PHASE-TRANSITION IN STATISTICAL PHYSICAL MODELS WITH DISCRETE AND CONTINUOUS SYMMETRIES

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ABSTRACT. We discuss the problem of existence or non-existence of phase-transition in statistical physical models. The main technical difficulties connected with this problem are formulated. We point out the difference between models with discrete and continouous symmetry. A particular model, Dyson's hierarchical model is considered in some more detail.

1. Introduction. In statistical physics the investigation of the existence and uniqueness of a random field, called equilibrium state, plays a most important role. This random field takes its values on the configurations $\sigma(j)$, $j \in \mathbf{Z}$, where the so-called spin-variables $\sigma(\cdot)$ are in some metric space S, generally S is a subset of the Euclidean space $\mathbf{R}^1$ or $\mathbf{R}^s$ with some $s \geq 2$, $\mathbf{Z}$ is a parameter set, generally the integer lattice $\mathbf{Z}^p$ in the p-dimensional Euclidean space with some $p \geq 1$. We have to define a probability measure on the space $S^{\mathbf{Z}}$. This measure depends on a Hamiltonian function, a physical parameter, the temperature, and a so-called free measure. For the sake of simpler notations we restrict ourselves to the case when the model contains only pair-interaction. In this case the Hamiltonian function is a formal series,

$$\mathcal{H}(\sigma) = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} \mathcal{U}_{j,k}(\{\sigma(j), \sigma(k)\}), \quad \sigma = \{\sigma(j);\ j \in \mathbf{Z}\}, \tag{1}$$

and

$$\sum_{k \in \mathbf{Z}} \mathcal{U}_{j,k}(\{\sigma(j), \sigma(k)\}) < \infty, \quad \text{for all } j \in \mathbf{Z} \tag{1$'$}$$

where $\mathcal{U}_{j,k}(\cdot, \cdot)$ are measurable functions on $S \times S$.

Let us fix some measure ν on S which is called the free measure in the literature. Given some finite set $V \subset \mathbf{Z}$ and a configuration $\bar{\sigma} = \{\bar{\sigma}(k);\ k \in \mathbf{Z} \setminus V\}$ and a parameter $T > 0$, the temperature, we define the conditional Gibbs distribution

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μ_V with some density $f_V(\sigma(j);\ j\in V|\bar\sigma)$ with respect to the product measure $\prod_{j\in V} d\nu(\sigma(j))$ by the formula

$$\begin{aligned}\frac{d\mu_V(\sigma(j);\ j\in V|\bar\sigma)}{\prod_{j\in V} d\nu(\sigma(j))}&=f_V(\sigma(j);\ j\in V),T|\bar\sigma)\\ &=\frac{1}{Z}\exp\left\{-\frac{1}{T}\sum_{j\in V,k\in\mathbf{Z}\setminus V}\mathcal{U}_{j,k}(\{\sigma(j),\bar\sigma(k)\})\right\},\end{aligned}\tag{2}$$

where the norming factor Z is defined as

$$Z=Z(V,\bar\sigma)=\int\exp\left\{-\frac{1}{T}\sum_{j\in V,k\in\mathbf{Z}\setminus V}\mathcal{U}_{j.k}(\{\sigma(j),\bar\sigma(k)\})\right\}\prod_{j\in V} d\nu(\sigma(j)),\tag{2$'$}$$

if formula (2) is meaningful.

We call a probability measure μ on $S^{\mathbf{Z}}$ an equilibrium state with a Hamiltonian $\mathcal{H}$ defined in (1) and free measure ν at temperature T, if for any finite set V the conditional distribution of $\mu(\{\sigma(j);j\in V\})$ with respect to the configurations $\bar\sigma$ in $\mathbf{Z}\setminus V$, or more precisely an appropriate version of it, is defined by the formula

$$\mu(\{\sigma(j);\ j\in V\}\in A|\bar\sigma)=\int_A f_V(\{\sigma(j);\ j\in V\}),T|\bar\sigma)\prod_{j\in V} d\nu(\sigma(j))\tag{3}$$

for all measurable $A\subset S^V$, where the function f_V is given by formula (2).

The first question to be clarified is whether such a measure μ exists and whether it is unique. The classical results of probability theory cannot be applied directly to answer this question. If there are several equilibrium states μ with the same Hamiltonian and free measure at the same temperature, then one speaks in the literature about phase-transition. A natural way to construct equilibrium states is to carry out the following procedure: Choose a sequence of finite sets $V_n\subset\mathbf{Z}$ such that $\lim_{n\to\infty}V_n=\mathbf{Z}$ and configurations $\bar\sigma_n=\{\bar\sigma(j);\ j\in\mathbf{Z}\setminus V_n\}$ on their complementary sets. Define the measures μ_{V_n} on the sets V_n with some boundary condition $\bar\sigma_n$ by formulas (2) and (2$'$). It is natural to expect that they are good approximations of the equilibrium states we are looking for. Hence, we try to prove that the sequence of measures μ_{V_n} has a convergent subsequence, i.e. the sequence μ_{V_n} is compact, and the limit of its convergent subsequences is an equilibrium state. The compactness of the sequence μ_{V_n} holds automatically, if S is a compact set. In more general cases, when S can be e.g. the Euclidean space $\mathbf{R}^s$, the proof of compactness may be a really hard problem. It is true under very general conditions that the limit of a subsequence of μ_{V_n} is an equilibrium state. Moreover, it is true that all equilibrium states are in the closure of the convex linear combination of measures obtained in such a way, if the boundary conditions $\bar\sigma_n$ can be chosen arbitrarily. Hence, the really hard problem is the problem of uniqueness. This is connected to the following question: Let us fix some $j\in\mathbf{Z}$ together with a large neighbourhood of V of j, and let us fix some configuration $\bar\sigma=\{\bar\sigma(k);\ k\in\mathbf{Z}\setminus V\}$ on $\mathbf{Z}\setminus V$. Take the measure $\mu_V(\cdot|\bar\sigma)$ on the set V. Does the distribution of $\sigma(j)$ with respect to this measure "feel strongly" the boundary condition $\bar\sigma$, i.e. can it have different limits for different boundary conditions? The investigation of this question requires a more refined analysis.

2. On translation invariant models. In this section we discuss some basic results about the uniqueness and non-uniqueness of models on the integer lattice $\mathbf{Z}^p$ of the Euclidean space $\mathbf{R}^p$ and such that $\mathcal{U}_{j,k}(x,y)=\mathcal{U}_{0,k-j}(x,y)$ for all j, $k\in\mathbf{Z}$ and x, $y\in S$. Such models are called translation invariant.

Let us remark that the multiplying term $\frac{1}{T}$ in the exponent of formula (2) plays an important role. Let us compare the density $f_V(\cdot|\bar\sigma)$ of two configurations $\sigma^{(1)}=\{\sigma^{(1)}(j);\ j\in V\}$ and $\sigma^{(2)}=\{\sigma^{(2)}(j);\ j\in V\}$ in the volume V with respect to the measure $\mu_V(\cdot|\bar\sigma)$. If one of them has less energy with respect to the boundary condition $\bar\sigma$, then its density is much greater for small T, but these configurations have almost the same density for large T. Hence, it is natural to expect that, since at high temperatures the μ_V probability of a configuration in a volume V weakly depends on the boundary condition $\bar\sigma$ on $\mathbf{Z}\setminus V$, there is a unique equilibrium state $\mu_V=\mu_V(T)$ at high temperatures. More precisely, at the temperature $T=\infty$ the spins $\sigma(j)$ are independent random variables with distribution ν with respect to the $\mu=\mu(\infty)$ measure. This measure is stable in the following sense: For large T the (unique) measure $\mu(T)$ is a small perturbation of the measure $\mu(\infty)$ under very general conditions. (See e.g. [5] or [8] Chapter 1.)

On the other hand, the situation at low temperatures is more complex. We shall call a configuration $\bar\sigma=\{\sigma(k);\ k\in\mathbf{Z}\}$ a configuration with (locally) minimal energy if for all configurations $\bar\sigma'=\{\sigma'(k);\ k\in\mathbf{Z}\}$ such that the configurations $\bar\sigma$ and $\bar\sigma'$ differ only at finitely many places $k\in\mathbf{Z}$ the conditional energy

$$\mathcal{H}(\bar\sigma'|\bar\sigma)=\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}}[\mathcal{U}_{j,k}(\{\sigma'(j),\sigma'(k)\})-\mathcal{U}_{j,k}(\{\sigma(j),\sigma(k)\})]\ge 0$$

is non-negative. (The sum in the last expression is finite because of (1′).)

The previous argument would suggest that for small $T>0$ and a configuration $\bar\sigma$ with minimal energy there is an equilibrium state $\mu(T)$ which is essentially concentrated on configurations that are small perturbations of $\bar\sigma$. Namely, if we take a sequence of volumes V_n such that $\lim_{n\to\infty}V_n=\mathbf{Z}$ and take the measures μ_{V_n} defined by formula (2) in V_n with the boundary condition $\bar\sigma$ on$\mathbf{Z}\setminus V_n$ then the limit of these measures should be such an equilibrium state. This would mean in particular, that if there exist several configurations with minimal energy, then there are several equilibrium states at low temperatures. Nevertheless, this is not always the case. Let us call a configuration with minimal energy stable if for small $T>0$ there is some equilibrium state $\mu(T)$ which is essentially concentrated on configurations that are close to it. Whether a configurations with minimal energy is stable or not that depends on how big is the conditional energy of a configuration differing of it on a large finite set with respect to this configuration. We can expect that a configuration with minimal energy is stable if this conditional energy is large, otherwise we expect that it is not stable. However, it is very hard to decide when this conditional energy is sufficiently large to ensure stability, and only partial results are available about this problem. An important result in this direction is the Pirogov-Sinai theorem, (see e.g. [8], Chapter 2). Here the situation is considered when the potential $\mathcal{U}$ has a finite range of interaction, i.e. $\mathcal{U}(j,k)=0$ if $|j-k|\le r$ with some $r>0$, and the set S where the spins $\sigma(\cdot)$ take their values is finite. The question whether a periodic configuration with minimal energy is stable is investigated. The situation is rather complex even in this particular case. Pirogov and Sinai gave a satisfactory sufficient condition for the stability of such configurations. We omit the exact formulation of their result, because it requires the introduction of some new notions, and the questions we are interested in in this paper are only loosely connected with this result. We only remark that the most important conditions they require are that the dimension of the parameter space $\mathbf{Z} = \mathbf{Z}^p$ must be at least $p \geq 2$ and the model must satisfy the so-called Peierls's condition, which says the following: The conditonal energy of a configuration that differs from a periodic configuration with minimal energy on a finite set B has a conditional energy with respect to this periodic configuration which is greater than ρ times the cardinality of the set B, where $\rho > 0$ is an appropriate fixed number. Peierls's condition should guarantee that the conditional energy of a configuration differing on a finite set from a periodic configuration with minimal energy must be sufficiently large.

The case when the state-space S is a connected set is much less known. The main problem is that in interesting cases there is no natural candidate for the analogue of the Peierls's condition. In this case only partial results are available, but there are some results which indicate that the situation in more general state space can be essentially different. Let us discuss some such models.

We consider models with the Hamiltonian function

$$\mathcal{H}(\sigma) = -\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} U_{j,k}\sigma(j)\sigma(k), \quad \sigma = \{\sigma(j);\ j \in \mathbf{Z}\},$$

where the numbers $U_{j,k} \geq 0$ is such that $U_{j,k} = U(j-k)$. First we want to compare the following two models.

1.a) $\mathbf{Z} = \mathbf{Z}^p$ with some $p \geq 2$, $U(j-k) = 1$ if j and k are neighbours in $\mathbf{Z}^p$, $S = \{-1, 1\}$, and the measure ν is defined as $\nu\{-1\} = \nu\{1\} = 1/2$.

1.b) $\mathbf{Z} = \mathbf{Z}^p$ with some $p \geq 2$, $U(j-k) = 1$ if j and k are neighbours in $\mathbf{Z}^p$, S is the unit-sphere in the Euclidean space $\mathbf{R}^s$ with some $s \geq 2$. Here $\sigma(j)\sigma(k)$ denotes scalar product, and ν is the Lebesgue measure on the unit sphere.

Let us compare the following two models too:

2.a) The same as model 1.a), only $\mathbf{Z} = \mathbf{Z}^1$, and $U(n) = n^{-\alpha}$, with some $\alpha > 1$.

2.b) The same as model 1.b), only $\mathbf{Z} = \mathbf{Z}^1$, and $U(n) = n^{-\alpha}$, with some $\alpha > 1$.

Models 1.a) and 2.a) have the following symmetry property: If we multiply all $\sigma(j)$ by -1 then both the Hamiltonian $\mathcal{H}(\sigma)$ and the free measure ν remain the same. Hence, we say that these models are invariant with respect to the multiplication group $\{-1, 1\}$. In the same way models 1.b) and 2.b) are invariant with respect to the group of rotations $U(s)$ of the s-dimensional space. The first invariance is called a discrete and the second one a continuous symmetry.

In models 1.a) and 2.a) the configurations $\sigma(j) = 1$ for all $j \in \mathbf{Z}$ or $\sigma(j) = -1$ for all $j \in \mathbf{Z}$ are configurations with minimal energies. It is proved that in model 1.a) for small $T > 0$ at any dimension $p \geq 2$ there is a translation invariant equilibrium state $\mu^+ = \mu^+(T)$ such that $\mu^+\{\sigma(j) = 1\} > \frac{1}{2}$ for all $j \in \mathbf{Z}$. Similarly, there is an equilibrium state μ^- such that $\mu^-\{\sigma(j) = -1\} > \frac{1}{2}$ for all $j \in \mathbf{Z}$. This means a phase transition which is connected to a break-down of symmetry, i.e. to the fact that the measures μ^+ and μ^- do not preserve the symmetry the Hamiltonian $\mathcal{H}$ and free measure ν have. They are in the vicinity of a configuration with minimal energy instead. In model 2.a) the same result holds if $\alpha \le 2$. On the other hand, the equilibrium state is unique for $\alpha > 2$ at any temperature T. In these results we should emphasize the phase-transition in the case $\alpha = 2$. This is a very delicate boundary case, and model 2.a) with this parameter has certain peculiar properties. (See [1].)

In models 1.b) and 2.b) the configurations $\sigma(j) = e$; $\forall j \in \mathbf{Z}$ with some $e \in S$ are configurations with minimal energies. One is interested in which of these models have a phase-transition at low temperatures and which have not. In model 1.b) there is a phase-transition at low temperatures if the dimension of the lattice $\mathbf{Z}^p$ is $p \ge 3$, (see [7]) and there is no phase-transition for $p = 2$. More precisely, the result for $p = 2$ is proved completely only in the case when the state space S is the unit circle, (see [4]). In the general case only the weaker result is proven that any equilibrium state is invariant with respect to rotations [6]. This excludes the possibility of such equilibrium states, where the configurations are in the vicinity of a configuration with minimal energy with probability almost one, and it is believed that there is no phase-transition, if this result holds. In model 2.b) there is a phase-transition for $\alpha < 2$, and there is no phase-transition for $\alpha \ge 2$. Models 1.a) and 2.a) behave differently in the case $p = 2$, and the difference between the behaviour of models 1.b) and 2.b) appears in the case $\alpha = 2$.

The above examples show that models with continuous and discrete symmetries behave differently. The heuristic explanation of this difference is clear. Let us fix a configuration with minimal energy in a neighbourhood of infinity, and let us look at how much energy is needed to change this configuration radically in a neighbourhood of zero. It may happen that in models with continuous symmetry we can achieve this change at the expense of less energy by rotating the configuration in such a way that the relative rotation between neighbour points is small. In models with discrete symmetry this cannot be done. However, this heuristic argument is not strong enough to give an orientation about what to expect in the general case. Hence, it may be interesting a model where these questions can be solved completely. We discuss a one-dimensional model, Dyson's hierarchical model in detail. This is a version of models 1.b) and 2.b). The main difference is that the number $U(i,j)$ appearing in the Hamiltonian of this model depends not on the usual distance $|i-j|$, but some different distance on $\mathbf{Z}$. Hence this model is not translation invariant, but it has some other symmetries which makes it simpler to handle.

3. Dyson's hierarchical model. Dyson's hierarchical model is a model on the positive integers $\mathbf{Z} = \{1, 2, \dots\}$ with Hamiltonian function

$$\mathcal{H}(\sigma) = -\sum_{i \in \mathbf{Z}} \sum_{\substack{j \in \mathbf{Z} \\ j > i}} \varphi(d(i,j))\sigma(i)\sigma(j)\,, \tag{4}$$

where the so-called hierarchical distance $d(\cdot,\cdot)$ is defined by the formula $d(i,j) = 2^{n(i,j)-1}$, and

$$n(i,j) = \min n,\ \exists \text{ some } k \text{ such that } (k-1)2^n < i, j \le k2^n,$$

$\varphi(\cdot)$ is a real function, and the free measure ν has the density function $p_0(x)$

$$p_0(x) = \frac{d\nu}{dx}(x) = C(t) \exp\left\{-\frac{x^2}{2} - \frac{t}{4}|x|^4\right\}, \tag{5}$$

where $t > 0$ is some small number. We consider both the scalar case when the spins $\sigma(\cdot)$ take values on the real line $\mathbf{R}^1$ and the vector case when they take values in $\mathbf{R}^s$ with some $s \geq 2$. In the latter case, the product $\sigma(j)\sigma(k)$ in formula (4) means scalar product. We are interested in the question for which functions $\varphi(\cdot)$ the model has a phase transition at low temperatures and for which one it has not.

The hierarchical distance $d(\cdot,\cdot)$ appearing in this model is a version of the usual distance. It is not translation invariant, but it has some other symmetries wich makes the model simpler. The density function $p_0(x)$ is a small perturbation of the normal density, and the condition $t > 0$ guarantees that all integrals we need are convergent. The scalar case is a version of problem 2.a) and the vector case of problem 2.b) of the previous section. The crucial step in solving the problem about phase transition consists of investigating the following question:

Put $V_n = \{1, 2, \dots, 2^n\}$, and

$$\mathcal{H}_{V_n}(x_1, \dots, x_{2^n}) = -\sum_{i \in V_n} \sum_{\substack{j \in V_n \\ j > i}} \varphi(d(i,j)) x_i x_j.$$

Define the probability measure $\mu_n = \mu_{n,T}$ on $\mathbf{R}^{V_n}$ (on $(\mathbf{R}^s)^{V_n}$ if we have a model with s-dimensional spins) with the density function $p_n(x_1, \dots, x_{2^n})$ by the following formula:

$$p_n(x_1, \dots, x_{2^n}) = \frac{d\mu_n(x_1, \dots, x_{2^n})}{dx_1 \dots dx_{2^n}} = C_n \exp\left\{-\frac{1}{T}\mathcal{H}_{V_n}(x_1, \dots, x_{2^n})\right\} \prod_{j=1}^{2^n} p(x_j).$$

Let $(\sigma(1), \sigma(2), \dots, \sigma(2^n))$ be a μ_n distributed random vector, and let $p_n(x)$ denote the density function of the average $2^{-n} \sum_{i=1}^{2^n} \sigma(i)$. Give a good asymptotic formula for $p_n(x)$.

This function $p_n(x)$ has the symmetry property $p_n(-x) = p_n(x)$ in the scalar case, and it is rotation invariant in the vector case. As some further analysis shows, there are two possibilities. Either the function $p_n(\cdot)$ is essentially concentrated in a small neighbourhood of the origin for large n or it is concentrated around some points $\pm M$, $M > 0$ in the scalar case and around the sphere $|R| = M$ in the vector case. In the second case there is a phase transition, and in the first case there is not. The last statement is far from trivial, it requires the investigation of the approximating measures μ_n with some boundary condition discussed in the first section of this paper. The main technical difficulty is to have a control on the finite dimensional projections (its dimension is independent of n) of these measures μ_n. This problem can be translated to a purely analytical question, where the formulas contain explicitly the above defined functions $p_n(\cdot)$. As a deeper analysis shows the existence or non-existence of phase transition depends on the behaviour of these functions. This question is discussed for instance in our paper [3], and here we omit the details.

The investigation of the function $p_n(x)$ also leads to a purely analytic question: Observe that

$$\mathcal{H}_{V_n}(x_1,\dots,x_{2^{n+1}}) = \mathcal{H}_{V_n}(x_1,\dots,x_{2^n}) + \mathcal{H}_{V_n}(x_{2^n+1},\dots,x_{2^{n+1}}) - \varphi(2^n)\left(\sum_{j=1}^{2^n} x_j\right)\left(\sum_{j=2^n+1}^{2^{n+1}} x_j\right).$$

This relation leads to the following recursive formula:

$$p_{n+1}(x) = C_n \int \exp\left\{\frac{4^n\varphi(2^n)}{T}(x^2-u^2)\right\} p_n(x-u)p_n(x+u)\,du, \tag{6}$$

where C_n is an appropriate norming constant, turning $p_{n+1}(x)$ into a density function. So we have to study the asymptotic behaviour of the functions $p_n(x)$ defined by the recursion formula (6). To complete the formulation of the problem we have to remark that the starting function $p_0(x)$ is defined in (5). The problem can be slightly simplified by introducing the following transformation of $p_n(x)$.

Put

$$A_n = 1 + \sum_{j=n+1}^{\infty} 2^j \frac{\varphi(2^{n+j})}{\varphi(2^n)},$$

and

$$q_n(x) = \exp\left\{\frac{A_n}{2(1+A_n)} 4^n\varphi(2^n)x^2\right\} p_n\left(\sqrt{\frac{T}{1+A_n}}x\right).$$

Some calculation shows that the above defined functions $q_n(x)$ satisfy the relation

$$\begin{aligned} q_{n+1}(x,T) = \bar{C}_n(T)\int \exp\left\{-4^n\varphi(2^n)u^2\right\} q_n\left(\sqrt{\frac{1+A_n}{1+A_{n+1}}}x-u, T\right) \\ q_n\left(\sqrt{\frac{1+A_n}{1+A_{n+1}}}x+u, T\right)\,du, \end{aligned} \tag{7}$$

and the starting function $q_0(x)$ is

$$q_0(x) = q_0(x,T) = C_0(T)\exp\left\{\frac{A_0-T}{1+A_0}\frac{x^2}{2} - \frac{tT^2}{(1+A_0)^2}\frac{|x|^4}{4}\right\}. \tag{7'}$$

We can study the functions $q_n(x)$ defined by formulas (7) and (7′) instead of the functions $p_n(x)$. Observe that the recursive relation (7) does not contain the parameter T, it appears only in the starting function $q_0(x)$. In formula (7′) the coefficient of $|x|^4$ is always negative and the coefficient of x^2 is negative for $T > A_0$, and it is positive for $T < A_0$. As a consequence, the function $q_0(x)$ has its minimum at zero in the first case and at $M_0 = \sqrt{\frac{A_0-T}{tT^2}(1+A_0)}$ in the second case. The integral operator in (7) is very similar to the convolution, the main difference between them is the appereance of the kernel $\exp\left\{-4^n\varphi(2^n)u^2\right\}$ in our case. The appereance of this kernel has far reaching consequences. It causes some localization in the following sense: The main contribution to the integral in (7) is given for small u. As a consequence, the original form of the starting function may be preserved, and this is the reason why the function $q_n(x)$ may have different behaviour for large and small T. For large T the starting function $q_0(x)$ is essentially concentrated in a small neighbourhood of the origin, and the integral (7) can be well approximated by convolution. For small T the situation is different, and actually we are interested in the following question: If the maximum $M_0 > 0$ around which point the function $q_0(|x|)$ is concentrated is very large, will this property be preserved for large n too? The answer depends on how large the coefficient $4^n\varphi(2^n)$ in the kernel is. Let us also remark that in the vector valued case formulas (7) and (7′) imply that the function $q_n(x)$ is in the space of rotation invariant functions for all n. This constraint implies a different behaviour in scalar and vector valued cases. We explain how to investigate these models at low temperatures.

In the scalar valued case at small T the function $q_0(x)$ is concentrated around some point $\pm M$, $M > 0$, and we are interested in whether this property will be preserved for all n. For this reason we make an appropriate rescaling. We try to define some new function $g_n(x) = A_n q_n(M_n + A_n x)$, where M_n is the place of maximum of the function $q_n(x)$, and the number A_n is chosen in such a way that the relation $g_{n+1}(x) = \mathbf{T} g_n(x)$ hold with some operator $\mathbf{T}$ not depending on n. Then the stability properties of this operator $\mathbf{T}$ must be investigated, and if it is stable enough, then the relation $\lim_{n\to\infty} g_n(x) = g^*(x)$ holds, where $g^*(x)$ is the solution of the fixed-point equation $g^*(x) = \mathbf{T} g^*(x)$. During these calculations we also get a recursive relation for the numerical sequence M_n, and the question whether there is a phase transition or not depends on whether $\lim_{n\to\infty} M_n > 0$ or not. It is relatively simple to carry out this program if $4^n\varphi(n) > \alpha^n$ with some $\alpha > 1$. In this case there is a phase-transition. The question is much harder in the case when although $4^n\varphi(n) \to \infty$, but it does not increase exponentially fast. Actually, this is the case we are first of all interested in.

In this case the kernel in the integral of (7) has a smaller effect, and to carry out the above sketched program one needs to study the behaviour of the function $q_n(x)$ in the interval $[-M_n, M_n]$ more carefully. In this interval the localization property of the integral (7) behaves differently. In the point zero the main contribution is given by $u = \pm M_n$, in the point $M/2$ by $u = \pm M_n/2$, etc. If the kernel in the integral in (7) tends to infinity slowly, then these contributions can be very essential. It may happen that new peaks appear in this interval which may be the new maximum. The above sketched program can be carried out, but it is much more sophisticated because of the intricate behaviour of the function $q_n(x)$ in the interval $[-M_n, M_n]$. We cannot explain the main ideas of this argument in this short note, we only give the recursive formula for M_n this procedure gives. (For some more discussion see [2].) It is:

$$M_{n+1} \sim M_n \left[1 - \exp\left\{-\frac{1}{T} 4^n \varphi(2^n) M_n^2\right\}\right]. \tag{8}$$

Formula (8) shows that in the case $4^n\varphi(2^n) >$ const. $\log n$ and $T > 0$ is small, (hence $M_0 > 0$ is large) $\lim_{n\to\infty} M_n > 0$, therefore there is a phase-transition. On the other hand, if $4^n\varphi(2^n)/\log n \to 0$ then $M_n \to 0$, and there is no phase transition. Moreover, the special case $4^n\varphi(2^n) = \text{const.} \log n$ has the following remarkable property: If $\frac{M_k^2}{T} < 1$ for some k, then $M_{n+1} < (1-n^{-\alpha})M_n$ for all $n \geq k$ with some $\alpha < 1$, hence $\lim_{n\to\infty} M_n = 0$. Define the function $M(T) = \lim_{n\to\infty} M_n(T)$, which is called in the literature the spontaneous magnetization. Since there is some $T_0 > 0$ such that $M(T) = 0$ for $T > T_0$, the above property implies that either $M^2(T) > T_0$ or $M(T) = 0$, i.e. the spontaneous magnetization as a function of the temperature has a discontinuity. This is called the Thouless effect in the literature. Model 2.a) of the previous section has a similar property in the special case $\alpha = 2$. (See [1].)

In the vector valued case the situations when $4^n\varphi(2^n)$ increases exponentially fast and when it increases slower can be similarly investigated. Since the function $q_n(x)$ is rotation invariant, it is useful to introduce the scalar valued function $Q_n(x) = q_n(x, 0, \dots, 0)$, where $x \in \mathbf{R}^1$ and $(x, 0, \dots, 0) \in \mathbf{R}^s$ and to study this function. For the sake of simpler notations let us assume that $s = 2$. The function $Q_n(x)$ is essentially concentrated in a small neighbourhood of the points $\pm M_n$, where M_n is defined as

$$M_n = \int_0^\infty x Q_n(x)\, dx.$$

We want to give a good asymptotic of the function $Q_n(x)$ in these neighbourhoods, together with a recursive formula on the sequence M_n. The main technical difficulty is that when rewriting formula (7) for the function $Q_n(x)$ the arguments $\sqrt{\frac{1+A_n}{1+A_{n+1}}}x \pm u$ turn into the upleasant expressions $\sqrt{(B_n x \pm u_1)^2 + u_2^2}$, with $B_n = \frac{1+A_n}{1+A_{n+1}}$. Since we are interested in a good asymptotic only in the case $x \sim M_n$, and the main contribution to the integral expressing $Q_n(x)$ for such x is given when u_1 and u_2 are small, we commit a small error by replacing this argument by the expression $B_n x \pm u_1 + \frac{u_2^2}{2B_n^2 M_n^2}$. Then we can continue our argument similarly to the scalar valued case. Let us introduce the functions $f_n(x) = 4^{-n}\varphi(2^n)^{-1} Q_n(M_n + 4^{-n}\varphi(2^n)^{-1}x)$. We cannot express $f_{n+1}(x)$ as an operator of $f_n(x)$ not depending on n, but the following weaker statement holds: We can write $f_{n+1}(x) = \mathbf{T} f_n(x) + \varepsilon_n(x)$, where $\varepsilon_n(x)$ is a small error term. The operator $\mathbf{T}$ is written down explicitly in [3], and its stability is also studied there. Here we omit the details. What is important for us is that this investigation gives us the relation

$$M_{n+1} \sim M_n - \frac{1}{M_n 4^n \varphi(2^n)},$$

or because of the relation $M_n \sim M_{n+1}$ we commit negligible error by rewriting the last formula as

$$M_{n+1}^2 - M_n^2 \sim \frac{2}{4^n\varphi(2^n)}.$$

This relation shows that in Dyson's vector valued model there is a phase-transition at low temperatures if the sum

$$\sum_{n=1}^\infty \frac{1}{4^n\varphi(2^n)}$$

is convergent, and there is no phase-transition if this sum is divergent.

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