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Abstract: Here I prove non-central limit theorems for non-linear functionals of vector valued stationary random fields under appropriate conditions. They are the multivariate versions of the results in paper [2]. Previously A. M. Arcones formulated a theorem in paper [1] which can be considered as the multivariate generalization of these results. But I found Arcones' discussion incomplete, and in my opinion to give a complete proof first a more profound foundation of the theory of vector valued Gaussian stationary random fields has to be worked out. This was done in my paper [4] which enabled me to adapt the method in paper [2] to the study of the vector valued case. Here I prove with its help the desired multivariate version of the results in paper [2].

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Non-central limit theorem for non-linear functionals of vector valued Gaussian stationary random fields

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1. Formulation of the results

Let us consider a d-dimensional vector valued Gaussian stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, where $\mathbb{Z}^\nu$ denotes the lattice points with integer coordinates in the ν-dimensional Euclidean space $\mathbb{R}^\nu$, and a function $H(x_1, \dots, x_d)$ of d variables. We define with their help the random variables $Y(p) = H(X_1(p), \dots, X_d(p))$ for all $p \in \mathbb{Z}^\nu$. Let us introduce for all $N = 1, 2, \dots$ the normalized sum

$$S_N = A_N^{-1} \sum_{p \in B_N} Y(p)$$

with an appropriate norming constant $A_N > 0$, where

$$B_N = \{p = (p_1, \dots, p_\nu)\colon\ 0 \le p_k < N \text{ for all } 1 \le k \le \nu\}. \tag{1}$$

We prove a non-Gaussian limit theorem for these normalized sums S_N with an appropriate norming constant A_N if this vector valued Gaussian stationary

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random field $X(p)$, $p \in \mathbb{Z}^\nu$, and function $H(x_1,\dots,x_d)$ satisfy certain conditions. In [2] we proved such limit theorems for non-linear functionals of scalar valued stationary Gaussian random fields, and now we are looking for their natural multivariate version.

A. M. Arcones formulated such a result in Theorem 6 of paper [1], but I found his discussion unsatisfactory. He applied several notions and results which were not worked out. More precisely, there were analogous definitions and results in scalar valued models, but in the vector valued case their elaboration was missing. My goal in this paper was to give a precise formulation and proof of Arcones' result.

The first step of this program was to work out the theory of vector valued Gaussian stationary random fields, to present the most important notions and results. This was done in my work [4]. Here I show how one can get the desired limit theorems for non-linear functionals of vector valued Gaussian stationary random fields with its help. The work [4] enabled me to adapt the proof of [2] in the study of the multivariate version of the result in that paper.

Remark: It was professor Herold Dehling who asked me to clarify the proof of Theorem 6 in Arcones' paper [1]. The goal of this work together with the preliminary paper [4] was to answer Dehling's question. It turned out that to settle this problem first the theory of vector valued stationary Gaussian random fields has to be worked out. This theory is similar to the theory of the scalar valued Gaussian random fields, but there are also some essential differences between them. Hence the theory of vector valued stationary Gaussian random fields cannot be considered as a simple generalization of the theory in the scalar valued case. I am grateful to Professor Dehling for calling my attention to this problem.

I start the discussion with a short overview of the results in [4] we need in our investigation.

We are working with d-dimensional vector valued Gaussian stationary random fields $X(p) = (X_1(p),\dots,X_d(p))$, $p \in \mathbb{Z}^\nu$, where $\mathbb{Z}^\nu$ denotes the lattice points with integer coordinates in the ν-dimensional Euclidean space $\mathbb{R}^\nu$ with expectation $EX_j(0) = 0$ for all $1 \le j \le d$. The distribution of such random fields is determined by the covariance function $r_{j,j'}(p) = EX_j(0)X_{j'}(p) = EX_j(m)X_{j'}(m+p)$, $1 \le j,j' \le d$, $m,p \in \mathbb{Z}^\nu$. Our first result in [4] was the representation of the functions $r_{j,j'}(p)$ as the Fourier transform $r_{j,j'}(p) = \int e^{i(p,x)}G_{j,j'}(\,dx)$, of the coordinates of a positive semidefinite matrix valued measure $G = (G_{j,j'})$, $1 \le j,j' \le d$, on the torus $[-\pi,\pi)^\nu$. (For a more detailed discussion of this result see Section 2 in [4].) Then we defined in Section 3 a vector valued random spectral measure $Z_G = (Z_{G,1},\dots,Z_{G,d})$ corresponding to a matrix valued spectral measure G together with a random integral with respect to it in such a way that the formula $X_j(p) = \int e^{i(p,x)}Z_{G,j}(\,dx)$, $p \in \mathbb{Z}^\nu$, $1 \le j \le d$, defines a d-dimensional stationary random field with matrix valued spectral measure G. In Section 4 of [4] we constructed so-called generalized vector valued Gaussian stationary random fields, and proved results analogous to the results of Sections 2 and 3 in [4] for them. In particular, we defined the vector valued random spectral measures corresponding to the matrix valued spectral measure of a generalized vector valued Gaussian stationary random field. We need these results, because we define the limit in our limit theorems by means of multiple Wiener–Itô integrals with respect to a vector valued random spectral measure corresponding to the matrix valued spectral measure of a generalized vector valued Gaussian stationary random field.

Our goal in the subsequent part of work [4] was to give a good representation of those random variables with finite second moments which are measurable with respect to the σ-algebra generated by the random variables of the underlying vector valued random field together with the shift transformation acting on them, because this gives a great help in the study of the limit theorems we are interested in. With such an aim we defined multiple Wiener–Itô integrals with respect to the coordinates of vector valued random spectral measures in Section 5, and studied non-linear functionals of vector valued stationary random fields with their help. In Section 6 we proved a technical result, called the diagram formula, about the calculation of the product of two multiple Wiener–Itô integrals. We needed this result to express Wick polynomials of the random variables in our vector valued Gaussian stationary random fields. The definition of Wick polynomials, which are the natural multivariate generalizations of Hermite polynomials, was recalled in Section 7 together with their most important properties. Section 7 of [4] also contains the formula about the expression of Wick polynomials by means of multiple Wiener–Itô integrals and an important formula about the calculation of the shift transformations of random variables given in the form of multiple Wiener–Itô integrals. This enabled us to reformulate our limit problems to a problem about limit theorems for the distribution of a sequence of sums of multiple Wiener–Itô integrals. To investigate such problems we proved a result in Section 8 of [4] which plays an important role in the investigation of this paper. We recalled it in Proposition 2A of this work.

First I formulate the conditions I impose on the stationary field $X(p)$, $p \in \mathbb{Z}^\nu$, and function $H(x_1, \dots, x_d)$ I am working with in this paper, and formulate the main result. I present the proof in the next section. In the proof I shall refer to [3] instead of [2], where the results I need in the proof are worked out in more detail.

In our results we impose a condition on the behaviour of the covariance function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, of the vector valued Gaussian stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$. In the corresponding result for scalar valued stationary random fields we worked with random variables of expectation zero, and we imposed a condition which meant that for large values p the covariance function $r(p) = EX(0)X(p)$ behaves like $|p|^{-\alpha}L(p)$ multiplied by a function depending on the direction $\frac{p}{|p|}$ of the vector p. Here $L(\cdot)$ is a slowly varying function at infinity. In the case of vector valued stationary random fields we impose the natural multivariate version of this condition. Namely, we demand that $EX_j(p) = 0$ for all $1 \le j \le d$, $p \in \mathbb{Z}^\nu$, and the covariance function $r_{j,j'}(p)=EX_j(0)X_{j'}(p)$ satisfies the relation

$$\lim_{T\to\infty}\sup_{p\colon\; p\in\mathbb{Z}^\nu,\,|p|\ge T}\frac{\left|r_{j,j'}(p)-a_{j,j'}(\frac{p}{|p|})|p|^{-\alpha}L(|p|)\right|}{|p|^{-\alpha}L(|p|)}=0 \tag{2}$$

for all $1\le j,j'\le d$, where $0<\alpha<\nu$, $L(t)$, $t\ge 1$, is a real valued function, slowly varying at infinity, bounded in all finite intervals, and $a_{j,j'}(t)$ is a real valued continuous function on the unit sphere $\mathcal{S}_{\nu-1}=\{x\colon\; x\in\mathbb{R}^\nu,\; |x|=1\}$, and the identity $a_{j',j}(x)=a_{j,j'}(-x)$ holds for all $x\in\mathcal{S}_{\nu-1}$ and $1\le j,j'\le d$.

Remark. I show that there are interesting vector valued Gaussian stationary random fields which satisfy condition (2). We can construct random fields with such a distribution by defining their matrix valued spectral measures in the following way. We get a correlation function satisfying condition (2) with the help of a matrix valued spectral measure whose coordinates $G_{j.j'}$, $1\le j,j'\le d$, have a density function of the form $g_{j,j'}(u)=|u|^{\alpha-\nu}b_{j,j'}(\frac{u}{|u|})h(u)$ with respect to the Lebesgue measure on the torus, $u\in[-\pi,\pi)^\nu$, where $b_{j,j'}(\cdot)$ is a non-negative smooth function on the unit sphere $\{u\colon\; u\in\mathbb{R}^\nu,\,|u|=1\}$, and $h(u)$ is a non-negative, smooth and even function on the torus $[-\pi,\pi)^\nu$ which does not disappear at the origin. The functions $b_{j,j'}(\cdot)$ must satisfy some additional conditions to guarantee that $G=(G_{j,j'})$, $1\le j,j'\le d$, is a matrix valued spectral measure. We get a good covariance matrix by defining $r_{j,j'}(p)=\int e^{i(p,x)}g_{j,j'}(x)\,dx$, and this function satisfies relation (1.2). Results of this type are studied in the theory of generalized functions.

Next we have to formulate a good condition on the function $H(x_1,\dots,x_d)$. In scalar valued models first the special case $H(x)=H_k(x)$ was considered, where $H_k(x)$ denotes the k-th Hermite polynomial with leading coefficient 1. Then it was shown that our limit problem with a function $H(x)$ whose expansion in the Hermite polynomials has the form $H(x)=\sum_{l=k}^\infty c_lH_l(x)$ with starting index k in the summation can be simply reduced to the special case when $H(x)=H_k(x)$. We shall prove a similar result in the multivariate case. In this case the Wick polynomials take the role of the Hermite polynomials. But we shall work with Wick polynomials only in an implicit way. We shall choose such models where the Wick polynomials can be simply calculated. We shall consider such vector valued Gaussian stationary random fields $X(p)=(X_1(p),\dots,X_d(p))$, $p\in\mathbb{Z}^\nu$, whose covariance function satisfies besides condition (2) also the relation

$$EX_j^2(0)=1 \text{ for all } 1\le j\le d, \text{ and } EX_j(0)X_{j'}(0)=0 \text{ if } j\ne j',\;\; 1\le j,j'\le d. \tag{3}$$

First I show that this new condition does not mean a real restriction of our problem.

Let $X(p)=(X_1(p),\dots,X_d(p))$, $p\in\mathbb{Z}^\nu$, be a vector valued Gaussian stationary random field with expectation $EX_j(p)=0$, $p\in\mathbb{Z}^\nu$, $1\le j\le d$, and take the random variables $X_1(0),\dots,X_d(0)$ in it. We can choose an appropriate number $1\le d'\le d$, and d' random variables $X'_j(0)=\sum_{l=1}^d c_{j,l}X_l(0)$, $1\le j\le d'$, with appropriate coefficients $c_{j,l}$, $1\le j\le d'$, $1\le l\le d$ in such a way that they have the following properties. $EX'_j(0)X'_{j'}(0)=\delta_{j,j'}$, $1\le j,j'\le d'$, where $\delta_{j,j'}=0$ if $j\ne j'$, and $\delta_{j,j}=1$, and also the random variables $X_j(0)$ can be expressed as the linear combination of the random variables $X'_l(0)$, $1\le l\le d'$, i.e. $X_j(0)=\sum_{l=1}^{d'} d_{j,l}X'_l(0)$ for all $1\le j\le d$ with appropriate coefficients $d_{j,l}$.

Let us define the vector valued random field $X'(p)=(X'_1(p),\dots,X'_{d'}(p))$ as $X'_j(p)=\sum_{l=1}^d c_{j,l}X_l(p)$, $1\le j\le d'$, with the same coefficients $c_{j,l}$ which appeared in the definition of $X'_j(0)$ for all $p\in\mathbb{Z}^\nu$. Then it is not difficult to see that $X'(p)$, $p\in\mathbb{Z}^\nu$, is a d'-dimensional Gaussian stationary random field whose elements have expectation zero, and it satisfies relation (3) (with parameter d' instead of d.) Moreover, if the covariance function of the original random field $X(p)$ satisfied relation (2), then this new random field also satisfies this condition with appropriate new functions $a'_{j,j'}(\frac{p}{|p|})$. Besides, it is not difficult to find such a function $H'(x_1,\dots,x_{d'})$ for which $H'(X'_1(p),\dots,X'_{d'}(p))=H(X_1(p),\dots,X_d(p))$ for all $p\in\mathbb{Z}^\nu$. This means that with the introduction of this new random field $X'(p)=(X'_1(p)\dots,X'_{d'}(p))$ we can reformulate our problem in such a way that our vector valued stationary Gaussian random field satisfies both relations (2) and (3). We shall work with such a new d'-dimensional random field $X'(p)$ and function $H'(x_1,\dots,x_{d'})$, only we shall omit the sign prime everywhere.

First we consider the case when our function $H(x_1,\dots,x_d)$ depends on a previously fixed constant k, and it has the form

$$\begin{aligned} H(x_1,\dots,x_d) &= H^{(0)}(x_1,\dots,x_d) \\ &= \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\cdots+k_d=k}} c_{k_1,\dots,k_d}H_{k_1}(x_1)\cdots H_{k_d}(x_d) \end{aligned} \tag{4}$$

with some coefficients $c_{k_1,\dots,k_d}$ where $H_k(\cdot)$ denotes the k-th Hermite polynomial with leading coefficient 1.

In scalar valued models we have proved a non-central limit theorem if $H(x)=H_k(x)$, $k\ge 2$, and the covariance function $r(n)=EX_0X_n$ satisfies condition (2) with $0<\alpha<\frac{\nu}{k}$ in the special case $d=1$. This result was formulated in Theorem 8.2 of [3]. In this result the limit was described with the help of a k-fold Wiener–Itô integral with respect to an appropriate random spectral measure. This random spectral measure corresponds to that spectral measure which appeared in Lemma 8.1 of [3] as the limit of a sequence consisting of appropriately normalized versions of the spectral measure of a stationary random field $X(p)$, $p\in\mathbb{Z}^\nu$, which satisfies condition (2)in the case $d=1$. I shall prove a multivariate version of Theorem 8.2 of [3]. But to do this first I present a multivariate version of Lemma 8.1 in [3].

This generalization of Lemma 8.1 in [3] describes the limit behaviour of the matrix valued spectral measure of a vector valued random field whose covariance function satisfies formula (2). This limit behaviour is described with the help of an appropriate spectral measure which is the spectral measure of a vector valued generalized stationary random field. This new spectral measure and a random spectral measure corresponding to it appears in the definition of th Wiener–Itô integrals that describe the limit in the multivariate version of the limit theorem in Theorem 8.2 of [3].

Given a vector valued stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, with expectation zero and covariance function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$ that satisfies relation (2) let us consider its matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-\pi, \pi)^\nu$. (The existence of this matrix valued spectral measure, and its characterization was proved e.g. in Theorem 2.2 of [4].) Take its rescaled version

$$G^{(N)}_{j,j'}(A) = \frac{N^\alpha}{L(N)} G_{j,j'}\left(\frac{A}{N}\right), \quad A \in \mathcal{B}^\nu, \quad N = 1, 2, \dots, \quad 1 \le j, j' \le d, \tag{5}$$

concentrated on $[-N\pi, N\pi)^\nu$ for all $N = 1, 2, \dots$, where $\mathcal{B}^\nu$ denotes the σ-algebra of the Borel measurable sets on $\mathbb{R}^\nu$. In the next Proposition I formulate a result which states that these complex measures $G^{(N)}_{j,j'}$ have a vague limit. This result also describes some properties of this limit. Before formulating it I recall the definition of vague limit.

Definition of vague convergence of complex measures on $\mathbb{R}^\nu$ with locally finite total variation. *Let $G^{(N)}$, $N = 1, 2, \dots$, be a sequence of complex measures on $\mathbb{R}^\nu$ with locally finite total variation. We say that the sequence $G^{(N)}$ vaguely converges to a complex measure $G^{(0)}$ on $\mathbb{R}^\nu$ with locally finite total variation if*

$$\lim_{N\to\infty} \int f(x)\, G^{(N)}(\,dx) = \int f(x)\, G^{(0)}(\,dx)$$

for all continuous functions f on $\mathbb{R}^\nu$ with a bounded support.

The definition of complex measures with locally finite total variation together with the notion of vector valued Gaussian stationary generalized random fields and their matrix valued spectral measures was introduced in Section 4 of [4]. In the next result I formulate the multivariate version of Lemma 8.1 in [3] with their help.

Proposition 1.1. *Let $G = (G_{j,j'})$ be the matrix valued spectral measure of a d-dimensional vector valued stationary random field whose covariance function $r_{j,j'}(p)$ satisfies relation (2) with some parameter $0 < \alpha < \nu$. Then for all pairs $1 \le j, j' \le d$ the sequence of complex measures $G^{(N)}_{j,j'}$ defined in (5) with the help of the complex measure $G_{j,j'}$ tends vaguely to a complex measure $G^{(0)}_{j,j'}$ on $\mathbb{R}^\nu$ with locally finite total variation. These complex measures $G^{(0)}_{j,j'}$, $1 \le j, j' \le d$,* *have the homogeneity property*

$$G^{(0)}_{j,j'}(A) = t^{-\alpha} G^{(0)}_{j,j'}(tA) \quad \text{for all bounded } A \in \mathcal{B}^\nu,\ 1 \le j, j' \le d, \text{ and } t > 0. \tag{6}$$

The complex measure $G^{(0)}_{j,j'}$ with locally finite variation is determined by the number $0 < \alpha < \nu$ and the function $a_{j,j'}(\cdot)$ on the unit sphere $S_{\nu-1}$ introduced in formula (2).

There exists a vector valued Gaussian stationary generalized random field on $\mathbb{R}^\nu$ with that matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, whose coordinates are the above defined complex measures $G^{(0)}_{j,j'}$, $1 \le j, j' \le d$.

Remark. The statement of Proposition 1.1 about the measures $G_{j,j}$ is actually contained in Lemma 8.1 of [3]. On the other hand, Proposition 1.1 states a similar result on the rescaled versions of the complex measures $G_{j,j'}$ also in the case $j \ne j'$. The main problem in the proof of this statement is that in this case $G_{j,j'}$ need not be a (real valued, positive) measure. On the other hand, it is the element of a matrix valued positive definite (spectral) measure, and this fact plays an important role in the proof of Proposition 1.1.

It was proved in [4] that to each matrix valued spectral measure corresponds a vector valued random spectral measure such that the Fourier transform of this random spectral measure defines a Gaussian stationary random field whose spectral measure equals this spectral measure. We can define multiple Wiener–Itô integrals with respect to the coordinates of this vector valued random spectral measure. This enables to formulate the multivariate version of Theorem 8.2 in [3] in the following Theorem 1.2A. This is a limit theorem where the limit is defined by means of a sum of multiple Wiener–Itô integrals with respect to the coordinates of a vector valued random spectral measure $Z_{G^{(0)}} = (Z_{G^{(0)},1}, \dots, Z_{G^{(0)},d})$ which corresponds to the matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, defined in Proposition 1.1.

Theorem 1.2A. *Fix some integer $k \ge 1$, and let $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, be a vector valued Gaussian stationary random field whose covariance function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, satisfies relation (2) with some $0 < \alpha < \frac{\nu}{k}$ and relation (3). Let $H(x_1, \dots, x_d)$ be a function of the form given in (4) with the parameter k we have fixed in the formulation of this result. Define the random variables $Y(p) = H(X_1(p), \dots, X_d(p))$ for all $p \in \mathbb{Z}^\nu$ together with their normalized partial sums*

$$S_N = \frac{1}{N^{\nu - k\alpha/2} L(N)^{k/2}} \sum_{p \in B_N} Y(p),$$

where the set B_N was defined in (1). These random variables S_N, $N = 1, 2, \dots$, satisfy the following limit theorem.

Let $Z_{G^{(0)}} = (Z_{G^{(0)},1}, \dots, Z_{G^{(0)},d})$ be a vector valued random spectral measure which corresponds to the matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, *defined in Proposition 1.1 with the help of the matrix valued spectral measure $G = (G_{j,j'})$, corresponding the covariance function $r_{j,j'}(p)$ we are working with. Then the sum of multiple Wiener–Itô integrals*

$$S_0 = \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\cdots+k_d=k}} c_{k_1,\dots,k_d} \int \prod_{l=1}^{\nu} \frac{e^{i(x_1^{(l)}+\cdots+x_k^{(l)})}-1}{i(x_1^{(l)}+\cdots+x_k^{(l)})} Z_{G^{(0)},j(1|k_1,\dots,k_d)}(\,dx_1)\dots Z_{G^{(0)},j(k|k_1,\dots,k_d)}(\,dx_k) \tag{7}$$

exists, where we use the notation $x_p = (x_p^{(1)},\dots,x_p^{(\nu)})$, $p = 1,\dots,k$, and we define the indices $j(s|k_1,\dots,k_d)$, $1 \le s \le k$, as $j(s|k_1,\dots,k_d) = r$ if $\sum_{u=1}^{s-1} k_u < r \le \sum_{u=1}^{s} k_u$, $1 \le s \le k$. (For $s = 1$ we apply the notation $\sum_{u=1}^{0} k_u = 0$ in the definition of $j(1|k_1,\dots,k_d)$.) The normalized sums S_N converge in distribution to the random variable S_0 defined in (7) as $N \to \infty$.

I explain the indexation of the terms $Z_{G^{(0)},j(s|k_1,\dots,k_d)}(\,dx_s)$ in formula (7) in a simpler way. In the first k_1 variables $x_1,\dots,x_{k_1}$ we wrote $Z_{G^{(0)},1}(\,dx_s)$, $1 \le s \le k_1$, in the next k_2 terms we wrote $Z_{G^{(0)},2}(\,dx_s)$, $k_1+1 \le s \le k_1+k_2$, and so on. In the last k_d terms we wrote $Z_{G^{(0)},d}(\,dx_s)$, $k_1+\cdots+k_{d-1}+1 \le s \le k$.

In Theorem 1.2A we described the limit of $A_N^{-1}\sum_{p\in B_N} H(X_1(p),\dots,X_d(p))$ if the expansion of the functions $H(x_1,\dots,x_d)$ in the product of Hermite polynomials contains only polynomials of order k. The next Theorem 1.2B which is the multivariate version of Theorem 8.2′ in [3] states that a result similar to Theorem 1.2A holds if the expansion of the function $H(x_1,\dots,x_d)$ in the product of Hermite polynomials may also contain polynomials of higher order.

Theorem 1.2B. *Let us consider the same vector valued Gaussian stationary random field $X(p) = (X_1(p),\dots,X_d(p))$, $p \in \mathbb{Z}^\nu$, as in Theorem 1.2A together with a function of the form $H(x_1,\dots,x_d) = H^{(0)}(x_1,\dots,x_d)+H^{(1)}(x_1,\dots,x_d)$, where $H^{(0)}(x_1,\dots,x_d)$ was defined in (4), and*

$$H^{(1)}(x_1,\dots,x_d) = \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\cdots+k_d\ge k+1}} c_{k_1,\dots,k_d} H_{k_1}(x_1)\cdots H_{k_d}(x_d) \tag{8}$$

with coefficients $c_{k_1,\dots,k_d}$ such that

$$\sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\cdots+k_d\ge k+1}} \frac{c^2_{k_1,\dots,k_d}}{k_1!\cdots k_d!} < \infty. \tag{9}$$

Define the random variables $Y(p) = H(X_1(p),\dots,X_d(p))$ for all $p \in \mathbb{Z}^\nu$ and their normalized partial sums

$$S_N = \frac{1}{N^{\nu-k\alpha/2}L(N)^{k/2}} \sum_{p\in B_N} Y(p), \quad N = 1,2,\dots,$$

with this function $H(x_1,\dots,x_d)$. The random variables S_N converge in distribution to the random variable S_0 defined in formula (7) as $N\to\infty$.

Actually condition (9) in Theorem 1.2B means that

$$EH^{(1)^2}(X_1(0),\dots,X_d(0))<\infty.$$

Finally I mention that Arcones formulated a more general result. To present it, more precisely its generalization when we are working with stationary random fields parametrized by the lattice points of $\mathbb{Z}^\nu$ with $\nu\ge 1$, let us define the following parameter sets for all $N=1,2,\dots$ and $t=(t_1,\dots,t_\nu)$, $0\le t_l\le 1$, for all $1\le l\le\nu$.

$$B_N(t)=B_N(t_1,\dots,t_\nu)=\{p=(p_1,\dots,p_\nu)\colon\ 0\le p_l<Nt_l \text{ for all } 1\le l\le\nu\}. \tag{10}$$

With this notation I formulate the following result.

Theorem 1.3. *Let us consider the same vector valued Gaussian stationary random field $X(p)=(X_1(p),\dots,X_d(p))$, $p\in\mathbb{Z}^\nu$, and function $H(x_1,\dots,x_d)$ as in Theorem 1.2B. Define the random variables $Y_p=H(X_1(p),\dots,X_d(p))$ for all $p\in\mathbb{Z}^\nu$ and the random fields*

$$S_N(t)=\frac{1}{N^{\nu-k\alpha/2}L(N)^{k/2}}\sum_{p\in B_N(t)}Y(p) \tag{11}$$

with parameter set $t=(t_1,\dots,t_\nu)$, $0<t_l\le 1$, $1\le l\le\nu$, for all $N=1,2,\dots$, where the set $B_N(t)$ was defined in (10). The finite dimensional distributions of the random fields $S_N(t)$ converge to that of the random field $S_0(t)$, $t=(t_1,\dots,t_\nu)$, $0\le t_l\le 1$, $1\le l\le\nu$, defined by the formula

$$\begin{aligned}S_0(t) \quad = \quad & \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\cdots+k_d=k}} c_{k_1,\dots,k_d}\int\prod_{l=1}^{\nu}\frac{e^{it_l(x_1^{(l)}+\cdots+x_k^{(l)})}-1}{i(x_1^{(l)}+\cdots+x_k^{(l)})} \\ & \qquad Z_{G^{(0)},j(1|k_1,\dots,k_d)}(\,dx_1)\dots Z_{G^{(0)},j(k|k_1,\dots,k_d)}(\,dx_k)\end{aligned} \tag{12}$$

if the limit $N\to\infty$ is taken. Similarly to Theorem 1.2A we use the notation $x_p=(x_p^{(1)},\dots,x_p^{(\nu)})$, $p=1,\dots,k$, and the indices $j(s|k_1,\dots,k_d)$, $1\le s\le k$, are defined as in formula (7).

Let us observe that the kernel function in the Wiener–Itô integrals expressing $S_0(t)$ equals $\varphi_t(x_1+\cdots+x_k)$, where $\varphi_t(u)$, $u\in\mathbb{R}^\nu$, is the Fourier transform of the Lebesgue measure on the rectangle $[0,t_1]\times\cdots\times[0,t_\nu]$. The integral in (12) is taken on the whole space. (The formulation of Arcones' result at this point is erroneous.)

We have formulated Theorem 1.3 in the form as Arcones did, but actually we could have formulated it in a slightly more general way. We could have defined the sets $B_N(t)$ in (10), the random variables $S_N(t)$, $N=1,2,\dots$, in (11) and $S_0(t)$ in (12) for all $t=(t_1,\dots,t_\nu)\in[0,\infty)^\nu$ and not only for $t=(t_1,\dots,t_\nu)\in[0,1]^\nu$. We could have proved, similarly to the proof of Theorem 1.3, that the finite dimensional distributions of the random fields $S_N(t)$ converge to the finite dimensional distributions of the random field $S_0(t)$ as $N\to\infty$ also in this more general case. We can also say that the random field $S_0(t)$, $t\in[0,\infty)^\nu$, is self-similar with parameter $\nu-k\alpha/2$, i.e $S_0(ut)\stackrel{\Delta}{=}u^{\nu-k\alpha/2}S_0(t)$ for all $u>0$, where $\stackrel{\Delta}{=}$ means that the finite dimensional distributions of the two random fields agree.

One can prove the self-similarity property of the random field $S_0(t)$, $t\in[0,\infty)^\nu$, by exploiting that $G^{(0)}(uA)=u^\alpha G^{(0)}(A)$ for the spectral measure $G^{(0)}$ for all $u>0$ and measurable sets $A\subset\mathbb{R}^\nu$ by formula (1.6) in Proposition 1.1, and this implies that

$$(Z_{G^{(0)},1}(uA_1),\dots,Z_{G^{(0)},d}(uA_d))\stackrel{\Delta}{=}(u^{\alpha/2}Z_{G^{(0)},1}(A_1),\dots,u^{\alpha/2}Z_{G^{(0)},d}(A_d))$$

for all $u>0$ and measurable sets $A_1\in\mathbb{R}^\nu,\dots,A_d\in\mathbb{R}^\nu$. We still have to exploit that the kernel functions

$$f_t(x_1,\dots,x_d)=\prod_{l=1}^{\nu}\frac{e^{it_l(x_1^{(l)}+\cdots+x_k^{(l)})}-1}{i(x_1^{(l)}+\cdots+x_k^{(l)})}$$

in the Wiener–Itô integrals in (1.12) (with the notation $t=(t_1,\dots,t_\nu)$) have the property

$$f_{ut}(x_1,\dots,x_k)=u^\nu f_t(ux_1,\dots,ux_k)$$

for all $u>0$, $t\in[0,\infty)^\nu$, $x_j\in\mathbb{R}^\nu$, $1\le j\le k$. The self-similarity property of the random field $S_0(t)$, $t\in[0,\infty)^\nu$, can be proved with the help of the above observations.

2. Proof of the results

Let us consider the proof of Theorem 1.2A. A most important point in it is to find a good representation of the normalized random sums S_N appearing in this result.

Let $X(p)=(X_1(p),\dots,X_d(p))$, $p\in\mathbb{Z}^\nu$, be a vector valued Gaussian stationary random field whose covariance function $r_{j,j'}(p)=EX_j(0)X_{j'}(p)$, $1\le j,j'\le d$, $p\in\mathbb{Z}^\nu$, satisfies relation (2) with some parameter $0<\alpha<\frac{\nu}{k}$ and relation (3). Let $G=(G_{j,j'})$, $1\le j,j'\le d$, be the matrix valued spectral measure of this stationary random field, and let us consider that vector valued random spectral measure $Z_G=(Z_{G,1},\dots,Z_{G,d})$ corresponding to this spectral measure for which $X_j(p)=\int e^{i(p,x)}Z_{G,j}(\,dx)$ for all $p\in\mathbb{Z}^\nu$ and $1\le j\le d$.

Because of relation (3) the random variable $Y(0)=H(X_1(0),\dots,X_d(0))$ defined with the function $H(x_1,\dots,x_d)=H^{(0)}(x_1,\dots,x_d)$ introduced in (4) is a Wick polynomial of order k of the random variables $X_1(0),\dots,X_d(0)$, (see Section 7 in [4]). This Wick polynomial can be rewritten because of the multivariate version of Itô's formula (see Corollary of Theorem 7.2 in [4]) and the identity

$X_j(0)=\int \mathbb{I}_1(y) Z_{G,j}(\,dy)$, $1\le j\le d$, where $\mathbb{I}_1(y)$ denotes the indicator function of the torus $[-\pi,\pi)^\nu$, as a sum of k-fold multiple Wiener–Itô integrals with respect to the vector valued random spectral measure $Z_G=(Z_{G,1},\dots,Z_{G,d})$. Let us remark that by Lemma 8B of [3] condition (2) implies that the diagonal elements $G_{j,j}$, $1\le j\le d$, of the matrix valued spectral measure $G=(G_{j,j'})$, $1\le j,j'\le d$, are non-atomic. Hence we can define the multiple Wiener–Itô integrals with respect to the coordinates of the vector valued random spectral measure $Z_G=(Z_{G,1},\dots,Z_{G,d})$. The above considerations yield the identity

$$\begin{aligned} Y_0 &= H(X_1(0),\dots,X_d(0))=H^{(0)}(X_1(0),\dots,X_d(0)) \\ &= \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} :c_{k_1,\dots,k_d}X_1(0)^{k_1}\cdots X_d(0)^{k_d}: \\ &= \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge0\ 1\le j\le d,\\ k_1+\dots+k_d=k}} c_{k_1,\dots,k_d}\int \mathbb{I}_1(y_1)\dots\mathbb{I}_1(y_k) \\ &\qquad\qquad \prod_{j=1}^d\left(\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dy_t)\right), \end{aligned}$$

where for $j=1$ we define $\prod\limits_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dy_t)=\prod\limits_{t=1}^{k_1} Z_{G,1}(\,dy_t)$, and if $k_j=0$ for some $1\le j\le d$, then we drop the term $\prod\limits_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dy_t)$ from this expression. (Here $:P(X_1(0),\dots,X_d(0)):$ denotes the Wick polynomial corresponding to $P(X_1(0),\dots,X_d(0))$, where $P(x_1,\dots,x_d)$ is a homogeneous polynomial.)

Since $Y(p)=T_pY(0)$ with the shift transformation T_p for all $p\in\mathbb{Z}^\nu$, the previous identity and Proposition 7.4 in [4] yield the formula

$$\begin{aligned} Y(p) &= T_pY(0) \\ &= \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} c_{k_1,\dots,k_d}\int e^{i(p,y_1+\dots+y_k)}\prod_{j=1}^d\left(\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dy_t)\right) \end{aligned}$$

for all $p\in\mathbb{Z}^\nu$. We get by summing up this formula for all $p\in B_N$ that

$$\begin{aligned} S_N=\frac{1}{N^{\nu-k\alpha/2}L(N)^{k/2}} &\sum_{\substack{(k_1,\dots,k_d),\ k_j\ge0,\ 1\le j\le d\\ k_1+\dots+k_d=k}} c_{k_1,\dots,k_d} \\ &\int\prod_{l=1}^\nu \frac{e^{i(N(y_1^{(l)}+\dots+y_k^{(l)})}-1}{e^{i((y_1^{(l)}+\dots+y_k^{(l)})}-1}\prod_{j=1}^d\left(\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dy_t)\right), \end{aligned}$$

where we write $y=(y^{(1)},\dots,y^{(\nu)})$ for all $y\in[-\pi,\pi)^\nu$.

We rewrite the above sum of Wiener–Itô integrals with the change of variables $x_s = Ny_s$, $1 \le s \le k$, in the form

$$S_N = \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\cdots+k_d=k}} \int f^N_{k_1,\dots,k_d}(x_1,\dots,x_k) \prod_{j=1}^d \left(\prod_{t=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} Z_{G^{(N)},j}(\,dx_t) \right), \tag{13}$$

where

$$f^N_{k_1,\dots,k_d}(x_1,\dots,x_k) = c_{k_1,\dots,k_d} \prod_{l=1}^{\nu} \frac{e^{i((x_1^{(l)}+\cdots+x_k^{(l)})}-1}{N(e^{i((x_1^{(l)}+\cdots+x_k^{(l)})/N}-1)} \tag{14}$$

is a function on $[-N\pi,N\pi)^\nu$, and $Z_{G^{(N)},j}(A) = \frac{N^{\alpha/2}}{L(N)^{1/2}} Z_{G,j}(\frac{A}{N})$ for all measurable sets $A \subset [-N\pi,N\pi)^\nu$ and $j=1,\dots,d$. Here we use the notation $x_s = (x_s^{(1)},\dots,x_s^{(\nu)})$ for all $1\le s\le k$ for a set of vectors $(x_1,\dots,x_k)$, $x_s\in\mathbb{R}^\nu$ for all $1\le s\le k$. Let us observe that $(Z_{G^{(N)},1},\dots,Z_{G^{(N)},d})$ is a vector valued random spectral measure on the torus $[-N\pi,N\pi)^\nu$ corresponding to the matrix valued spectral measure $G^{(N)} = (G^{(N)}_{j,j'})$, $1\le j,j'\le d$, on the torus $[-N\pi,N\pi)^\nu)$, defined by the formula $G^{(N)}_{j,j'}(A) = \frac{N^\alpha}{L(N)} G_{j,j'}(\frac{A}{N})$, $1\le j,j'\le d$, for all measurable sets $A\subset[-N\pi,N\pi)^\nu$, where $G=(G_{j,j'})$, $1\le j,j'\le d$, is the matrix valued spectral measure of the original vector valued stationary random field $X(p)=(X_1(p),\dots,X_d(p))$, $p\in\mathbb{Z}^\nu$.

In formulas (13) and (14) we gave a useful representation of the normalized random sum S_N investigated in Theorem 1.2A in the form of a sum of k-fold multiple Wiener–Itô integrals. Let us observe that the kernel functions $f^N_{k_1,\dots,k_d}(x_1,\dots,x_k)$ of these Wiener–Itô integrals satisfy the relation

$$\lim_{N\to\infty} f^N_{k_1,\dots,k_d}(x_1,\dots,x_k) = f^0_{k_1,\dots,k_d}(x_1,\dots,x_k) \tag{15}$$

for all indices $k_1,\dots,k_d$ such that $k_j\ge 0$, $1\le j\le d$, and $k_1+\cdots+k_d=k$ with the function

$$f^0_{k_1,\dots,k_d}(x_1,\dots,x_k) = c_{k_1,\dots,k_d} \prod_{l=1}^{\nu} \frac{e^{i((x_1^{(l)}+\cdots+x_k^{(l)})}-1}{i(x_1^{(l)}+\cdots+x_k^{(l)})} \tag{16}$$

defined on $\mathbb{R}^{k\nu}$, and this convergence is uniform in all bounded subsets of $\mathbb{R}^{k\nu}$.

On the other hand, Proposition 1.1 states that the matrix valued spectral measures $G^{(N)} = (G^{(N)}_{j,j'})$ converge to a matrix valued spectral measure $G^{(0)} = (G^{(0)}_{j,j'})$ on $\mathbb{R}^\nu$, and in (13) we integrate with respect to a vector valued random spectral measure corresponding to the matrix spectral measure $(G^{(N)}_{j,j'})$, $1 \le$

$j, j' \le N$. Hence it is natural to expect that the random variables S_N converge in distribution to the random variable

$$S_0 = \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\cdots+k_d=k}} \int f^0_{k_1,\dots,k_d}(x_1,\dots,x_k) \prod_{j=1}^d \left(\prod_{t=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} Z_{G^{(0)},j}(\,dx_t) \right), \qquad (17)$$

where $(Z_{G^{(0)},1},\dots,Z_{G^{(0)},d})$ is a vector valued random spectral measure on $\mathbb{R}^\nu$ corresponding to the matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1\le j,j'\le d$. This is actually the statement of Theorem 1.2A with a slightly different notation.

First I prove Proposition 1.1, and then Theorem 1.2A by justifying the above heuristic argument with the help of Proposition 2A presented later, which is a reformulation of Proposition 8.1 in [4].

Proof of Proposition 1.1. Let us remark that for the diagonal elements $G_{j,j}$, $1\le j\le d$, of the matrix valued spectral measure G the measures $G^{(N)}_{j,j}$ converge vaguely to a locally finite measure $G^{(0)}_{j,j}$ which satisfies relation (6). This follows from Lemma 8.1 of [3]. Simply we have to apply this lemma for the measures $G_{j,j}$.

In the case of the non-diagonal elements $G_{j,j'}$, $j\ne j'$, this argument does not work in itself, because a complex measure $G_{j,j'}$ with finite total variation may be not a measure. In this case we exploit that G is a positive semidefinite matrix valued measure, hence the 2×2 matrix

$$G(A|j,j') = \begin{pmatrix} G_{j,j}(A), & G_{j,j'}(A) \\ G_{j',j}(A), & G_{j',j'}(A) \end{pmatrix}$$

is positive semidefinite for all pairs $1\le j,j'\le d$, $j\ne j'$, and measurable sets $A\subset\mathbb{R}^\nu$. Hence the quadratic forms $(1,1)G(A|j,j')(1,1)^* = G_{j,j}(A) + G_{j',j'}(A)+G_{j,j'}(A)+G_{j',j}(A)$ and $(1,i)G(A|j,j')(1,-i)^* = G_{j,j}(A)+G_{j',j'}(A)-i[G_{j,j'}(A)-G_{j',j}(A)]$ are non-negative numbers for all measurable sets $A\subset\mathbb{R}^\nu$. This implies that the set-functions $R_{j,j'}(\cdot)$ and $S_{j,j'}(\cdot)$ defined as $R_{j,j'}(A) = G_{j,j}(A)+G_{j',j'}(A)+G_{j,j'}(A)+G_{j',j}(A)$ and $S_{j,j'}(A) = G_{j,j}(A)+G_{j',j'}(A)-i[G_{j,j'}(A)-G_{j',j}(A)]$ for all measurable sets A are finite measures. Moreover, they satisfy relation (2) (with a new function $a_{j,j'}(\cdot)$). Hence Lemma 8.2 of [3] can be applied for them. This fact together with the property $G_{j',j}(A) = \overline{G_{j,j'}(A)}$ and the result about the behaviour of $G^{(N)}_{j,j}$ and $G^{(N)}_{j',j'}$ imply that the sequence $G^{(N)}_{j,j'}$ vaguely converges to a complex measure $G^{(0)}_{j,j'}$ with locally finite total variation which satisfies relation (6). (A similar argument was applied in the proof of Proposition 8.1 in [4].)

Lemma 8.1 in [3] (applied for the measures $G_{j,j}$, $1\le j\le d$) implies that $\mu^{(0)}_{j,j}(\,dx) = \prod_{l=1}^\nu \frac{1-\cos x^{(l)}}{(x^{(l)})^2}\, G^{0)}_{j,j}(\,dx)$ is a finite measure whose Fourier transform can be expressed by means of the parameter α and the function $a_{j,j}(x)$ on the unit sphere $\mathcal{S}_{\nu-1}$. This fact together with relation (6) imply that these quantities determine the distribution of the measure $G^{(0)}_{j,j}$. By refining this argument one can prove that $\mu^{(0)}_{j,j'}(\,dx) = \prod_{l=1}^{\nu} \frac{1-\cos x^{(l)}}{(x^{(l)})^2}\, G^{0)}_{j,j'}(\,dx)$ is a complex measure with finite total variation whose Fourier transform is determined by the parameter α and the function $a_{j,j'}(\cdot)$ on the unit sphere $\mathcal{S}_{\nu-1}$. Hence they determine the distribution of the complex measure $G^{(0)}_{j,j'}$. In a detailed proof the Fourier transform of the complex measures $\mu^{(0)}_{j,j'}$ should be written down. In Lemma 8.1 of [3] this is done for the corresponding formula. Here I omitted this part of the proof, because actually this result is not needed in our investigation.

We have to show that $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$ is a matrix valued spectral measure on $\mathbb{R}^\nu$. (In Section 4 of [4] the matrix valued spectral measures on $\mathbb{R}^\nu$ are defined as the positive semidefinite matrix valued even measures on $\mathbb{R}^\nu$ with moderately increasing distribution at infinity.) In Lemma 8.2 of [4] it is proved that they are positive semidefinite matrix valued even measures on $\mathbb{R}^\nu$, since their coordinates $G^{(0)}_{j,j'}$ are the vague limits of the coordinates $G^{(N)}_{j,j'}$ of the matrix valued spectral measures $(G^{(N)}_{j,j'})$, $1 \le j, j' \le d$ as $N \to \infty$. It follows from relation (6) that they have moderately increasing distribution at infinity. (This property is defined in formula (4.1) in [4].) Proposition 1.1 is proved.

To prove Theorem 1.2A I recall Proposition 8.1 of [4] in the following Proposition 2A. It states that under certain conditions a sequence of sums of k-fold Wiener–itô integrals converges in distribution to a sum of k-fold Wiener–Itô integrals. Before formulating this result I recall that in Section 5 of [4], in the definition of multiple Wiener–Itô integrals I have defined a real Hilbert space $\mathcal{K}_{k,j_1,\dots,j_k} = \mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1},\dots,G_{j_k,j_k})$ depending on the diagonal elements $G_{1,1},\dots,G_{d,d}$ of a matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$, and a sequence of length k of integers $(j_1,\dots,j_k)$ such that $1 \le j_s \le d$ for all $1 \le s \le k$. (In paper [4] I worked with Wiener–Itô integrals of order n, while here I work with Wiener–Itô integrals of order k. Hence I use here a slightly different notation.) I shall refer to this Hilbert space in the formulation of our result. This Hilbert space appears in this result, because it contains those functions $f(x_1,\dots,x_k)$ for which we defined the k-fold Wiener–Itô integral $\int f(x_1,\dots,x_k) Z_{G,j_1}(\,dx_1)\dots Z_{G,j_k}(\,dx_k)$ with respect to a vector valued random spectral measure $(Z_{G,1},\dots,Z_{G,d})$ corresponding to the matrix valued spectral measure $(G_{j,j'})$, $1 \le j, j' \le d$.

We take for all $N = 1, 2, \dots$ a sequence of matrix valued non-atomic spectral measures $(G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-A_N\pi, A_N\pi)^\nu$ with a parameter A_N such that $A_N \to \infty$ as $N \to \infty$. We also take some functions

$$h^N_{j_1,\dots,j_k}(x_1,\dots,x_k) \in \mathcal{K}_{k,j_1,\dots,j_k} = \mathcal{K}_{k,j_1,\dots,j_k}(G^{(N)}_{j_1,j_1},\dots,G^{(N)}_{j_k,j_k})$$

on the torus $[-A_N\pi, A_N\pi)^\nu$ for all sequences $(j_1,\dots,j_k)$ such that $1 \le j_s \le d$, $1 \le s \le k$, and $N = 1, 2, \dots$. Besides, we fix for all $N = 1, 2, \dots$ a vector valued random spectral measure $(Z_{G^{(N)},1},\dots,Z_{G^{(N)},d})$ on the torus $[-A_N\pi,A_N\pi)^\nu$ corresponding to the matrix valued spectral measure $(G^{(N)}_{j,j'})$, $1\le j,j'\le d$, and we define with the help of these quantities the sums of k-fold Wiener–Itô integrals

$$S_N=\sum_{\substack{(j_1,\dots,j_k)\\ 1\le j_s\le d,\text{ for all }1\le s\le k}}\int h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k),\tag{18}$$

$N=1,2,\dots$. We want to find some good conditions under which these random variables S_N converge in distribution to a random variable S_0 expressed in the form of a sum of multiple Wiener–Itô integrals.

The following result supplies such conditions.

Proposition 2A. *Let us consider the sums of k-fold Wiener–Itô integrals S_N defined in formula (18) with the help of certain vector valued random spectral measures $(Z_{G^{(N)},1},\dots,Z_{G^{(N)},d})$ corresponding to some non-atomic matrix valued spectral measures $(G^{(N)}_{j,j'})$, $1\le j,j'\le d$ defined on tori $[-A_N,A_N)^\nu$ such that $A_N\to\infty$ as $N\to\infty$ and functions*

$$h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)\in\mathcal{K}_{k,j_1,\dots,j_k}(G^{(N)}_{j_1,j_1},\dots,G^{(N)}_{j_k,j_k}).$$

Let the coordinates $G^{(N)}_{j,j'}$, $1\le j,j'\le d$, of the matrix valued spectral measures $(G^{(N)}_{j,j'})$, $1\le j,j'\le d$, converge vaguely to the coordinates $G^{(0)}_{j,j'}$ of a non-atomic matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1\le j,j'\le d$, on $\mathbb{R}^\nu$ for all $1\le j,j'\le d$ as $N\to\infty$, and let $(Z_{G^{(0)},1},\dots,Z_{G^{(0)},d})$ be a vector valued random spectral measure on $\mathbb{R}^\nu$ corresponding to the matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1\le j,j'\le d$. Let us also have some functions $h^0_{j_1,\dots,j_k}$ for all $1\le j_s\le d$, $1\le s\le k$, such that these functions and matrix valued spectral measures satisfy the following conditions.

(a) *The functions $h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)$ are continuous on $\mathbb{R}^{k\nu}$ for all $1\le j_s\le d$, $1\le s\le k$, and for all $T>0$ and indices $1\le j_s\le d$, $1\le s\le k$, the functions $h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)$ converge uniformly to the function $h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)$ on the cube $[-T,T]^{k\nu}$ as $N\to\infty$.*

(b) *For all $\varepsilon>0$ there is some $T_0=T_0(\varepsilon)>0$ such that*

$$\int_{\mathbb{R}^{k\nu}\setminus[-T,T]^{k\nu}}|h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)|^2G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(dx_k)<\varepsilon^2\tag{19}$$

for all $1\le j_s\le d$, $1\le s\le k$, and $N=1,2\dots$ if $T>T_0$.

Then inequality (19) holds also for $N=0$,

$$h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)\in\mathcal{K}_{k,j_1,\dots,j_k}=\mathcal{K}_{k,j_1,\dots,j_k}(G^{(0)}_{j_1,j_1},\dots G^{(0)}_{j_k,j_k}),$$

the sum of random integrals

$$S_0 = \sum_{\substack{(j_1,\dots,j_k) \\ 1\le j_s\le d, \text{ for all } 1\le s\le k}} \int h^0_{j_1,\dots,j_k}(x_1,\dots,x_k) Z_{G^{(0)},j_1}(\,dx_1)\dots Z_{G^{(0)},j_k}(\,dx_k). \tag{20}$$

exists, and the random variables S_N defined in (18) converge to S_0 in distribution as $N\to\infty$.

I shall prove Theorem 1.2A with the help of Proposition 2A and the representation of the random variables S_N, $N=1,2,\dots$, in formulas (13) and (14). I shall apply Proposition 2A with an appropriate choice of the random variables S_N, $N=1,2,\dots$, in formula (18) and of the random variable S_0 in (20). In this application I define the random variables S_N in (18) and the random variable S_0 in (20) by rewriting first the random variables defined in (13) and (14) and then the random variable defined in (16) and (17) in an appropriate way. I shall rewrite them in such a special form of formulas (18) and (20) where the summation is going only for such sequences $(j_1,\dots,j_k)$ whose elements j_s, $1\le s\le k$, go up in increasing order from 1 to d, i.e. $1\le j_1\le j_2\le\dots\le j_k\le d$. Given a sequence $(j_1,\dots,j_k)$ with this property I define with its help the numbers

$$k_s(j_1,\dots j_k) \;=\; \text{the number of such elements } j_p \text{ in the sequence } j_1,\dots,j_k \text{ for which } j_p=s \tag{21}$$

for all indices $1\le s\le d$.

I shall rewrite the sums in (13) and (17) with which I shall work by replacing the indices of summation $k_1,\dots,k_d$ in them by appropriately chosen indices $1\le j_1\le\dots\le j_k\le d$. I choose these indices $j_1,\dots,j_k$ in such a way that the quantities $k_s=k_s(j_1,\dots,j_k)$, $1\le s\le d$, defined in (21) agree with the original sequence $(k_1,\dots,k_d)$ in (13) or (17). I exploit that there is a natural one to one correspondence between the sequences $j_1,\dots,j_k$ such that $1\le j_1\le\dots\le j_k\le d$ and the sequences $k_1,\dots,k_d$ of integers with the property $k_s\ge0$ for all $1\le s\le d$ and $k_1+\dots+k_d=k$. With the help of this correspondence the random sums S_N in (13) can be rewritten in the form

$$S_N = \sum_{\substack{(j_1,\dots,j_k), \\ 1\le j_1\le\dots\le j_k\le d}} \int f^N_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)}(x_1,\dots,x_k) Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k) \tag{22}$$

for all $N=1,2,\dots$ with the functions $f^N_{k_1,\dots,k_d}(x_1,\dots,x_k)$ defined in (14) and the functions $k_s(j_1,\dots,j_k)$, $1\le s\le d$ defined in (21). This means that relation (22) can be rewritten in the form

$$S_N = \sum_{\substack{(j_1,\dots,j_k), \\ 1\le j_1\le\dots\le j_k\le d}} \int h^N_{j_1,\dots,j_k}(x_1,\dots,x_k) Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k) \tag{23}$$

with

$$h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)=f^N_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)}(x_1,\dots,x_k), \tag{24}$$

where the functions $k_s(j_1,\dots,j_k)$, $1\le s\le d$, are defined in (21). Similarly, the random sum S_0 in (17) can be rewritten in the form

$$S_0 = \sum_{\substack{(j_1,\dots,j_k)\\ 1\le j_1\le\cdots\le j_k\le d}} \int f^0_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)}(x_1,\dots,x_k) \\ Z_{G^{(0)},j_1}(\,dx_1)\dots Z_{G^{(0)},j_k}(\,dx_k)$$

with the function $f^0_{k_1,\dots,k_d}(x_1,\dots,x_k)$ defined in (16) or in the following equivalent form

$$S_0=\sum_{\substack{(j_1,\dots,j_k),\\ 1\le j_1\le\cdots\le j_k\le d}}\int h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)Z_{G^{(0)},j_1}(\,dx_1)\dots Z_{G^{(0)},j_k}(\,dx_k) \tag{25}$$

with

$$h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)=f^0_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)}(x_1,\dots,x_k). \tag{26}$$

I shall apply Proposition 2A for the sequences S_N, defined in (23), (14) and (24) for $N=1,2,\dots$, and in (25), (16) and (26) for $N=0$. I shall integrate with respect to the coordinates of vector valued spectral measures $(Z_{G^{(N)},1},\dots,Z_{G^{(N)},d})$, $N=0,1,2,\dots$, corresponding to the matrix valued spectral measures $(G^{(N)}_{j.j'})$, $1\le j,j'\le d$, $N=0,1,2,\dots$, defined in formula (5) for $N\ge1$ and in Proposition 1.1 for $N=0$. In the proof of Theorem 1.2A I have to show that the conditions of Proposition 2A are satisfied with such a choice.

It follows from Proposition 1.1 that the (non-atomic) complex measures $G^{(N)}_{j,j'}$ with finite total variation vaguely converge to the (non-atomic) complex measure $G^{(0)}_{j,j'}$ with locally finite total variation as $N\to\infty$ for all $1\le j,j'\le d$. It is also clear that $h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)\in\mathcal{K}_{k,j_1,\dots,j_k}(G^{(N)}_{j_1,j_1},\dots,G^{(N)}_{j_k,j_k})$ for all $1\le j_1\le\cdots\le j_k\le d$ and $N=1,2,\dots$ with $h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)$ defined in (24).

It follows from (15), (24) and (26) that condition (a) of Proposition 2A holds with the choice of the functions and measures we chose in the proof of Proposition 1.2. We still have to prove relation (19) in condition (b). This will be done with the help of the following Proposition 2.1. (Actually we shall prove in Proposition 2.1 a slightly stronger result than we need.)

Proposition 2.1. *Let us fix some integer $k\ge1$, and let $(G_{j,j'})$, $1\le j,j'\le d$, be the matrix valued spectral measure of a vector valued stationary field $X(p)=(X_1(p),\dots,X_p(d))$, $p\in\mathbb{Z}^\nu$, defined on the torus $[-\pi,\pi)^\nu$, whose correlation function $r_{j,j'}(p)=EX_j(0)X_{j'}(p)$, $1\le j,j'\le d$, $p\in\mathbb{Z}^\nu$, satisfies relation (2) with some $0<\alpha<\frac1k$. For all $N=1,2,\dots$, let us consider the measures $G^{(N)}_{j,j}$,*

$1 \le j \le d$, defined in formula (5) together with the measures $\mu^{(N)}_{j_1,\dots,j_k}$ defined for all sets of indices $j_1,\dots,j_k$, $1 \le j_s \le d$, $1 \le s \le k$, on $\mathbb{R}^{k\nu}$ by the formula

$$\mu^{(N)}_{j_1,\dots,j_k}(A) = \int_A |h_N(x_1,\dots,x_k)|^2 G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k), \quad A \in \mathcal{B}^{k\nu}, \tag{27}$$

with

$$h_N(x_1,\dots,x_k) = \prod_{l=1}^{\nu} \frac{e^{i((x_1^{(l)}+\cdots+x_k^{(l)})}-1}{N(e^{i((x_1^{(l)}+\cdots+x_k^{(l)})/N}-1)}, \tag{28}$$

where we use the notation $x=(x^{(1)},\dots,x^{(\nu)})$ for a vector $x \in \mathbb{R}^\nu$. The measures $\mu^{(N)}_{j_1,\dots,j_k}$ converge weakly to a finite measure $\mu^{(0)}_{j_1,\dots,jk}$ on $\mathbb{R}^{k\nu}$.

Proof of Theorem 1.2A with the help of Proposition 2.1. It is enough to show that the complex measures $G^{(N)}_{j,j'}$, $1 \le j,j' \le d$, $N=0,1,2,\dots$, defined in (5) and in the result of Proposition 1.1 together with the functions $h^N_{j_1,\dots,j_k}$, $1 \le j_s, \le d$, for all $1 \le s \le k$ and $N=0,1,2,\dots$ defined in (14), (23) and (24) for $N=1,2,\dots$ and in (16), (25) and (26) for $N=0$ satisfy the conditions of Proposition 2A. We have proved the validity of all these conditions expect formula (19) in condition (b). But even this condition is simply follows from Proposition 2.1 which implies that the measures $\mu^{(N)}_{j_1,\dots,j_k}$, $N=1,2,\dots$, are tight for all indices $N=1,2,\dots$. This fact together with the definition of the measures $\mu^{(N)}_{j_1,\dots,j_k}$ and the identity $h^N_{j_1,\dots,j_k}(x_1,\dots,x_k) = c_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)} h_N(x_1,\dots,x_k)$ imply that relation (19) holds. Theorem 1.2A is proved.

It remained to prove Proposition 2.1.

Proof of Proposition 2.1. Most calculations needed in the proof of Proposition 2.1 were actually carried out in the proof of Theorem 8.2 of [3] with some slight modifications. I shall omit the details of these calculations.

I compute for all $N=1,2,\dots$ the Fourier transform

$$\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k) = \int e^{i((t_1,x_1)+\cdots+(t_k,x_k))} \mu^{(N)}_{j_1,\dots,j_k}(\,dx_1,\dots,dx_k)$$

of the measures $\mu^{(N)}_{j_1,\dots,j_k}$ defined in (27), and give a good asymptotic formula for them. More precisely, I do this only for such coordinates $(t_1,\dots,t_k)$ of the function $\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)$ which have the form $t_l = \frac{p_l}{N}$ with some $p_l \in \mathbb{Z}^\nu$, $l=1,\dots,k$. But even such a result will be sufficient for us. In the calculation of the formula expressing $\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)$ I exploit that the function $h_N(x_1,\dots,x_k)$ defined in (28) can be written in the form

$$h_N(x_1,\dots,x_k) = \frac{1}{N^\nu} \sum_{u \in B_N} \exp\left\{ i\frac{1}{N}(u, x_1+\cdots+x_k) \right\}.$$

Hence

$$\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)=\frac{1}{N^{2\nu}}\int \exp\left\{i\frac{1}{N}((p_1,x_1)+\cdots+(p_k,x_k))\right\}$$
$$\sum_{u\in B_N}\sum_{v\in B_N}\exp\left\{i\left(\frac{u-v}{N},x_1+\cdots+x_k\right)\right\}$$
$$G^{(N)}_{j_1,j_k}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k)$$
$$=\frac{1}{N^{2\nu-k\alpha}L(N)^k}\sum_{u\in B_N}\sum_{v\in B_N} r_{j_1,j_1}(u-v+p_1)\cdots r_{j_k,j_k}(u-v+p_k)$$

if $t_l=\frac{p_l}{N}$ with some $p_l\in\mathbb{Z}^\nu$, $1\le l\le k$. This identity can be rewritten by taking the summation at the right-hand side of the last formula first for such pairs of (u,v) for which $u-v=y$ with some fixed value $y\in\mathbb{Z}^\nu$ and then for the lattice points $y\in\mathbb{Z}^\nu$. By working with $x=\frac{y}{N}$ instead of y we get that

$$\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)=\int_{[-1,1]^\nu} f^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)\lambda_N(\,dx)$$

with

$$f^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)$$
$$=\left(1-\frac{|x^{(1)}N|}{N}\right)\cdots\left(1-\frac{|x^{(\nu)}N|}{N}\right)\frac{r_{j_1,j_1}(N(x+t_1))}{N^{-\alpha}L(N)}\cdots\frac{r_{j_k,j_k}(N(x+t_k))}{N^{-\alpha}L(N)},$$

where λ_N is the measure concentrated in the points of the form $x=\frac{p}{N}$ with such points $p=(p_1,\dots,p^\nu)\in\mathbb{Z}^\nu$ for which $-N<p_l<N$ for all $1\le l\le\nu$, and $\lambda_N(x)=N^{-\nu}$ for such points x.

Let us extend the definition of $\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)$ to $(t_1,\dots,t_k)\in\mathbb{R}^{k\nu}$ by defining it as

$$\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)=\varphi^{(N)}_{j_1,\dots,j_d}\left(\frac{p_1}{N},\dots,\frac{p_k}{N}\right),\quad t_l\in\mathbb{R}^\nu \text{ for all } 1\le l\le k,$$

if $\frac{p^{(s)}_l}{N}-\frac{1}{2N}\le t^{(s)}_l<\frac{p^{(s)}_l}{N}+\frac{1}{2N}$ with $p_l=(p^{(1)}_l,\dots,p^{(\nu)}_l)\in\mathbb{Z}^\nu$, $1\le l\le k$. Let us extend similarly the definition of the function $f^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)$ to $(t_1,\dots,t_k,x)\in\mathbb{R}^{k\nu}\times[-1,1]^\nu$ by means of the formula

$$f^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)$$
$$=\left(1-\frac{|q^{(1)}N|}{N}\right)\cdots\left(1-\frac{|q^{(\nu)}N|}{N}\right)\frac{r_{j_1,j_1}(N(q+p_1))}{N^{-\alpha}L(N)}\cdots\frac{r_{j_k,j_k}(N(q+p_k))}{N^{-\alpha}L(N)},$$

for $t_l\in\mathbb{R}^\nu$, $1\le l\le k$, and $x\in[-1,1]^\nu$, where $\frac{p^{(s)}_l}{N}-\frac{1}{2N}\le t^{(s)}_l<\frac{p^{(s)}_l}{N}+\frac{1}{2N}$, $1\le s\le\nu$, $1\le l\le k$, with $p_l=(p^{(1)}_l,\dots,p^{(\nu)}_l)\in\mathbb{Z}^\nu$, $1\le l\le k$, and $\frac{q^{(s)}}{N}-\frac{1}{2N}\le x^{(s)}<\frac{q^{(s)}}{N}+\frac{1}{2N}$, $1\le s\le\nu$ with $q=(q^{(1)},\dots,q^{(\nu)})\in\mathbb{Z}^\nu$.

It follows from relation (2) that for all parameters $t_1,\dots,t_k$ and $\varepsilon>0$

$$f^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)\to f^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)$$

holds uniformly with the limit function

$$\begin{aligned}&f^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)\\&\qquad=(1-|x^{(1)}|)\dots(1-|x^{(\nu)}|)\frac{a_{j_1,j_1}\left(\frac{x+t_1}{|x+t_1|}\right)}{|x+t_1|^\alpha}\dots\frac{a_{j_k,j_k}\left(\frac{x+t_k}{|x+t_k|}\right)}{|x+t_k|^\alpha}\end{aligned}$$

on the set $x\in[-1,1]^\nu\setminus\bigcup\limits_{l=1}^k\{x\colon\ |x+t_l|>\varepsilon\}$.

Some additional calculation shows that

$$\varphi^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k)\to\varphi^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k)=\int_{[-1,1]^\nu}f^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)\,dx \tag{29}$$

as $N\to\infty$ for all $(t_1,\dots,t_k)\in\mathbb{R}^{k\nu}$, and $\varphi^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k)$ is a continuous function. This calculation was carried out in the proof of Theorem 8.2 in [3] (in the calculations made after the end of the proof of Lemma 8.4), hence I omit it.

By a classical result of probability theory if the Fourier transforms of a sequence of finite measures on $\mathbb{R}^{k\nu}$ converge to a function continuous at the origin, then the limit function is also the Fourier transform of a finite measure on $\mathbb{R}^{k\nu}$, and the sequence of probability measures whose Fourier transforms were taken converge to this measure. In the proof of Theorem 2.1 we cannot apply this result, because we have a control on the Fourier transform of $\mu^{(N)}_{j_1,\dots,j_k}$ only in points of the form $(t_1,\dots,t_k)$ with $t_l=\frac{p_l}{N}$ with $p_l\in\mathbb{Z}^\nu$, $1\le l\le k$. But the measures $\mu^{(N)}_{j_1,\dots,j_k}$ have the additional property that they are concentrated in the cube $[-N\pi,N\pi)^{k\nu}$. Lemma 8.4 of [3] contains such a result that enables us to deduce from relation (29) (together with the continuity of the limit function $\varphi^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k)$) and the above mentioned property of the measures $\mu^{(N)}_{j_1,\dots,j_k}$ the convergence of the measures $\mu^{(N)}_{j_1,\dots,j_k}$ to a finite measure $\mu^{(0)}_{j_1,\dots,j_k}$. We have also proved that this finite measure $\mu^{(0)}_{j_1,\dots,j_k}$, has the Fourier transform $\varphi^{(0)}(t_1,\dots,t_k)$. Proposition 2.1 is proved.

I prove Theorem 1.2B by showing that if a function $H^{(1)}(\cdot)$ satisfies (8) and (9), and the Gaussian stationary process $Z(p)=(X_1(p),\dots,X_d(p))$ satisfies (2) and (3), then

$$\frac{1}{N^{\nu-k\alpha/2}L(N)^{k/2}}\sum_{p\in B_N}H^{(1)}(X_1(p),\dots,X_d(p))\Rightarrow 0\quad\text{as }N\to\infty, \tag{30}$$

where $\Rightarrow$ denotes convergence in probability. I shall prove that even the second moments of the normalized sums in(30) tend to zero. The following Lemma 2B which agrees with Lemma 1 of [1] (only with a slightly different notation) helps in the proof of this statement.

Lemma 2B. *Let* $X_1, \dots, X_d$ *and* $Y_1, \dots Y_d$ *be jointly Gaussian random variables with expectation zero such that* $EX_jX_{j'} = EY_jY_{j'} = \delta_{j,j'}$, $1 \le j, j' \le d$, *and let* $r_{j,j'} = EX_jY_{j'}$, $1 \le j, j' \le d$. *Take a number* $k \ge 1$ *and a function* $H^{(1)}(x_1, \dots, x_d)$ *satisfying relations (8) and (9). Assume that*

$$\psi = \max\left(\left(\sup_{1\le j\le d}\sum_{j'=1}^d |r_{j,j'}|\right), \left(\sup_{1\le j'\le d}\sum_{j=1}^d |r_{j,j'}|\right)\right) \le 1.$$

Then

$$|EH^{(1)}(X_1, \dots, X_d)H^{(1)}(Y_1, \dots, Y_d)| \le \psi^{k+1} E{H^{(1)}}^2(X_1, \dots, X_d).$$

Proof of Theorem 1.2. It follows from relation (2) and Lemma 2B with the choice $X_j = X_j(p)$, $Y_j = X_j(q)$, $1 \le j \le d$, that there exists some threshold index $n_0 \ge 1$ and constant $0 < C < \infty$ such that

$$|EH^{(1)}(X_1(p), \dots X_d(p))H^{(1)}(X_1(q), \dots, X_d(q))| \le C|p-q|^{-(k+1)\alpha} L(|p-q|)^{k+1}$$

if $|p - q| \ge n_0$. On the other hand,

$$\begin{aligned}|EH^{(1)}(X_1(p), \dots X_d(p))H^{(1)}(X_1(q), \dots, X_d(q))|\\ \le E{H^{(1)}}^2(X_1(0), \dots X_d(0)) \le C_1\end{aligned}$$

for all $p, q \in \mathbb{Z}^\nu$ with some $C_1 < \infty$ by the Schwarz inequality and relation (9). Hence

$$\begin{aligned}\left|EH^{(1)}(X_1(p), \dots, X_d(p))\left(\sum_{q\in B_N} H^{(1)}(X_1(q), \dots, X_d(q)\right)\right|\\ \le C_2(1 + N^{\nu+\varepsilon-(k+1)\alpha})\end{aligned}$$

for all $p \in B_N$ and $\varepsilon > 0$ with an appropriate $C_2 = C_2(\varepsilon) > 0$, and since $\nu - k\alpha > 0$

$$\frac{1}{N^{2\nu-k\alpha}L(N)^k} E\left[\sum_{p\in B_N} H^{(1)}(X_1(p), \dots, X_d(p))\right]^2 \to 0 \quad \text{as } N \to \infty.$$

The last relation implies formula (30). Formula (30) together Theorem 1.2A yield Theorem 1.2B. Theorem 1.2B is proved.

The proof of Theorem 1.3 is very similar to that of Theorems 1.2A and 1.2B, hence I shall explain it briefly.

It is enough to prove that for any positive integer K, parameters $t_1,\dots,t_K$, $t_p\in[0,1]^\nu$, $1\le p\le K$ and real constants $C_1,\dots,C_K$ the linear combinations $\sum_{p=1}^K C_pS_N(t_p)$ converge to $\sum_{p=1}^K C_pS_0(t_p)$ in distribution as $N\to\infty$, since this implies that the random vectors $(S_N(t_1),\dots,S_N(t_K))$ converge in distribution to the random vector $(S_0(t_1),\dots,S_0(t_K))$ as $N\to\infty$. Moreover, similarly to the proof of Theorem 1.2B we can reduce the proof of the result to the case when $H(x_1,\dots,x_d)=H^{(0)}(x_1,\dots,x_d)$ with a function $H^{(0)}(x_1,\dots,x_d)$ of the form given in (4).

In the first step of the proof I write the linear combinations $\sum_{p=1}^K C_pS_N(t_p)$, $N=0,1,2,\dots$, in the form of a sum of k-fold Wiener-Itô integrals with respect to the coordinates of an appropriate vector valued random spectral measure. We can do this if we can write down the random variables $S_N(t)$ in the desired form for all $t\in[0,1]^\nu$. The random variables $S_0(t)$ are given in this form in (12). In the case $N=1,2,\dots$ we can find the right formulation of $S_N(t)$ similarly to the method applied in the proof of Theorem 1.2A. We get similarly to the proof of formulas (13) and (14) that

$$\begin{aligned}S_N(t) \quad &= \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} \int f^N_{k_1,\dots,k_d}(t,x_1,\dots,x_k)\\ &\qquad\qquad \prod_{j=1}^d\left(\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G^{(N)},j}(\,dx_t)\right)\end{aligned}$$

with

$$f^N_{k_1,\dots,k_d}(t,x_1,\dots,x_k)=c_{k_1,\dots,k_d}\prod_{l=1}^{\nu}\frac{\exp\left\{i\frac{]t^{(l)}N[}{N}(x_1^{(l)}+\dots+x_k^{(l)})\right\}-1}{N\left(\exp\left\{i\frac1N(x_1^{(l)}+\dots+x_k^{(l)})\right\}-1\right)},$$

where $t=(t^{(1)},\dots,t^{\nu})$, the number $]t^{(l)}N[$ (in the definition of the function $f^N_{k_1,\dots,k_d}(t,x_1,\dots,x_k)$) is the smallest integer which is larger than $t^{(l)}N$, and $Z_{G^{(N)},j}$ agrees with the spectral measure that appeared in (13) under the same notation.

It is not difficult to see that similarly to relations (15) and (16)

$$\lim_{N\to\infty} f^N_{k_1,\dots,k_d}(t,x_1,\dots,x_k)=f^0_{k_1,\dots,k_d}(t,x_1,\dots,x_k)$$

for all indices $k_1,\dots,k_d$ such that $k_j\ge0$, $1\le j\le d$, and $k_1+\dots+k_d=k$ with the function

$$f^0_{k_1,\dots,k_d}(t,x_1,\dots,x_k)=c_{k_1,\dots,k_d}\prod_{l=1}^{\nu}\frac{e^{it^{(l)}(x_1^{(l)}+\dots+x_k^{(l)})}-1}{i(x_1^{(l)}+\dots+x_k^{(l)})},$$

defined for $(x_1.\dots,x_k)\in\mathbb{R}^{k\nu}$, and this convergence is uniform in all bounded subsets of $\mathbb{R}^{k\nu}$.

With the help of the above considerations the proof of Theorem 1.3 can be reduced, similarly to the proof of Theorem 1.2A, to the following statement. Fix some number K, real constants $C_1,\dots,C_K$ and points $t_1,\dots t_K$ with $t_p\in[0,1]^\nu$, $1\le p\le K$ together with some constants $c_{k_1,\dots,k_d}$ with parameters $k_j\ge 0$, $1\le j\le d$, and $k_1+\dots+k_d=k$ as the coefficients in the sum (4). Let us define with their help the random sums

$$S_N = \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} \int \left(\sum_{p=1}^K C_p c_{k_1,\dots,k_d} f^N_{k_1,\dots,k_d}(t_p,x_1,\dots,x_k)\right) \prod_{j=1}^d \left(\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G^{(N)},j}(\,dx_t)\right) \tag{31}$$

with the above defined functions $g^{(N)}_{k_1,\dots,k_d}(t,x_1,\dots,x_k)$ for all $N=1,2,\dots$, and

$$S_0 = \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} \int \left(\sum_{p=1}^K C_p c_{k_1,\dots,k_d} f^0_{k_1,\dots,k_d}(t_p,x_1,\dots,x_k)\right) \prod_{j=1}^d \left(\prod_{t=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G^{(0)},j}(\,dx_t)\right) \tag{32}$$

with the previously defined function $g^0_{k_1,\dots,k_d}(t,x_1,\dots,x_k)$. I claim that the sequence of random variables S_N defined in (31) converge in distribution to S_0 defined in (32) as $N\to\infty$.

This statement can be proved similarly to Theorem 1.2A with the help of Proposition 2A. In this proof we rewrite the random variables S_N, $N=1,2,\dots$, and S_0 in a form in which we can apply Proposition 2A. We rewrite them as a sum of multiple Wiener–Itô integrals indexed by sequences of integers $j_1,\dots,j_k$ such that $1\le j_1\le\dots\le j_k\le d$. This can be done similarly to the reformulation of formulas (13) and (17) in formulas (23), (24) and (25). (26) with the help of the expressions $k_s(j_1\dots,j_k)$ defined in (21). We rewrite (31) as

$$S_N = \sum_{\substack{(j_1,\dots,j_k),\\ 1\le j_1\le\dots\le j_k\le d}} \int \left(\sum_{p=1}^K C_p c_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)} h^N_{j_1,\dots,j_k}(t_p,x_1,\dots,x_k)\right) Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k) \tag{33}$$

with

$$h^N_{j_1,\dots,j_k}(t_p,x_1,\dots,x_k) = f^N_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)}(t_p,x_1,\dots,x_k)$$

for all $N = 1, 2, \dots$, and (32) as

$$S_0 = \sum_{\substack{(j_1,\dots,j_k),\\ 1\le j_1\le\dots\le j_k\le d}} \int \left(\sum_{p=1}^{K} C_p c_{k_1(j_1,\dots,j_k),\dots,k_d(j,\dots,j_k)} h^0_{j_1,\dots,j_k}(t_p, x_1, \dots, x_k) \right) Z_{G^{(0)},j_1}(\,dx_1)\dots Z_{G^{(0)},j_k}(\,dx_k) \tag{34}$$

with

$$h^0_{j_1,\dots,j_k}(t_p, x_1, \dots, x_k) = f^0_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)}(t_p, x_1, \dots, x_k)$$

where the functions $k_s(j_1, \dots, j_k)$, $1 \le s \le d$, are defined in (21).

The random integrals in formulas (33) and (34) have kernel functions of the form

$$\begin{aligned} h^N_{j_1\dots,j_k}(x_1, \dots, x_k) &= h^N_{j_1\dots,j_k,t_1,\dots,t_K}(x_1, \dots, x_k) \\ &= \sum_{p=1}^{K} C_p c_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)} h^N_{j_1,\dots,j_k}(t_p, x_1, \dots, x_k) \end{aligned} \tag{35}$$

for all $N = 1, 2, \dots$. Let us introduce the measures $\mu_{N,j_1,\dots,j_k}$, $N = 0, 1, 2, \dots$, defined by the formula

$$\begin{aligned} \mu^{(N)}_{j_1,\dots,j_k}(A) &= \mu^{(N)}_{j_1,\dots,j_k,t_1,\dots,t_K}(A) \\ &= \int_A |h_{N,j_1,\dots,j_K,t_1,\dots,t_K}(x_1, \dots, x_k)|^2 G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k) \end{aligned} \tag{36}$$

for all measurable sets $A \in \mathcal{B}^{k\nu}$.

We want to show with the help of Proposition 2A that the random variables S_N, $N = 1, 2, \dots$, defined in (33) converge weakly to the random variable S_0 defined in (34). This implies Theorem 1.3.

To prove this convergence we have to show that the functions $h^N_{j_1,\dots,j_k}$, $n = 0, 1, 2, \dots$, defined in (35) and the measures $G^{(N)}_{j,j'}$, $1 \le j, j' \le d$, $N = 0, 1, 2, \dots$, satisfy the conditions of Proposition 2A. The main point is to prove relation (19) in condition (b) of Proposition 2.1. To prove this we show that the measures $\mu^{(N)}_{j_1,\dots,j_k}$, $N = 1, 2, \dots$, defined in (36) are tight, i.e. for all $\varepsilon > 0$ there exists a $T = T(\varepsilon, j_1, \dots, j_k, t_1, \dots, t_K)$ such that

$$\mu^N_{j_1,\dots,j_k,t_1,\dots,t_k}(\mathbb{R}^{k\nu} \setminus [-T, T]^{k\nu}) < \varepsilon \quad \text{for all } N = 1, 2, \dots.$$

Because of the Schwarz inequality and the definition of the functions $h^N_{j_1\dots,j_k,t_1,\dots,t_K}(x_1, \dots, x_k)$ the proof of this tightness property can be reduced to the justification of the following inequality.

Let us define for all $t = (t_1, \dots, t_\nu) \in [0, 1]^\nu$, and $N = 1, 2, \dots$ the measure

$\mu_{N,t}$ on $\mathbb{R}^{k\nu}$ by the formula

$$\mu_{N,t}(A) = \int_A \left| \prod_{l=1}^{\nu} \frac{\exp\left\{ i\frac{]t^{(l)}N[}{N}(x_1^{(l)}+\cdots+x_k^{(l)})\right\}-1}{N\left(\exp\left\{i\frac{1}{N}(x_1^{(l)}+\cdots+x_k^{(l)})\right\}-1\right)}\right|^2 G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k)$$

for all $A\in\mathcal{B}^{k\nu}$. Then the inequality

$$\mu_{N,t}(\mathbb{R}^{k\nu}\setminus[-T,T]^{k\nu})<\varepsilon$$

holds for all $N=1,2,\dots$, if $T\ge T_0(\varepsilon,t)$ with an appropriate threshold index $T_0(\varepsilon,t)>0$.

I claim that the measures $\mu_{N,t}$ converge weakly to a measure $\mu_{0,t}$ on $\mathbb{R}^{k\nu}$ as $N\to\infty$. This convergence implies the above inequality. This convergence can be proved similarly to Proposition 2.1. Namely we can write

$$\prod_{l=1}^{\nu} \frac{\exp\left\{ i\frac{]t^{(l)}N[}{N}(x_1^{(l)}+\cdots+x_k^{(l)})\right\}-1}{N\left(\exp\left\{i\frac{1}{N}(x_1^{(l)}+\cdots+x_k^{(l)})\right\}-1\right)} = \frac{1}{N^\nu}\sum_{u\in B_N(t)}\exp\left\{i\frac{1}{N}(u,x_1+\cdots+x_k)\right\},$$

where $B_N(t)$ was defined in (10), and then we can calculate the Fourier transform of the measure $\mu_{N,t}$ in all points of the form $u=(u_1,\dots,u_k)$ with some $u_s=\frac{p_s}{N}$, $p_s\in\mathbb{Z}^\nu$, $1\le s\le k$, similarly to the corresponding calculation in Proposition 2.1. Then we can give a good asymptotic on this Fourier transform with the help of relation (2), and this yields the proof of the above mentioned convergence. Here again we apply a natural adaptation of the proof in Proposition 2.1. I omit the details of these calculations.

Thus we have proved that in our model condition (b) of Proposition 2A holds. The proof of the remaining conditions is much simpler. Similarly to the proof of Proposition 1.1 we can refer to Proposition 1.1 when we want to check that the spectral measures $G^N_{j,j'}$ satisfy the demanded convergence property. Finally, it is not difficult to check that the functions $h^N_{j_1,\dots,j_k}$ defined in (35) satisfy condition A of Proposition 2A. Theorem 1.3 is proved.

Let me finally remark that some simple and natural modification in the proof of Theorem 1.3 shows that this results also holds if the random variables $S_0(t)$ in it are defined for all $t\in[0,\infty)^\nu$, (in the way as it is explained at the end of Section 1) and not only for $t\in[0,1]^\nu$.

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