,p})$, $\nu_{n,p}(\cdot)$ the distribution of the random variable $V_{n,p}$, and

$$\begin{aligned}H_{n,p}(u,v) &= E\left[f(U_{n,p})\{D(y^{-k_n}V_n) - ED(y^{-k_n}V_n)\} \mid U_{n,p} = u, V_{n,p} = v\right] \\ &= f(u)E\left(\left\{D\left(y^{-k_n}v \prod_{j=k_n-p+1}^{k_n} Y_j^{(n)}\right) - ED(y^{-k_n}V_n)\right\}\right)\end{aligned}$$

because of our independence properties and the identity $V_n = V_{n,p} \prod_{j=k_n-p+1}^{k_n} Y_j^{(n)}$.

The relations $\operatorname{supp}(Y_j^{(n)}) \subset a\mathbf{K}' = \mathbf{K}'a$ for all $1 \le j \le k_n$ and $y \in a\mathbf{K}' = \mathbf{K}'a$ imply that $\operatorname{supp}(V_{n,p}) = \operatorname{supp}\left(\prod_{j=1}^{k_n-p} Y_j^{(n)}\right) \subset a^{k_n-p}\mathbf{K}'$ and $y^{-k_n} \in a^{-k_n}\mathbf{K}'$. Hence if $v = V_{n,p} \in \operatorname{supp} V_{n,p}$ then $y^{-k_n}v \in y^{-p}\mathbf{K}' = \mathbf{K}'y^{-p}$, and

$$\begin{aligned}\gamma(n,p) &\le \|f\|_\infty \sup_{v\in\mathbf{K}'} \left\|ED\left(vy^{-p} \prod_{j=k_n-p+1}^{k_n} Y_j^{(n)}\right) - ED(y^{-k_n}V_n)\right\| \\ &= \|f\|_\infty \sup_{v\in\mathbf{K}'} \left\|\mathcal{F}_{vy^{-p}V_{n,p}^{-1}V_n}(D) - \mathcal{F}_{y^{-n}V_n}(D)\right\|.\end{aligned}$$

Let us take a random variable V with uniform distribution on the subgroup $\mathbf{K}'$. The last relation implies that

$$\gamma(n,p) \le \|f\|_\infty \sup_{v\in\mathbf{K}'} \left\| \mathcal{F}_{vy^{-p}V_{n,p}^{-1}V_n}(D) - \mathcal{F}_V(D) \right\| + \|f\|_\infty \|\mathcal{F}_V(D) - \mathcal{F}_{y^{-n}V_n}(D)\|. \quad (2.10)$$

For all $p = 1, 2, \dots$ and $v \in \mathbf{K}'$ put

$$g_p(v) = \sup_{n\colon\, k_n \ge p} \left\| \mathcal{F}_{vy^{-p}V_{n,p}^{-1}V_n}(D) - \mathcal{F}_V(D) \right\|.$$

We claim that $g_p(\cdot)$ is a continuous function on the space $\mathbf{K}'$. Indeed, if $v, v' \in \mathbf{K}'$ then

$$\begin{aligned} \left\| \mathcal{F}_{vy^{-p}V_{n,p}^{-1}V_n}(D) - \mathcal{F}_V(D) \right\| &\le \left\| \mathcal{F}_{vy^{-p}V_{n,p}^{-1}V_n}(D) - \mathcal{F}_{v'y^{-p}V_{n-p}^{-1}V_n}(D) \right\| \\ &\quad + \left\| \mathcal{F}_{v'y^{-p}V_{n,p}^{-1}V_n}(D) - \mathcal{F}_V(D) \right\| \\ &\le \sup_{x\in\mathbf{K}} \|D(vx) - D(v'x)\| + g_p(v') \end{aligned}$$

for all $p \le k_n$, hence

$$g_p(v) \le g_p(v') + \sup_{x\in\mathbf{K}} \|D(vx) - D(v'x)\|.$$

Then because of the symmetric role of v and v'

$$|g_p(v) - g_p(v')| \le \sup_{x\in\mathbf{K}} \|D(vx) - D(v'x)\|,$$

and the function $g_p(\cdot)$ is continuous on the group $\mathbf{K}'$ because of the uniform continuity of the group representations $D \in \operatorname{Irr}\mathbf{K}$.

Because of property (ii) $\lim\limits_{p\to\infty} g_p(v) = 0$ for all $v \in \mathbf{K}'$. Hence

$$\bigcup_{p=1}^{\infty} \{v \in K'\colon\, g_p(v) < \varepsilon\} = \mathbf{K}'$$

for all $\varepsilon > 0$, and the compactness of the group $\mathbf{K}'$ implies that there exists an index $p(\varepsilon)$ such that

$$\bigcup_{p=1}^{p(\varepsilon)} \{v \in \mathbf{K}'\colon\, g_p(v) < \varepsilon\} = \mathbf{K}',$$

that is

$$\lim_{p\to\infty} \sup_{n\ge p} \sup_{v\in\mathbf{K}'} \left\| \mathcal{F}_{vy^{-p}V_{n,p}^{-1}V_n}(D) - \mathcal{F}_V(D) \right\| = 0.$$

Let us also observe that by taking only $p = k_n$ instead of $\sup\limits_{k_n \ge p}$ in Condition (ii) we get with the choice $v = e$, the unit element of the group that $\lim\limits_{n\to\infty} \|\mathcal{F}_V(D) - \mathcal{F}_{y^{-k_n}V_n}(D)\| = 0$. Hence the last relation together with formula (2.10) imply that

$$\lim_{p\to\infty} \sup_{n\ge p} \gamma(n,p) = 0. \quad (2.11)$$

Relations (2.7), (2.9) and (2.11) imply formula (2.5), hence Proposition 2.2.

3. Limit theorems on compact groups.

The results about limit theorems on compact groups are fairly well understood. In this Section we show that a result of the paper of Stromberg [7] formulated in Theorem 3.3.5 which he called in his paper the Main Theorem has some interesting consequences. We formulate Stromberg's result in a slightly different form.

Proposition 3.1. (Stromberg) *Let $Y_1, Y_2, \dots$ be a sequence of independent, identically distributed random variables on a compact group $\mathbf{K}$. Let us assume that the support of the distribution of the random variable Y_1,* $\operatorname{supp}(Y_1)$ *is not contained in any proper closed subgroup of the group $\mathbf{K}$. Then the random products $V_n = \prod_{j=1}^{n} Y_j$ converge in distribution as $n \to \infty$ if and only if the support* $\operatorname{supp}(Y_1)$ *of Y_1 is not contained in any coset of any proper closed normal subgroup of $\mathbf{K}$. If the limit distribution exists, then it is the Haar measure $\mu_{\mathbf{K}}$ of the group $\mathbf{K}$.*

A necessary and sufficient condition of the convergence in distribution of the random products V_n to the Haar measure $\mu_{\mathbf{K}}$ of the group $\mathbf{K}$ can be expressed with the help of the Fourier transform of the random variable Y_1 in the following way: This convergence holds if and only if for all irreducible group representations $D \in \operatorname{Irr} \mathbf{K}$ such that $D \neq D_0$, where D_0 denotes the identity group representation (i.e. $D_0(g) = 1$ for all $g \in \mathbf{K}$), the absolute values of all eigenvalues of the Fourier transform $\mathcal{F}_{Y_1}(D) = ED(Y_1)$ are strictly less than 1.

Stromberg formulated his result in the language of probability measures instead of random variables. Beside this, he formulated a slightly more general result, because he also discussed the case when the smallest closed subgroup $\mathbf{K}_0$ containing the support of the random variable Y_1 may be a proper subgroup of the group $\mathbf{K}$. But it is not hard to reduce this general case to the case described in Proposition 3.1, and actually this is done in Stromberg's paper. Stromberg did not formulate explicitly the statement of the second paragraph in Proposition 3.1, but he proved it. Actually the core of the proof of the sufficiency part of the convergence in distribution in Proposition 3.1 consists of the verification of this statement. We formulated this statement explicitly, because it plays an important role in our subsequent discussion.

Let us remark that a sequence of random variables V_n on a compact group $\mathbf{K}$ converges in distribution to the Haar measure $\mu_{\mathbf{K}}$ of this group if and only if for all irreducible group representations $D \in \operatorname{Irr} \mathbf{K}$, $D \neq D_0$, where D_0 is the identity group representation, $\lim_{n\to\infty} \mathcal{F}_{V_n}(D) = 0$. By Proposition 3.1, if these random variables are of the form $V_n = \prod_{j=1}^{n} Y_j$, where Y_j, $j = 1, 2, \dots$, are independent, identically distributed random variables, then this relation can hold only if the Fourier transform of Y_1 satisfies the property formulated in Proposition 3.1. This fact has deep consequences. Such consequences will be formulated in the following Corollary of Proposition 3.1

Corollary of Proposition 3.1. *Let $Y_1, Y_2, \dots,$ be a sequence of independent, identically distributed random variables on a compact group $\mathbf{K}$ such that the support* $\operatorname{supp}(Y_1)$

of the random variable Y_1 is not contained in any proper closed subgroup or any coset of any proper closed normal subgroup of $\mathbf{K}$*. Then the random products* $V_n = \prod\limits_{j=1}^{n} Y_j$ *converge in distribution to the Haar measure* $\mu_{\mathbf{K}}$ *of the group* $\mathbf{K}$*, and the random variables* Y_j *also satisfy property (ii) formulated in Proposition (2.2) with* $k_n = n$*,* $Y_j^{(n)} = Y_j$*,* $j = 1, \dots, n$*,* $\mathbf{K} = \mathbf{K}'$ *and* $y = e$*, where* e *is the unit element the group* $\mathbf{K}$*. Also the following generalization of the above statement holds.*

For all $n = 1, 2, \dots$*, let* $Y_j^{(n)}$*,* $j = 1, \dots, n$*, be a sequence of independent, identically distributed random variables on a compact group* $\mathbf{K}$ *such that the distributions of the random variables* $Y_1^{(n)}$*,* $\operatorname{dist} Y_1^{(n)}$ *converge weakly to the distribution of a random variable* Y *whose support* $\operatorname{supp}(Y)$ *is not contained in any proper closed subgroup or any coset of any closed normal subgroup of* $\mathbf{K}$*. Then these random variables* $Y_j^{(n)}$ *also satisfy property (ii) of Proposition 2.2 with* $k_n = n$*,* $\mathbf{K}' = \mathbf{K}$ *and* $y = e$*, and the random products* $V_n = \prod\limits_{j=1}^{n} Y_j^{(n)}$ *converge weakly to the Haar measure* $\mu_{\mathbf{K}}$ *of the group* $\mathbf{K}$ *as* $n \to \infty$*. These statements also hold if we do not assume that the independent random variables* $Y_j^{(n)}$ *are identically distributed, we only assume that if* ρ *is such a metric on the space of probability measures* μ *on the group* $\mathbf{K}$ *which metrizes weak convergence of probability distributions on* $\mathbf{K}$ *(such a metric on the space of probability measures on* $\mathbf{K}$ *exists if* $\mathbf{K}$ *is a separable metric space), then* $\lim\limits_{n\to\infty} \sup\limits_{1\le j\le n} \rho(\operatorname{dist} Y_j^{(n)}, \operatorname{dist} Y) = 0$*.*

This corollary states that if the products of independent and identically distributed random variables converge to the Haar measure of the group, then they also satisfy property (ii) of Proposition 2.2. Moreover, the same relation also holds for their small perturbations.

Proof of the Corollary of Proposition 3.1. It is enough to prove formula (2.3) with $k_n = n$ in the case $D \in \operatorname{Irr} G$, $D \ne D_0$, where D_0 is the identity group representation of $\mathbf{K}$. Then in the case investigated in this corollary $\mathcal{F}_V(D) = 0$, and $\mathcal{F}_{vy^{-p} \prod\limits_{j=n-p+1}^{n} Y_j^{(n)}}(D) = D(v) \prod\limits_{j=n-p+1}^{n} ED\left(Y_j^{(n)}\right)$, and we have to show that

$$\lim_{p\to\infty} \sup_{n\ge p} \left\| D(v) \prod_{j=n-p+1}^{n} ED\left(Y_j^{(n)}\right) \right\| = 0.$$

If $Y_j = Y_j^{(n)}$, $j = 1, 2, \dots, n$, are independent, identically distributed random variables satisfying the conditions of the first paragraph of this Corollary, then because of Proposition 3.1 there exists an index $m = m(D)$ such that

$$\left\| \prod_{j=l}^{l+m} ED\left(Y_j^{(n)}\right) \right\| \le \frac{1}{2} \quad \text{if} \quad 1 \le l \le l+m \le n. \tag{3.1}$$

Since $\|D(v)\| = 1$, $\left\| ED\left(Y_j^{(n)}\right)\right\| \le 1$ relation (3.1) implies that

$$\left\| D(v) \prod_{j=n-p+1}^{n} ED\left(Y_j^{(n)}\right)\right\| \le \left(\frac{1}{2}\right)^{p/m-1} \le \varepsilon \quad \text{if} \quad p \ge p_0 = p_0(D, \varepsilon)$$

for all $n \ge p$ with an appropriate number p_0. Since this relation holds for all $\varepsilon > 0$ it implies condition (ii) of Proposition 2.2 if the conditions in the first paragraph of this Corollary holds.

If the conditions of the second paragraph hold, then a slight modification of this argument yields the proof of formula (ii). Indeed, let Y_j, $j = 1, \dots, n$, be a sequence of independent, identically random variables with the same distribution as the random variable Y. Then there exists an index $n_0 = n_0(m, D)$ such that

$$\left\| \prod_{j=l}^{l+m} ED\left(Y_j^{(n)}\right) - \prod_{j=l}^{l+m} ED\left(Y_j\right)\right\| \le \frac{1}{6} \quad \text{for all} \quad 1 \le l \le l+m \le n$$

if $n \ge n_0$. This implies that a slight modification of formula (3.1), where the upper bound $\frac{1}{2}$ is replaced by $\frac{2}{3}$ holds in this case. This fact implies the validity of formula (2.3) also in this case. Finally, formula (2.3) with the choice $v = e$ and $n = p$ instead of $\sup_{n \ge p}$ implies that $\lim_{n\to\infty} ED(V_n) = 0$ if $D \ne D_0$. Hence the distributions of the random variables V_n converge to the Haar measure $\mu_{\mathbf{K}}$.

The following Theorem 3.2 can be obtained as a consequence of the already proved results.

Theorem 3.2. *Let* $\mathbf{N}$ *be a locally compact and* $\mathbf{K}$ *a compact group. Let* $\mathbf{G} = \mathbf{N} \times \mathbf{K}$ *denote their direct product. Let us consider the triangular array* $(X_j^{(n)}, Y_j^{(n)})$ *of random variables on the group* $\mathbf{G}$, $n = 1, 2, \dots$, $1 \le j \le k_n$, $k_n \to \infty$ *if* $n \to \infty$ *which are independent for a fixed* n *for all indices* $1 \le j \le k_n$. *Let us define the random products*

$$U_n = \prod_{j=1}^{k_n} X_j^{(n)}, \quad V_n = \prod_{j=1}^{k_n} Y_j^{(n)}, \qquad n = 1, 2, \dots.$$

Let us assume that Corollary 3.1 can be applied for the independent $\mathbf{K}$ *valued random variables* $Y_j^{(n)}$, $j = 1, 2, \dots, k_n$, *i.e.* $\lim_{n\to\infty} \sup_{1 \le j \le k_n} \rho\left(\operatorname{dist} Y_j^{(n)}, \operatorname{dist} Y\right) = 0$ *with a random variable* Y *on the group* $\mathbf{K}$ *which is not contained in any proper closed subgroup or any coset of a closed proper normal subgroup of* $\mathbf{K}$.

Let us also assume that the distributions of the random variables U_n *converge weakly to a probability measure* ν *on the group* $\mathbf{N}$, *and the random variables* $X_j^{(n)}$, $1 \le j \le n_k$, *satisfy the following smallness property: For all fixed positive integers* j $X_{k_n - j}^{(n)} \Rightarrow e$ *as*

$n \to \infty$, where $\Rightarrow$ denotes stochastic convergence, and e is the unit element of the group $\mathbf{N}$.

Then the distributions of the random vectors (U_n, V_n), $n = 1, 2, \dots$, converge weakly to the direct product $\nu \times \mu_{\mathbf{K}}$ on the group $\mathbf{G}$ as $n \to \infty$, where $\mu_{\mathbf{K}}$ denotes the Haar measure on the group $\mathbf{K}$.

Remark: In classical limit theorems for the products of independent random variables on a Lie group we also assume that the random variables whose normalized products converge in distribution satisfy the uniform smallness condition $\sup\limits_{1 \le j \le k_n} X_j^{(n)} \Rightarrow e$, and this is an essentially stronger condition than the condition imposed in Theorem 3.2.

Proof of Theorem 3.2. Let us first observe that to prove Theorem 3.2 it is enough to show that under its conditions the random variables $X_j^{(n)}$ satisfy condition (i) of Proposition 2.2. Indeed, this relation together with the Corollary of Proposition 3.1 imply that Proposition 2.2 can be applied, hence formula (2.4) holds with $y = e$, where e is the unit element of the group of $\mathbf{K}$. This relation together with the weak convergence of the random products U_n and V_n imply the validity of formula (2.1) with the choice $X = \mathbf{N}$, $Y = \mathbf{K}$ if P_n is the distribution of the random vector (U_n, V_n), and the pair of measures (μ, ν) is replaced by the pair of measures $(\nu, \mu_{\mathbf{K}})$. Hence Proposition 2.1 yields the desired statement.

To prove condition (i) of Proposition 2.2 observe that because of the smallness condition imposed on the random variables $X_j^{(n)}$ in Theorem 3.2 and the continuity of multiplication and inverse $U_p^{(n)} = \prod\limits_{j=k_n-p+1}^{k_n} X_j^n \Rightarrow e$ and also its inverse satisfies the relation $\left(U_p^{(n)}\right)^{-1} \Rightarrow e$ as $n \to \infty$ for all fixed positive integers p. Beside this, as the random variables U_n are weakly convergent, they are also tight, i.e. for all $\varepsilon > 0$ there is a compact set $K = K(\varepsilon) \subset \mathbf{N}$ such that $P(U_n \in K) > 1 - \varepsilon$ for all $n \ge n_0(\varepsilon, K)$. Since $\prod\limits_{j=1}^{k_n-p} X_j^{(n)} = U_n \left(\prod\limits_{j=k_n-p+1}^{k_n} X_j^n\right)^{-1}$, the above relations together with the uniform continuity of the product on a compact subset of $\mathbf{N} \times \mathbf{N}$ imply that

$$\rho\left(\prod_{j=1}^{k_n} X_j^{(n)}, \prod_{j=1}^{k_n-p} X_j^{(n)}\right) \Rightarrow 0 \tag{3.2}$$

for all fixed p as $n \to \infty$. Relation (3.2) follows from the above facts and the observation that for all $\varepsilon > 0$ and compact sets K there is a number $\delta = \delta(\varepsilon, K) > 0$ such that $\rho(x, xy^{-1}) \le \varepsilon$ if $x \in K$ and $\rho(y, e) < \delta$.

Since the products $\prod\limits_{j=1}^{k_n-p} X_j^{(n)}$ have a limit distribution (actually we only need that these random variables are tight) relation (3.2) also implies that

$$f\left(\prod_{j=1}^{k_n} X_j^{(n)}\right) - f\left(\prod_{j=1}^{k_n-p} X_j^{(n)}\right) \Rightarrow 0$$

for all continuous and bounded functions f on $\mathbf{N}$. Hence relation (2.2) holds. Theorem 3.2 is proved.

4. On limit theorems on Lie groups. Some open problems.

To apply Theorem 3.2 we still need some limit theorems for the products of the (independent) elements of a row in a triangular array of independent random variables taking values in a Lie group. In certain applications limit theorems for normalized random products of independent random variables are useful. Normalization of random products $\prod_{j=1}^{n} X_j$ of independent random variables X_j, $j = 1, 2, \dots,$ taking values in a Lie group means the application of a sequence of homomorphisms τ_n, $n = 1, 2, \dots,$ of the Lie group that is the definition of the expressions $\tau_n\left(\prod_{j=1}^{n} X_j\right) = \prod_{j=1}^{n} \tau_n(X_j)$. If we study such normalized products it is natural to restrict our attention to some special Lie groups to the so-called stratified groups where the homomorphisms τ_n can be defined in a natural way.

Fortunately, several non-trivial and useful central and other kind of limit theorems are known both for the products of the elements in a row of a triangular array and for the normalized products of independent random variables which take their values in a Lie group. Wehn [8] considered a triangular array of random variables $X_j^{(n)}$, $1 \le j \le k_n$, $k_n \to \infty$ if $n \to \infty$, taking values in a Lie group such that the elements of the random variables in the same row are not only independent, but also exchangeable in the sense that $X_j^{(n)} X_k^{(n)}$ and $X_k^{(n)} X_j^{(n)}$ have the same distribution, and gave sufficient conditions (including uniform smallness) for the convergence of the distribution of the products $U_n = \prod_{j=1}^{k_n} X_j^{(n)}$ towards a Gaussian measure. (Pap [5] proved that these conditions are also necessary under some extra assumption.) Pap has also proved in [4] the Lindeberg theorem for a triangular array of random variables in a stratified Lie group such that the elements of the random variables in the same row are exchangeable, and the limit distribution is a Gaussian measure which is stable with respect to the natural dilations. Moreover, [4] contains a Lindeberg theorem for the normalized products of independent, exchangeable random variables in the Heisenberg group such that the limit distribution is the standard Gaussian measure. (It is not known whether this theorem can be generalized for all stratified Lie groups.) There are some further results about functional limit theorems on Lie groups. But since such problems do not appear in our context we only refer to the paper Heyer and Pap [2] and the reference list therein.

The behaviour of the simplest and best understood case the behaviour of sums of independent real valued random variables suggests some natural conjectures and problems whose solution seems to be hard. We formulate some of them.

Let X_j, $j = 1, 2, \dots$, be a sequence of independent (stratified) Lie-group valued random variables, τ_n a sequence of homomorphisms of the Lie group such that the random variables $\tau_n(X_j)$ satisfy the uniform smallness condition, i.e.

$$\lim_{n\to\infty} \sup_{1\le j\le n} P(\tau_n(X_j) \notin G) = 0$$

for all open neighbourhoods G of the unit element of the group, and the sequences $\tau_n\left(\prod_{j=1}^n X_j\right)$ have a limit distribution for $n \to \infty$. We are interested when we can state that the normalized products of a small perturbation of these random variables also satisfy a limit theorem. More explicitly, we formulate the following problem. Let $\bar X_j$, $j = 1, 2, \dots$, be a new sequence of independent random variables on the same Lie-group which is a small perturbation of the original sequence X_j, i.e. the sequence $\bar X_j X_j^{-1}$ converges stochastically to the unit element of the Lie group as $j \to \infty$. When can we state that also the products $\tau_n\left(\prod_{j=1}^n \bar X_j\right)$ or their appropriate normalizations $\tau_n\left(\prod_{j=1}^n \bar{\bar X}_j\right)$ have a limit distribution where $\bar{\bar X}_j = \bar X_j x_j$ with an appropriate element x_j of the Lie group? What can be said if the random variables X_j are not only independent but also identically distributed?

If real valued random variables are considered then the answer to the above questions is fairly well understood. Let us consider the most important special case when the partial sums of the independent random variables X_j, $j = 1, 2, \dots$, divided by the square root of the number of summands satisfy the central limit theorem. If the new random variables $\bar X_j$ satisfy the relation $\lim_{j\to\infty} E(\bar X_j - X_j)^2 = 0$ then the normalized partial sums of the random variables $\bar X_j$ may not converge in distribution but the normalized partial sums of their appropriate scaling $\bar{\bar X}_j = \bar X_j - E\bar X_j$ satisfy the same central limit theorem as the normalized partial sums of the original random variables X_j. This is the reason why we asked in the case of general Lie groups not only about the possible limit distribution of the normalization of the products of the random variables $\bar X_j$ but also about the limit distribution of the normalized products of their appropriate shift $\bar{\bar X}_j = \bar X_j x_j$.

We are interested in the question how the above statement can be generalized to the case of general Lie group valued random variables. We can prove only some similar results in this direction if the Lie group is special, it is a stratified Lie group, and the homomorphisms are also special, they are the natural dilations of the group. A similar question can be asked also in the case of other limit theorems when the limit distribution may be a non normal law. But we cannot handle this problem in the general case. The main cause of the difficulty in this problem is that the notion of expected value and variance of a Lie group valued random variable cannot be defined in the case of general Lie groups.

Let us consider a triangular array of random variables $X_j^{(n)}$, $1 \le j \le k_n$, $k_n \to \infty$ if $n \to \infty$, taking values in a Lie group such that the elements of the random variables in the same row are not only independent, but also identically distributed, and the products $U_n = \prod\limits_{j=1}^{k_n} X_j^{(n)}$ have a non-degenerated limit distribution. We are interested in the question when we can state that this convergence of the products in distribution implies that the random variables $X_j^{(n)}$ satisfy the uniform smallness condition, i.e. when the random variables $X_1^{(n)}$ converge stochastically to the unit element of the group. It is known that if the Lie group is the real line then this property holds. The question arises for which Lie groups this result can be generalized. It is natural to expect such a result for such Lie groups which have no compact subgroup beside the trivial subgroup consisting of the unit element of the group. But the classical proof of this result on the real line exploits the special properties of the trigonometrical functions, hence the proof of such a result demands new ideas.

A natural generalization of the problem investigated in this paper is the study the condition under which the products $\left(A_n \prod\limits_{j=1}^{k_n} X_j^{(n)}, B_n \prod\limits_{j=1}^{k_n} Y_j^{(n)}\right)$ have a limit distribution, where $X_j^{(n)}$, $1 \le j \le k_n$, is a triangular array of random variables in a non-compact Lie group $\mathbf{N}$, $Y_j^{(n)}$, $1 \le j \le k_n$, is a triangular array of random variables taking values in a compact group $\mathbf{K}$, the random variables $X_j^{(n)}$, $1 \le j \le k_n$, are independent for a fixed n, the same relation holds for the random variables $Y_j^{(n)}$, and finally $A_n \in \mathbf{N}$ and $B_n \in \mathbf{K}$ are appropriate norming constants in the groups $\mathbf{N}$ and $\mathbf{K}$ respectively. Formally this question can be reduced to the original question studied in this paper, since we can get rid of the norming constants A_n and B_n by replacing the random variables $X_n^{(n)}$ by $\bar{X}_n^{(n)} = A_n X_n^{(n)}$, $X_k^{(n)}$ by $\bar{X}_k^{(n)} = A_n X_k^{(n)} A_n^{-1}$ for $1 \le k \le n-1$, $Y_n^{(n)}$ by $\bar{Y}_n^{(n)} = B_n Y_n^{(n)}$ and $Y_k^{(n)}$ by $\bar{Y}_k^{(n)} = B_n Y_k^{(n)} B_n^{-1}$ for $1 \le k \le n-1$. However, this observation in itself is not enough to handle the more general problem, since the conditions are formulated for the original random variables $X_k^{(n)}$ and $Y_n^{(k)}$ and not for their transforms $\bar{X}_k^{(n)}$ and $\bar{Y}_k^{(n)}$. We know very little about this problem.

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