Non-central limit theorem for non-linear functionals of vector valued Gaussian stationary random fields

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Abstract

Here I prove non-central limit theorems for non-linear functionals of vector valued stationary random fields under appropriate conditions. They are the multivariate versions of the results in paper [6]. Previously A. M. Arcones formulated such a result in Theorem 6 of his paper [1]. But there are serious problems with his result. Even its formulation must be corrected. I explain the problems related to Arcones' paper in the main text. In this paper I present the right formulation of the multivariate version of the non-central limit theorem in paper [6] together with its correct proof. To do this first the theory of the Gaussian stationary random fields described in the work [9] had to be generalized to the case of vector valued random fields. This was done in my work published in two subsequent papers [10] and [11]. Here I prove the multivariate version of the result about non-central limit theorems in paper [6] with their help.

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1. On the motivation for this research.

In this paper the following problem is considered.

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Let us have a d-dimensional vector valued Gaussian stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, where $\mathbb{Z}^\nu$ denotes the lattice points with integer coordinates in the ν-dimensional Euclidean space $\mathbb{R}^\nu$ and a function $H(x_1, \dots, x_d)$ of d variables with arguments $x_s \in \mathbb{R}^\nu$, $1 \le s \le d$. We define with their help the random variables $Y(p) = H(X_1(p), \dots, X_d(p))$ for all $p \in \mathbb{Z}^\nu$. Let us introduce for all $N = 1, 2, \dots$ the normalized sum

$$S_N = A_N^{-1} \sum_{p \in B_N} Y(p)$$

with an appropriate norming constant $A_N > 0$, where

$$B_N = \{p = (p_1, \dots, p_\nu) \in \mathbb{Z}^\nu \colon\ 0 < p_k \le N \text{ for all } 1 \le k \le \nu\}. \tag{1.1}$$

In this paper a non-Gaussian limit theorem is proved for these normalized sums S_N with an appropriate norming constant A_N if this vector valued Gaussian stationary random field $X(p)$, $p \in \mathbb{Z}^\nu$, and the function $H(x_1, \dots, x_d)$ satisfy certain conditions. Paper [6] contains such limit theorems for non-linear functionals of scalar valued stationary Gaussian random fields, and here their natural multivariate generalizations are presented.

A. M. Arcones formulated such a result in Theorem 6 of paper [1], but I found his discussion unsatisfactory. Here I explain the main problems related to his proof.

In the proof of the limit theorem in Theorem 6 of [1] the spectral representation of the covariance function of a vector valued stationary random process is needed. This representation is presented in formula (3.2) of [1]. But the properties of the measures (actually complex measures) of $G^{(p,q)}$ are not discussed. The same can be told about the random spectral measures $Z_{G^{(p,p)}}$ in the next formula (3.3). These objects were defined in the scalar valued case, and their basic properties were also proved. But the generalization of these definitions to the vector valued case and the proof of their properties are far from trivial.

The same can be said about the statements of paper [1] in formulas (3.5), (3.6) and (3.7). Here first the statements must be corrected. In formula (3.5) the arguments A_i, $1 \le i \le d$, must be replaced by $\frac{A_i}{n}$. Then it must be explained what kind of limit is taken in this formula. Finally, it must also be explained what kind of limit $(Z_{G_0^{(1,1)}}, \dots, Z_{G_0^{(d,d)}})$ random spectral measures appear here as the limit. It is not a random spectral measure in the classical sense, it can be interpreted only as the random spectral measure of a

generalized random field. The necessary definitions and proofs are missing again.

Formulas (3.6) and (3.7) in [1] contain a limit theorem which is actually a special case of Theorem 6. The main step in Arcones' proof consists in the reduction of the result in Theorem 6 to this special case. But the proof of the result formulated in (3.6) and (3.7) is missing. This is a limit theorem for a sequence of random vectors. The convergence of the single coordinates of these vectors follows from the already proved result in [6] which deals with the one-dimensional version of this problem. (More precisely, this one-dimensional convergence would follow from this already proved result if formula (3.7) were written in the correct form. The random integrals defined in it should be taken on $\mathbb{R}^\tau$ instead of $[-\pi, \pi]^\tau$.) But I do not see how the results proven in the one-dimensional case could help in the proof the convergence of the random vectors, i.e. how the result formulated in (3.6) and (3.7) could be proved by the methods of [1]. I have the impression that the proof of these formulas is not simpler than a direct proof of Theorem 6 in [1].

An appropriate proof of the non-central limit theorem should start with a good and complete description of the spectral representation of the covariance function of vector valued stationary processes. This is done e.g. in paper [5] of Cramer or in paper [13] of Rozanov. This result is missing from Arcones' paper.

In the present paper I recall the multivariate version of this result where a stationary random field is considered with elements indexed by the lattice points $p \in \mathbb{Z}^\nu$. I do this in an overview about the results in [10] and [11]. In this overview I also speak about the random spectral measure of a vector valued Gaussian stationary random field which yields a spectral representation of the (vector valued) elements of these random fields. This is a natural vector valued counterpart of the result about the spectral representation of scalar valued Gaussian stationary random fields.

There are also some other notions and results related to vector valued Gaussian stationary random fields whose discussion is needed in the proof of the non-central limit theorem for non-linear functionals of such random fields. Such notions are the generalized vector valued Gaussian random fields, their spectral and random spectral measures, and the multiple Wiener–Itô integral with respect to the coordinates of a vector valued random spectral measure. They are introduced, and their most important properties are proved in papers [10] and [11]. The goal of this paper is to give a correct proof of the multivariate generalization of the results in [6] with their help.

In short, in my opinion the proof about the multivariate generalization of the result in [6] must be started from the very beginning. First the basic results about the behavior of vector valued stationary random fields must be worked out. This is missing from Arcones' paper. Moreover, it seems to me that a direct proof of Theorem 6 in [1] would be not more difficult than the proof of its reduced version presented in formulas (3.6) and (3.7) of [1].

Let me remark that although Arcones' proof of the non-central limit theorem for non-linear functionals of vector valued Gaussian random fields was problematic, the proof of its counterpart about the central limit theorem for such linear functionals under appropriate conditions was correct. Moreover, in the study of this result he proved such an estimate in Lemma 1 of his paper which was applied also in this work.

This paper consists of five sections and two appendices. In Section 2 the basic notions and results of papers [10] and [11] are recalled. Section 3 contains the main results of this paper. In Section 4 the preparatory lemmas needed in the proof of the basic theorems are presented. Section 5 contains the proof of these theorems. In Appendix A the background of the limit theorems of this paper is discussed. In Appendix B I prove that not only the finite dimensional distributions of the stochastic processes considered in Theorem 3.4 converge, but these processes also weakly converge to their limit.

Remark: It was professor Herold Dehling who asked me to clarify the proof of Theorem 6 in Arcones' paper [1]. The goal of this work together with the preliminary papers [10] and [11] was to answer Dehling's question. It turned out that to settle this problem first the theory of vector valued stationary Gaussian random fields has to be worked out. This theory is similar to the theory of scalar valued Gaussian random fields, but there are also some essential differences between them. Hence the theory of vector valued stationary Gaussian random fields cannot be considered as a simple generalization of the theory in the scalar valued case. I am grateful to professor Dehling for calling my attention to this problem.

2. On some properties of vector valued Gaussian stationary random fields.

In this section I present the most important results of [10] and [11] needed in this paper. At this point I do not give their detailed formulation. I shall present them in a more detailed form when they appear in our investigation.

We are working with a d-dimensional vector valued Gaussian stationary random fields $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, where $\mathbb{Z}^\nu$ denotes the lattice points with integer coordinates in the ν-dimensional Euclidean space $\mathbb{R}^\nu$ with expectation $EX_j(0) = 0$ for all $1 \le j \le d$. The distribution of such random fields is determined by their covariance function $r_{j,j'}(p) = EX_j(0)X_{j'}(p) = EX_j(m)X_{j'}(m+p)$, $1 \le j, j' \le d$, $m, p \in \mathbb{Z}^\nu$.

In a result of [10] it was shown that this covariance function $r_{j,j'}(p)$, $1 \le j, j' \le d$, can be presented in the following way. For all $1 \le j, j' \le d$ there is a complex measure $G_{j,j'}$ on the torus $[-\pi, \pi)^\nu$ with finite total variation such that $r_{j,j'}(p) = \int e^{i(p,x)} G_{j,j'}(\, dx)$ for all $p \in \mathbb{Z}^\nu$, and $G = (G_{j,j'})$, $1 \le j, j' \le d$, is an even, positive semidefinite matrix valued measure on the torus $[-\pi, \pi)^\nu$. G is called the spectral measure of the random field $X(p)$, $p \in \mathbb{Z}^\nu$. (A d-dimensional matrix valued measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, is called even if $G_{j,j'}(-A) = \overline{G_{j,j'}(A)}$ for all $1 \le j, j' \le d$ and measurable sets A.) For a more detailed discussion see Section 2 in [10].

In Section 3 of [10] I also defined a d-dimensional vector valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ corresponding to a d-dimensional matrix valued spectral measure G together with a random integral with respect to it in such a way that the random integrals $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\, dx)$, $p \in \mathbb{Z}^\nu$, $1 \le j \le d$, define a d-dimensional Gaussian stationary random field with matrix valued spectral measure G. Besides, I gave the basic properties of a random spectral measure Z_G corresponding to a spectral measure G. These properties determine the distribution of the random spectral measure as a function of the spectral measure to which it corresponds.

Once, these results are proved it is not difficult to generalize them to the case of vector valued Gaussian stationary random fields defined on the lattice $\frac{1}{K}\mathbb{Z}^\nu$ with some $K > 0$. We define the covariance function $r_{j,j'}(p) = EX_j((0)X_{j'}(p) = EX_j(m)X_{j'}(p+m)$, $1 \le j, j' \le d$, $p, m \in \frac{1}{K}\mathbb{Z}^\nu$ also in this case. There exists a spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, defined on the torus $[-K\pi, K\pi)^\nu$ which is a d-dimensional matrix valued even measure, and satisfies the identity $r_{j,j'}(p) = \int e^{i(p,x)} G_{j,j'}(\, dx)$ for all $p \in \frac{1}{K}\mathbb{Z}^\nu$ and $1 \le j, j' \le d$. There is also a vector valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G.d})$ corresponding to this spectral measure G such that $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\, dx)$, $1 \le j \le d$, $p \in \frac{1}{K}\mathbb{Z}^\nu$, is a vector valued Gausssian stationary random field on $\frac{1}{K}\mathbb{Z}^\nu$ with expectation zero and spectral measure G.

It is useful also to consider vector valued Gaussian stationary random fields defined in the space $\mathbb{R}^\nu$. It turned out that it is even more useful to

work with vector valued generalized Gaussian stationary random fields which can be considered as their generalization. They are random fields

$$X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi)) \text{ with parameter set } \varphi \in \mathcal{S}$$

(instead of $\mathbb{Z}^\nu$ or $\mathbb{R}^\nu$), where $\mathcal{S}$ denotes the class of real valued functions in the ν-dimensional Schwartz space. The definitions applied in the theory of generalized random fields were explained in Section 4 of [10] together with the notions needed to understand them.

In paper [10] generalized vector valued Gaussian stationary random fields were also constructed, and their properties were explained. Results similar to those of Sections 2 and 3 in [9] about ordinary vector valued Gaussian stationary random fields were proved for them. Generalized vector valued stationary Gaussian random fields were constructed with the help of their spectral measure which were also defined.

The spectral measure of a generalized vector valued stationary random field $X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}$, has properties similar to that of an ordinary vector valued stationary random field, but there are some important differences between them. It is a $d \times d$ even, positive definite matrix valued function $G(A) = (G_{j.j'}(A))$, $1 \le j, j' \le d$, $A \subset \mathbb{R}^\nu$, defined on the bounded, measurable subsets of the ν-dimensional Euclidean space $\mathbb{R}^\nu$ whose restriction to the measurable subsets of any finite cube $[-K, K]^\nu$ is a matrix valued measure with coordinates that are complex measures with finite total variation. On the other hand, $\sup_A |G_{j,j}(A)|$, where supremum is taken for all bounded, measurable sets A need not be finite. Only the weaker condition

$$\int (1 + |x|)^{-r} G_{j,j}(\, dx) < \infty \text{ for all } 1 \le j \le d \text{ with some number } r > 0, \tag{2.1}$$

is imposed. This property is called moderate increase at infinity.

The definition and construction of spectral measures of vector valued generalized Gaussian stationary random fields was done in the following way.

Let us consider an even, positive definite matrix valued function $G(A) = (G_{j.j'}(A))$, $1 \le j, j' \le d$, defined on the bounded and measurable sets $A \subset \mathbb{R}^\nu$ with moderate increase at infinity, and such that the restriction of its coordinates to a finite cube $[-K, K]^\nu$ is a complex measure with finite total variation. If there exists a generalized vector valued Gaussian stationary random field $X(\varphi) = (X_1(\varphi), \dots, X_d(\varphi))$, $\varphi \in \mathcal{S}$, with the additional property

$EX_j(\varphi) = 0$, $1 \le j \le d$, for all $\varphi \in \mathcal{S}$ such that the identity

$$EX_j(\varphi)X_{j'}(\psi) = \int \tilde{\varphi}(x)\overline{\tilde{\psi}}(x)G_{j.j'}(\,dx), \quad 1 \le j, j' \le d, \quad \text{for all } \varphi, \psi \in \mathcal{S}$$

holds, where $\tilde{}$ denotes Fourier transform, and overline means complex conjugate, then this set of matrix valued functions $G(A) = (G_{j.j'}(A))$, $1 \le j, j' \le d$, is called the spectral measure of this generalized random field $X(\varphi)$, $\varphi \in \mathcal{S}$.

For any set of matrix valued functions $G(A) = (G_{j,j'}(A))$, $1 \le j, j' \le d$, with the above properties there exists a generalized vector valued Gaussian stationary random field $X(\varphi)$, $\varphi \in \mathcal{S}$, with expectation zero whose covariance function $EX_j(\varphi)X_{j'}(\psi)$, $\varphi, \psi \in \mathcal{S}$, satisfies the above conditions. This means that a matrix valued function with the above properties is the spectral measure of a generalized, vector valued Gaussian stationary random field. Moreover, the distribution of this random field is determined by its spectral measure.

Given the spectral measure $G = (G_{j,j'}(\cdot))$ of a generalized random field, such a vector valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ can be constructed for which $X_j(\varphi) = \int \tilde{\varphi}(x)Z_{G,j}(\,dx)$, $1 \le j \le d$, $\varphi \in \mathcal{S}$, is a generalized Gaussian stationary random field with spectral measure G and $EX_j(\varphi) = 0$ for all $1 \le j \le d$ and $\varphi \in \mathcal{S}$. We say that such a random spectral measure is adapted to the generalized spectral measure G. The basic properties of the random spectral measures adapted to a generalized spectral measure also were proved. Their distribution is determined by the spectral measure to which they are adapted.

The introduction of the random spectral measures corresponding to the spectral measures of generalized Gaussian stationary random fields turned out to be useful for us. This class of random spectral measures is much larger than the class of random spectral measures corresponding to the spectral measure of a classical vector valued Gaussian random field. The limit in the limit theorems of this paper could be expressed by means of a sum of multiple Wiener–Itô integrals with respect to such a random spectral measure.

In the subsequent part of the works [10] and [11] my goal was to give a good representation of those random variables with finite second moment which are measurable with respect to the σ-algebra generated by the random variables of the underlying vector valued random field and to present a useful formula for their shift transforms. Such results turned out to be very useful in the study of the limit theorems I was interested in. A good representation

can be given with the help of multiple Wiener–Itô integrals with respect to vector valued random spectral measures introduced in Section 5 of [10].

To define multiple Wiener-Itô integrals I considered the matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, of a d-dimensional Gaussian stationary random field, (ordinary or generalized one), and took a random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ corresponding to it. In Section 5 of [10] I defined for all $n \ge 1$ and sequences of integers $j_1, \dots, j_n$ with the property $1 \le j_s \le d$ for all $1 \le s \le n$ a set $K_{n,j_1,\dots,j_n} = K_{n,j_1,\dots,j_n}(G_{j_1,j_1}, \dots, G_{j_n,j_n})$ of complex number valued functions with arguments in $\mathbb{R}^{n\nu}$. (In the terminology of this paper $K_{n,j_1,\dots,j_n}$ is a subset of the class of complex valued functions $f(x_1, \dots, x_n)$ of n variables with arguments $x_s \in \mathbb{R}^\nu$, $1 \le s \le n$). I defined the n-fold Wiener–Itô integral

$$I_n(f|j_1, \dots, j_n) = \int f(x_1, \dots, x_n) Z_{G,j_1}(\,dx_1) \dots Z_{G,j_n}(\,dx_n)$$

for the functions $f \in K_{n,j_1,\dots,j_n}$. (The definition of the set of functions $K_{n,j_1,\dots,j_n}$ is recalled in Section 4 of this paper before the formulation of Proposition 4A.) Then I proved the most important properties of these random integrals.

In Section 6 of paper [10] I proved a technical result, called the diagram formula about the expression of the product of two multiple Wiener–Itô integrals as a sum of multiple Wiener–Itô integrals.

These results were exploited in paper [11]. Here I recalled the notion of Wick polynomials which turned out to be a useful tool in our investigations. Wick polynomials are natural multivariate generalizations of Hermite polynomials. Their definition together with their most important properties was recalled from [9] in Section 2 of [11]. Section 2 of [11] also contains an important formula about the expression of Wick polynomials by means of multiple Wiener–Itô integrals and another important formula about the calculation of the shift transforms of a random variable presented in the form of a sum of multiple Wiener–Itô integrals. This made possible to reformulate our limit problems to limit problems about sums of multiple Wiener–Itô integrals. A result in Section 3 of [11] was proved in order to investigate such problems. It plays an important role in the investigation of this paper, hence I recalled it in Proposition 4A of this paper.

3. Formulation of the main results.

In this section I present the main results of this paper. I shall compare both the formulation of the conditions and the proof of the results with those appearing in the study of the analogous results in the scalar valued case. But I shall refer to [9] instead of [6] in this comparison, because in that work the proofs are worked out in more detail.

I shall work with such random fields for which $EX_j(p) = 0$ for all $1 \le j \le d$ and $p \in \mathbb{Z}^\nu$. Besides this property I shall impose two kinds of conditions in this paper. The first of them deals with the covariance function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, of the vector valued Gaussian stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, considered in this paper, the second one with the function $H(x_1, \dots, x_d)$ which appears in the definition of the random sums whose limit behavior is investigated.

The following condition is imposed about the covariance function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$.

$$\lim_{T \to \infty} \sup_{p\colon\ p \in \mathbb{Z}^\nu,\, |p| \ge T} \frac{\left| r_{j,j'}(p) - a_{j,j'}(\frac{p}{|p|})|p|^{-\alpha} L(|p|) \right|}{|p|^{-\alpha} L(|p|)} = 0 \tag{3.1}$$

for all $1 \le j, j' \le d$, where $0 < \alpha < \nu$, $L(t)$, $t \ge 1$, is slowly varying at infinity, bounded in all finite intervals, and $a_{j,j'}(t)$ is a real valued continuous function on the unit sphere $\mathcal{S}_{\nu-1} = \{x\colon\ x \in \mathbb{R}^\nu,\ |x| = 1\}$, which satisfies the identity $a_{j,j'}(x) = a_{j',j}(-x)$ for all $x \in \mathcal{S}_{\nu-1}$ and $1 \le j, j' \le d$.

I construct a vector valued Gaussian stationary random field which satisfies relation (3.1). This example indicates that the covariance functions which satisfy (3.1) have some additional properties, too. These properties will be discussed in Appendix A, because they may help in a better understanding of the picture about the limit theorems of this paper.

Example for a stationary random field with a covariance function that satisfies relation (3.1). I shall construct a stationary random field whose covariance function satisfies (3.1). I will do this by defining the spectral measure of such a random field. To do this I recall some results about the Fourier transform of generalized functions from the literature.

In the ν-dimensional space $\mathbb{R}^\nu$ the Fourier transform of the homogeneous function $|x|^\lambda$ (as the Fourier transform of a generalized function) equals $C|p|^{-\lambda-\nu}$ with some coefficient $C = C(\lambda, \nu) > 0$. (See the list of Fourier

transforms at the end of the book [7].) The value of this coefficient $C(\lambda, \nu)$ is known, but it has no importance for us.

On the other hand, if $u(x)$, $x \in \mathbb{R}^\nu$, is a sufficiently smooth function, concentrated in a compact domain, and $u(0) = 1$, then the Fourier transform of $|x|^\lambda u(x)$ equals $\int e^{i(x,p)}|x|^\lambda u(x)\, dx = C(\lambda, \nu)|p|^{-\lambda-\nu}(1 + o(1))$.

In the following construction the above property of the Fourier transform of $|x|^\lambda u(x)$ will be exploited. Define some functions $g_{j.j'}(x)$, $1 \le j, j' \le d$, $x \in [-\pi, \pi)^\nu$, in the following way. Take a non-negative, smooth function $u(x)$ concentrated in the cube $[-\pi, \pi]^\nu$ such that $u(-x) = u(x)$, and $u(0) = 1$. Put $g_{j,j}(x) = |x|^{\alpha-\nu}u(x)$, for $1 \le j \le d$, and $g_{j,j'}(x) = \varepsilon_{j,j'}|x|^{\alpha-\nu}u(x)$ for $1 \le j, j' \le d$ if $j \ne j'$ with a sufficiently small real valued coefficient $\varepsilon_{j,j'}$ such that $\varepsilon_{j,j'} = \varepsilon_{j',j}$. (One could choose a complex valued coefficient $\varepsilon_{j,j'}$ too, but this would demand a more complicated argument.) I claim that $(g_{j,j'}(x))$, $1 \le j, j' \le d$, $x \in [\pi, \pi)^\nu$, with the above defined functions $g_{j,j'}(\cdot)$ is a spectral density function, and the covariance function $r_{j,j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, of a stationary random field with this spectral density satisfies relation (3.1) with $|p|^{-\alpha}$, $L(p) = 1$, $a_{j,j}\left(\frac{p}{|p|}\right) = C(\alpha-\nu, \nu)$, and $a_{j,j'}\left(\frac{p}{|p|}\right) = \varepsilon_{j,j'}C(\alpha-\nu, \nu)$ for $j \ne j'$.

Indeed, relation (3.1) holds with such a choice, because $r_{j,j'}(p)$ is the Fourier transform of $g_{j,j'}(x)$. We still have to check that $(g_{j,j'}(x))$, $1 \le j, j' \le d$, is a spectral density matrix. The main point is to show that this matrix is positive definite. This property holds, since this matrix has the form $|x|^{\alpha-\nu}u(x)(I + D(\varepsilon))$ with a small matrix $D(\varepsilon)$, where I denotes the identity matrix.

Observe that the function $a_{j,j'}\left(\frac{p}{|p|}\right)|p|^{-\alpha}L(|p|) = a_{j,j'}\left(\frac{p}{|p|}\right)|p|^{-\alpha}$ appearing in formula (3.1) with the functions defined in the above example is the Fourier transform of $g^{(0)}_{j,j}(x) = |x|^{\alpha-\lambda}$ if the indices j and j' of the above function agree, and $g^{(0)}_{j,j'}(x) = \varepsilon_{j,j'}|x|^{\alpha-\lambda}$ if $j \ne j'$. Besides, the matrix $(g^{(0)}_{j,j'}(x))$, $1 \le j, j' \le d$, is the spectral density of a vector valued, stationary, generalized random field. This spectral density has the homogeneity property $(g^{(0)}_{j,j'}(tx)) = t^{\alpha-\nu}(g^{(0)}_{j,j'}(x))$, $1 \le j, j' \le d$, for all $t > 0$. The spectral density $(g_{j.j'}(x))$, $1 \le j, j' \le d$, is in some sense close to this spectral density $(g^{(0)}_{j,j'}(x))$. In Appendix A I show that the spectral measure of a vector valued stationary random field whose covariance matrix satisfies relation (3.1) has a similar behavior. It is close in some sense to such a spectral measure which has some homogeneity property. (This spectral measure belongs to a gener-

alized random field.) This homogeneity property has deep consequences in the theory of the limit theorems we are interested in.

Remark: There is a natural generalization of the results of the present paper. One may consider such vector valued stationary Gaussian random fields, where the partial sums of different coordinates have a limit with different normalization. They satisfy limit theorems similar to those of the present paper, but the different behavior of the different coordinates must be taken into consideration in the choice of the normalization.

Such more general models were considered in the paper [14] of Sanchez de Naranjo, who considered models whose covariance matrices satisfy a generalized version of relation (3.1). Namely, they satisfy the relation

$$r_{j,j'}(p) \sim |p|^{\alpha_{j,j'}} a_{j,j'} \left(\frac{p}{|p|} \right) L_{j,j'}(|p|)$$

with such an exponent $\alpha_{j,j'}$ and slowly varying function $L_{j,j'}(\cdot)$ which may depend on the indices j and j'. With a good choice of these quantities an interesting generalization of the results of the present paper can be obtained. Such results can be proved by means of a natural generalization of the arguments of the present paper, but since this would demand a lot of space and the introduction of many new quantities I omit their discussion here.

Next I explain the condition imposed on the function $H(x_1, \dots, x_d)$ that appears in the limit theorems of this paper. In scalar valued models first the special case $H(x) = H_k(x)$ was considered, where $H_k(x)$ denotes the k-th Hermite polynomial with leading coefficient 1. Then it was shown that our limit problem with a function $H(x)$ whose expansion by the Hermite polynomials has the form $H(x) = \sum_{l=k}^{\infty} c_l H_l(x)$ with starting index k in the summation can be simply reduced to the special case when $H(x) = c_k H_k(x)$. Similar results will be proved in the multivariate case. In this case Wick polynomials take the role of the Hermite polynomials. But Wick polynomials appear in this work only in an implicit way. In the models studied in this paper the Wick polynomials can be simply calculated. Such vector valued Gaussian stationary random fields $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, are considered whose covariance functions satisfy besides condition (3.1) also the relation

$$EX_j^2(0) = 1 \text{ for all } 1 \le j \le d, \text{ and } EX_j(0)X_{j'}(0) = 0 \text{ if } j \ne j', \ \ 1 \le j, j' \le d. \tag{3.2}$$

First I show that this new condition does not mean a real restriction of our problem.

Let $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, be a vector valued Gaussian stationary random field with expectation $EX_j(p) = 0$, $p \in \mathbb{Z}^\nu$, $1 \le j \le d$, and take the random variables $X_1(0), \dots, X_d(0)$ in it. An appropriate number $1 \le d' \le d$ can be chosen, and d' random variables $X'_j(0) = \sum_{l=1}^{d} c_{j,l} X_l(0)$, $1 \le j \le d'$, can be defined with appropriate coefficients $c_{j,l}$, $1 \le j \le d'$, $1 \le l \le d$, with the following properties. $EX'_j(0)X'_{j'}(0) = \delta_{j,j'}$, $1 \le j, j' \le d'$, where $\delta_{j,j'} = 0$ if $j \ne j'$, and $\delta_{j,j} = 1$, and the random variables $X_j(0)$, $1 \le j \le d$, can be expressed as the linear combinations of the random variables $X'_l(0)$, $1 \le l \le d'$, i.e. $X_j(0) = \sum_{l=1}^{d'} d_{j,l} X'_l(0)$ for all $1 \le j \le d$ with appropriate coefficients $d_{j,l}$.

Let us define the vector valued random field $X'(p) = (X'_1(p), \dots, X'_{d'}(p))$ as $X'_j(p) = \sum_{l=1}^{d} c_{j,l} X_l(p)$, $1 \le j \le d'$, with the same coefficients $c_{j,l}$ as in the definition of $X'_j(0)$ for all $p \in \mathbb{Z}^\nu$. Then it is not difficult to see that $X'(p)$, $p \in \mathbb{Z}^\nu$, is a d'-dimensional Gaussian stationary random field whose elements have expectation zero, and it satisfies relation (3.2) (with parameter d' instead of d.) Moreover, if the covariance function of the original random field $X(p)$ satisfied relation (3.1), then the covariance function of this new random field also satisfies this condition with appropriate new functions $a'_{j,j'}(\frac{p}{|p|})$. Besides, it is not difficult to find such a function $H'(x_1, \dots, x_{d'})$ for which $H'(X'_1(p), \dots, X'_{d'}(p)) = H(X_1(p), \dots, X_d(p))$ for all $p \in \mathbb{Z}^\nu$. This means that with the introduction of this new random field $X'(p) = (X'_1(p) \dots, X'_{d'}(p))$ our problem can be reformulated in such a way that our vector valued stationary Gaussian random field satisfies both relations (3.1) and (3.2). We shall work with such a new d'-dimensional random field $X'(p)$ and function $H'(x_1, \dots, x_{d'})$, only the sign prime will be omitted everywhere.

First we consider the case when we fix a positive integer k, and the function $H(x_1, \dots, x_d)$ has the form

$$\begin{aligned} H(x_1, \dots, x_d) &= H^{(0)}(x_1, \dots, x_d) \\ &= \sum_{\substack{(k_1,\dots,k_d),\ k_j \ge 0,\ 1 \le j \le d, \\ k_1 + \dots + k_d = k}} c_{k_1,\dots,k_d} H_{k_1}(x_1) \cdots H_{k_d}(x_d) \end{aligned} \tag{3.3}$$

with the previously fixed number k, the coefficients $c_{k_1,\dots,k_d}$ are real numbers, and $H_{k_j}(\cdot)$ denotes the Hermite polynomial of order k_j with leading coefficient 1. The function $H'(x_1, \dots, x_{d'})$ preserves this property of the function

$H(x_1, \dots, x_d)$ when the previously mentioned transformation is applied, only different coefficients $c'_{k_1,\dots,k_d}$ appear in its expansion.

Remark: Although I shall not apply the observation of this remark, it may be worth mentioning that if $(X_1, \dots, X_d)$ is a d-dimensional random vector with standard normal distribution then $H(X_1, \dots, X_d)$ with a function $H(x_1, \dots, x_d)$ having the form (3.3) is a Wick polynomial of order k of the random vector $(X_1, \dots, X_d)$. (See e.g. Corollary 2C in [11] or Corollary 2.3 in [9].) In general, one can say that Hermite polynomials play an important role in limit theorems for non-linear functionals of scalar valued Gaussian random fields. In the case of non-linear functionals of vector valued Gaussian random fields Wick polynomials take their role.

In scalar valued models, i.e. in the case $d = 1$ a non-central limit theorem was proved if $H(x) = H_k(x)$, $k \ge 2$, and the covariance function $r(n) = EX_0X_n$ satisfies condition (3.1) (with $d = 1$) with $0 < \alpha < \frac{\nu}{k}$. This result was formulated in Theorem 8.2 of [9]. In this result the limit was described by means of a k-fold Wiener–Itô integral with respect to an appropriate random spectral measure. This random spectral measure corresponds to the spectral measure that appeared in Lemma 8.1 of [9] as the limit of a sequence of appropriately normalized versions of the spectral measure of a stationary random field $X(p)$, $p \in \mathbb{Z}^\nu$, whose covariance function satisfies condition (3.1) with $d = 1$. Here I prove a multivariate version of Theorem 8.2 of [9] with the help of a multivariate version of Lemma 8.1 in [9] formulated below.

This generalization of Lemma 8.1 in [9] is a limit theorem for a sequence of appropriately rescaled versions of the coordinates $G_{j.j'}$ of a spectral measure $G = (G_{j.j'})$, $1 \le j, j' \le d$, with some nice properties. In this limit theorem the vague convergence of complex measures is considered. Before the formulation of this result I recall the definition of this convergence from Section 3 of [11]. In this definition the notion of complex measures with locally finite total variation appears. I explain its meaning in a remark after the definition.

Definition of vague convergence of complex measures on $\mathbb{R}^\nu$ with locally finite total variation. *Let $G^{(N)}$, $N = 1, 2, \dots$, be a sequence of complex measures on $\mathbb{R}^\nu$ with locally finite total variation. We say that this sequence $G^{(N)}$ vaguely converges to a complex measure $G^{(0)}$ on $\mathbb{R}^\nu$ with locally finite total variation if*

$$\lim_{N\to\infty} \int f(x)\, G^{(N)}(\, dx) = \int f(x)\, G^{(0)}(\, dx)$$

for all continuous functions f on $\mathbb{R}^\nu$ with a bounded support.

Remark: In the above definition the notion of complex measures with locally finite total variation appeared. This notion was introduced in Section 4 of [10] together with the notion of vector valued Gaussian stationary generalized random fields and their matrix valued spectral measures. A complex measure with locally finite total variation is such a complex valued function on the bounded measurable subsets of $\mathbb{R}^\nu$ whose restriction to the measurable subsets of a cube $[-T,T]^\nu$ is a complex measure with finite total variation for all $T>0$.

The above definition of vague convergence slightly differs from the classical one presented e.g. in Section 8 of [9] (before Lemma 8.1 of this paper), where the vague convergence of locally finite (non-negative) measures is considered. The locally finite measures were defined on all measurable subsets of $\mathbb{R}^\nu$. Here we deal with complex measures, because we also want to study the non-diagonal elements $G_{j,j'}$, $j\neq j'$, of a matrix valued spectral measure, and they are complex (i.e. not necessary real valued) measures. A non-negative locally finite measure always can be extended to a measure on all measurable subsets of $\mathbb{R}^\nu$, while there are locally finite complex measures which do not have this property. This fact was taken into account in the introduction of the above definition.

The next Proposition 3.1 contains the multivariate version of Lemma 8.1 in [9].

Proposition 3.1. *Let $G=(G_{j,j'})$ be the matrix valued spectral measure of a d-dimensional vector valued stationary random field whose covariance function $r_{j,j'}(p)$ satisfies relation (3.1) with some parameter $0<\alpha<\nu$ and slowly varying function $L(\cdot)$. Let us define the following rescaled versions of the coordinates $G_{j,j'}$, $1\le j,j'\le d$, of this matrix valued spectral measure:*

$$G^{(N)}_{j,j'}(A)=\frac{N^\alpha}{L(N)}G_{j,j'}\left(\frac{A}{N}\right),\quad A\in\mathcal{B}^\nu,\quad 1\le j,j'\le d, \tag{3.4}$$

for all $N=1,2,\dots,$where $\mathcal{B}^\nu$ denotes the σ-algebra of the Borel measurable sets on $\mathbb{R}^\nu$. Then $G^{(N)}_{j,j'}$ is a complex measure with finite total variation concentrated in $[-N\pi,N\pi)^\nu$.

For all pairs $1\le j,j'\le d$ the sequence of complex measures $G^{(N)}_{j,j'}$ defined in (3.4) tends vaguely to a complex measure $G^{(0)}_{j,j'}$ on $\mathbb{R}^\nu$ with locally finite total

variation. These complex measures $G^{(0)}_{j,j'}$, $1 \le j, j' \le d$, *have the homogeneity property*

$$G^{(0)}_{j,j'}(A) = t^{-\alpha} G^{(0)}_{j,j'}(tA) \text{ for all bounded sets } A \in \mathcal{B}^\nu, \quad 1 \le j, j' \le d, \text{ and } t > 0. \tag{3.5}$$

The complex measure $G^{(0)}_{j,j'}$ *is determined by the number* $0 < \alpha < \nu$ *and functions* $a_{j,j}(\cdot)$, $a_{j,j'}(\cdot)$, $a_{j',j}(\cdot)$ *and* $a_{j',j'}(\cdot)$ *defined in formula (3.1) on the unit sphere* $S_{\nu-1}$. *This implies that for all spectral measures* G *that satisfy relation (3.1) with the same parameter* α *and functions* $a_{j,j'}(\cdot)$, $1 \le j, j' \le d$ *the vague limit of the complex measures* $G^{(N)}_{j,j'}$ *is the same for all* $1 \le j, j' \le d$.

Finally, there exists a vector valued generalized Gaussian stationary random field on $\mathbb{R}^\nu$ *whose matrix valued spectral measure is* $G^{(0)} = (G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, *with the complex measures* $G^{(0)}_{j,j'}$, $1 \le j, j' \le d$, *defined in this Proposition.*

In the following Theorem 3.2 I formulate the multivariate version of Theorem 8.2 in [9]. In its formulation the result of Proposition 3.1 is applied where a matrix valued spectral measure $G^{(0)} = (G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, is constructed under some conditions which are imposed also in Theorem 3.2. Theorem 3.2 is a limit theorem where the limit is defined by means of a sum of multiple Wiener–Itô integrals with respect to a vector valued random spectral measure that corresponds to the matrix valued spectral measure $G^{(0)}$ constructed in Proposition 3.1. Let me remark that I formulated this result also in paper [10]. But in that work it was not proved. That work contained only a heuristic argument which indicated why it is natural to expect such a result. Its goal was to indicate the usefulness of the theory worked out in [10] and [11].

Theorem 3.2. *Fix an integer* $k \ge 1$, *and let* $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, *be a vector valued Gaussian stationary random field whose covariance matrix* $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, *satisfies both relation (3.1) with some number* α *such that* $0 < \alpha < \frac{\nu}{k}$ *and relation (3.2). Let* $H(x_1, \dots, x_d)$ *be a function of the form given in (3.3) also with the previously fixed number* k. *Define the random variables* $Y(p) = H(X_1(p), \dots, X_d(p))$ *for all* $p \in \mathbb{Z}^\nu$ *together with their normalized partial sums*

$$S_N = \frac{1}{N^{\nu - k\alpha/2} L(N)^{k/2}} \sum_{p \in B_N} Y(p),$$

where the set B_N was defined in (1.1). These random variables S_N, $N = 1, 2, \dots$, satisfy the following limit theorem.

Let $Z_{G^{(0)}} = (Z_{G^{(0)},1}, \dots, Z_{G^{(0)},d})$ be a vector valued random spectral measure which corresponds to the matrix valued spectral measure $G^{(0)} = (G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, defined in Proposition 3.1 with the help of the matrix valued spectral measure $G = (G_{j,j'})$ of a vector valued Gaussian stationary random field with covariance function $r_{j,j'}(s)$, $1 \le j, j' \le d$, $s \in \mathbb{R}^\nu$, satisfying relation (3.1). Then the sum of multiple Wiener–Itô integrals with the coefficients $c_{k_1,\dots,k_d}$ appearing in (3.3)

$$S_0 = \sum_{\substack{(k_1,\dots,k_d),\ k_j \ge 0,\ 1 \le j \le d,\\ k_1+\dots+k_d=k}} c_{k_1,\dots,k_d} \int \prod_{l=1}^{\nu} \frac{e^{i(x_1^{(l)}+\dots+x_k^{(l)})}-1}{i(x_1^{(l)}+\dots+x_k^{(l)})} \, Z_{G^{(0)},j(1|k_1,\dots,k_d)}(\,dx_1) \dots Z_{G^{(0)},j(k|k_1,\dots,k_d)}(\,dx_k) \tag{3.6}$$

exists, where the notation $x_p = (x_p^{(1)}, \dots, x_p^{(\nu)}) \in \mathbb{R}^\nu$, $p = 1, \dots, k$, is applied, and the indices $j(s|k_1, \dots, k_d)$, $1 \le s \le k$, are defined as $j(s|k_1, \dots, k_d) = r$ if $\sum_{u=1}^{r-1} k_u < s \le \sum_{u=1}^{r} k_u$, $1 \le s \le k$, $1 \le r \le d$. (For $r-1 = 0$ the convention $\sum_{u=1}^{0} k_u = 0$ is applied in this definition.) The normalized sums S_N converge in distribution to the random variable S_0 defined in (3.6) as $N \to \infty$.

The indexation of the terms $Z_{G^{(0)},j(s|k_1,\dots,k_d)}(\,dx_s)$ in formula (3.6) can be described in a simpler form. In the first k_1 arguments $x_1, \dots, x_{k_1}$, i.e. for $1 \le s \le k_1$ $Z_{G^{(0)},1}(\,dx_s)$, is written, in the next k_2 arguments, i.e. for $k_1 + 1 \le s \le k_1 + k_2$ $Z_{G^{(0)},2}(\,dx_s)$ is written, and so on. In the last k_d arguments, i.e, when $k_1 + \dots + k_{d-1} + 1 \le s \le k$, $(k = k_1 + \dots + k_d)$, $Z_{G^{(0)},d}(\,dx_s)$ is written.

In Theorem 3.2 the limit of $A_N^{-1} \sum_{p \in B_N} H(X_1(p), \dots, X_d(p))$ is described if the expansion of the function $H(x_1, \dots, x_d)$ is a linear combination of products of Hermite polynomials with different arguments, and all these products are polynomials of order k. The next Theorem 3.3 which is the multivariate version of Theorem 8.2′ in [9] states that a similar result holds if the function $H(x_1, \dots, x_d)$ is the linear combination of products of Hermite polynomials, but some of these products may be polynomials of order higher than k.

Theorem 3.3. *Let us consider a vector valued Gaussian stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, that satisfies the conditions of Theorem 3.2 and a function of the form $H(x_1, \dots, x_d) = H^{(0)}(x_1, \dots, x_d) +$*

$H^{(1)}(x_1, \dots, x_d)$, *where* $H^{(0)}(x_1, \dots, x_d)$ *was defined in (3.3), and*

$$H^{(1)}(x_1, \dots, x_d) = \sum_{\substack{(k_1,\dots,k_d),\ k_j \ge 0,\ 1 \le j \le d, \\ k_1 + \dots + k_d \ge k+1}} c_{k_1,\dots,k_d} H_{k_1}(x_1) \cdots H_{k_d}(x_d) \tag{3.7}$$

with real valued coefficients $c_{k_1,\dots,k_d}$ *such that*

$$\sum_{\substack{(k_1,\dots,k_d),\ k_j \ge 0,\ 1 \le j \le d, \\ k_1 + \dots + k_d \ge k+1}} \frac{c^2_{k_1,\dots,k_d}}{k_1! \cdots k_d!} < \infty. \tag{3.8}$$

Define the random variables $Y(p) = H(X_1(p), \dots, X_d(p))$ *for all* $p \in \mathbb{Z}^\nu$ *and their normalized partial sums*

$$S_N = \frac{1}{N^{\nu - k\alpha/2} L(N)^{k/2}} \sum_{p \in B_N} Y(p), \quad N = 1, 2, \dots,$$

with this function $H(x_1, \dots, x_d)$. *The random variables* S_N *converge in distribution to the random variable* S_0 *defined in formula (3.6) as* $N \to \infty$.

Actually condition (3.8) in Theorem 3.3 means that

$$E\left[H^{(1)}(X_1(0), \dots, X_d(0))^2\right] < \infty.$$

Finally I mention that Arcones formulated a more general result. To present it, more precisely to present its generalization to the case when we are working with stationary random fields parametrized by the lattice points of $\mathbb{Z}^\nu$ with some $\nu \ge 1$ let us define the following parameter sets for all $N = 1, 2, \dots$ and $t = (t_1, \dots, t_\nu)$, $0 \le t_l \le 1$, for all $1 \le l \le \nu$.

$$\begin{aligned} B_N(t) &= B_N(t_1, \dots, t_\nu) \\ &= \{p = (p_1, \dots, p_\nu) \in \mathbb{Z}^\nu \colon\ 0 < p_l \le N t_l \text{ for all } 1 \le l \le \nu\}. \end{aligned} \tag{3.9}$$

With this notation the following result holds.

Theorem 3.4. *Let us consider the same vector valued Gaussian stationary random field* $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, *and function* $H(x_1, \dots, x_d)$ *as in Theorem 3.3. Define the random variables* $Y(p) = H(X_1(p), \dots, X_d(p))$ *for all* $p \in \mathbb{Z}^\nu$ *together with the random fields*

$$S_N(t) = \frac{1}{N^{\nu - k\alpha/2} L(N)^{k/2}} \sum_{p \in B_N(t)} Y(p) \tag{3.10}$$

with parameter set $t = (t_1, \dots, t_\nu)$, $0 \le t_l \le 1$, $1 \le l \le \nu$, *for all* $N = 1, 2, \dots$, *where the set* $B_N(t)$ *was defined in (3.9). The finite dimensional distributions of the random fields* $S_N(t)$ *converge to that of the random field* $S_0(t)$, $t = (t_1, \dots, t_\nu)$, $0 \le t_l \le 1$, $1 \le l \le \nu$, *defined by the formula*

$$S_0(t) = \sum_{\substack{(k_1,\dots,k_d),\ k_j \ge 0,\ 1 \le j \le d, \\ k_1 + \dots + k_d = k}} c_{k_1,\dots,k_d} \int \prod_{l=1}^{\nu} \frac{e^{it_l(x_1^{(l)} + \dots + x_k^{(l)})} - 1}{i(x_1^{(l)} + \dots + x_k^{(l)})} \tag{3.11}$$
$$Z_{G^{(0)},j(1|k_1,\dots,k_d)}(\,dx_1) \dots Z_{G^{(0)},j(k|k_1,\dots,k_d)}(\,dx_k)$$

if the limit $N \to \infty$ *is taken. Similarly to Theorem 3.2 the notation* $x_p = (x_p^{(1)}, \dots, x_p^{(\nu)})$, $p = 1, \dots, k$, *is applied, and the indices* $j(s|k_1, \dots, k_d)$, $1 \le s \le k$, *are defined in the same way as in formula (3.6).*

A referee proposed to show that also a strengthened form of Theorem 3.4 formulated in the next Corollary holds. I shall present the proof of this result in Appendix B. I shall omit some technical details of the proof, and in the case $\nu > 1$ I shall apply a result whose formulation I did not find in the literature. I chose such an approach, because a detailed proof would demand the elaboration of many complicated technical details which are not related to the subject of this paper.

Corollary of Theorem 3.4. *Under the conditions of Theorem 3.4 not only the finite dimensional distributions of the random fields* $S_N(t)$, $t = (t_1, \dots, t_\nu)$, $0 \le t_l \le 1$, $1 \le l \le \nu$, *introduced in (3.10) converge to those of the random field* $S_0(t)$ *defined in (3.11), but even the distribution of the random fields* $S_N(\cdot)$, *converge weakly to the distribution of* $S_0(\cdot)$ *in the Skorochod space on* $[0, 1]^\nu$ *as* $N \to \infty$. *Moreover, the trajectories of* $S_0(\cdot)$ *are continuous functions on* $[0, 1]^\nu$.

Let us observe that the kernel functions in the Wiener–Itô integrals appearing in the sum which defines $S_0(t)$ in (3.11) equal $\varphi_t(x_1 + \dots + x_k)$, where $\varphi_t(u)$, $u \in \mathbb{R}^\nu$, is the Fourier transform of the Lebesgue measure on the rectangle $[0, t_1] \times \dots \times [0, t_\nu]$. The integral in (3.11) is taken on the whole space.

Theorem 3.4 was formulated in that form as Arcones did, but it could have been formulated in a slightly more general form. The sets $B_N(t)$ in (3.9), the random variables $S_N(t)$, $N = 1, 2, \dots$, in (3.10) and $S_0(t)$ in (3.11) could have been defined for all $t = (t_1, \dots, t_\nu) \in [0, \infty)^\nu$ and not only for

$t = (t_1, \dots, t_\nu) \in [0,1]^\nu$. After the introduction of these objects it could have been proved, similarly to the proof of Theorem 3.4, that the finite dimensional distributions of the random fields $S_N(t)$ converge to the finite dimensional distributions of the random field $S_0(t)$ as $N \to \infty$ also in this more general case. This more general form of the result is useful, because it makes possible to formulate an important property of the limit field $S_0(t)$, called the self-similarity property. The limit random field $S_0(t)$, $t \in [0,\infty)^\nu$, is self-similar with parameter $\nu - k\alpha/2$, which means that $S_0(ut) \stackrel{\Delta}{=} u^{\nu - k\alpha/2} S_0(t)$ for all $u > 0$, where $\stackrel{\Delta}{=}$ means that the finite dimensional distributions of the two random fields agree.

The self-similarity property of the random field $S_0(t)$, $t \in [0,\infty)^\nu$, can be proved by exploiting that by formula (3.5) in Proposition 3.1 $G^{(0)}(uA) = u^\alpha G^{(0)}(A)$ for the spectral measure $G^{(0)}$ for all $u > 0$ and measurable sets $A \subset \mathbb{R}^\nu$. This implies that

$$(Z_{G^{(0)},1}(uA_1), \dots, Z_{G^{(0)},d}(uA_d)) \stackrel{\Delta}{=} (u^{\alpha/2} Z_{G^{(0)},1}(A_1), \dots, u^{\alpha/2} Z_{G^{(0)},d}(A_d))$$

for all $u > 0$ and measurable sets $A_1 \in \mathbb{R}^\nu, \dots, A_d \in \mathbb{R}^\nu$. We still have to exploit that the kernel functions

$$f_t(x_1, \dots, x_k) = \prod_{l=1}^{\nu} \frac{e^{it_l(x_1^{(l)} + \dots + x_k^{(l)})} - 1}{i(x_1^{(l)} + \dots + x_k^{(l)})}$$

in the Wiener–Itô integrals in (1.13) (with the notation $t = (t_1, \dots, t_\nu)$) have the property

$$f_{ut}(x_1, \dots, x_k) = u^\nu f_t(ux_1, \dots, ux_k)$$

for all $u > 0$, $t \in [0,\infty)^\nu$, $x_j \in \mathbb{R}^\nu$, $1 \le j \le d$. The self-similarity property of the random field $S_0(t)$, $t \in [0,\infty)^\nu$, can be proved with the help of the above observations.

4. Preparatory results for the proof of the main theorems.

This section contains the proof of Proposition 3.1 and the elaboration of a method that helps in proving the theorems of this paper. In the application of this method the normalized random sums S_N appearing in the formulation of Theorem 3.2 are rewritten in the form of a sum of multiple Wiener-Itô integrals with respect to a vector valued random spectral measure. Then

Proposition 3.1 of paper [11] is recalled, and it is shown how the sums of Wiener–Itô integrals expressing the random sums S_N can be investigated with its help.

First I prove Proposition 3.1.

Proof of Proposition 3.1. Proposition 3.1 is proved by means of an adaptation of the proof of Lemma 8.1 in [9]. The same argument works, only some steps of the proof must be modified in a natural way. I do not work out all details, I only briefly remark what kind of modifications are needed.

The diagonal elements $G_{j,j}$, $1 \le j \le d$, of the matrix valued spectral measure G are spectral measures. Hence Lemma 8.1 of [9] implies that for any $1 \le j \le d$ the measures $G_{j,j}^{(N)}$ converge vaguely to a locally finite measure $G_{j,j}^{(0)}$ determined by the function $a_{j,j}(\cdot)$ and the number α which appears in relation (3.5).

For the non-diagonal elements $G_{j,j'}$, $j \neq j'$, this argument cannot be applied, because $G_{j,j'}$ is a complex measure with finite total variation which may be not a (positive) measure. In this case it can be exploited that G is a positive semidefinite matrix valued measure. Hence the 2×2 matrix

$$G(A|j,j') = \begin{pmatrix} G_{j,j}(A), & G_{j,j'}(A) \\ G_{j',j}(A), & G_{j',j'}(A) \end{pmatrix}$$

is positive semidefinite for all pairs $1 \le j, j' \le d$, $j \neq j'$, and measurable sets $A \subset \mathbb{R}^\nu$. This implies that the quadratic forms

$$(1,1)G(A|j,j')(1,1)^* = G_{j,j}(A) + G_{j',j'}(A) + G_{j,j'}(A) + G_{j',j}(A)$$

and

$$(1,i)G(A|j,j')(1,-i)^* = G_{j,j}(A) + G_{j',j'}(A) - i[G_{j,j'}(A) - G_{j',j}(A)]$$

are non-negative numbers for all measurable sets $A \subset \mathbb{R}^\nu$. Therefore the set-functions $R_{j,j'}(\cdot)$ and $S_{j,j'}(\cdot)$ defined as $R_{j,j'}(A) = G_{j,j}(A) + G_{j',j'}(A) + G_{j,j'}(A) + G_{j',j}(A)$ and $S_{j,j'}(A) = G_{j,j}(A) + G_{j',j'}(A) - i[G_{j,j'}(A) - G_{j',j}(A)]$ for all measurable sets $A \in [-\pi,\pi)^\nu$ are finite measures. Their Fourier transforms equal $r_{j,j'}^{(1)}(p) = \int e^{i(p,x)} R_{j,j'}(\,dx) = r_{j,j}(p) + r_{j',j'}(p) + r_{j.j'}(p) + r_{j',j}(p)$ and $r_{j,j'}^{(2)}(p) = \int e^{i(p,x)} S_{j,j'}(\,dx) = r_{j,j}(p) + r_{j',j'}(p) + i[r_{j.j'}(p) - r_{j',j}(p)]$, $p \in \mathbb{Z}^\nu$. These Fourier transforms satisfy the following relation, similar to formula (3.1).

$$\lim_{T\to\infty} \sup_{p:\ p\in\mathbb{Z}^\nu,\,|p|\ge T} \frac{\left| r_{j,j'}^{(s)}(p) - a_{j,j'}^{(s)}(\frac{p}{|p|})|p|^{-\alpha} L(|p|) \right|}{|p|^{-\alpha} L(|p|)} = 0 \qquad (4.1)$$

both for $s=1$ and $s=2$ with some functions $a^{(s)}_{j,j'}(\cdot)$ which can be expressed by means of the functions $a_{j,j}(\cdot)$, $a_{j.j'}(\cdot)$, $a_{j',j}(\cdot)$ and $a_{j',j'}(\cdot)$. The only difference from formula (3.1) is that the continuous function, $a^{(2)}_{j,j'}(\cdot)$ may be complex valued. (I also remark that the symmetry property $a_{j,j'}(u)=a_{j',j}(-u)$ yields that $a^{(2)}_{j,j'}(-u)=\overline{a^{(2)}_{j',j}(u)}$. On the other hand, $a^{(1)}_{j,j'}(\cdot)$ is a real valued function, for which $a^{(1)}_{j,j'}(-u)=a^{(1)}_{j,j}(u)$. These relations correspond to the fact that $r^{(s)}_{j,j'}(p)$, $s=1,2$ are Fourier transforms of real valued measures.)

A natural adaptation of the proof of Lemma 8.1 in [9] shows that the measures $R_{j.j'}(\cdot)$ and $S_{j,j'}(\cdot)$ have properties similar to $G_{j,j}(\cdot)$, only the function $a_{j,j}(\cdot)$ must be replaced by $a^{(1)}_{j,j'}(\cdot)$ and $a^{(2)}_{j,j'}(\cdot)$ in them. Moreover, the proof of Lemma 8.1 in [9] can be applied to show this. To understand this let us remark that $R_{j,j'}(\cdot)$ and $S_{j,j'}(\cdot)$ are measures on the torus $[-\pi.\pi)^\nu$, and their Fourier transforms satisfy relation (4.1). The spectral measure $G(\cdot)$ investigated in Lemma 8.1 of [9] has similar properties, and the proof was based on them.

More explicitly, define the measures $R^{(N)}_{j,j'}(\cdot)$, and $S^{(N)}_{j,j'}(\cdot)$ as

$$R^{(N)}_{j,j'}(A)=\frac{N^\alpha}{L(N)}R_{j,j'}\left(\frac{A}{N}\right), \qquad S^{(N)}_{j,j'}(A)=\frac{N^\alpha}{L(N)}S_{j,j'}\left(\frac{A}{N}\right)$$

for all measurable sets $A\subset[-N\pi,N\pi)^\nu$ and $N=1,2,\dots$. I claim that these measures converge vaguely to some locally finite measures $R^{(0)}_{j.j'}(\cdot)$ and $S^{(0)}_{j,j'}(\cdot)$ with some homogeneity property on $\mathbb{R}^\nu$.

To prove this homogeneity property let us introduce, similarly to the proof of Lemma 8.1 in [9] the measures $\mu^{(1)}_N$ and $\mu^{(2)}_N$, $N=1,2,\dots$ as

$$\mu^{(1)}_N(A)=\int_A |K_N(x)|^2\, R^{(N)}_{j,j'}(\,dx), \quad A\in\mathcal{B}^\nu, \quad N=1,2,\dots$$

and

$$\mu^{(2)}_N(A)=\int_A |K_N(x)|^2\, S^{(N)}_{j,j'}(\,dx), \quad A\in\mathcal{B}^\nu, \quad N=1,2,\dots$$

with the function $K_N(\cdot)$ defined as

$$K_N(x)=\frac{1}{N}\sum_{p\in\mathbb{B}_N} e^{i(p,x/N)}=\prod_{j=1}^{\nu}\frac{e^{ix^{(j)}}-1}{N(e^{ix^{(j)}/N}-1)}, \quad N=1,2,\dots.$$

I claim that both for $s = 1$ and $s = 2$ the sequence of measures $\mu_N^{(s)}$ converge weakly to a measure $\mu_0^{(s)}$ as $N \to \infty$ whose Fourier transform depends on the function $a_{j,j'}^{(s)}(\cdot)$ and parameter α appearing in formula (4.1).

This can be proved by calculating the Fourier transforms of the measures $\mu_N^{(s)}$ and by showing that they have a limit which is a continuous function. More precisely, Lemma 8.4 in [9] must be applied, which yields a modified version of this method. The reason for this modification is that we can calculate the Fourier transforms of the measures $\mu_N^{(s)}$, $s = 1, 2$, only in the points $\frac{p}{N}$, $p \in \mathbb{Z}^\nu$. On the other hand, the measures $\mu_N^{(s)}$ are concentrated in the cube $[-N\pi, N\pi)^\nu$. Lemma 8.4 in [9] provides such a version of the characteristic function method which works in such cases.

The calculations needed to prove the above properties of the measures $\mu_N^{(s)}$, $s = 1, 2$, are carried out in the proof of Theorem 8.2 in [9]. Actually, a more general result is proved there. I omit the details.

Let us define the continuous function $K_0(x) = \prod_{j=1}^{\nu} \frac{e^{ix^{(j)}}-1}{ix^{(j)}}$ on $\mathbb{R}^\nu$. In all compact subsets of $\mathbb{R}^\nu$ the functions $K_N(x)$ converge to $K_0(x)$ in the supremum norm as $N \to \infty$. The proof of Lemma 8.1 in [9] shows on the basis of this property that the limit measures $\mu_0^{(s)}$, $s = 1, 2$, have the following representation. There are measures $H_{j,j'}^{(0)}$ and $K_{j,j'}^{(0)}$ such that $\mu_0^{(1)}(A) = \int_A |K_0(x)|^2 H_{j,j'}^{(0)}(\,dx)$, and $\mu_0^{(2)}(A) = \int_A |K_0(x)|^2 K_{j,j'}^{(0)}(\,dx)$ for all measurable sets $A \subset \mathbb{R}^\nu$. Moreover, $H_{j,j'}^{(0)}$ and $K_{j,j'}^{(0)}$ are locally finite measures, and they are the vague limits of the sequences of measures $R_{j,j'}^{(N)}$ and $S_{j,j'}^{(N)}$ respectively. The measures $H_{j,j'}^{(0)}$ and $K_{j,j'}^{(0)}$ are determined by the limit measures $\mu_0^{(1)}$ and $\mu_0^{(2)}$, hence also by the parameter α and functions $a_{j,j'}^{(1)}(\cdot)$ and $a_{j,j'}^{(2)}(\cdot)$ in(4.1). The argument of the proof in Lemma 8.1 of [9] enables us to show that (3.5) holds if the complex measure $G_{j,j}^{(0)}$ is replaced by the measure $H_{j,j'}^{(0)}$ or $K_{j,j'}^{(0)}$ in it.

Since the complex measure $G_{j,j'}$ can be expressed as a linear combination of the measures $G_{j,j}$, $G_{j',j'}$, $R_{j,j'}$ and $S_{j,j'}$ the properties proved for them imply the statements formulated about the behavior of $G_{j,j'}$ in Proposition 3.1.

We still have to show that $(G_{j,j'}^{(0)})$, $1 \le j, j' \le d$, is the spectral measure of a generalized vector valued stationary Gaussian random field. By Theorem 4.1 of [10] $(G_{j,j'}^{(0)})$, $1 \le j, j' \le d$, is the spectral measure of a vector valued generalized Gaussian stationary random field if it is a positive definite matrix valued even measure on $\mathbb{R}^\nu$ whose distribution is moderately increasing at

infinity, i.e it satisfies relation (2.1). It follows from Lemma 3.2 in [11] and the already proved part of Proposition 3.1 that this system is a positive semidefinite matrix valued even measure on $\mathbb{R}^\nu$. The validity of relation (2.1) follows from the fact that $G^{(0)}_{j,j}$ has locally finite total variation, and it satisfies relation (3.5). $\square$

Now I turn to the representation of the normalized random sums in the form of a sum of multiple Wiener–itô integrals. To do this let us first consider the random variable $Y(0) = H(X_1(0), \dots, X_d(0))$ defined with the help of the function $H(x_1, \dots, x_d) = H^{(0)}(x_1, \dots, x_d)$ introduced in (3.3) and a vector valued Gaussian stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, with covariance function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, which satisfies relation (3.1) with some parameter $0 < \alpha < \frac{\nu}{k}$ together with its shifts $Y(p) = H(X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, and express them as a sum of Wiener–Itô integrals.

This will be done with the help of the results in [10] and [11].

Let $G = (G_{j,j'})$, $1 \le j, j' \le d$, be the matrix valued spectral measure of the stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$, and let us consider that vector valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ corresponding to this spectral measure for which $X_j(p) = \int e^{i(p,x)} Z_{G,j}(\, dx)$ for all $p \in \mathbb{Z}^\nu$ and $1 \le j \le d$. By the results of [10] there exists such a vector valued random spectral measure.

The random variable $Y(0) = H(X_1(0), \dots, X_d(0))$ will be rewritten in the form of a sum of Wiener–Itô integrals with the help of the multiple version of Itô's formula presented in Theorem 2.2 of [11], more precisely by the corollary of this result. As $Y(0)$ is a Wick polynomial of the (independent) random variables $X_j(0)$ with standard Gaussian distribution, and $X_j(0) = \int Z_{G,j}(\, dy)$, $1 \le j \le d$, this formula yields the desired expression for $Y(0)$. Let me remark that by Lemma 8B of [9] condition (3.1) implies that the diagonal elements $G_{j,j}$, $1 \le j \le d$, of the matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, are non-atomic. Hence the multiple Wiener–Itô integrals with respect to the coordinates of the vector valued random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ whose sum expresses $Y(0)$ in the next formula are meaningful.

The above results yield the identity

$$\begin{aligned} Y(0) &= H(X_1(0),\dots,X_d(0)) = H^{(0)}(X_1(0),\dots,X_d(0)) \\ &= \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} :\, c_{k_1,\dots,k_d} X_1(0)^{k_1}\cdots X_d(0)^{k_d} : \\ &= \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0\ 1\le j\le d,\\ k_1+\dots+k_d=k}} c_{k_1,\dots,k_d} \int \prod_{j=1}^{d} \left(\prod_{s=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dy_s) \right), \end{aligned}$$

where for $j=1$ we define $\prod_{s=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dy_s) = \prod_{s=1}^{k_1} Z_{G,1}(\,dy_s)$, and if $k_j = 0$ for some $1\le j\le d$, then we drop the term $\prod_{s=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dy_s)$ from this expression. (Here $:P(X_1(0),\dots,X_d(0)):$ denotes the Wick polynomial corresponding to $P(X_1(0),\dots,X_d(0))$, where $P(x_1,\dots,x_d)$ is a homogeneous polynomial.)

Since $Y(p) = T_p Y(0)$ with the shift transformation T_p for all $p\in\mathbb{Z}^\nu$, the previous identity and Proposition 2.4 in [11] yield the formula

$$\begin{aligned} Y(p) &= T_p Y(0) \\ &= \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\dots+k_d=k}} c_{k_1,\dots,k_d} \int e^{i(p,y_1+\dots+y_k)} \prod_{j=1}^{d} \left(\prod_{s=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dy_s) \right) \end{aligned}$$

for all $p\in\mathbb{Z}^\nu$. By summing up this formula for all $p\in B_N$ we get that

$$\begin{aligned} S_N = \frac{1}{N^{\nu-k\alpha/2}L(N)^{k/2}} &\sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d\\ k_1+\dots+k_d=k}} c_{k_1,\dots,k_d} \\ &\int \prod_{l=1}^{\nu} \frac{e^{i(N(y_1^{(l)}+\dots+y_k^{(l)})}-1}{e^{i((y_1^{(l)}+\dots+y_k^{(l)})}-1} \prod_{j=1}^{d} \left(\prod_{s=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G,j}(\,dy_s) \right), \end{aligned}$$

where we write $y=(y^{(1)},\dots,y^{(\nu)})$ for all $y\in[-\pi,\pi)^\nu$.

The above sum of Wiener–Itô integrals can be rewritten with the change

of variables $x_s = Ny_s$, $1 \le s \le k$, in the form

$$S_N = \sum_{\substack{(k_1,\dots,k_d),\ k_j \ge 0,\ 1 \le j \le d,\\ k_1+\dots+k_d=k}} \int c_{k_1,\dots,k_d} f^N(x_1+\dots+x_k) \prod_{j=1}^{d} \left(\prod_{s=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G^{(N)},j}(\,dx_s) \right), \quad (4.2)$$

where

$$f^N(x) = \prod_{l=1}^{\nu} \frac{e^{ix^{(l)}} - 1}{N(e^{ix^{(l)}/N} - 1)} \quad (4.3)$$

is a function on $[-N\pi, N\pi)^\nu$, and $Z_{G^{(N)},j}(A) = \frac{N^{\alpha/2}}{L(N)^{1/2}} Z_{G,j}(\frac{A}{N})$ for all measurable sets $A \subset [-N\pi, N\pi)^\nu$ and $j = 1, \dots, d$. In formula (4.3) the notation $x = (x^{(1)}, \dots, x^{(\nu)})$ is applied for all $x \in \mathbb{R}^\nu$. Let us observe that $(Z_{G^{(N)},1}, \dots, Z_{G^{(N)},d})$ is a vector valued random spectral measure on the torus $[-N\pi, N\pi)^\nu$ which corresponds to the matrix valued spectral measure $G^{(N)} = (G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-N\pi, N\pi)^\nu)$, defined by the formula $G^{(N)}_{j,j'}(A) = \frac{N^\alpha}{L(N)} G_{j,j'}(\frac{A}{N})$, $1 \le j, j' \le d$, on the sets $A \subset [-N\pi, N\pi)^\nu$, where $G = (G_{j,j'})$, $1 \le j, j' \le d$, is the matrix valued spectral measure of the original vector valued stationary random field $X(p) = (X_1(p), \dots, X_d(p))$, $p \in \mathbb{Z}^\nu$.

In formulas (4.2) and (4.3) the normalized random sum S_N investigated in Theorem 3.2 is written in the form of a sum of k-fold multiple Wiener–Itô integrals. Let us observe that the kernel functions $c_{k_1,\dots,k_d} f^N(x_1+\dots+x_k)$ of these Wiener–Itô integrals satisfy the relation

$$\lim_{N\to\infty} c_{k_1,\dots,k_d} f^N(x_1+\dots+x_k) = c_{k_1,\dots,k_d} f^0(x_1+\dots+x_k) \quad (4.4)$$

for all indices $k_1, \dots, k_d$ such that $k_j \ge 0$, $1 \le j \le d$, and $k_1 + \dots + k_d = k$ with the function

$$f^0(x) = \prod_{l=1}^{\nu} \frac{e^{ix^{(l)}} - 1}{ix^{(l)}} \quad (4.5)$$

defined on $\mathbb{R}^\nu$, and this convergence is uniform in all bounded subsets of $\mathbb{R}^{k\nu}$.

On the other hand, Proposition 3.1 states that the elements of the matrix valued spectral measures $G^{(N)} = (G^{(N)}_{j,j'})$ vaguely converge to the elements of

a matrix valued spectral measure $G^{(0)} = (G^{(0)}_{j,j'})$ on $\mathbb{R}^\nu$. In (4.2) we integrate with respect to a vector valued random spectral measure corresponding to the matrix valued spectral measure $(G^{(N)}_{j,j'})$, $1 \le j, j' \le N$ of a generalized vector valued Gaussian stationary random field. Hence it is natural to expect that the random variables S_N converge in distribution to the random variable

$$S_0 = \sum_{\substack{(k_1,\dots,k_d),\ k_j \ge 0,\ 1 \le j \le d, \\ k_1+\cdots+k_d=k}} \int c_{k_1,\dots,k_d} f^0(x_1 + \cdots + x_k) \prod_{j=1}^{d} \left(\prod_{s=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} Z_{G^{(0)},j}(\, dx_s) \right), \quad (4.6)$$

where $(Z_{G^{(0)},1}, \dots, Z_{G^{(0)},d})$ is a vector valued random spectral measure on $\mathbb{R}^\nu$ corresponding to the matrix valued spectral measure $(G^{(0)}_{j,j'})$, $1 \le j, j' \le d$. This is actually the statement of Theorem 3.2 with a slightly different notation.

I shall formulate such a result in the following Proposition 4A which helps to justify the above heuristic argument. It states that this argument yields a correct result if some additional conditions are also satisfied. Theorem 3.2 will be proved with the help of this Proposition 4A which is a reformulation of Proposition 3.1 in [11].

Before the presentation of Proposition 4A I recall from Section 5 of [10] the definition of that class of functions which can be chosen for the kernel function of a multiple Wiener–Itô integral with respect to a vector valued random spectral measure. This class of functions appears in the formulation of Proposition 4A.

Let us consider the matrix valued spectral measure $G = (G_{j,j'})$, $1 \le j, j' \le d$, with non-atomic measures $G_{j,j}$, $1 \le j \le d$, of a vector valued Gaussian stationary random field. (We can consider the spectral measure both of an ordinary or of a generalized random field.) In [10] I have defined a real Hilbert space $\mathcal{K}_{k,j_1,\dots,j_k} = \mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1}, \dots, G_{j_k,j_k})$ depending on the diagonal elements $G_{1,1}, \dots, G_{d,d}$ of the spectral measure G and on a sequence of integers $(j_1, \dots, j_k)$ of length k such that $1 \le j_s \le d$ for all $1 \le s \le k$. This Hilbert space has the property that the k-fold Wiener–Itô integral

$$\int f(x_1, \dots, x_k) Z_{G,j_1}(\, dx_1) \dots Z_{G,j_k}(\, dx_k)$$

with respect to a vector valued random spectral measure

$$Z_G = (Z_{G,1}, \dots, Z_{G,d})$$

corresponding to the matrix valued spectral measure G is defined for the kernel functions

$$f(x_1, \dots, x_k) \in \mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1}, \dots, G_{j_k,j_k}).$$

(In papers [10] and [11] I worked with Wiener–Itô integrals of order n, while here I work with Wiener–Itô integrals of order k. Hence I use here a slightly different notation.)

We have $f \in \mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1}, \dots, G_{j_k,j_k})$ for a complex number valued function $f(x_1, \dots, x_k)$ with arguments $x_s \in \mathbb{R}^\nu$, $1 \le s \le k$, if it satisfies the following conditions (a) and (b):

(a) $f(-x_1, \dots, -x_k) = \overline{f(x_1, \dots, x_k)}$ for all $(x_1, \dots, x_k) \in \mathbb{R}^{k\nu}$,

(b) $\|f\|^2 = \int |f(x_1, \dots, x_k)|^2 G_{j_1,j_1}(\,dx_1) \dots G_{j_k,j_k}(\,dx_n) < \infty$.

The scalar product in $\mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1}, \dots, G_{j_k,j_k})$ is defined in the usual way. If f, $g \in \mathcal{K}_{k,j_1,\dots,j_k}(G_{j_1,j_1}, \dots, G_{j_k,j_k})$, then

$$\langle f, g \rangle = \int f(x_1, \dots, x_k) \overline{g(x_1, \dots, x_k)} G_{j_1,j_1}(\,dx_1) \dots G_{j_k,j_k}(\,dx_k).$$

In the formulation of Proposition 4A we take for all $N = 1, 2, \dots$ a matrix valued non-atomic spectral measure $G^{(N)} = (G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, on the torus $[-A_N\pi, A_N\pi)^\nu$ with a parameter A_N such that $A_N \to \infty$ as $N \to \infty$. We also take some functions

$$h^N_{j_1,\dots,j_k}(x_1, \dots, x_k) \in \mathcal{K}_{k,j_1,\dots,j_k} = \mathcal{K}_{k,j_1,\dots,j_k}(G^{(N)}_{j_1,j_1}, \dots, G^{(N)}_{j_k,j_k})$$

on the torus $[-A_N\pi, A_N\pi)^\nu$ for all sets of indices $(j_1, \dots, j_k)$ such that $1 \le j_s \le d$, $1 \le s \le k$, and $N = 1, 2, \dots$. Besides, we fix for all $N = 1, 2, \dots$ a vector valued random spectral measure $Z_{G^{(N)}} = (Z_{G^{(N)},1}, \dots, Z_{G^{(N)},d})$ on the torus $[-A_N\pi, A_N\pi)^\nu$ corresponding to the matrix valued spectral measure

$G^{(N)} = (G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, and we define with the help of these quantities the sums of k-fold Wiener–Itô integrals

$$S_N = \sum_{\substack{(j_1,\dots,j_k) \\ 1 \le j_s \le d, \text{ for all } 1 \le s \le k}} \int h^N_{j_1,\dots,j_k}(x_1,\dots,x_k) Z_{G^{(N)},j_1}(\,dx_1) \dots Z_{G^{(N)},j_k}(\,dx_k), \tag{4.7}$$

$N = 1, 2, \dots$. We want to find some good conditions under which these random variables S_N converge in distribution to a random variable S_0, expressed similarly as a sum of k-fold multiple Wiener–Itô integrals.

This will be done with the help of the following Proposition 4A which agrees with Proposition 3.1 in paper [11].

Proposition 4A. *Let us consider for all $N = 1, 2, \dots$ the sum of k-fold Wiener–Itô integrals S_N defined in formula (4.7) with the help of a vector valued random spectral measure $Z_{G^{(N)}} = (Z_{G^{(N)},1}, \dots, Z_{G^{(N)},d})$ corresponding to some non-atomic matrix valued spectral measure $G^{(N)} = (G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, defined on a torus $[-A_N, A_N)^\nu$ such that $A_N \to \infty$ as $N \to \infty$ and functions*

$$h^N_{j_1,\dots,j_k}(x_1,\dots,x_k) \in \mathcal{K}_{k,j_1,\dots,j_k}(G^{(N)}_{j_1,j_1}, \dots, G^{(N)}_{j_k,j_k}).$$

Let the coordinates $G^{(N)}_{j,j'}$, $1 \le j, j' \le d$, of the matrix valued spectral measures $G^{(N)} = (G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, converge vaguely to the coordinates $G^{(0)}_{j,j'}$ of a non-atomic matrix valued spectral measure $G^{(0)} = (G^{(0)}_{j,j'})$, $1 \le j, j' \le d$, on $\mathbb{R}^\nu$ for all $1 \le j, j' \le d$ as $N \to \infty$, and let $Z_{G^{(0)}} = (Z_{G^{(0)},1}, \dots, Z_{G^{(0)},d})$ be a vector valued random spectral measure on $\mathbb{R}^\nu$ corresponding to the matrix valued spectral measure $G^{(0)} = (G^{(0)}_{j,j'})$, $1 \le j, j' \le d$. Let us also have some functions $h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)$ on $\mathbb{R}^{k\nu}$ for all $1 \le j_s \le d$, $1 \le s \le k$, such that these functions and matrix valued spectral measures satisfy the following conditions (a) and (b).

(a) *The functions $h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)$ are continuous on $\mathbb{R}^{k\nu}$ for all $1 \le j_s \le d$, $1 \le s \le k$, and for all $T > 0$ and indices $1 \le j_s \le d$, $1 \le s \le k$, the functions $h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)$ converge uniformly to the function $h^0_{j_1,\dots,j_k}(x_1,\dots,x_k)$ on the cube $[-T,T]^{k\nu}$ as $N \to \infty$.*

(b) *For all $\varepsilon > 0$ there is some $T_0 = T_0(\varepsilon) > 0$ such that*

$$\int_{\mathbb{R}^{k\nu} \setminus [-T,T]^{k\nu}} |h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)|^2 G^{(N)}_{j_1,j_1}(\,dx_1) \dots G^{(N)}_{j_k,j_k}(dx_k) < \varepsilon^2 \tag{4.8}$$

for all $1 \le j_s \le d$, $1 \le s \le k$, *and* $N = 1, 2 \dots$ *if* $T > T_0$.

Then

$$h^0_{j_1,\dots,j_k}(x_1, \dots, x_k) \in \mathcal{K}_{k,j_1,\dots,j_k} = \mathcal{K}_{k,j_1,\dots,j_k}(G^{(0)}_{j_1,j_1}, \dots G^{(0)}_{j_k,j_k}),$$

inequality (4.8) holds also for $N = 0$, *the sum of random integrals*

$$S_0 = \sum_{\substack{(j_1,\dots,j_k) \\ 1 \le j_s \le d, \text{ for all } 1 \le s \le k}} \int h^0_{j_1,\dots,j_k}(x_1, \dots, x_k) Z_{G^{(0)},j_1}(\, dx_1) \dots Z_{G^{(0)},j_k}(\, dx_k) \tag{4.9}$$

exists, and the random variables S_N *defined in (4.7) converge to* S_0 *in distribution as* $N \to \infty$.

(In the formulation of Proposition 4A I took the natural identification of the torus $[-A_N, A_N)^\nu$ with the cube $[-A_N, A_N)^\nu$ in the space $\mathbb{R}^\nu$. Thus, I considered the functions $h_{j_1,\dots,j_k}(\cdot)$ as functions on $\mathbb{R}^{k\nu}$ which disappear outside $[-A_N, A_n)^{k\nu}$, and the complex measures $G^{(N)}_{j,j'}$ as complex measures on $\mathbb{R}^\nu$, concentrated on $[-A_N, A_N)^\nu$. In such a way the vague convergence mentioned in the formulation of Proposition 4A is meaningful.)

In the proof of Theorem 3.2 we want to show with the help of Proposition 4A that the sequence of random variables S_N, $N = 1, 2, \dots$, defined in (4.2) converge to the random variable S_0 defined in (4.6) as $N \to \infty$. To do this we rewrite these formulas with a different indexation in such a way that the indices in the definition of the random variables S_N and S_0 fit to the indices in the definition of the random variable of the random variables S_N and S_0 appearing in the formulation of Proposition 4A. These random variables were defined in formulas (4.7) and (4.9).

In formulas (4.2) and (4.6) summation is taken for terms with indices $(k_1, \dots, k_d)$ such that $k_s \ge 0$, $1 \le s \le d$ and $k_1 + \dots + k_d = k$, while in the corresponding expressions in formulas (4.7) and (4.9) in Proposition 4A it is taken for terms with indices $(j_1, \dots, j_k)$ such that $1 \le j_s \le d$, $1 \le s \le k$.

An important difference between the indexation in the two cases is that in (4.2) and (4.6) only a special subset of the indices in formulas (4.7) and (4.9) appear. Namely, if $s < s'$, and we compare the indices j and j' in the terms $Z_{G^{(N)},j}(\, dx_s)$ and $Z_{G^{(N)},j'}(\, dx_{s'})$ belonging to these indices s and s' in formula (4.2) or (4.6), then we find that $j \le j'$. Hence such a

reindexation of the indices in (4.2) and (4.6) will be made, where the set $\mathcal{J}$ of the new indices is only a subset of the indices $(j_1, \dots, j_k)$ appearing in formulas (4.7) and (4.9). Summation will be taken only for the elements of $\mathcal{J}$ in these formulas.

More explicitly, the terms in the sums in (4.2) and (4.6) will be reindexed with such indices $(j_1, \dots, j_k)$ for which the relation $1 \le j_1 \le j_2 \le \cdots \le j_k \le d$ holds. This is a subset of the set of indices $(j_1, \dots, j_k)$ appearing in formulas (4.7) and (4.9). To carry out the desired reindexation a one to one map will be defined between the sets

$$\mathcal{J} = \{(j_1, \dots, j_k)\colon\ 1 \le j_1 \le j_2 \le \cdots \le j_k \le d\}$$

and

$$\mathcal{K} = \{(k_1, \dots, k_d)\colon\ k_s \ge 0 \text{ for all } 1 \le s \le d,\ \ k_1 + \cdots + k_d = k\}.$$

Put

$$\text{for all } (j_1, \dots, j_k) \in \mathcal{J} \qquad k_s(j_1, \dots j_k) = \text{the number of such elements } j_p \text{ for which } j_p = s, \text{ for all } 1 \le s \le d. \tag{4.10}$$

This is a one to one map from $\mathcal{J}$ to $\mathcal{K}$ whose inverse is

$$\text{for all } (k_1, \dots, k_d) \in \mathcal{K} \qquad j_s(k_1, \dots, k_d) = \min p\colon\ k_1 + \cdots + k_p \ge s, \text{ for all } 1 \le s \le k. \tag{4.11}$$

We shall apply these maps.

With the help of this correspondence between the sets $\mathcal{J}$ and $\mathcal{K}$ the random sums S_N in (4.2) can be rewritten in a form where summation is taken for the sequences $(j_1, \dots, j_k) \in \mathcal{J}$ instead of the sequences $(k_1, \dots, k_d) \in \mathcal{K}$, and $k_s(j_1, \dots, j_k)$ is written instead of k_s, $1 \le s \le d$.

The expression S_N defined in (4.2) can be rewritten as

$$S_N = \sum_{\substack{(j_1,\dots,j_k),\\ 1 \le j_1 \le \cdots \le j_k \le d}} \int c_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)} f^N(x_1 + \cdots + x_k) \prod_{s=1}^{k} Z_{G^{(N)},j_s}(\,dx_s) \tag{4.12}$$

for all $N = 1, 2, \dots$ with the functions $f^N(x)$ defined in (4.3) and the indices $k_s(j_1, \dots, j_k)$, $1 \le s \le d$ defined in (4.10).

To show why formula (4.12) holds let us rewrite formula (4.2) (with the help of the one to one map we defined between the sets $\mathcal{K}$ and $\mathcal{J}$) in the form

$$S_N = \sum_{\substack{(k_1,\dots,k_d),\ k_j\ge 0,\ 1\le j\le d,\\ k_1+\cdots+k_d=k}} \int c_{k_1,\dots,k_d} f^N(x_1+\cdots+x_k) \prod_{s=1}^{k} Z_{G^{(N)},j_s(k_1,\dots,k_d)}(\,dx_s). \tag{4.13}$$

To understand why formula (4.13) holds we have to show that in (4.13) the term $Z_{G^{(N)},u}(\,dx_s)$ with $u = j_s(k_1,\dots,k_d)$ had to be chosen. It can be seen from (4.2) that this number u must be chosen in such a way that $k_1+\cdots+k_{u-1}+1 \le s \le k_1+\cdots+k_u$. Then a comparison of this condition with the definition of the mapping from $\mathcal{K}$ to $\mathcal{J}$ in (4.11) shows that $u = j_s(k_1\dots,k_d)$.

Then if we rewrite the formula at the right-hand side of (4.12) by replacing the arguments $(k_1,\dots.k_d) \in \mathcal{K}$ by the arguments $(j_1,\dots,j_k) \in \mathcal{J}$ with the help of the transformation we defined from $\mathcal{J}$ to $\mathcal{K}$, and then we exploit that the transformation we defined from $\mathcal{K}$ to $\mathcal{J}$ is its inverse transformation, we get that formula (4.13) implies (4.12).

Relation (4.12) can be rewritten in the form

$$S_N = \sum_{\substack{(j_1,\dots,j_k),\\ 1\le j_1\le\cdots\le j_k\le d}} \int h^N_{j_1,\dots,j_k}(x_1,\dots,x_k) Z_{G^{(N)},j_1}(\,dx_1)\dots Z_{G^{(N)},j_k}(\,dx_k) \tag{4.14}$$

with

$$h^N_{j_1,\dots,j_k}(x_1,\dots,x_k) = c_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)} f^N(x_1+\cdots+x_k), \tag{4.15}$$

where the indices $k_s(j_1,\dots,j_k)$, $1 \le s \le d$, are defined in (4.10). Similarly, the random sum S_0 in (4.6) can be rewritten in the form

$$S_0 = \sum_{\substack{(j_1,\dots,j_k),\\ 1\le j_1\le\cdots\le j_k\le d}} \int c_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)} f^0(x_1+\cdots+x_k) \prod_{s=1}^{k} Z_{G^{(0)},j_s}(\,dx_s)$$

with the function $f^0(x)$ defined in (4.5) or in the following equivalent form.

$$S_0 = \sum_{\substack{(j_1,\dots,j_k),\\ 1\le j_1\le\cdots\le j_k\le d}} \int h^0_{j_1,\dots,j_k}(x_1,\dots,x_k) Z_{G^{(0)},j_1}(\,dx_1)\dots Z_{G^{(0)},j_k}(\,dx_k) \tag{4.16}$$

with

$$h^0_{j_1,\dots,j_k}(x_1,\dots,x_k) = c_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)} f^0(x_1+\dots+x_k). \tag{4.17}$$

5. Proof of the main theorems.

Theorem 3.2 will be proved by means of the application of Proposition 4A for the sequences S_N defined in (4.14), (4.3), (4.15) and (4.10) for $N = 1, 2, \dots$, and in (4.16), (4.5), (4.17) and (4.10) for $N = 0$. To do this we have to show that under the conditions of Theorem 3.2 the conditions of Proposition 4A are also satisfied with such a choice. Then the application of Proposition 4A implies Theorem 3.2. (I would remark that the random variable S_0 defined in formula (3.6) as the limit in Theorem 3.2 agrees with the random variable S_0 defined in (4.6), which is the same as the limit we get in the application of Proposition 2A with the above written choice. Only it is written there in a different form.)

To check the conditions of Proposition 4A let us first observe that it follows from Proposition 3.1 that the (non-atomic) elements $G^{(N)}_{j,j'}$ of the spectral measures $G^{(N)}$ vaguely converge to the (non-atomic) complex measures $G^{(0)}_{j,j'}$ of a spectral measure $G^{(0)}$ as $N \to \infty$ for all $1 \le j, j' \le d$. It is also clear that the functions $h^N_{j_1,\dots,j_k}(x_1,\dots,x_k)$ defined in (4.15) for all $1 \le j_1 \le \dots \le j_k \le d$ and $N = 1, 2, \dots$ satisfy the condition $h^N_{j_1,\dots,j_k}(x_1,\dots,x_k) \in \mathcal{K}_{k,j_1,\dots,j_k}(G^{(N)}_{j_1,j_1},\dots,G^{(N)}_{j_k,j_k})$.

It follows from (4.4), (4.15) and (4.17) that condition (a) of Proposition 4A holds with the functions and measures chosen in the proof of Theorem 3.2. We still have to prove relation (4.8) in condition (b) of Proposition 4A. This will be done with the help of the following Proposition 5.1. (Actually in Proposition 5.1 we prove a result slightly sharper than we need.)

Proposition 5.1. *Let us fix an integer $k \ge 1$, and let $G = (G_{j,j'})$, $1 \le j, j' \le d$, be the matrix valued spectral measure of a vector valued stationary random field $X(p) = (X_1(p),\dots,X_p(d))$, $p \in \mathbb{Z}^\nu$, defined on the torus $[-\pi,\pi)^\nu$ with such correlation function $r_{j,j'}(p) = EX_j(0)X_{j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$, that satisfies relation (3.1) with some $0 < \alpha < \frac{\nu}{k}$. For all $N = 1, 2, \dots$ let us consider the measures $G^{(N)}_{j,j}$, $1 \le j \le d$, defined in formula (3.4) together with the measures $\mu^{(N)}_{j_1,\dots,j_k}$ defined for all sets of indices $j_1,\dots,j_k$ such that*

$1 \le j_s \le d$, $1 \le s \le k$, *on* $\mathbb{R}^{k\nu}$ *by the formula*

$$\mu^{(N)}_{j_1,\dots,j_k}(A) = \int_A |h_N(x_1,\dots,x_k)|^2 G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k), \quad A \in \mathcal{B}^{k\nu}, \tag{5.1}$$

with

$$h_N(x_1,\dots,x_k) = f^N(x_1+\dots+x_k) = \prod_{l=1}^{\nu} \frac{e^{i((x_1^{(l)}+\dots+x_k^{(l)})} - 1}{N(e^{i((x_1^{(l)}+\dots+x_k^{(l)})/N} - 1)}, \tag{5.2}$$

where we use the notation $x = (x^{(1)},\dots,x^{(\nu)})$ *for a vector* $x \in \mathbb{R}^\nu$. *These measures* $\mu^{(N)}_{j_1,\dots,j_k}$ *converge weakly to a finite measure* $\mu^{(0)}_{j_1,\dots,jk}$ *on* $\mathbb{R}^{k\nu}$.

Proof of Theorem 3.2 with the help of Proposition 5.1. As we have seen to prove Theorem 3.2 it is enough to check that the measures $G^{(N)} = (G^{(N)}_{j,j'})$, $1 \le j, j' \le d$, and functions $h^N_{j_1,\dots,j_k}$ defined before satisfy the conditions of Proposition 4A, since this enables us to apply this result. Moreover, we have proved the validity of all of these conditions except formula (4.8) in condition (b) of Proposition 4A. But the validity of this condition follows from Proposition 5.1, since this result implies that the measures $\mu^{(N)}_{j_1,\dots,j_k}$, $N = 1, 2, \dots$, defined in (5.1) and (5.2) are uniformly tight. This fact together with the definition of the measures $\mu^{(N)}_{j_1,\dots,j_k}$ and the identity $h^N_{j_1,\dots,j_k}(x_1,\dots,x_k) = c_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)} h_N(x_1,\dots,x_k)$ imply that relation (4.8) holds. □

It remained to prove Proposition 5.1.

Proof of Proposition 5.1. Most calculations needed in the proof of Proposition 5.1 were actually carried out in the proof of Theorem 8.2 of [9]. Only some slight modifications are needed in the proof. In some steps I shall refer to the corresponding part in the proof of Theorem 8.2 in [9] and omit the details of the calculation.

I compute for all $N = 1, 2, \dots$ the Fourier transform

$$\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k) = \int e^{i((t_1,x_1)+\dots+(t_k,x_k))} \mu^{(N)}_{j_1,\dots,j_k}(\,dx_1,\dots,dx_k)$$

of the measures $\mu^{(N)}_{j_1,\dots,j_k}$ defined in (5.1) and give a good asymptotic formula for it. More precisely I do this only for such coordinates $(t_1,\dots,t_k)$ of the function $\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)$ which have the form $t_l = \frac{p_l}{N}$ with some $p_l \in \mathbb{Z}^\nu$, $l = 1,\dots,k$. But as it is explained at the end of this proof, even such

a result is sufficient for us. In the calculation of the formula expressing $\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)$ I exploit that the function $h_N(x_1,\dots,x_k)$ defined in (5.2) can be written in the form

$$h_N(x_1,\dots,x_k)=\frac{1}{N^\nu}\sum_{u\in B_N}\exp\left\{i\frac{1}{N}(u,x_1+\dots+x_k)\right\}.$$

Hence, and because of the definition of the spectral measures $G^{(N)}_{j,j}(\cdot)$ in (3.4)

$$\begin{aligned}\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)&=\frac{1}{N^{2\nu}}\int\exp\left\{i\frac{1}{N}((p_1,x_1)+\dots+(p_k,x_k))\right\}\\ &\sum_{u\in B_N}\sum_{v\in B_N}\exp\left\{i\left(\frac{u-v}{N},x_1+\dots+x_k\right)\right\}G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k)\\ &=\frac{1}{N^{2\nu}}\sum_{u\in B_N}\sum_{v\in B_N}\left(\prod_{s=1}^k\int\exp\left\{i\left(\frac{u-v+p_s}{N},x_s\right)\right\}G^{(N)}_{j_s,j_s}(\,dx_s)\right)\\ &=\frac{1}{N^{2\nu-k\alpha}L(N)^k}\sum_{u\in B_N}\sum_{v\in B_N}r_{j_1,j_1}(u-v+p_1)\cdots r_{j_k,j_k}(u-v+p_k)\end{aligned}$$

if $t_l=\frac{p_l}{N}$ with some $p_l\in\mathbb{Z}^\nu$, $1\le l\le k$. This identity can be rewritten by taking the summation at the right-hand side of the last formula first for such pairs (u,v) for which $u-v=y$ with a fixed point $y\in\mathbb{Z}^\nu$ and then for the lattice points $y\in\mathbb{Z}^\nu$. By working with $x=\frac{y}{N}$ instead of y we get that

$$\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)=\int_{[-1,1]^\nu}f^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)\lambda_N(\,dx)$$

with

$$\begin{aligned}&f^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)\\ &\quad=\left(1-\frac{|x^{(1)}N|}{N}\right)\cdots\left(1-\frac{|x^{(\nu)}N|}{N}\right)\frac{r_{j_1,j_1}(N(x+t_1))}{N^{-\alpha}L(N)}\cdots\frac{r_{j_k,j_k}(N(x+t_k))}{N^{-\alpha}L(N)},\end{aligned}$$

where λ_N is the measure concentrated in the points of the form $x=\frac{p}{N}$ with such points $p=(p_1,\dots,p_\nu)\in\mathbb{Z}^\nu$ for which $-N<p_l<N$ for all $1\le l\le\nu$, and $\lambda_N(x)=N^{-\nu}$ for each point x with this property. (Here such a calculation is applied which is similar to that in the proof of Theorem 8.2 of [9] when formula (8.20) of that work was rewritten in another form.)

Let us extend the definition of $\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)$ to all $(t_1,\dots,t_k)\in\mathbb{R}^{k\nu}$ by defining it as

$$\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)=\varphi^{(N)}_{j_1,\dots,j_d}\left(\frac{p_1}{N},\dots,\frac{p_k}{N}\right),\quad t_l\in\mathbb{R}^\nu \text{ for all } 1\le l\le k,$$

where $p_l=p_l(t_l)$ is defined as the integer part $[t_lN]$ of t_lN, $1\le l\le k$, i.e. $p_l\in\mathbb{Z}^\nu$, and $p_l^{(s)}\le t_l^{(s)}N<p_l^{(s)}+1$ if $t_l^{(s)}>0$, and $p_l^{(s)}-1<t_l^{(s)}N\le p_l^{(s)}$ if $t_l^{(s)}\le 0$, $1\le s\le\nu$.

Let us also extend the definition of the function $f^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)$ to $(t_1,\dots,t_k,x)\in\mathbb{R}^{k\nu}\times[-1,1]^\nu$ by means of the formula

$$\begin{aligned}&f^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)\\&\quad=\left(1-\frac{|q^{(1)}|}{N}\right)\cdots\left(1-\frac{|q^{(\nu)}|}{N}\right)\frac{r_{j_1,j_1}(q+p_1)}{N^{-\alpha}L(N)}\cdots\frac{r_{j_k,j_k}(q+p_k)}{N^{-\alpha}L(N)}\end{aligned}$$

for $t_l\in\mathbb{R}^\nu$, $1\le l\le k$, and $x\in[-1,1]^\nu$, where $p_l=p_l(t)$ is defined as before, and $q=q(x)$ is defined as $q=(q^{(1)},\dots,q^{(\nu)})\in\mathbb{Z}^\nu$ with $q_l\in\mathbb{Z}^\nu$, and $q^{(s)}\le x_l^{(s)}<q^{(s)}+1$.

We have

$$\varphi^{(N)}_{j_1,\dots,j_d}(t_1,\dots,t_k)=\int_{[-1,1]^\nu}f^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)\,dx \tag{5.3}$$

for the functions $\varphi^{(N)}(\cdot)$ and $f^{(N)}(\cdot)$ with this extended domain of definition, where dx denotes integration with respect to the Lebesgue measure.

It follows from relation (3.1) and the fact that $\frac{q}{N}$ is very close to x, and $\frac{p_l}{N}$ is very close to t_l, for all $1\le l\le k$ if N is large that for all parameters $t_1,\dots,t_k$ and $\varepsilon>0$

$$f^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)\to f^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)$$

holds uniformly with the limit function

$$\begin{aligned}&f^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)\\&\quad=(1-|x^{(1)}|)\dots(1-|x^{(\nu)}|)\frac{a_{j_1,j_1}\left(\frac{x+t_1}{|x+t_1|}\right)}{|x+t_1|^\alpha}\cdots\frac{a_{j_k,j_k}\left(\frac{x+t_k}{|x+t_k|}\right)}{|x+t_k|^\alpha}\end{aligned}$$

on the set $x\in[-1,1]^\nu\setminus\bigcup_{l=1}^k\{x\colon\ |x+t_l|<\varepsilon\}$.

Some additional calculation shows that for small $\varepsilon > 0$ integration on the domain

$$[-1,1]^\nu \setminus \left([-1,1]^\nu \setminus \bigcup_{l=1}^k \{x\colon\ |x+t_l| < \varepsilon\} \right) = [-1,1]^\nu \cap \left(\bigcup_{l=1}^k \{x\colon\ |x+t_l| < \varepsilon\} \right)$$

gives a negligible contribution to the integral in formula (5.3) (with parameters $j_1, \dots, j_k$ and $t_1, \dots, t_l$), or to the integral that we get if the kernel function $f^{(N)}_{j_1,\dots,j_k}$ is replaced by $f^{(0)}_{j_1,\dots,j_k}$ in the integral in (5.3). Hence the relation

$$\varphi^{(N)}_{j_1,\dots,j_k}(t_1,\dots,t_k) \to \varphi^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k) = \int_{[-1,1]^\nu} f^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k,x)\,dx \tag{5.4}$$

holds for all $(t_1,\dots,t_k) \in \mathbb{R}^{k\nu}$ as $N \to \infty$, and $\varphi^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k)$ is a continuous function. This calculation was carried out in that part of the proof of Theorem 8.2 in [9] which followed the discussion of Lemma 8.4. Hence here I omit it.

By a classical result of probability theory if the Fourier transforms of a sequence of finite measures on $\mathbb{R}^{k\nu}$ converge to a function continuous at the origin, then the limit function is also the Fourier transform of a finite measure on $\mathbb{R}^{k\nu}$, and the sequence of probability measures whose Fourier transforms were taken converge to this measure. In the proof of Proposition 5.1 this result cannot be applied, because we have a control on the Fourier transform of $\mu^{(N)}_{j_1,\dots,j_k}$ only in points of the form $(t_1,\dots,t_k)$ with $t_l = \frac{p_l}{N}$ and $p_l \in \mathbb{Z}^\nu$, $1 \le l \le k$. But the measures $\mu^{(N)}_{j_1,\dots,j_k}$ have the additional property that they are concentrated in the cube $[-N\pi, N\pi)^{k\nu}$. Lemma 8.4 of [9] can be applied, and it shows that relation (5.4) and the continuity of the limit function $\varphi^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k)$ together with the above mentioned concentration property of the measures $\mu^{(N)}_{j_1,\dots,j_k}$ imply the weak convergence of the measures $\mu^{(N)}_{j_1,\dots,j_k}$ to a finite measure $\mu^{(0)}_{j_1,\dots,j_k}$. This result also implies that this finite measure $\mu^{(0)}_{j_1,\dots,j_k}$ has the Fourier transform $\varphi^{(0)}_{j_1,\dots,j_k}(t_1,\dots,t_k)$. □

To prove Theorem 3.3 with the help of Theorem 3.2 it is enough to show that if a function $H^{(1)}(\cdot)$ satisfies (3.7) and (3.8), and the Gaussian stationary

random field $X(p) = (X_1(p), \dots, X_d(p))$ satisfies (3.1) and (3.2), then

$$\frac{1}{N^{\nu - k\alpha/2} L(N)^{k/2}} \sum_{p \in B_N} H^{(1)}(X_1(p), \dots, X_d(p)) \Rightarrow 0 \quad \text{as } N \to \infty, \tag{5.5}$$

where $\Rightarrow$ denotes convergence in probability. I shall prove that even the second moments of the normalized sums in (5.5) tend to zero as $N \to \infty$. The following Lemma 5A which agrees with Lemma 1 of [1] (only with a slightly different notation) helps in the proof of this statement.

Lemma 5A. *Let $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots Y_d)$ be two Gaussian random vectors with expectation zero such that $EX_jX_{j'} = EY_jY_{j'} = \delta_{j,j'}$, $1 \le j, j' \le d$, and let $r_{j,j'} = EX_jY_{j'}$, $1 \le j, j' \le d$. Take a number $k \ge 1$ and a function $H^{(1)}(x_1, \dots, x_d)$ that satisfies relations (3.7) and (3.8). Assume that*

$$\psi = \max\left(\left(\sup_{1 \le j \le d} \sum_{j'=1}^{d} |r_{j,j'}| \right), \left(\sup_{1 \le j' \le d} \sum_{j=1}^{d} |r_{j,j'}| \right) \right) \le 1.$$

Then

$$|EH^{(1)}(X_1, \dots, X_d) H^{(1)}(Y_1, \dots, Y_d)| \le \psi^{k+1} E\left[H^{(1)}(X_1, \dots, X_d)\right]^2.$$

Proof of Theorem 3.3. It follows from relations (3.1), (3.2) and Lemma 5A together with the inequality $E\left[H^{(1)}(X_1(0), \dots, X_d(0))\right]^2 < \infty$ which holds because of (3.8) that for two elements $X(p) = (X_1(p), \dots, X_d(p))$ and $X(q) = (X_1(q), \dots, X_d(q))$, $p, q \in \mathbb{Z}^\nu$, of our vector valued Gaussian stationary random field there exists some threshold index $n_0 \ge 1$ and constant $0 < C < \infty$ such that

$$\begin{aligned} &|EH^{(1)}(X_1(p), \dots X_d(p)) H^{(1)}(X_1(q), \dots, X_d(q))| \\ &\qquad \le C|p-q|^{-(k+1)\alpha} L(|p-q|)^{k+1} \end{aligned}$$

if $|p - q| \ge n_0$. On the other hand,

$$\begin{aligned} &|EH^{(1)}(X_1(p), \dots X_d(p)) H^{(1)}(X_1(q), \dots, X_d(q))| \\ &\qquad \le E{H^{(1)}}^2(X_1(0), \dots X_d(0)) \le C_1 \end{aligned}$$

for all $p, q \in \mathbb{Z}^\nu$ with some $C_1 < \infty$ by the Schwarz inequality and relation (3.8). Hence we get by summing up the above two inequalities for all

$q \in B_N$ with a fixed $p \in B_N$, and applying the first inequality if $|p-q| > n_0$ and the second one if $|p-q| \le n_0$ that

$$\left| EH^{(1)}(X_1(p), \dots, X_d(p)) \left(\sum_{q \in B_N} H^{(1)}(X_1(q), \dots, X_d(q) \right) \right| \le C_2(1 + N^{\nu+\varepsilon-(k+1)\alpha})$$

for all $p \in B_N$ and $\varepsilon > 0$ with an appropriate $C_2 = C_2(\varepsilon) > 0$. Since $\nu - k\alpha > 0$ we get by summing up the last inequality for all $p \in B_N$ that

$$\frac{1}{N^{2\nu-k\alpha}L(N)^k} E \left[\sum_{p \in B_N} H^{(1)}(X_1(p), \dots, X_d(p)) \right]^2 \to 0 \quad \text{as } N \to \infty.$$

Indeed, it can be seen that for all $\varepsilon > 0$ the expression in the last formula can be bounded from above by $C(\varepsilon)N^{-\delta+\varepsilon}$ with $\delta = \min(\nu - k\alpha, \alpha) > 0$ and a constant $C(\varepsilon) > 0$ depending only on ε. This implies formula (5.5). Formula (5.5) together Theorem 3.2 yield Theorem 3.3. □

Proof of Theorem 3.4. The proof of Theorem 3.4 is very similar to that of Theorems 3.2 and 3.3. Hence I only briefly explain it.

It is enough to show that for any positive integer K, parameters $t_1, \dots, t_K$, $t_p \in [0,1]^\nu$, $1 \le p \le K$ and real constants $C_1, \dots, C_K$ the linear combinations $\sum_{p=1}^K C_p S_N(t_p)$ converge to $\sum_{p=1}^K C_p S_0(t_p)$ in distribution as $N \to \infty$, since this implies that the random vectors $(S_N(t_1), \dots, S_N(t_K))$ converge in distribution to the random vector $(S_0(t_1), \dots, S_0(t_K))$ as $N \to \infty$. Moreover, similarly to the proof of Theorem 3.3 the proof of Theorem 3.4 can be reduced to the case $H(x_1, \dots, x_d) = H^{(0)}(x_1, \dots, x_d)$ with a function $H^{(0)}(x_1, \dots, x_d)$ which satisfies relation (3.3).

In the first step of the proof the linear combinations $\sum_{p=1}^K C_p S_N(t_p)$, $N = 0, 1, 2, \dots$, are written in the form of a sum of k-fold Wiener-Itô integrals with respect to the coordinates of an appropriate vector valued random spectral measure. This can be done, first by writing the random variables $S_N(t)$ for all $t \in [0,1]^\nu$ in the desired form. The random variables $S_0(t)$ are written in such a form in (3.11). In the case $N = 1, 2, \dots$ the right expression of $S_N(t)$ in the form of a sum of Wiener–Itô integrals can be found similarly to the method applied in the proof of Theorem 3.2. We can write, similarly to the

proof of formulas (4.2) and (4.3)

$$S_N(t) = \sum_{\substack{(k_1,\dots,k_d),\, k_j\ge 0,\, 1\le j\le d,\\ k_1+\dots+k_d=k}} \int c_{k_1,\dots,k_d} f^N(t, x_1+\dots+x_k) \prod_{j=1}^{d} \left(\prod_{l=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G^{(N)},j}(\, dx_l) \right)$$

with

$$f^N(t,x) = \prod_{l=1}^{\nu} \frac{\exp\left\{ i \frac{]t^{(l)}N[}{N}(x^{(l)}) \right\} - 1}{N\left(\exp\left\{ i\frac{1}{N}(x^{(l)})\right\} - 1\right)},$$

where $t = (t^{(1)}, \dots, t^{(\nu)})$, the number $]t^{(l)}N[$ in the definition of the function $f^N(t, x_1, \dots, x_k))$ is the smallest integer which is not smaller than $t^{(l)}N$, and $Z_{G^{(N)},j}$ agrees with the spectral measure that appeared in formula (4.2).

It is not difficult to see that, similarly to relations (4.4) and (4.5)

$$\lim_{N\to\infty} f^N(t, x_1+\dots+x_k) = f^0(t, x_1+\dots+x_k)$$

with the function

$$f^0(t,x) = \prod_{l=1}^{\nu} \frac{e^{it^{(l)}(x^{(l)})} - 1}{i(x^{(l)})}$$

for all $(x_1, \dots, x_k) \in \mathbb{R}^{k\nu}$, and for a fixed parameter t this convergence is uniform in all bounded subsets of $\mathbb{R}^{k\nu}$.

With the help of the above considerations the proof of Theorem 3.4 can be reduced, similarly to the proof of Theorem 3.2 to the following statement.

Fix some number K, real constants $C_1, \dots, C_K$ and points $t_1, \dots t_K$ with $t_p \in [0,1]^\nu$, $1 \le p \le K$ together with some constants $c_{k_1,\dots,k_d}$ with parameters $k_j \ge 0$, $1 \le j \le d$, and $k_1 + \dots + k_d = k$ which agree with the coefficients in the sum (3.3). Let us define with their help the random sums

$$S_N = \sum_{\substack{(k_1,\dots,k_d),\, k_j\ge 0,\, 1\le j\le d,\\ k_1+\dots+k_d=k}} \int \left(\sum_{p=1}^{K} C_p c_{k_1,\dots,k_d} f^N(t_p, x_1+\dots+x_k) \right) \prod_{j=1}^{d} \left(\prod_{l=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G^{(N)},j}(\, dx_l) \right) \tag{5.6}$$

with the above defined functions $f^{(N)}(t, x_1, \dots, x_k)$ for all $N = 1, 2, \dots$, and

$$S_0 = \sum_{\substack{(k_1,\dots,k_d),\, k_j \ge 0,\, 1 \le j \le d, \\ k_1+\dots+k_d=k}} \int \left(\sum_{p=1}^{K} C_p c_{k_1,\dots,k_d} f^0(t_p, x_1 + \dots + x_k) \right) \prod_{j=1}^{d} \left(\prod_{l=k_1+\dots+k_{j-1}+1}^{k_1+\dots+k_j} Z_{G^{(0)},j}(\, dx_l) \right) \tag{5.7}$$

with the previously defined function $f^0(t, x_1, \dots, x_k)$. The sequence of random variables S_N defined in (5.6) converge in distribution to S_0 defined in (5.7) as $N \to \infty$.

This statement can be proved, similarly to Theorem 3.2 with the help of Proposition 4A. First the random variables S_N, $N = 1, 2, \dots$, and S_0 must be rewritten in a form in which Proposition 4A can be applied. They can be rewritten in the form of a sum of multiple Wiener–Itô integrals indexed by sequences of integers $j_1, \dots, j_k$ such that $1 \le j_1 \le \dots \le j_k \le d$. This can be done similarly to the rewriting of formulas (4.2) and (4.6) in formulas (4.14), (4.15) and (4.16), (4.17) with the help of the expressions $k_s(j_1 \dots, j_k)$ defined in (4.10). The random variable S_N in (5.6) can be rewritten as

$$S_N = \sum_{\substack{(j_1,\dots,j_k), \\ 1 \le j_1 \le \dots \le j_k \le d}} \int \left(\sum_{p=1}^{K} C_p c_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)} f^N(t_p, x_1 + \dots + x_k) \right) Z_{G^{(N)},j_1}(\, dx_1) \dots Z_{G^{(N)},j_k}(\, dx_k) \tag{5.8}$$

for all $N = 1, 2, \dots$, and the random variable in (5.7) as

$$S_0 = \sum_{\substack{(j_1,\dots,j_k), \\ 1 \le j_1 \le \dots \le j_k \le d}} \int \left(\sum_{p=1}^{K} C_p c_{k_1(j_1,\dots,j_k),\dots,k_d(j,\dots,j_k)} f^0(t_p, x_1 + \dots + x_k) \right) Z_{G^{(0)},j_1}(\, dx_1) \dots Z_{G^{(0)},j_k}(\, dx_k), \tag{5.9}$$

where the indices $k_s(j_1, \dots, j_k)$, $1 \le s \le d$, are defined in (4.10).

The random integrals in formulas (5.8) and (5.9) have kernel functions of the form

$$h^N_{j_1,\dots,j_k}(x_1, \dots, x_k) = h^N_{j_1\dots,j_k,t_1,\dots,t_K}(x_1, \dots, x_k) = \sum_{p=1}^{K} C_p c_{k_1(j_1,\dots,j_k),\dots,k_d(j_1,\dots,j_k)} f^N(t_p, x_1 + \dots + x_k) \tag{5.10}$$

for all $N = 0, 1, 2, \dots$. Let us define for all $N = 0, 1, 2, \dots$ the measures $\mu_{N,j_1,\dots,j_k}$ as

$$\begin{aligned} \mu^{(N)}_{j_1,\dots,j_k}(A) &= \mu^{(N)}_{j_1,\dots,j_k,t_1,\dots,t_K}(A) \qquad (5.11) \\ &= \int_A |h^N_{j_1,\dots,j_k,t_1,\dots,t_K}(x_1,\dots,x_k)|^2 G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k) \end{aligned}$$

where integral is taken for all measurable sets $A \in \mathcal{B}^{k\nu}$.

We want to show with the help of Proposition 4A that the distributions of the random variables S_N, $N = 1, 2, \dots$, defined in (5.8) converge weakly to the distribution of the random variable S_0 defined in (5.9). This implies Theorem 3.4.

To prove this convergence we have to show that the functions $h^N_{j_1,\dots,j_k}$, $N = 0, 1, 2, \dots$, defined in (5.10) and the measures $G^{(N)}_{j,j}$, $1 \le j \le d$, $N = 0, 1, 2, \dots$, satisfy the conditions of Proposition 4A. The main point is to prove relation (4.8) in condition (b) of Proposition 4A. To prove this we show that the measures $\mu^{(N)}_{j_1,\dots,j_k}$, $N = 1, 2, \dots$, defined in (5.11) are tight, i.e. for all $\varepsilon > 0$ there exists a $T = T(\varepsilon, j_1, \dots, j_k, t_1, \dots, t_K)$ such that

$$\mu^N_{j_1,\dots,j_k,t_1,\dots,t_K}(\mathbb{R}^{k\nu} \setminus [-T, T]^{k\nu}) < \varepsilon \quad \text{for all } N = 1, 2, \dots.$$

Because of the Schwarz inequality and the definition of the functions $h^N_{j_1\dots,j_k,t_1,\dots,t_K}(x_1, \dots, x_k)$ the proof of this tightness property can be reduced to the justification of the following inequality.

Let us define for all $t = (t_1, \dots, t_\nu) \in [0, 1]^\nu$, and $N = 1, 2, \dots$ the measure $\mu_{N,t}$ on $\mathbb{R}^{k\nu}$ by the formula

$$\begin{aligned} \mu_{N,t}(A) &= \int_A |f^N(t, x_1 + \dots + x_k)|^2 G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k) \\ &= \int_A \left| \prod_{l=1}^{\nu} \frac{\exp\left\{ i \frac{]t^{(l)}N[}{N} (x_1^{(l)} + \dots + x_k^{(l)}) \right\} - 1}{N\left(\exp\left\{ i \frac{1}{N} (x_1^{(l)} + \dots + x_k^{(l)}) \right\} - 1 \right)} \right|^2 \\ &\qquad\qquad G^{(N)}_{j_1,j_1}(\,dx_1)\dots G^{(N)}_{j_k,j_k}(\,dx_k) \end{aligned}$$

for all $A \in \mathcal{B}^{k\nu}$. The inequality

$$\mu_{N,t}(\mathbb{R}^{k\nu} \setminus [-T, T]^{k\nu}) < \varepsilon$$

holds for all $N = 1, 2, \dots$, if $T \ge T_0(\varepsilon, t)$ with an appropriate threshold index $T_0(\varepsilon, t) > 0$.

I claim that the measures $\mu_{N,t}$ converge weakly to a measure $\mu_{0,t}$ on $\mathbb{R}^{k\nu}$ as $N \to \infty$. This convergence implies the above inequality. This convergence can be proved similarly to Proposition 5.1. Namely, we can write

$$\prod_{l=1}^{\nu} \frac{\exp\left\{i\frac{]t^{(l)}N[}{N}(x_1^{(l)} + \dots + x_k^{(l)})\right\} - 1}{N\left(\exp\left\{i\frac{1}{N}(x_1^{(l)} + \dots + x_k^{(l)})\right\} - 1\right)} = \frac{1}{N^\nu} \sum_{u \in B_N(t)} \exp\left\{i\frac{1}{N}(u, x_1 + \dots + x_k)\right\},$$

where the set $B_N(t)$ was defined in (3.9). With the help of this formula the Fourier transform of the measure $\mu_{N,t}$ can be calculated in all points of the form $u = (u_1, \dots, u_k)$, $u_s = \frac{p_s}{N}$, $p_s \in \mathbb{Z}^\nu$, $1 \le s \le k$, This can be done similarly to the corresponding calculation in Proposition 5.1. Then a good asymptotic formula can be proved for this Fourier transform with the help of relation (3.1), and this implies the above mentioned convergence. Here again the method of proof in Proposition 5.1 is applied. I omit the details.

This implies that condition (b) of Proposition 4A holds in our model. The proof of the remaining conditions is much simpler. Similarly to the proof of Theorem 3.2 it can be shown with the help of Proposition 3.1 that the spectral measures $G_{j,j'}^{(N)}$ satisfy the required convergence property. Finally, it is not difficult to check that the functions $h_{j_1,\dots,j_k}^N$ defined in (5.10) satisfy condition (a) of Proposition 4A. □

Let me finally remark that a simple and natural modification in the proof of Theorem 3.4 shows that this result also holds if the random variables $S_0(t)$ in it are defined for all $t \in [0, \infty)^\nu$, (in the way as it is explained at the end of Section 3) and not only for $t \in [0, 1]^\nu$.

Appendix A. On the background of the limit theorems of this paper.

In the example after formula (3.1) I constructed a vector valued stationary random field with a spectral density function in such a way that its covariance function satisfies relation (3.1) with some appropriately defined functions $a_{j,j'}\left(\frac{p}{|p|}\right)$ and $L(p)$. The spectral density of this random field is close in

some sense to the spectral density of a vector valued generalized stationary random field. Moreover, the spectral density of this generalized random field has some homogeneity property.

I would like to point out that the spectral measures of all stationary Gaussian random fields whose covariance function satisfy (3.1) show a similar behavior. Indeed, take the spectral measure $(G_{j,j'}(\cdot))$, $1 \le j, j' \le d$, of such a random field whose covariance function $r_{j,j'}(p)$, $1 \le j, j' \le d$, $p \in \mathbb{Z}^\nu$ satisfies relation (3.1). Let us recall the results of Proposition 3.1 about the properties of this spectral measure.

The elements, $G^{(N)}_{j,j'}(\cdot)$, defined as $G^{(N)}_{j,j'}(A) = \frac{N^\alpha}{L(N)} G_{j,j'}\left(\frac{A}{N}\right)$, $1 \le j, j' \le d$, $N = 1, 2, \dots$, $A \subset \mathbb{R}^\nu$, of the rescaled versions of the spectral measure $G = (G_{j,j'}(\cdot))$, $1 \le j, j' \le d$, of a random field, whose covariance matrix satisfies (3.1) have a vague limit $G^{(0)}_{j,j'}(\cdot)$, when $N \to \infty$. These vague limits have the homogeneity property $G^{(0)}_{j,j'}(tA) = t^\alpha G^{(0)}_{j,j'}(A)$, $1 \le j, j' \le d$, for all $t > 0$ and measurable, bounded sets $A \subset \mathbb{R}^\nu$. Moreover, $(G^{(0)}_{j,j'}(\cdot))$, $1 \le j, j' \le d$, is the spectral measure of a generalized, stationary random field.

The above mentioned homogeneity property of the measures $G^{(0)}_{j,j'}$ is important for us, because it enables us to construct self-similar random fields, and in our limit theorems self-similar random fields appear as the limit. Here I recall the definition of self-similarity in a slightly more general situation than in the main text. In this definition vector valued random fields are considered. A vector valued random field $S(t) = (S_1(t), \dots, S_m(t))$, $t \in [0, \infty)^\nu$, of dimension m is called self-similar with parameter β, $\beta > 0$, if $S(ut) \stackrel{\Delta}{=} u^\beta S(t)$ for all $u > 0$, where $\stackrel{\Delta}{=}$ means that the finite dimensional distributions of the two random fields agree.

To understand how a vector valued self-similar random field can be constructed with the help of the spectral measure $(G_{j,j'}(\cdot))$ of a vector valued stationary generalized random field whose elements have the homogeneity property $G_{j,j'}(tA) = t^\alpha G_{j,j'}(A)$ with some $\alpha > 0$ let us first recall that the set of functions φ for which the random variable $Z_G(\varphi)$ of a generalized, vector valued Gaussian random field with spectral measure G is defined can be enlarged. Indeed, let $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ be a random spectral measure corresponding to the spectral measure $(G_{j,j'}(\cdot))$. In the original definition the elements of the vector valued, generalized Gaussian stationary random field corresponding to this random spectral measure are the random vectors $(Z_{G,1}(\varphi), \cdots, Z_{G,d}(\varphi))$, with $Z_{G,j}(\varphi) = \int \tilde{\varphi}(x) Z_{G,j}(\, dx)$, $1 \le j \le d$, where

the function $\varphi(\cdot)$ is an element of the Schwartz space $\mathcal{S}$, and $\tilde{\varphi}$ denotes its Fourier transform. These random integrals $Z_{G,j}(\varphi)$ can be defined for a larger class of functions. They can be defined for those real valued functions $\varphi(x)$, for which $\int |\tilde{\varphi}(x)|^2 \, G_{j,j}(\, dx) < \infty$ for all $1 \le j \le d$.

The spectral measure $(G_{j,j'}(\cdot))$, $1 \le j, j' \le d$, of those generalized random fields are important for us for which the domain of arguments of the random variables $Z_{G,j}(\varphi)$, $1 \le j \le d$, can be extended with the indicator functions of the rectangles $[0,t] = \prod_{s=1}^{\nu} [0,t_s]$ for all $t = (t_1, \dots, t_\nu] \in \mathbb{R}^\nu$, i.e.

$$\int \widetilde{|I_{[0,t]}}(x)|^2 G_{j,j}(\, dx) = \int \left(\prod_{s=1}^{\nu} \frac{2(1-\cos(t_s x_s))}{x_s^2} \right) G_{j,j}(\, dx) < \infty$$

for all $1 \le j \le d$. This inequality holds if $G_{j,j}(tA) = t^\alpha G_{j,j}(A)$ with some $0 < \alpha < 2\nu$ for all $t > 0$ and $1 \le j \le d$.

Let us consider a spectral measure $(G_{j,j'}(\cdot))$, $1 \le j, j' \le d$, such that

$$G_{j,j'}(tA) = t^\alpha G_{j,j'}(A) \text{ with some } 0 < \alpha < 2\nu$$

for all $t > 0$ and $1 \le j, j' \le d$, and let $(Z^\alpha_{G,1}, \dots, Z^\alpha_{G.d})$ be a random spectral corresponding to this spectral measure. (Here I put the homogeneity parameter α of the spectral measure in the upper index of the elements of the random spectral measure.) Consider for all pairs of vectors $t^{(1)} = (t^{(1)}_1, \dots, t^{(1)}_\nu) \in \mathbb{R}^\nu$ and $t^{(2)} = (t^{(2)}_1, \dots, t^{(2)}_\nu) \in \mathbb{R}^\nu$ such that $t^{(1)}_s < t^{(2)}_s$ for all $1 \le s \le \nu$ the rectangle $[t^{(1)}, t^{(2)}] = \prod_{s=1}^{\nu} [t^{(1)}_s, t^{(2)}_s]$, and define the random vectors

$$Z^\alpha_G([t^{(1)}, t^{(2)}]) = (Z^\alpha_{G,1}([t^{(1)}, t^{(2)}]), \dots, Z^\alpha_{G.d}([t^{(1)}, t^{(2)}]))$$

with coordinates

$$\begin{aligned} Z^\alpha_{G,j}([t^{(1)}, t^{(2)}]) &= \int \widetilde{I_{[t^{(1)}, t^{(2)}]}}(x) Z^\alpha_{G,j}(\, dx) \\ &= \int \left(\prod_{s=1}^{\nu} \frac{e^{i(t^{(2)}_s x_s} - e^{it^{(1)}_s) x_s}}{i x_s} \right) Z^\alpha_{G,j}(\, dx), \quad 1 \le j \le d, \end{aligned}$$

for all these rectangles.

Introduce the vectors $S_0(t) = Z^\alpha_G([0,t])$ for all $t \in \mathbb{R}^\nu$ with positive coordinates, where 0 denotes the origin in $\mathbb{R}^\nu$, and $X_0(p) = Z^\alpha([p-1, p])$ for all

$p \in \mathbb{Z}^\nu$, where $p-1 = (p_1-1, \dots, p_\nu-1)$ for $p = (p_1, \dots, p_\nu)$. Then $S_0(\cdot)$ is a vector valued self-similar random field with self-similarity parameter $\nu - \frac{\alpha}{2}$, $X_0(p)$, $p \in \mathbb{Z}^\nu$, is a vector valued stationary, Gaussian random field, and for a fixed vector $p = (p_1, \dots p_\nu) \in \mathbb{Z}^\nu$ and all $N = 1, 2, \dots$

$$\frac{1}{N^{\nu-\alpha/2}} \sum_{\substack{j=(j_1,\dots,j_\nu) \\ 1 \le j_s \le Np_s \text{ for all } 1 \le s \le \nu}} X_0(j) = \frac{1}{N^{\nu-\alpha/2}} S_0(Np) \stackrel{\Delta}{=} S_0(p).$$

A similar relation holds also for the linear combinations of the coordinates of the vector valued random field $X_0(p)$, $p \in \mathbb{Z}^\nu$. This means that these random fields satisfy the limit theorems of Theorems 3.2—3.4 for $k = 1$. Let me remark that the covariance function $r_{j,j'}(p) = EX_0(0)X_0(p))$, $p \in \mathbb{Z}^\nu$, satisfies relation (3.1). Indeed, it can be proved that

$$\begin{aligned} r_{j,j'}(p) &= \int \widetilde{I_{[0,1]}}(x)\widetilde{I_{[p,p+1]}}(x) G_{j,j'}(\,dx) \\ &= \int e^{i(p,x)} \left(\prod_{s=1}^{\nu} \frac{2(1-\cos x_s)}{x_s^2} \right) G_{j,j'}(\,dx) \\ &= C_{j,j'}\left(\frac{p}{|p|}\right) |p|^{-\alpha}(1+o(1)) \end{aligned}$$

with some function $C_{j,j'}\left(\frac{p}{|p|}\right)$ because of the homogeneity property of $G_{j,j'}(\cdot)$.

Theorems 1.2—1.4 in the case $k = 1$ state that the corresponding limit theorems also hold for models which satisfy Condition (3.1) with $0 < \alpha < \nu$. The restriction of the value of α to $0 < \alpha < \nu$ instead of $0 < \alpha < 2\nu$ in these results has a good reason. The partial sums which were normalized in these theorems have variances of order $N^{2\nu-\alpha}L(N)$. In the case $\alpha > \nu$ this means an exponent smaller than ν. So in the case $\alpha > \nu$ we can get a limit theorem (for $k = 1$) only in such models where both positive and negative covariances appear, and their effects compensate each other in a very special way.

In the case $k > 1$ a similar picture arises. Here again, we take random spectral measures corresponding to such spectral measures which have homogeneity property. We define the self-similar random fields we are working with in this case by means of k-fold Wiener–Itô integrals with respect to random spectral measures corresponding to them.

Given an integer $k \ge 2$ let us take the spectral measure $(G_{j,j'}(\cdot))$ of a generalized random field which has the homogeneity property $G_{j,j'}(tA) =$

$t^\alpha G_{j,j'}(A)$ for all measurable sets with finite diameter, $t > 0$, $1 \le j, j' \le d$, and a number $\alpha > 0$ whose possible value will be given later. Let us consider a random spectral measure $Z_G = (Z_{G,1}, \dots, Z_{G,d})$ corresponding to this spectral measure, and define with its help a random vector $Z_G([t^{(1)}, t^{(2)}])$ for all rectangles $[t^{(1)}, t^{(2)}]$ introduced in the previous construction with coordinates $Z_{G,j_1,\dots,j_k}([t^{(1)}, t^{(2)}])$, where $(j_1, \dots, j_k)$ is a sequence with the property $1 \le j_s \le d$ for all $1 \le s \le k$. These random variables are defined as k-fold Wiener–Itô integrals by the following formula.

$$\begin{aligned} Z_{G,j_1,\dots,j_k}([t^{(1)}, t^{(2)}]) &= \int \widetilde{I_{[t^{(1)},t^{(2)}]}}(x^{(1)} + \dots + x^{(k)}) \\ &\qquad Z_{G,j_1}(\,dx^{(1)}) \dots Z_{G,j_k}(\,dx^{(k)}) \\ &= \int \left(\prod_{s=1}^{\nu} \frac{e^{it_s^{(2)}(x_s^{(1)} + \dots + x_s^{(k)})} - e^{it_s^{(1)}(x_s^{(1)} + \dots + x_s^{(k)})}}{i\left(x_s^{(1)} + \dots + x_s^{(k)}\right)} \right) \\ &\qquad Z_{G,j_1}(\,dx^{(1)}) \dots Z_{G,j_k}(\,dx^{(k)}). \end{aligned}$$

(Actually, the value of this random integral depends only on the multiplicity of the numbers $1, 2, \dots, d$ in the sequence $(j_1, \dots, j_k)$. The order of these numbers in the sequence $j_1, \dots, j_k$ does not count.)

With the help of the above defined Wiener–Itô integrals let us define, similarly to the case $k = 1$, the random vector

$$Z_G([t_1, t_2]) = \{Z_{G,j_1,\dots,j_k}([t^{(1)}, t^{(2)}]) \colon\ 1 \le j_s \le d \text{ for all } 1 \le s \le l\}.$$

Naturally, we have to choose the spectral measure $(G_{j,j'}(\cdot)$ in such a way that the above k-fold Wiener–itô integral be meaningful. There is an integral which must be finite for the existence of these random integrals. In the case of k-fold Wiener–Itô this relation holds if the homogeneity parameter α of the underlying spectral measure satisfies the inequality $0 < \alpha < \frac{\nu}{k}$. The next calculation is an estimation which implies the existence of the above Wiener–itô integrals with such a choice of α. I omit the explanation why this calculation is correct, although this is not self-evident. Actually the existence of the random integrals I am considering here also follows from the results of the main text.

In the case $\alpha < \frac{\nu}{k}$

$$\int \left(\prod_{s=1}^{\nu} \frac{1 - \cos(t_s^{(2)} - t_s^{(1)})(x_s^{(1)} + \cdots + x_s^{(k)})}{\left(x_s^{(1)} + \cdots + x_s^{(k)}\right)^2} \right) G_{j_1,j_1}(\, dx^{(1)}) \dots G_{j_k,j_k}(\, dx^{(k)})$$

$$\le C \int \left(\prod_{s=1}^{\nu} \frac{1}{\left(1 + \left(x_s^{(1)} + \cdots + x_s^{(k)}\right)^2\right)} \right) |x^{(1)}|^{\alpha-\nu} \cdots |x^{(k)}|^{\alpha-\nu}\, dx^{(1)} \dots\, dx^{(k)}$$

$$\le C' \prod_{s=1}^{\nu} \left(\int \frac{|x_s^{(1)}|^{-1+\alpha/\nu} \cdots |x_s^{(k)}|^{-1+\alpha/\nu}}{\left(1 + \left(x_s^{(1)} + \cdots + x_s^{(k)}\right)^2\right)}\, dx_s^{(1)} \dots\, dx_s^{(k)} \right) < \infty.$$

Similarly to the case $k = 1$ such a special limit theorem can be presented which shows some similarity to the results of Theorems 3.2—3.4. Define, with similar notation as in the case $k = 1$, the random vectors $S_0(t) = Z_G([0, t])$ for all $t \in \mathbb{R}^\nu$ with positive coordinates, and $X_0(p) = Z^\alpha([p-1, p])$ for all $p \in \mathbb{Z}^\nu$. Then $S_0(\cdot)$ is a vector valued self-similar random field with self-similarity parameter $\nu - \frac{k\alpha}{2}$, $X_0(p)$, $p \in \mathbb{Z}^\nu$, is a vector valued stationary, Gaussian random field, and for a fixed vector $p = (p_1, \dots p_\nu) \in \mathbb{Z}^\nu$, $p_s > 0, \le \nu$, and all $N = 1, 2, \dots$

$$\frac{1}{N^{\nu - k\alpha/2}} \sum_{\substack{j=(j_1,\dots,j_\nu) \\ 1 \le j_s \le N p_s \text{ for all } 1 \le s \le \nu}} X_0(j) = \frac{1}{N^{\nu - k\alpha/2}} S_0(Np) \stackrel{\Delta}{=} S_0(p).$$

This is a limit theorem, where the limit is the above constructed self-similar random field $S_0(\cdot)$. Theorems 3.2–3.4 are limit theorems with the same limit. They hold for such vector valued Gausian stationary random fields whose covariance matrices are similar to the covariance matrix of a random field with that spectral measure ($G_{j,j'}(\cdot)$, $1 \le j, j' \le d$, which was applied in the definition of the self-similar random field $S_0(\cdot)$.

The goal of this Appendix was to explain the background of the results in this paper. Here I concentrated on the explanation of the definition of the

self-similar random fields which appear as the limit in our limit theorems. Their construction was based on the theory of multiple Wiener–Itô integrals for several dimensional stationary Gaussian random fields, in particular on the properties of the random spectral measures of generalized Gaussian random fields.

The proof of the results consisted of two steps. In the first step the random sums we wanted to study were rewritten as a sum of Wiener–Itô integrals. This could be done with the help of the multivariate version of Itô's formula formulated in Theorem 2.2 of [11]. Then limit theorems for sequences of sums of Wiener–Itô integrals had to be proved. This could be done with the help of Proposition 4A. Naturally, the Wiener–Itô integrals which define the limit random variables in Theorems 3.2–3.4 must exist. The condition $0 < \alpha < \frac{\nu}{k}$ in these theorems appear because of this condition. They are to guarantee the existence of the k-fold Wiener–Itô integrals which define the limit random variables in these results.

If the conditions of Theorems 3.2–3.4 hold, but with a parameter $\alpha \geq \frac{\nu}{k}$, then the random integrals defining the limit in these theorems do not exist. In such cases the central limit theorem holds with the classical normalization. This follows from the result of [3] in the scalar valued and from its multivariate generalization in Theorem 4 of [1] in the vector valued case. This problem is discussed in Appendix B of [11].

In this paper limit theorems were proved for non-linear functionals of stationary Gaussian random fields. I try to give a short overview about papers which deal with similar problems. The scalar valued version of the results in this paper was proved in [6]. M. S. Taqqu proved in paper [16] similar results. Both papers contain non-central limit theorems for sequences of non-linear functionals of (scalar valued) stationary Gaussian random fields. Taqqu's result has no multivariate version, and I do not know how such a result can be proved.

Paper [2] contains a result which tells when the classical central limit theorem holds for sequences of non-linear functionals of stationary Gaussian random fields similar to those considered in [6]. The book [12] generalizes the result of this paper. A. M. Arcones wanted to generalize both the central limit theorem of [2] and the non-central limit theorem of [6] for non-linear functionals of vector valued Gaussian random fields. He proved the central limit theorem part of these results in Theorem 4 of his paper [1]. He claimed to have also proved the multivariate version of the non-central limit theorem

in [6] in Theorem 6 of his paper. But I found the proof (and also the formulation) of this result problematic. Sanchez de Naranjo dealt also with this problem in his paper [14]. But his discussion contains some serious gaps. Hence I cannot consider his result a valid proof. The goal of the present paper was to formulate and prove the right multivariate version of the result in [6].

Let me also remark that H. C. Ho and T. C. Sun proved an interesting result in [8]. They proved a result which can be considered as an interesting mixture of the central and non-central limit theorem for non-linear functionals of stationary Gaussian random fields. They considered a two-dimensional vector valued stationary Gaussian random process (X_m, Y_m), $m \in \mathbb{Z}$, together with two non-linear functionals which are of such type as the non-linear functionals applied in [6]. They applied the first functionals for the process X_m, and the second functional for the process Y_m. In Theorem 1 of their paper they investigated the case when the sequence of the linear functionals defined with the help of the elements in the first coordinate satisfy a non-central, and the corresponding sequence defined with the help of the elements in the second coordinate satisfy the central limit theorem. They proved under some additional conditions that the joint distributions of these sequences also have a limit, and the two coordinates of this limit are independent. (The processes X_m and Y_m are not independent.) Let me remark that the usual proofs of the central and non-central limit theorem apply different methods. This indicates that the proof of this result in [8] demanded new ideas.

Donatas Surgailis proved results about similar problems, and they are also worth mentioning. He proved, together with some coauthors such results which can be considered as a generalization of the central and non-central limit theorems proved for non-linear functionals of stationary Gaussian random fields. He proved limit theorems for non-linear functionals of a new class of stationary stochastic processes which contains non-Gaussian processes, too. He worked with scalar valued random processes, but probably these results can be generalized also to vector valued random processes.

Surgailis has several articles about this subject. I would mention his paper [16], where he investigated non-linear functionals of moving average processes which may be non-Gaussian. He proved limit theorems for non-linear functionals of such processes. These results are very similar to those proved for Gaussian processes. The main point in the proofs of this paper is that Surgailis considered the Appel polynomials related to the moving

average process he was working with, and showed that the arguments applied in the Gaussian case can be adapted to the problems investigated by him if the Hermite polynomials are replaced by these Appel polynomials. (In the case of Gaussian moving averages the Appel polynomials are Hermite polynomials.)

Finally I briefly mention a field of research where similar limit theorems are proved with the help of essentially different arguments. This research deserves special attention because of its importance. This is the theory about the KPZ (Kardar–Parisi–Zhang) universality classes and the problems related to them. Many important problems can be studied with their help. On the other hand, the application of this theory demands hard analysis. An overview about it can be found in paper [4] together with a long list of literature.

Appendix B. Proof of the corollary of Theorem 3.4.

Proof of the corollary of Theorem 3.4. We want to show that for all pairs $\varepsilon > 0$ and $\eta > 0$ there exists some $\delta = \delta(\varepsilon, \eta) > 0$ and threshold index $N_0 = N_0(\varepsilon, \eta)$ such that for all $N \ge N_0$ the inequality

$$P\left(\sup_{\substack{(s,t):\ s,t\in[0,1]^\nu \\ |t-s|<\delta}} |S_N(t) - S_N(s)| > \varepsilon \right) < \eta \tag{B.1}$$

holds for the random field $S_N(t)$, $t \in [0,1]^\nu$, defined in (3.9) and (3.10).

Inequality (B.1) means that the random fields $S_N(t)$, $t \in [0,1]^\nu$, introduced in Theorem 3.4 satisfy besides the limit theorem formulated in Theorem 3.4 also the tightness condition for probability measures in the space of continuous functions $C([0,1]^\nu, \mathcal{C})$. It can be seen that these two properties together imply the desired functional limit theorem.

First I show that relation (B.1) can be replaced with the following set of simpler inequalities.

Define for all $1 \le j \le \nu$ and $\delta > 0$ the following set $V(j,\delta)$ consisting of pairs of vectors (s,t), $s \in [0,1]^\nu$ and $t \in [0,1]^\nu$.

$$\begin{aligned} V(j,\delta) \ = \ \{(s,t)\colon\ & s = (s_1,\dots,s_\nu),\ t = (s_1,\dots,s_{j-1},t_j,s_{j+1},\dots,s_\nu)\colon \\ & 0 \le s_l \le 1 \text{ for all } 1 \le l \le \nu, \text{ and } s_j \le t_j \le \min(1, s_j + \delta)\}. \end{aligned}$$

Then

$$P\left(\sup_{(s,t)\in V(j,\delta)}|S_N(t)-S_N(s)|>\varepsilon\right)<\eta \quad \text{for all } 1\le j\le\nu \tag{B.2}$$

if $\delta\le\delta(\varepsilon,\eta)$ and $N\ge N_0(\varepsilon,\eta)$ with some $\delta(\varepsilon,\eta)>0$ and $N_0(\varepsilon,\eta)$.

To see the possibility of such a reduction let us first observe that inequality (B.1) follows from its following formally weaker version.

For all $\varepsilon>0$ and $\eta>0$ there exists some $\delta=\delta(\varepsilon,\eta)>0$ and $N_0=N_0(\varepsilon,\eta)$ such that

$$P\left(\sup_{\substack{(s,t)\colon\ 0\le s_j<t_j\le 1,\\ \text{for all } 1\le j\le\nu,\ |t-s|<\delta}}|S_N(t)-S_N(s)|>\varepsilon\right)<\eta \tag{B.3}$$

if $N\ge N_0$.

Indeed, for a pair of vectors (s,t), $s\in[0,1]^\nu$, $t\in[0,1]^\nu$, define the vector

$$s^*=s^*(s,t)=(\min(s_1,t_1),\dots,\min(s_\nu,t_\nu)),$$

and consider the pairs (s^*,s) and (s^*,t). Let us apply relation (B.3) with parameters δ and N_0 corresponding to the parameters $\frac{\varepsilon}{2}$ and $\frac{\eta}{2}$. If the pair (s,t) satisfies the conditions appearing in the supremum of (B.1) with these parameters, then the pairs (s^*,s) and (s^*,t) satisfy the conditions in the supremum of (B.3) with the same parameter δ. Also the relation $|S_N(t)-S_N(s)|\le|S_N(s)-S_N(s^*)|+|S_N(t)-S_N(s^*)|$ holds. Hence inequality (B.3) with the above chosen δ and N_0 implies that

$$\begin{aligned}\sup_{\substack{(s,t)\colon\ s,t\in[0,1]^\nu\\ |t-s|<\delta}}|S_N(t)-S_N(s)|\le &\sup_{\substack{(s^*,s)\colon\ 0\le s_j^*<s_j\le 1,\\ \text{for all } 1\le j\le\nu,\ |s-s^*|<\delta}}|S_N(s)-S_N(s^*)|\\ +&\sup_{\substack{(s^*,t)\colon\ 0\le s_j^*<t_j\le 1,\\ \text{for all } 1\le j\le\nu,\ |t-s^*|<\delta}}|S_N(t)-S_N(s^*)|\le\varepsilon\end{aligned}$$

with probability more than $1-\eta$ if $N>N_0$. This means that relation (B.3) implies relation (B.1).

Relation (B.2) can be reduced to reation (B.3) in a similar way. To do this let us first define for a pair of vectors $s=(s_1,\dots,s_\nu)\in[0,1]^\nu$, $t=(t_1,\dots,t_\nu)\in[0,1]^\nu$ and number $1\le j\le\nu$ the vector $s(j)=s(j,s,t)=$

$(t_1, \dots, t_{j-1}, s_j, \dots, s_\nu)$ (for $j = 1$ $s(1) = (s_1, \dots, s_\nu) = s$), and consider the pairs of vectors $(s(j), s(j+1))$, $1 \le j \le \nu - 1$. Observe that $(s(j), s(j+1)) = (s(j,s,t), s(j+1,s,t)) \in V(j,\delta)$ if the pair (s,t) satisfies the relations $|t-s| < \delta$ and $0 \le s_j < t_j \le 1$ for all $1 \le j \le \nu$. Let us choose a $\delta > 0$ and N_0 in such a way that inequality (B.2) holds with parameters $\frac{\varepsilon}{\nu}$ and $\frac{\eta}{\nu}$ with this number δ and $N \ge N_0$. Let us take those pairs of vectors (s,t) which satisfy the conditions imposed in the supremum of formula (B.3) with this number δ. We have seen that

$$\{(s(j,s,t), s(j+1,s,t))\colon\ 0 \le s_l < t_l \le 1, \\ \text{for all } 0 \le l \le \nu, |t-s| \le \delta\} \subset V(j,\delta).$$

The identity $S_N(t) - S_N(s) = \sum_{j=1}^{\nu-1} [S_N(s(j+1,s,t)) - S_N(s(j,s,t))]$ also holds. These relations imply that with our choice of δ

$$\sup_{\substack{(s,t)\colon\ 0 \le s_j < t_j \le 1, \\ \text{for all } 1 \le j \le \nu,\ |t-s| < \delta}} |S_N(t) - S_N(s)| \le \sum_{j=1}^{\nu-1} \sup_{(s,t) \in V(j,\delta)} |S_N(t) - S_n(s)| \le \varepsilon$$

with probability more than $1 - \eta$ if $N > N_0$. This means that (B.2) implies (B.3).

Next I present an inequality with the help of the random variables $H(X_1(p), \dots, X_d(p))$ instead of $S_N(t)$ which implies inequality (B.2). For this goal I introduce the following notations.

Let us define the rectangle $D_N(r,s)$ for all pairs of vectors $r = (r_1, \dots, r_\nu)$ and $s = (s_1, \dots, s_\nu)$ with integer coordinates such that $0 \le r_j < s_j \le N$ for all $1 \le j \le \nu$ by the formula

$$D_N(r,s) = \{p = (p_1, \dots, p_\nu)\colon\ p \in \mathbb{Z}^\nu, \quad r_j < p_j \le s_j \text{ for all } 1 \le j \le \nu\},$$

and introduce for all $\delta > 0$ and $1 \le j \le \nu$ a set $\mathcal{D}_N(\delta, j)$ consisting of the above defined rectangles $D_N(r,s)$ with some additional properties. We define

$$\begin{aligned} \mathcal{D}_N(\delta, j) \ = \ & \{D_N(r,s)\colon\ r_l = 0 \text{ for } l \ne j,\ \ 0 \le r_j \le N, \\ & 0 < s_l \le N \text{ for all } 1 \le l \le \nu, \text{ and } 0 < s_j - r_j \le \delta N\}. \end{aligned}$$

Inequality (B.2) follows from the relation

$$P\left(\sup_{D_N(r,s)\in\mathcal{D}_N(\delta,j)}\left|\frac{\sum\limits_{p\in D_N(r,s)}H(X_1(p),\dots,X_d(p))}{N^{\nu-k\alpha/2}L(N)^{k/2}}\right|>\varepsilon\right)\le\eta$$
$$\text{for all } 1\le j\le\nu \tag{B.4}$$

if $\delta\le\delta(\varepsilon,\eta)$ with some $\delta(\varepsilon,\eta)>0$. Here, and also in the remaining part of the proof $H(x_1,\dots,x_d)$ is a sum of the form

$$H(x_1,\dots,x_d)=H^{(0)}(x_1,\dots,x_d)+H^{(1)}(x_1,\dots,x_d),$$

with functions $H^{(0)}(\cdot)$ and $H^{(1)}(\cdot)$ defined in formulas (3.3) and (3.8).

In formulas (B.4) and (B.2) very similar expressions are estimated. The main difference between them is that in (B.2) random variables of the form $|S_N(t)-S_N(s)|$ are considered with arguments $s,t\in[0,1]^\nu$, while in (B.4) random variables of the form $\left|S_N\left(\frac{s}{N}\right)-S_N\left(\frac{r}{N}\right)\right|$ with arguments $\frac{s}{N}$ and $\frac{r}{N}$, where s and r are vectors with integer coordinates with values between 0 and N. This is a sort of discretization, and in the reduction of (B.4) to (B.2) it has to be shown that this discretization has a negligible effect in the case of a large sample size N.

This can be seen with the help of the following observation. If N is large, then because of the definition of the random field $S_N(\cdot)$ for all $t\in[0,1]^\nu$ there exists a vector $r=(r_1,\dots,r_\nu)$ with integer coordinates r_j, $0\le r_j\le N$, such that $S_N(t)=S_N\left(\frac{r}{N}\right)$, and t and $\frac{r}{N}$ are very close to each other.

Inequality (B.4) will be proved by means of a good estimate on the tail distribution of the random variables $\sum\limits_{p\in D_N(r,s)}H(X_1(p),\dots,X_d(p))$ for the rectangles $D_N(r,s)$. These expressions will be estimated by means of an argument similar to the proof of Theorem 3.3. To do this let us first remark that Lemma 1 of [1] implies the following result, too. The inequality in Lemma 5A holds also in the case when the function $H^{(1)}(x_1,\dots,x_d)$ is replaced in it by $H(x_1,\dots,x_d)$, and the coefficient ψ^{k+1} in the upper bound is replaced by ψ^k.

This modified version of Lemma 5A yields that there exists a threshold index n_0 and some constant $C>0$ such that if the parameters p and q of two elements $X(p)=(X_1(p),\dots,X_d(p))$ and $X(q)=(X_1(q),\dots,X_d(q))$ of our random field satisfy the inequality $|p-q|\ge n_0$, then

$$|EH(X_1(p),\dots X_d(p))H(X_1(q),\dots,X_d(q))|\le C|p-q|^{-k\alpha}L(|p-q|)^k.$$

Let us observe that for such pairs p and q also the inequality

$$|p-q|^{-k\alpha}L(|p-q|)^k \le C \prod_{\substack{j\colon\ 1\le j\le\nu\\ p_j-q_j\neq 0}} |p_j-q_j|^{-k\alpha/\nu}(L(|p_j-q_j|)^{k/\nu}+I_{\{|p_j-q_j|<D\}})$$

holds with some $C > 0$ and $D > 0$, where $I_{\{x<D\}}$ denotes the indicator function of the set $\{x\colon\ x < D\}$. Hence the previous estimate has the consequence

$$\begin{aligned}|EH(X_1(p),\dots X_d(p))H(X_1(q),\dots,X_d(q))|\\ \le C \prod_{\substack{j\colon\ 1\le j\le\nu\\ p_j-q_j\neq 0}} |p_j-q_j|^{-k\alpha/\nu}(L(|p_j-q_j|)^{k/\nu}+I_{\{|p_j-q_j|\le D\}})\end{aligned}$$

if $|p-q| \ge n_0$. This inequality is more appropriate for us than the previous one.

On the other hand, the inequality

$$\begin{aligned}|EH(X_1(p),\dots,X_d(p))H(X_1(q),\dots,X_d(q))|\\ \le EH^2(X_1(0),\dots X_d(0)) \le C_1\end{aligned}$$

also holds for all $p, q \in \mathbb{Z}^\nu$ with some $C_1 < \infty$ because of the Schwarz inequality and relation (3.8).

The last two inequalities imply that for any rectangular $D_N(r,s) \subset B_N$ and $p \in D_N(r,s)$

$$\begin{aligned}\left|EH(X_1(p),\dots,X_d(p))\left(\sum_{q\in D_N(r,s)} H(X_1(q),\dots,X_d(q))\right)\right| \qquad \text{(B.5)}\\ \le C_2\prod_{j=1}^{\nu}\left(1+(s_j-r_j)^{1-k\alpha/\nu}L(s_j-r_j)^{k/\nu}\right)\end{aligned}$$

with an appropriate constant C_2. Indeed, these inequalities imply that

$$\begin{aligned}\left|EH(X_1(p),\dots,X_d(p))\left(\sum_{q\in D_N(r,s)} H(X_1(q),\dots,X_d(q))\right)\right|\\ \le C\prod_{j=1}^{\nu}\left(1+2\sum_{q_j=1}^{s_j-r_j} q_j^{-k\alpha/\nu}(L(q_j)^{k/\nu}+I_{\{q_j\le D\}})\right)\end{aligned}$$

and $\sum_{q_j=1}^{s_j-r_j} q_j^{-k\alpha/\nu}(L(q_j)^{k/\nu} + I_{\{q_j<\le D\}}) \le C'(1 + (s_j - r_j)^{1-k\alpha/\nu}L(s_j - r_j)^{k/\nu})$ with some $C' > 0$, since $k\alpha/\nu < 1$ by the conditions of Theorem 3.4. These relations imply (B.5).

By summing up inequality (B.5) for all $p \in D_N(r, s)$, and applying an appropriate normalization we get that

$$\frac{1}{N^{2\nu-k\alpha}L(N)^k} E\left[\sum_{p\in D_N(r,s)} H(X_1(p), \dots, X_d(p)\right]^2 \le C_2 \prod_{j=1}^{\nu} \left(\frac{(s_j - r_j) + (s_j - r_j)^{2-k\alpha/\nu}L(s_j - r_j)^{k/\nu}}{N^{2-k\alpha/\nu}L(N)^{k/\nu}}\right). \tag{B.6}$$

I claim that if $1 \le s_j - r_j \le N$ for some $1 \le j \le \nu$ and $\eta > 0$ chosen so small that $\beta = \frac{k\alpha}{\nu} + \eta < 1$, then

$$\frac{(s_j - r_j) + (s_j - r_j)^{2-k\alpha/\nu}L(s_j - r_j)^{k/\nu}}{N^{2-k\alpha/\nu}L(N)^{k/\nu}} \le C\left(\frac{s_j - r_j}{N}\right)^{2-k\alpha/\nu-\eta} \tag{B.7}$$

with some $C = C(\eta) > 0$.

Indeed,

$$\begin{aligned}\frac{(s_j - r_j)}{N^{2-k\alpha/\nu}L(N)^{k/\nu}} &= \left(\frac{s_j - r_j}{N}\right)^{2-k\alpha/\nu-\eta} (s_j - r_j)^{k\alpha/\nu+\eta-1} \frac{N^{-\eta}}{L(N)^{k/\nu}} \\ &\le C\left(\frac{s_j - r_j}{N}\right)^{2-k\alpha/\nu-\eta}\end{aligned}$$

if $\eta > 0$ is chosen so small that $\frac{k\alpha}{\nu} + \eta < 1$, and

$$\frac{(s_j - r_j)^{2-k\alpha/\nu}L(s_j - r_j)^{k/\nu}}{N^{2-k\alpha/\nu}L(N)^{k/\nu}} = \left(\frac{s_j - r_j}{N}\right)^{2-k\alpha/\nu-\eta} \left(\frac{s_j - r_j}{N}\right)^{\eta} \left(\frac{L(s_j - r_j)}{L(N)}\right)^{k/\nu} \le C\left(\frac{s_j - r_j}{N}\right)^{2-k\alpha/\nu-\eta}.$$

These two inequalities imply (B.7).

Let us choose a sufficiently large number $D > 0$ (whose value does not depend on N and δ), and introduce the quantity $d_N(p) = \frac{D}{N}$ for all $1 \le p \le N$.

With such a notation we can get the following inequality with the help of relations (B.6) and (B.7).

Take some rectangle $D_N(r,s) \subset B_N$. Then we have for any $\lambda > 0$

$$P\left(\frac{\left| \sum\limits_{p \in D_N(r,s)} H(X_1(p), \dots, X_d(p)) \right|}{N^{\nu - k\alpha/2} L(N)^{k/2}} > \lambda \right) \tag{B.8}$$

$$\le \frac{1}{\lambda^2} \frac{E\left[\sum\limits_{p \in D_N(r,s)} H(X_1(p), \dots, X_d(p)) \right]^2}{N^{2\nu - k\alpha} L(N)^k}$$

$$\le \frac{C_3}{\lambda^2} \prod_{j=1}^{\nu} \left(\frac{s_j - r_j}{N} \right)^{2 - k\alpha/\nu - \eta} \le \frac{1}{\lambda^2} \prod_{j=1}^{\nu} \left(\sum_{p_j = r_j + 1}^{s_j} d_N(p_j) \right)^{2-\beta}$$

with $\beta = \frac{k\alpha}{\nu} + \eta < 1$.

With the help of formula (B.8) one can get such a maximum-type inequality which implies formula (B.4).

In the case $\nu = 1$ Theorem 10.2 of Billingsley's book [2] can be applied. In this case this result together with formula (B.8) imply that

$$P\left(\sup_{(u,v):\ r < u < v \le s} \left| \frac{\sum\limits_{u < p \le v} H(X_1(p), \dots, X_d(p))}{N^{\nu - k\alpha/2} L(N)^{k/2}} \right| > \lambda \right)$$

$$\le \frac{K}{\lambda^2} \left(\sum_{r < p \le s} d_N(p) \right)^{2-\beta} = \frac{D^{2-\beta} K n}{\lambda^2} \left(\frac{s - r}{N} \right)^{2-\beta}$$

for any pairs $0 \le r < s \le N$ with some $K > 0$. In particular,

$$P\left(\sup_{(u,v):\ r < u < v \le r + \delta} \left| \frac{\sum\limits_{u < \frac{p}{N} \le v} H(X_1(p), \dots, X_d(p))}{N^{\nu - k\alpha/2} L(N)^{k/2}} \right| > \lambda \right) \le \frac{K'}{\lambda^2} \delta^{2-\beta}$$

for any interval $[r, r + \delta] \subset [0,1]$ with $\delta > 0$,

Since the exponent of δ in the last inequality equals $2 - \beta > 1$ it is not difficult to see that this relation implies inequality (B.4) in the case $\nu = 1$.

Indeed, we get (B.4) by applying this inequality with the choice $\lambda = \varepsilon$ for the intervals $[k\delta, (k+2)\delta]$ for all $0 \le k < \frac{1}{\delta}$ with a sufficiently small $\delta > 0$. Then inequality (B.4) implies inequality (B.1), too.

There is a multivariate version of the inequality cited from Billingsley's book [2] also in the case $\nu > 1$ which, together with formula (B.8) imply inequality (B.4) in the general case. This inequality implies for any $\nu \ge 1$ that if inequality (B.8) holds for all rectangles $D_N(r,s)$, then

$$P\left(\sup_{\substack{D_N(u,v):\\ D_N(u,v)\subset D_N(r,s)}} \left| \frac{\sum\limits_{p\in D_N(u,v)} H(X_1(p),\dots,X_d(p))}{N^{\nu-k\alpha/2}L(N)^{k/2}} \right| > \lambda \right) \tag{B.9}$$
$$\le \frac{K}{\lambda^2}\prod_{j=1}^{\nu}\left(\sum_{r_j<p_j\le s_j} d_N(p_j)\right)^{2-\beta} = \frac{D^{(2-\beta)\nu}K}{\lambda^2}\prod_{j=1}^{\nu}\left(\frac{s_j-r_j}{N}\right)^{2-\beta}$$

with some $K > 0$, $D > 0$ and $\beta = \frac{k\alpha}{\nu} + \eta < 1$. (Here we are working with the previously defined $d_N(p) = \frac{D}{N}$.)

Indeed, although I did not find this result in the literature there is such a generalized version of the inequality quoted from Billingsley's book which states that if inequality (B.8) holds, then it implies inequality (B.9), too. This can be proved for instance by means of induction with respect to the dimension ν by exploiting that this result holds for $\nu = 1$. In the proof we have to exploit that the upper bound in (B.8) has a special product form.

Let us fix some parameter $1 \le j \le \nu$, a number $0 < \delta \le 1$ an integer $0 \le r \le N$, and define with their help, similarly to the definition of $\mathcal{D}_N(\delta, j)$ the set of of rectangles

$$\begin{aligned}\mathcal{D}_N(\delta,j,r) \;=\; \{D_N(u,v)\colon\; & u_l = 0, \text{ and } 0 < v_l \le N \text{ for } l \ne j,\\ & \text{and } \; r \le u_j < v_j \le r + N\delta\}.\end{aligned}$$

The following inequality is a special case of (B.9).

$$P\left(\sup_{D(u,v)\in\mathcal{D}_N(\delta,j,r)} \left| \frac{\sum\limits_{p\in D_N(u,v)} H(X_1(p),\dots,X_d(p))}{N^{\nu-k\alpha/2}L(N)^{k/2}} \right| > \lambda \right)$$
$$\le \frac{K'}{\lambda^2}\left(\frac{s_j-r_j}{N}\right)^{2-\beta}.$$

Inequality (B.4), hence inequality (B.1) can be proved with the help of the last inequality in the same way as it was done for $\nu = 1$. Actually, in that proof a special case of this inequality was applied. Since, as it was mentioned at the start of the proof relation (B.1) together with Theorem 3.4 imply the desired weak convergence the corollary is proved. □

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References

[1] Arcones, A. M. (1994) Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors. Annals of Probability **22**, 2242–2274

[2] Billingsley P. (1999) Convergence of Probability Measures, Second Edition. Wiley Series in Probability and Statistics, John Wiley & Sons, Inc.

[3] Breuer, P. and Major, P. (1983) Central limit theorems for non-linear functionals of Gaussian fields. Journal of Multivariate Analysis **13**, 425–441

[4] Corwin, I. (2012) The Kardar–Parisi–Zhang equation and university class. Random Matrices: Theory and Applications **1**, No. 1

[5] Cramer, H. (1940) On the theory of stationary random processes. Ann. Math. **91**, 215–230

[6] Dobrushin, R. L. and Major, P. (1979) Non-central limit theorems for non-linear functionals of Gaussian fields. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 27–52.

[7] Gelfand I. M. and Shilov G. E. (1964) Generalized Functions, Volume 1. Properties and Operations. Academic Press, New York and London

[8] Ho, H. C. and Sun, T. C. (1990) Limit distributions of non-linear vector functions of stationary Gaussian processes. Annals of Probability **18**, 1159–1173

[9] Major, P. (2014) Multiple Wiener–Itô Integrals. *Lecture Notes in Mathematics* **849**, Revised (augmented) version, Springer Verlag, Berlin–Heidelberg–New York.

[10] Major, P. (2020) The theory of Wiener–Itô integrals in vector valued Gaussian stationary random fields. Part I. Moscow Math. Journal, **20**, 4, 749–812

[11] Major, P. (2023) The theory of Wiener–Itô integrals in vector valued Gaussian stationary random fields. Part II. Moscow Math, Journal, **23**, 3, 331–367

[12] Nourdin I. and Peccati G.(2012) Normal Approximations with Malliavin calculus. Cambridge University Press

[13] Rozanov, Yu. A. (1958) Spectral theory of n-dimensional stationary stochastic processes with discrete time. Uspekhi Math. Nauk 13:2**80**, 93–142 (in Russian)

[14] Sanchez de Naranjo, M. V. (1993) Non-central limit theorems for non-linear functionals of k Gaussian fields. Journal of Multivariate Analysis **44**, 227–255

[15] Surgailis, D. (2000) Long-range dependence and Appel rank. Annals of Probability **28**, 478–497

[16] Taqqu, M. S. (1979) Convergence of iterated process of arbitrary Hermite rank. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 53–83.