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A Local Limit Theorem for the Convolution of Probability Measures on a Compact Connected Group

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Dedicated to Professor Leopold Schmetterer on his sixtieth Birthday

Summary. Given a sequence of probability measures on a compact connected group we give an estimate on the speed of convergence to 1 of the density function of their n-fold convolution. Our method is an adaptation of the characteristic function technique to this case.

1. Introduction

The convolution of probability measures $\mu_1, \ldots, \mu_n$ on a compact group G tends under quite general conditions to the Haar measure of this group. (See e.g. [1, 4, 5].) In this paper, we investigate the speed of this convergence.

Let us introduce some notation. Given a measure μ, we decompose it as $\mu = \lambda + \nu$, where λ is the singular and ν is the absolute continuous part of μ with respect to the Haar measure χ. Let

$$p = \frac{d\nu}{d\chi}.$$

We define the quantities

$$M_\mu(x) = \chi(g \in G, p(g) \geqq x)$$

and

$$\tilde{S}(\mu) = \int_0^\infty [M_\mu(x)]^3 \, dx.$$

It was shown in [7] and in [8] that $\tilde{S}(\mu)$ is an important characteristic of the measure μ. We introduce another quantity

$$S(\mu) = \int_0^{\frac{\pi}{4}} x^2 N_\mu(x) \, dx$$

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where

$$N_\mu(x) = \inf\{u: M_\mu(u) < x\}$$

i.e. N_μ is the inverse function of M_μ. The main result of this paper is the following

Theorem. *Let $\mu_1, \ldots, \mu_n$, $n \geqq 2$ be a finite sequence of probability measures on a compact connected group G. Let us assume that there are two indices i, j, $1 \leqq i < j \leqq n$, such that μ_i and μ_j are absolutely continuous with respect to the Haar measure χ of G with density functions p_i and p_j belonging to the class $L^2(G, \chi)$. Then the measure $\mu_1 * \mu_2 * \ldots * \mu_n$ has a density function q_n satisfying the inequality*

$$\sup_{g \in G} |q_n(g) - 1| \leqq \|p_i - 1\|_{L^2(G, \chi)} \|p_j - 1\|_{L^2(G, \chi)} \prod_{\substack{k=1 \\ k \neq i, j}}^{n} \left(1 - \tfrac{11}{24} S(\mu_k)\right).$$

The following relation holds between the quantities $S(\mu)$ and $\tilde{S}(\mu)$ introduced in [7].

$$\int_0^\infty [M_\mu(x)]^3 \, dx = \int_0^1 N_\mu(x) \, d(x^3) = \int_0^1 3x^2 N_\mu(x) \, dx.$$

Since N_μ is a monotone decreasing function, the last relation implies that

$$S(\mu) \leqq \frac{1}{3} \tilde{S}(\mu) \leqq \frac{1}{3} \left(\frac{4}{\pi}\right)^3 S(\mu).$$

2. Discussion of the Theorem

Remark 1. Our theorem has the following

Corollary. *Given a sequence of density functions $p_1, p_2, \ldots$ on a compact connected topological group G, we define their convolutions $q_n = p_1 * \ldots * p_n$. If $\sup_{g \in G} p_n(g) \leqq C_n$ then*

$$\sup_{g \in G} |q_n(g) - 1| \leqq (C_1 - 1)(C_2 - 1) \exp\left(-C \sum_{j=3}^{n} C_n^{-2}\right)$$

for every $n \geqq 2$. In particular if $\sum_{j=1}^{\infty} C_n^{-2} = \infty$ then q_n tends uniformly to 1. (C is a universal constant, it can be chosen e.g. as $C = 1/200$.)

Proof of the Corollary. As $N_\mu(u) \geqq x$ if and only if $M_\mu(x) \leqq u$ the set $\{u, u \in [0, 1], N_\mu(u) \geqq x\}$ has Lebesgue measure $M_\mu(x)$. Thus we have

$$\int N_{\mu_k}(x) \, dx = \int p_k(g) \, d\chi(g) = 1$$

and

$$N_{\mu_k}(x) \leqq C_k, \quad 0 \leqq x \leqq 1,$$

where μ_k is the measure with the density function p_k. Hence

$$\int_0^1 x^2 N_{\mu_k}(x)\,dx \geqq \int_0^{\frac{1}{C_k}} C_k x^2\,dx = \frac{1}{3C_k^2},$$

and

$$S(\mu_k) \geqq \left(\frac{\pi}{4}\right)^3 \frac{1}{3C_k^2}.$$

As $\|p_k - 1\|_{L^2(G,\chi)} \leqq C_k - 1$, the theorem together with the estimate on $S(\mu_k)$ imply the corollary. At the end of this paper in Remark 4 we will investigate whether the result of the theorem and its corollary are sharp. We will show that if G is commutative, or more generally, it has a one-dimensional representation and the functions M_k, $M_k(1) < 1$, are fixed then the following estimates from below hold true: There exists a sequence of measures $\mu_1, \mu_2, \ldots$ in such a way that the absolute continuous part of μ_k has a density function p_k with distribution M_k i.e. $\chi(g \in G,\ p_k(g) \geqq x) = M_k(x)$ and the density function q_n of the measure $\mu^{(n)} = \mu_1 * \ldots * \mu_n$ satisfies the inequality

$$\sup_{g \in G} |q_n(g) - 1| \geqq \prod_{k=1}^{n} \max(1 - 11\, S(\mu_k), 0) \tag{2.2}$$

(if the density q_n exists at all). If $\Sigma S(\mu_k) < \infty$ then the sequence $\mu^{(n)}$ does not tend to the Haar measure even in the weak-star topology. In particular in the exponent in formula (2.1) only the constant factor can be improved, and if the C_j are given in the corollary in such a way that $\Sigma C_j^{-2} < \infty$, then q_n may not tend uniformly to 1.

Remark 2. Let p be a density function on the group G. Let $p_n = p * \ldots * p$ denote the n-fold convolution of this function. Our theorem implies that if

$$p \in L^2(G, \chi),$$

then

$$\sup |p_n(g) - 1| \leqq K q^n$$

with some $K > 0$, $0 < q < 1$ for every $n \geqq 2$. The same results holds for sufficiently large n if we assume only that $p \in L^{1+\varepsilon}(G, \chi)$ with some $\varepsilon > 0$. Indeed, if $q_1 \in L^\alpha(G, \chi)$, $q_2 \in L^\beta(G, \chi)$, $\alpha, \beta > 1$ then $q_1 * q_2 \in L^{\alpha\beta}(G, \chi)$, and this statement implies that $p_k \in L^2(G, \chi)$ for sufficiently large k. The last statement follows from the following estimates:

$$\begin{aligned} q_1 * q_2(g) &= \int q_1(h)\, q_2(h^{-1} g)\, d\chi(h) \leqq \left[\int q_1(h)\, q_2(h^{-1} g)^\beta\, d\chi(h)\right]^{\frac{1}{\beta}} \\ &\leqq \left[\int q_2(h^{-1} g)^\beta\, d\chi(h)\right]^{\frac{1}{\beta}\left(1-\frac{1}{\alpha}\right)} \left[\int q_1(h)^\alpha\, q_2(h^{-1} g)^\beta\, d\chi(h)\right]^{\frac{1}{\alpha\beta}} \end{aligned}$$

by Hölder's inequality.

Therefore

$$\int [q_1 * q_2(g)]^{\alpha\beta} \, d\chi(h) \leqq C \iint q_1(h)^\alpha \, q_2(h^{-1} g)^\beta \, d\chi(h) \, d\chi(g)$$
$$= C \int q_1(h)^\alpha \, d\chi(h) \int q_2(h)^\beta \, d\chi(h)$$

with $C = [\int q_2(h)^\beta \, d\chi(h)]^{(\alpha-1)}$.

On the other hand, there exists a density function p such that p_n does not tend uniformly to 1. The following example shows this:

Let G be the interval $[-\frac{1}{2}, \frac{1}{2})$ with addition modulo 1. We define

$$f_m(g) = \begin{cases} 2^{m^4} & \text{if} \quad |g| < 2^{-(m^4+m+1)} \\ 0 & \text{otherwise} \end{cases}.$$

Set $p(g) = \sum_{m=1}^{\infty} f_m(g)$. The functions p_n do not tend umformly to 1 since the inequality $p_n(g) > 2^{\frac{1}{2}n^2}$ if $|g| < 2^{-n^4-2n}$ holds for every large n. (Actually, this inequality holds even for the n-fold convolution of the function f_n.)

The results in [7] and [8] imply that the measures μ_n, $\mu_n(A) = \int_A p_n(g) \, d\chi(g)$ tend to the Haar measure in the variational metric. Actually the speed of convergence is exponential. But, as the last example shows, $p_n(g)$ does not necessarily tend uniformly to 1.

The argument of Remark 2 shows that the condition about the existence of two square integrable density functions can be substituted by some other condition. On the other hand if we have no moment type conditions on the density functions then the statement of the theorem may cease holding.

Remark 3. If the quantity $S(\mu)$ is large then the absolute continuous part of the measure μ cannot be concentrated on a small subset of G. Thus the intuitive meaning of our theorem is the following: If the measures μ_k are not concentrated on very small subsets of G then their convolutions tend to the Haar measure of G. The condition about the existence of two square integrable density functions was imposed in order to guarantee a local limit theorem, i.e. a limit theorem for the density functions of the convolutions. If we drop this condition we can still state the following result. If a sequence of probability measures $\mu_1, \mu_2, \ldots, \mu_n \ldots$ is such that $\Sigma S(\mu_k) = \infty$ then the sequence of measures $\mu^{(n)} = \mu_1 * \ldots * \mu_n$ tends to the Haar measure in the variational metric. This result can simply be deduced from our theorem, and actually in [8] it was deduced from a weaker statement. One can also give an estimate on the distance of the measures $\mu^{(n)}$ and χ in the variational metric. We show this is an example.

Let μ be a probability measure on G, $S(\mu) > 0$ and let $\mu^{(n)}$ be its n-fold convolution. Then we can choose two appropriate probability measures μ_1 and μ_2 in such a way that $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ with some $0 < \alpha < 1$, $S(\mu_1) > 0$, $S(\mu_2) > 0$, μ_1 is absolute continuous with respect to the Haar measure χ and $\frac{d\mu_1}{d\chi} \in L^2(G, \chi)$. We have

$$\mu^{(n)} = \Sigma \, \alpha^{k(i_1, \ldots, i_n)} (1-\alpha)^{n-k(i_1, \ldots, L_n)} \mu_{i_1} * \ldots * \mu_{i_n}$$

where $i_j=1$ or $2, j=1,2,\ldots,n$ and $k(i_1,i_2,\ldots,i_n)$ is the number of $1-s$ among the indices $i_1, i_2,\ldots,i_n$. If $k(i_1,i_2,\ldots,i_n)\geqq 2$ then because of the theorem $\mu_{i_1}*\mu_{i_2}*\ldots*\mu_{i_n}$ has a density function which tends to 1 exponentially fast. The sum of the coefficients of those term for which $k(i_1,i_2,\ldots,i_n)<2$ is exponentially small. Therefore we can conclude that the measures $\mu^{(n)}$ tend to the Haar measure χ in the variational metric expontially fast. As the condition $S(\mu)>0$ is equivalent to saying that μ has an absolute continuous part actually we obtained a strenghtened form of a result of Bhattacharya (theorem [3] in 1). The other results of [1, 5] and [6] concerning compact connected groups can also be deduced from our result. In case of a non-connected group the conditions of our theorem are not sufficient to imply convergence to the Haar measure of this group. In this case some additional conditions must be imposed in order to exclude the possibility that the measures are concentrated on a closed proper subgroup.

Let us finally remark that the paper [2] had a great impact on this work. In that paper a special case of the corollary of this work would have been needed. The result needed in [2] was reduced to a much weaker statement about the convolution of measures on the unit circle, but it did not follow from previously known results about compact groups. Finally it was deduced from a local central limit theorem on the real line. Later the authors of the present paper found a simple proof for the corollary of this paper in the case when the group G is the unit circle. The present work, just as papers [7] and [8], was made in an attempt to understand the more general law behind this result. Paper [7] contains no proof. In paper [8] the case of a general compact connected group is reduced to the case of the unit circle. It is done by means of some structure theorems about Lie groups. In the present work we applied Fourier analysis in the general situation. This method enabled us to generalize the results and to simplify the proof.

3. Proof of the Theorem

First we recall some facts from the harmonic analysis on compact groups (see [3]). Given a (locally) compact group G and a complex Hilbert space H a mapping $\sigma: G\to L(H)$, where $L(H)$ is the group of unitary transformations on H, is called a representation of G in H if it has the properties (I) $\sigma(g_1 g_2) =\sigma(g_1)\sigma(g_2)$ for all $g_1, g_2\in G$ and (II) for all $x\in H$ the mapping $x\to\sigma(g)x$ is continuous in $g\in G$. The representation σ is called irreducible if there is no proper closed subspace of H invariant under all $\sigma(g)$, $g\in G$. Every representation of a compact group G is the direct product of some irreducible representations, and every irreducible representation is finite dimensional. If G is commutative, then every irreducible representation is one-dimensional. Two representations σ_1 and σ_2 are called unitarily equivalent if there exists a unitary transformation U such that $\sigma_1(g)=U\sigma_2(g)U^*$ for every $g\in G$.

Let $\Sigma=\{\sigma\}$ denote the set of irreducible unitarily non-equivalent group representations of the group G. If μ is a finite measure on G, its Fourier transform $\hat{\mu}$ is the operator valued function defined by the equation

$$\hat{\mu}(\sigma)=\int_G \sigma(g)\,d\mu(g), \qquad \sigma\in\Sigma.$$

The following relation holds true:

$$(\mu_1 * \mu_2)(\sigma)=\hat{\mu}_1(\sigma)\,\hat{\mu}_2(\sigma). \tag{3.1}$$

Let $d(\sigma)$ denote the dimension of the representation σ, and let $e_1^\sigma, \ldots e_{d(\sigma)}^\sigma$ be an orthonormal basis in the space of the representation. By the Peter-Weyl theorem the set of functions $d(\sigma)^{\frac{1}{2}}\langle e_i^\sigma, \sigma(g)\, e_j^\sigma\rangle$, $1\leqq i, j\leqq d(\sigma)$, $\sigma\in\Sigma$ is a complete orthonormal system in the space $L^2(G,\chi)$. Thus for every $f\in L^2(G,\chi)$

$$\begin{aligned}\int_G |f(g)|^2\, d\chi(g) &= \sum_\sigma d(\sigma) \sum_{1\leqq i,\, j\leqq d(\sigma)} |\langle e_i^\sigma, f(\sigma)\, e_j^\sigma\rangle|^2\\ &=\sum_\sigma d(\sigma)\sum_{j=1}^{d(\sigma)} \|\hat{f}(\sigma)\, e_j^\sigma\|^2=\sum_\sigma d(\sigma)\sum_{j=1}^{d(\sigma)} \|\hat{f}^*(\sigma)\, e_j^\sigma\|^2,\end{aligned} \tag{3.2}$$

where $\hat{f}$ is the Fourier transform of the measure $f\cdot\chi$ and $\hat{f}^*(\sigma)$ denotes the adjoint of $\hat{f}(\sigma)$. Let us define the norm of the measure μ as

$$\|\mu\|=\sup_{\sigma\in\Sigma-\{\mathrm{Id}\}} \|\hat{\mu}(\sigma)\|$$

where $\|\hat{\mu}(\sigma)\|$ is the usual norm of the linear transformation $\hat{\mu}(\sigma)$ in the $d(\sigma)$ dimensional complex Euclidean space.

Let $p\in L^2(G,\chi)$ be a density function, and μ a probability measure on G. We consider the density functions q_1 and q_2 of the measures $\mu * p\cdot\chi$ and $p\cdot\chi*\mu$.

As $\mu*p\cdot\chi-\chi=\mu*(p-1)\cdot\chi$ and $p\cdot\chi*\mu-\chi=(p-1)\cdot\chi*\mu$ relations (3.1) and (3.2) imply that

$$\|q_i-1\|_{L^2(G,\chi)}\leqq \|p-1\|_{L^2(G,\chi)}\cdot\|\mu\|. \tag{3.3}$$

With the help of relation (3.3) we prove the following basic lemma. We preserve the notation of the theorem.

Lemma 1. *Let G be a (not necessarily connected) compact group. Let the sequence $\mu_1, \ldots, \mu_n$ of measures satisfy the conditions of the theorem. Then we have*

$$\sup_{g\in G} |q_n(g)-1|\leqq \|p_i-1\|_{L^2(G,\chi)}\cdot\|p_j-1\|_{L^2(G,\chi)} \prod_{\substack{k=1\\ k\neq i,\, j}}^n \|\mu_k\|.$$

Proof. Let us define

$$\nu_1=\mu_i*\ldots*\mu_i, \qquad \nu_2=\mu_{i+1}*\ldots*\mu_n, \qquad q_i=\frac{d\nu_i}{d\chi}, \qquad i=1,2.$$

Relation (3.3) implies that

$$\|q_1-1\|_{L^2(G,\chi)}\leqq \|p_i-1\|_{L^2(G,\chi)} \prod_{k=1}^{i-1} \|\mu_k\|$$

and

$$\|q_2-1\|_{L^2,(G,\chi)} \leqq \|p_j-1\|_{L^2(G,\chi)} \prod_{\substack{k=i+1 \\ k \neq j}}^{n} \|\mu_k\|.$$

Applying the Cauchy-Schwarz inequality we obtain that

$$\begin{aligned}|q_n(g_0)-1| &= |[q_1-1]*[q_2-1](g_0)| \\ &= |\int [q_1(g)-1][q_2(g^{-1}g_0)-1]\, d\chi(g)| \\ &\leqq \|q_1-1\|_{L^2(G,\chi)} \|q_2-1\|_{L^2(G,\chi)}\end{aligned}$$

for every $g_0 \in G$. These inequalities imply the lemma.

In order to complete the proof of the theorem we have to estimate $\|\mu_k\|$.

Let us now consider an n-dimensional unitary irreducible representation $\pi: G \to U(n)$ of the compact connected group G. Let $S^{2n-1} \subset \mathbb{C}^n$ denote the unit sphere in the space of the representation, and let $\mathscr{B}^{2n-1}$ be the Borel σ-algebra on S^{2n-1}. Let τ be a (G,π)-invariant probability measure on S^{2n-1} i.e. let $\tau(E) = \tau(\pi(g)E)$ for every $E \in \mathscr{B}^{2n-1}$ and $g \in G$.

We identify $\mathbb{C}^n$ with the $2n$ dimensional real Euclidean space R^{2n} by defining the scalar product in the new space as the real part of the scalar product in $\mathbb{C}^n$. We identify also the unit sphere $S^{2n-1} \subset R^{2n}$ with $S^{2n-1} \subset \mathbb{C}^n$ and the σ-algebras on these spheres. We introduce the metric $\rho(x,y) = \arccos \langle x, y\rangle$, $x, y \in S^{2n-1}$. Let us finally fix an $x \in S^{2n-1}$ and define the set

$$\mathscr{D}_x(r) = \{y,\ y \in S^{2n-1},\ \rho(x,y) < r\}.$$

We need the following estimate on $\tau(\mathscr{D}_x(r))$.

Lemma 2. *The inequality*

$$\tau(\mathscr{D}_x(r)) \leqq \left[\frac{\pi}{2r}\right]^{-1}$$

holds true, where [ ] *denotes integer part.*

Proof. Let us consider the set

$$O(x) = \{\pi(g)x : g \in G\}$$

It is enough to show that there exist $m = \left[\frac{\pi}{2r}\right]$ points $x_1, \ldots, x_m$ such that $x_1 = x$, $x_j \in O(x)$ and $g(x_i, x_j) \geqq 2r$ if $i \neq j$. Indeed, these properties imply that $\mathscr{D}_{x_i}(r) \cap \mathscr{D}_{x_j}(r) = \emptyset$ if $i \neq j$, and $\tau(\mathscr{D}_{x_i}(r)) = \tau(\mathscr{D}_{x_j}(r))$ because of the (G,π)-invariance of τ.

Thus

$$\tau(\mathscr{D}_x(r)) \leqq \frac{1}{m}\, \tau\left(\bigcup_{l=1}^{m} \mathscr{D}_{x_i}(r)\right) \leqq \frac{1}{m}$$

as we claimed.

Because of the orthogonality of the functions $\langle e, \mathrm{Id}\, e\rangle = 1$ and $\langle e_i, \pi e_j\rangle$ $i, j = 1, \ldots, n$ in the space $L^2(G, \chi)$ we have

$$\int_G \pi(g)\, x\, d\chi(g) = 0.$$

Thus the set $O(x)$ cannot be contained in any half-sphere of S^{2n-1}. On the other hand $O(x)$ is a connected set, being the image of a continuous mapping from a connected topological space. Now we begin the construction of the sequence $x_1, \ldots, x_m$ with the required properties. Let $x_1 = x$. If $x_1, \ldots, x_j$ are already given we look for a point $x_{j+1} \in O(x)$ such that $\rho(x_{j+1}, \{x_1, \ldots, x_j\}) = 2r$. It is sufficient to show that there exists such a point x_{j+1}, if $j < m$. It is not difficult to see by induction that if $j < m$, there exists a segment containing $x_1, \ldots, x_j$ with diameter less than or equal to $(2j-1)r$. As $O(x)$ is contained in no half-sphere, there exists a point $y \in O(x)$ such that $\rho(y, \{x_1, \ldots, x_j\}) \geqq 2r$. Because of the connectedness of $O(x)$ there exists also an $x_{j+1} \in O(x)$ such that $\rho(x_{j+1}, \{x_1, \ldots, x_j\}) = 2r$. The lemma is proven.

We will use the following somewhat weaker statement:

$$\tau(\mathscr{D}_x(r)) \leqq r \quad \text{if } r \leqq \frac{\pi}{4}. \tag{3.4}$$

Now we prove the following

Lemma 3. *Let μ be a probability measure on G, and π a non-trivial irreducible unitary representation of G. Then the inequality*

$$\|\hat{\mu}(\pi)\| < 1 - \tfrac{11}{24} S(\mu)$$

holds true.

Lemmas 1 and 3 together imply the theorem.

Proof of Lemma 3. Given two points $x, y \in S^{2n-1}$ we give an estimate from below on $1 - \langle x, \hat{\mu}(\pi)\, y\rangle$. Let $\mu = \lambda + \nu$, where λ is the singular and ν is the absolute continuous part of μ. We have

$$\begin{aligned} 1 - \langle x, \hat{\mu}(\pi)\, y\rangle &= \int [1 - \langle x, \pi(g)\, y\rangle]\, d\mu(g) \\ &\geqq \int_G [1 - \langle x, \pi(g)\, y\rangle]\, d\nu(g) = \int_0^\pi [1 - \cos u]\, d\psi(u) \end{aligned} \tag{3.5}$$

where

$$\psi(A) = \nu(g\colon g \in G, \rho(x, \pi(g)\, y) \in A), \qquad \text{for } A \subset [0, \pi].$$

The last identity can be shown on applying the transformation

$$T\colon G \to [0, \pi], \qquad T(g) = \rho(x, \pi(g)\, y).$$

Let us define the measure $\bar{\psi}$ on the interval $[0, \pi]$ by the formula

$$\bar{\psi}([0, a)) = \begin{cases} \int_0^a N_\mu(x)\, dx & \text{if } a < \frac{\pi}{4} \\ \nu(G) & \text{if } a \geqq \frac{\pi}{4} \end{cases}$$

We have

$$\psi([0, \pi]) = \bar{\psi}([0, \pi]) = \nu(G). \tag{3.6}$$

We are going to show that

$$\psi([0, a)) \leqq \bar{\psi}([0, a)) \quad \text{for all } 0 \leqq a \leqq \pi. \tag{3.7}$$

First we show that if $C \subset G$ is a measurable set, and $\chi(C) \leqq a$ then

$$\nu(C) \leqq \int_0^a N_\mu(x)\, dx.$$

Let us define the functions

$$M_{\mu, C}(x) = \chi\left(g \in C, \frac{d\nu}{d\chi}(g) \geqq x\right)$$

and

$$N_{\mu, C}(x) = \inf\{u : M_{\mu, C}(u) < x\}.$$

Then we have

$$M_{\mu, C}(x) \leqq M_\mu(x), \qquad N_{\mu, C}(x) \leqq N_\mu(x)$$

and

$$N_{\mu, C}(a) = 0.$$

Hence

$$\nu(C) = \int_0^a N_{\mu, C}(u)\, du \leqq \int_0^a N_\mu(u)\, du.$$

Now in order to prove formula (3.7) it is enough to show that for all sets

$$C(a) = \{g \in G, \rho(x, \pi(g)\, y) < a\}, \qquad a \leqq \frac{\pi}{4}$$

we have $\chi(C(a)) \leqq a$.

But defining the measure τ on the unit sphere by

$$\tau(B) = \chi(g : \pi(g)\, y \in B) \qquad \text{for } B \in \mathscr{B}^{2n-1}$$

it is not difficult to see that τ is a (G, π) invariant measure. Hence, applying relation (3.4), we obtain that

$$\chi(C(a)) = \tau(\mathscr{D}_x(a)) \leqq a \qquad \text{for } a \leqq \frac{\pi}{4}.$$

Since $1-\cos u$ is a monotone increasing function on the interval $[0,\pi]$, relations (3.6) and (3.7) imply the inequality

$$\int_0^\pi [1-\cos u]\, d\psi(u) \geqq \int_0^\pi [1-\cos u]\, d\bar{\psi}(u).$$

This fact can easily be seen, e.g. on integrating by parts. Therefore relation (3.5) yields the inequality

$$\begin{aligned}1-\langle x, \hat{\mu}(\pi)\, y\rangle &\geqq \int_0^\pi [1-\cos u]\, d\bar{\psi}(u) \geqq \int_0^{\frac{\pi}{4}} [1-\cos u]\, N_\mu(u)\, du\\ &\geqq \tfrac{11}{24}\int_0^{\frac{\pi}{4}} u^2 N_\mu(u)\, du.\end{aligned}$$

In other words

$$\langle x, \hat{\mu}(\pi)\, y\rangle \leqq 1-\tfrac{11}{24} S(u)$$

for every $x, y \in S^{2n-1}$, therefore the lemma is proven.

Remark 4. The estimate in Lemma 3, and hence also in the theorem can be improved. If G is a commutative compact connected group we can give a sharp estimate on $\|\hat{\mu}(\pi)\|$. In this case every irreducible non-trivial representation is one-dimensional, and if π is such a representation, the only (G,π)-invariant probability measure m on S^1 is the uniform distribution on the unit circle.

Indeed, every rotation $\pi(g)$, $g\in G$ leaves the measure m invariant. The set $A=\{\pi(g)\colon g\in G\}$ is a connected subset of the unit circle, containing at least two points. Thus A contains an arc. The only invariant measure with respect to these rotations is the uniform distribution.

This means that in the commutative case we can write

$$\tau(\mathscr{D}_x(r))=\frac{r}{\pi}, \qquad 0\leqq r\leqq \pi$$

in Lemma 2.

Thus we obtain the estimate

$$\|\mu\| \leqq 1-\bar{S}(\mu)$$

instead of Lemma 3, where

$$\bar{S}(\mu)=\int_0^1 [1-\cos \pi u]\, N_\mu(u)\, du.$$

This implies that the estimate in the theorem can be improved to

$$\sup_{g\in G} |q_n(g)-1| \leqq \|p_i-1\|_{L^2(G,\chi)}\, \|p_j-1\|_{L^2(G,\chi)} \prod_{\substack{k=1\\ k\neq i,j}}^{n} (1-\bar{S}(\mu_k)). \tag{3.8}$$

On the other hand if a representation π of the group G and the distributions M_k of the functions p_k are fixed, where p_k is the density function of the absolute continuous part of the measure μ_k appearing in the theorem, then fixing an arbitrary $x \in S$ the measures μ_k can be chosen so that

$$\hat{\mu}_k(\pi)\,x=(1-S(\mu_k))\,x. \tag{3.9}$$

This implies that μ_k can be chosen in such a way that

$$\hat{\mu}_k(\pi)=(1-\bar{S}(\mu_k))\cdot \mathrm{Id}. \tag{3.10}$$

A measure μ_k satisfying (3.9) can be obtained by concentrating its singular part λ_k on the unit element of G and defining the density p_k of its absolute continuous part in such a way that the distribution function of p_k is M_k and $p_k(g_1) \geqq p_k(g_2)$ if $\langle x, \pi(g_1)x\rangle \geqq \langle x, \pi(g_2)x\rangle$. The density function g_n can be written as

$$q_n(g)=1+\sum_{\sigma} a_n(\sigma)\,\sigma(g),$$

where

$$a_n(\sigma)=\prod_{k=1}^{n} \hat{\mu}_k(\sigma),$$

and $\sigma-s$ denote the non-trivial irreducible representations of the group G.

Because of the orthogonality of the $\sigma-s$ we may write

$$\int |g_n(g)-1|^2\,d\chi(g)=\sum_{\sigma} |a_n(\sigma)|^2 \geqq |a_n(\pi)|^2=\prod_{k=1}^{n} [1-\bar{S}(\mu_k)]^2.$$

Therefore

$$\sup_{g\in G} |q_n(g)-1| \geqq \prod_{k=1}^{n} [1-\bar{S}(\mu_k)]. \tag{3.11}$$

It is not difficult to see that

$$2S(\mu) \leqq \bar{S}(\mu) \leqq 11\,S(\mu). \tag{3.12}$$

This estimate together with (3.11) imply that the relation (2.2) holds for an appropriate construction in case of commutative groups. The same argument works if G has a one-dimensional representation π. A sequence of measures $\mu^{(n)}$ on G does not tend to the Haar measure of G in the weak-star topology if there exists a non-trivial (one-dimensional) representation π such that $\lim\limits_{n\to\infty} \hat{\mu}^{(n)}(\pi)=0$ does not hold. Under the conditions mentioned at the end of Corollary 1 one can choose a sequence of measures μ_k and a one-dimensional representation π in such a way that they satisfy (3.10). This relation implies that

$$\hat{\mu}^{(n)}(\pi)=\prod_{k=1}^{n} (1-\bar{S}(\mu_k))\,\mathrm{Id}.$$

Since we assumed that $\Sigma S(\mu_k) < \infty$ and $M_k(1) < 1$ for every k (the latter relation implies that $1 - S(\mu_k) \neq 0$) relation (3.12) gives that $\hat{\mu}^{(n)}(\pi) \not\to 0$, consequently the sequence $\mu^{(n)}$ does not tend to the Haar measure in the weak-star topology.

In case of a commutative group, formula (3.8) improves the result of the theorem. We have also proved, that the essential part in the inequality (3.8), $\prod(1-\bar{S}(\mu_k))$ cannot be substituted by a smaller quantity. The question whether formula (3.8) holds for an arbitrary compact connected group, remains an open problem.

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