A counter–example in ergodic theory

Péter Major
Mathematical Institute of the Hungarian Academy of Sciences
and
Bolyai College of the Eötvös Loránd University, Budapest

address: H–1364, Budapest, P.O.B. 127
(Mathematical Institute of the Hungarian Academy of Sciences)
e-mail address: major@math-inst.hu
AMS Subject Classification (1991) 47A35

Abstract: We construct a (non-integrable) function f and two measure preserving, ergodic transformations $\mathbf{S}$ and $\mathbf{T}$ on a measure space $(\mathcal{X},\mathcal{A},\mu)$, $\mu(\mathcal{X})=1$, in such a way that the ergodic means $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n} f(\mathbf{S}^k x)$ and $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n} f(\mathbf{T}^k x)$ exist for almost all x, they are finite constants not depending on x, but these constants differ when we are averaging with respect to the operators $\mathbf{S}$ and $\mathbf{T}$. This means that in the case of a non-integrable function f and an ergodic transformation $\mathbf{T}$ the ergodic mean depends not only on the function f, but also on the transformation $\mathbf{T}$. The construction applies some probabilistic arguments.

The aim of this paper is to construct a probability space $(\mathcal{X},\mathcal{A},\mu)$, a function f and two ergodic transformations $\mathbf{T}$ and $\mathbf{S}$ on it such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbf{T}^k f=0 \quad \text{a. s.} \tag{1}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathbf{S}^k f=a \quad \text{a. s.} \tag{2}$$

with some constant $0<a<\infty$. Here and in the sequel the same symbol is used for a measure preserving transformation and the linear operator it induces on measurable functions by composition. (Because of the ergodic theorem such an example is possible only with a function f such that $\int f^+\,d\mu=\infty$ and $\int f^-\,d\mu=-\infty$.)

The question about the possibility of such an example was raised by Zoltán Buczolich. He studied the problem that for how large class of functions the classical notion of integral can be extended. He was interested in the relation of this problem to the ergodic theorem (see [1]), and this was the reason for asking such a question. But we hope that this question can also be interesting for its own sake.

We define the probability space $(\mathcal{X},\mathcal{A},\mu)$ as $\mathcal{X}=[0,1]^{\mathbf{Z}}$, where $\mathbf{Z}$ is the set of integers, $\mathcal{A}$ is the Borel σ-algebra on $\mathcal{X}$ and $\mu=\lambda^{\mathbf{Z}}$, the direct product of different copies of the

Lebesgue measure on $[0,1]$. Let $\mathbf{T}$ be the shift operator to the left on $\mathcal{X}$, i.e. for $x \in \mathcal{X}$, $x = (\dots, x_{-1}, x_0, x_1, \dots)$

$$\mathbf{T}(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, x_0, x_1, x_2, \dots)\ .$$

To define the function f we introduce the sequences

$$A_n = \left[\frac{2^{n+10}}{n^2}\right], \quad n = 1, 2, \dots\ ,$$

$$B_0 = 0, \quad B_n = 2(A_1 + \dots + A_n), \quad n = 1, 2, \dots\ ,$$

where $[\cdot]$ denotes integer part. Put $f_k(u) = 0$ for $k \le 0$, $0 \le u \le 1$, define the functions $f_k(u)$ for $B_{n-1} < k \le B_n$, $n = 1, 2, \dots$, on $[0,1]$ as

$$f_k(u) = \begin{cases} n^3 & \text{for } 0 \le u < 2^{-n} \\ 0 & \text{otherwise} \end{cases} \qquad \text{if } k \text{ is odd}$$

$$f_k(u) = \begin{cases} -n^3 & \text{for } 0 \le u < 2^{-n} \\ 0 & \text{otherwise} \end{cases} \qquad \text{if } k \text{ is even}\ ,$$

and set

$$f(x) = \sum_{k=-\infty}^{\infty} f_k(x_k)\ , \qquad x = (\dots, x_{-1}, x_0, x_1, \dots)\ .$$

The relation

$$\sum_{k=-\infty}^{\infty} \mu(f_k(x_k) \neq 0) = \sum_{n=1}^{\infty} 2A_n 2^{-n} \le 2^{11} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

holds, hence the Borel–Cantelli lemma implies that for almost all x with respect to the measure μ the sum defining $f(x)$ contains only finitely many non-zero terms. Hence this sum is meaningful. Moreover, since the sum defining $f(x)$ contains finitely many non-zero terms with probability one, the rearrangements of summation appearing in this article are legitimate.

First we show that the above defined $\mathbf{T}$ and f satisfy (1), then define an operator $\mathbf{S}$ which satisfies (2).

Write

$$\frac{1}{n}\sum_{k=1}^{n} \mathbf{T}^k f(x) = \frac{1}{n}\sum_{m=-\infty}^{\infty} H_{m,n}(x_m) = \frac{1}{n}\sum_{m=-n+1}^{B_L} H_{m,n}(x_m)$$
$$+\frac{1}{n}\sum_{m=B_L+1}^{\infty} H_{m,n}(x_m) = Z_n^{(1)}(L) + Z_n^{(2)}(L)$$

with arbitrary $L > 0$ and

$$H_{m,n}(u) = \sum_{k=m+1}^{m+n} f_k(u)\ , \quad 0 \le u \le 1\ .$$

The relation

$$\mu\left(Z_n^{(2)}(L)\neq 0 \text{ for some } n\geq 1\right)\leq \sum_{k=L}^{\infty}\sum_{m=B_k+1}^{B_{k+1}}\mu(x_m\in[0,2^{-k-1}]) \tag{3}$$
$$=2\sum_{k=L}^{\infty}A_{k+1}2^{-k}<\frac{\text{const.}}{L}\ , \quad \text{where } x=(x_m,\ m\in\mathbf{Z})$$

holds.

Observe that the functions $H_{m,n}(x_m)$ are independent for fixed n, and $H_{m,n}(u)=j(n+m)f_{n+m}(u)-j(m)f_{m+1}(u)$, where $j(m)=1$ if m is odd, and $j(m)=0$ if m is even. Define the moments $E_n^{(p)}(m)=\int H^p_{m,n}(x_m)\,d\mu(x)$ of the random variables $H_{m,n}(x_m)$, $p=1,2\ldots$. The identity $E_n^{(1)}(m)=j(n+m)\ell^3(n+m)2^{-\ell(n+m)}-j(m)\ell^3(m+1)2^{-\ell(m+1)}$ holds, where $\ell(m)$ denotes the number l for which $B_{l-1}<m\leq B_l$ if $m\geq 1$, and $\ell(m)=0$ if $m\leq 0$. Also the inequalities $|E_n^{(p)}(m)|\leq C(p)$, $p=1,2,\ldots$, hold with some constant $C(p)$. These relations together with the independence of the variables $H_{m,n}(x_m)$ imply that

$$\lim_{n\to\infty} EZ_n^{(1)}(L)=\lim_{n\to\infty}\frac{1}{n}\sum_{m=-n+1}^{B_L}E_n^{(1)}(m)=0\ , \tag{4}$$

and

$$E\left(Z_n^{(1)}(L)-EZ_n^{(1)}(L)\right)^4\leq\frac{\text{const.}}{n^2}\ .$$

Hence

$$\mu\left(\left|Z_n^{(1)}(L)-EZ_n^{(1)}(L)\right|>\varepsilon\right)=\mu\left(\left|Z_n^{(1)}(L)-EZ_n^{(1)}(L)\right|^4>\varepsilon^4\right)\leq\frac{\text{const.}}{n^2\varepsilon^4}\ ,$$

and since the right-hand side of the last inequality is summable in n, the Borel–Cantelli lemma implies that

$$\lim_{n\to\infty} Z_n^{(1)}(L)-EZ_n^{(1)}(L)=0 \quad \text{a. s.}\ . \tag{5}$$

Since L can be chosen arbitrary large, relations (3), (4) and (5) imply relation (1).

The operator $\mathbf{S}$ we shall construct is an appropriate conjugate of $\mathbf{T}$. First we conjugate $\mathbf{T}$ with an operator $\mathbf{U}_\pi$ which induces a permutation of the coordinates x_k. Then we apply another conjugation with an operator $\mathbf{G}$ which places the range of the functions f_k to other intervals. We describe this construction in more detail.

Let π be a permutation of the positive integers $\mathbf{Z}^+=\{1,2,\ldots\}$. We denote its extension to $\mathbf{Z}$ defined by the formula $\pi(k)=k$ for $k\leq 0$ again by π and define the transformation $\mathbf{U}_\pi$ on $\mathcal{X}$ as

$$\mathbf{U}_\pi x=(x_{\pi(k)},\ k\in\mathbf{Z})\,,\ \text{for } x=(x_k,\ k\in\mathbf{Z})\,.$$

For all $k\in\mathbf{Z}$ define a transformation $g_k\colon[0,1]\to[0,1]$ such that either $g_k(u)=u$ or it is an interval exchange transformation defined in the following way: We consider two disjoint intervals $I_k^{(1)} = [a(k), b(k))$ and $I_k^{(2)} = [c(k), d(k))$ such that $0 \le a(k) < b(k) \le 1$, $0 \le c(k) < d(k) \le 1$, $b(k) - a(k) = d(k) - c(k)$, and put

$$g_k(u) = \begin{cases} u & \text{if } u \in [0,1] \setminus (I_k^{(1)} \cup I_k^{(2)}) \\ u + c(k) - a(k) & \text{if } u \in I_k^{(1)} \\ u + a(k) - c(k) & \text{if } u \in I_k^{(2)} \end{cases} . \tag{6}$$

Given the above defined set of functions $g_k(u)$, $k \in \mathbf{Z}$, we introduce the following transformation $\mathbf{G}$ of $\mathcal{X}$:

$$\mathbf{G}(x) = (g_k(x_k),\ k \in \mathbf{Z}), \quad \text{for } x = (x_k,\ k \in \mathbf{Z}) .$$

We remark that $\mathbf{G}^{-1} = \mathbf{G}$ and define with the help of the above considered permutation π and $\mathbf{G}$ the transformation

$$\mathbf{S} = \mathbf{S}(\pi,\, g_k,\, k \in \mathbf{Z}) = \mathbf{G}\mathbf{U}_\pi \mathbf{T} \mathbf{U}_\pi^{-1} \mathbf{G} .$$

Clearly,

$$\mathbf{S}^k = \mathbf{G}\mathbf{U}_\pi \mathbf{T}^k \mathbf{U}_\pi^{-1} \mathbf{G} .$$

Observe that

$$\mathbf{S}^k f(x) = \sum_{m=-\infty}^{\infty} f_{\pi(k+\pi^{-1}(m))}\big(g_{\pi(k+\pi^{-1}(m))}(g_m(x_m))\big) \quad \text{for } x = (x_k,\ k \in \mathbf{Z}) ,$$

and this formula together with the relations $\pi(k) = k$ and $f_k(u) \equiv 0$ for $k \le 0$ imply that

$$\frac{1}{n} \sum_{k=1}^{n} \mathbf{S}^k f(x) = \sum_{m=-n+1}^{\infty} U_{m,n}(x_m) \tag{7}$$

with

$$U_{m,n}(u) = \frac{1}{n} \sum_{k=1}^{n} f_{\pi(k+\pi^{-1}(m))}\big(g_{\pi(k+\pi^{-1}(m))}(g_m(u))\big) .$$

We want to construct the operator $\mathbf{S}$ in such a way that

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} U_{m,n}(x_m) = 0 \quad \text{a. s.} , \tag{8}$$

and

$$\lim_{n\to\infty} \sum_{m=-n+1}^{0} U_{m,n}(x_m) = a > 0 \quad \text{a. s.} . \tag{9}$$

Relations (7), (8) and (9) imply (2). We want to guarantee (8) by defining π and $\mathbf{G}$ in such a way that for large m the probability of the set where $U_{n,m}(x_m) \neq 0$ is negligibly small and to deduce (9) from the law of large numbers. To accomplish this goal we make such a construction where the cancellations between different functions f_k guarantee that $U_{m,n}(x_m)$ is strictly positive and relatively small. Before this construction we show that a transformation $\mathbf{S}$ defined in the above way is ergodic.

The transformation $\mathbf{S}$ is measure preserving. Given a measurable set $A \subset \mathcal{X}$ define the set B such that $A = \mathbf{G}\mathbf{U}_\pi B$. Then $\mathbf{S}^{-1}A = \mathbf{G}\mathbf{U}_\pi \mathbf{T}^{-1}B$. Hence $A = \mathbf{S}^{-1}A$ if and only if $B = \mathbf{T}^{-1}B$. The latter relation can hold only if $\mu(B)$, hence $\mu(A)$ equals zero or one because of the ergodicity of the operator $\mathbf{T}$.

To define the permutation π and functions g_k such that the operator $\mathbf{S}$ given with their help satisfies relations (8) and (9) we introduce some sequences.

Put $M_n = n^3 2^{-n}$ and

$$\begin{aligned} &P_1 = A_1, \quad P_n = 2\left\lceil \frac{P_{n-1}}{2}\frac{M_{n-1}}{M_n}\right\rceil, \quad n = 2,3,\dots\,, \\ &R_0 = 0, \quad R_n = B_{n-1} + P_n\,, \quad S_n = R_n + 2(A_n - P_n) = B_n - P_n\,, \quad n = 1,2,\dots\,, \end{aligned}$$

where $\lceil x \rceil$ denotes the smallest integer larger than or equal to x. Simple induction shows that $P_n \le A_n$ for all positive integers n, and $P_n < A_n$ for $n \ge 2$. Really, this relation holds for $n = 1$, and by induction

$$P_n \le 2\left(\frac{A_{n-1}}{2}\frac{M_{n-1}}{M_n} + 1\right) = \frac{2(n-1)^3}{n^3}A_{n-1} + 2 \le 2^{n+10}\frac{(n-1)}{n^3} + 2 < A_n$$

for $n > 1$. This relation implies that $R_n \le S_n \le B_n \le R_{n+1}$ for all $n = 1, 2, \dots$. Now we define the value of the permutation π for $1 \le k \le P_1 = S_1$, $R_n < k \le S_n$, $n \ge 2$, and $S_n < k \le R_{n+1}$, $n \ge 1$, together with some functions $b(k)$ and $j(k)$ that we need for the definition of the functions $g_k(u)$. Since $R_1 = S_1$, these sets cover $\mathbf{Z}^+$.

For $1 \le k \le P_1$ let $\pi(k) = 2k - 1$, $j(k) = 0$ and $b(k) = 0$. For $R_n < k \le S_n$, $n \ge 2$, put $j(k) = 0$, $b(k) = n$ and define

$$\begin{aligned} \pi(k) &= k + P_n \quad \text{if } k - R_n \text{ is odd} \\ \pi(k) &= k - P_n \quad \text{if } k - R_n \text{ is even.} \end{aligned}$$

(The numbers $\pi(k)$, k and $k - R_n$ have the same parity.) We have

$$\{\pi(k),\ 1 \le k \le P_1\} = \{k,\ 1 \le k < B_1\} \cap \{\text{odd numbers}\}$$

and

$$\begin{aligned} \{\pi(k),\ R_n < k \le S_n\} = {}& [\{k,\ B_{n-1} + 2P_n + 1 \le k \le B_n - 1\} \cap \{\text{odd numbers}\}] \\ & \bigcup [\{k,\ B_{n-1} + 2 \le k \le B_n - 2P_n\} \cap \{\text{even numbers}\}]\,. \end{aligned}$$

For $S_n < k \le R_{n+1}$, $n \ge 1$, we define $\pi(k)$ as a map from $K^{(n)} = [S_n + 1, R_{n+1}]$ to $J_1^{(n)} \cup J_2^{(n)}$ with

$$\begin{aligned} J_1^{(n)} &= [B_n + 1,\, B_n + 2P_{n+1} - 1] \cap \{\text{odd numbers}\} \\ J_2^{(n)} &= [B_n - 2P_n + 2,\, B_n] \cap \{\text{even numbers}\}\ . \end{aligned}$$

The set $J_1^{(n)}$ has cardinality P_{n+1} and $J_2^{(n)}$ cardinality P_n. The cardinality of $K^{(n)}$ equals $R_{n+1} - S_n = P_n + P_{n+1}$. We define the functions $\pi(k)$, $b(k)$ and $j(k)$, $S_n < k \le R_{n+1}$, by induction. The function $j(k)$ equals either 0 or 1 and $b(k)$ takes the values n or $n+1$. Let us assume that we have already defined these functions for all $S_n < k \le L$ with some $L \ge S_n + 1$, but not for $L < k \le R_{n+1}$. Put

$$K_1^{(n)}(L) = \{k\colon\ S_n < k \le L,\, b(k) = n+1\}$$

and

$$K_2^{(n)}(L) = \{k\colon\ S_n < k \le L,\, b(k) = n\}\ .$$

Set

$$\Pi_1^{(n)}(L) = \max_{k \in K_1^{(n)}(L)} \pi(k) \quad \text{and} \quad \Pi_2^{(n)}(L) = \max_{k \in K_2^{(n)}(L)} \pi(k)\ .$$

If $M_{n+1}|K_1^{(n)}(L)| \ge M_n(|K_2^{(n)}(L)| + 1)$, where $|K_i^{(n)}(L)|$ denotes the cardinality of the set $K_i^{(n)}(L)$, then define $\pi(L+1) = \Pi_2^{(n)}(L) + 2$, $j(L+1) = 0$ and $b(L+1) = n$. In the other case define $\pi(L+1) = \Pi_1^{(n)}(L) + 2$, $\pi(L+2) = \Pi_1^{(n)}(L) + 4$, $j(L+1) = 0$, $j(L+2) = 1$ and $b(L+1) = b(L+2) = n+1$. If the set $K_1^{(n)}(L)$ or $K_2^{(n)}(L)$ is empty, then we define $\Pi_1^{(n)}(L) = B_n - 1$ and $\Pi_2^{(n)}(L) = B_n - 2P_n$ respectively.

We have to show that the above definition is correct, i.e. the iteration can be stopped in such a way that $\Pi_1^{(n)}(L) = B_n + 2P_{n+1} - 1$ and $\Pi_2^{(n)}(L) = B_n$, what is equivalent to saying that $|K_1^{(n)}(L)| = P_{n+1}$ and $|K_2^{(n)}(L)| = P_n$. To prove the correctness of this definition first we make the following observation:

$$\begin{aligned} P_n M_n = 2\frac{P_n}{2}\frac{M_n}{M_{n+1}} M_{n+1} &\le P_{n+1} M_{n+1} \\ &< 2\left(\frac{P_n}{2}\frac{M_n}{M_{n+1}} + 1\right) M_{n+1} = P_n M_n + 2M_{n+1}\ . \end{aligned} \tag{10}$$

Let $\bar{L}$ be a value for which one of the relations $|K_1^{(n)}(\bar{L})| = P_{n+1}$ or $|K_2^{(n)}(\bar{L})| = P_n$ holds. It is enough to show that if the cardinality of the other set is less than the corresponding number, then in the next step of the iteration this set increases.

If $|K_1^{(n)}(\bar{L})| = P_{n+1}$ and $|K_2^{(n)}(\bar{L})| < P_n$, then $|K_2^{(n)}(\bar{L})| \le P_n - 1$ and $|K_1^{(n)}(\bar{L})| M_{n+1} = M_{n+1}P_{n+1}$. Hence by the left side of relation (10),

$$M_n(|K_2^{(n)}(\bar{L})| + 1) \le P_n M_n \le P_{n+1} M_{n+1} = M_{n+1}|K_1^{(n)}(\bar{L})|\ ,$$

and the value of $|K_2^{(n)}(\cdot)|$ is increasing in the next step of the iteration.

If $|K_2^{(n)}(\bar{L})| = P_n$ and $|K_1^{(n)}(\bar{L})| < P_{n+1}$, then $|K_1^{(n)}(\bar{L})| \le P_{n+1} - 2$. Hence we get, using the right-hand side of (10), that

$$M_{n+1}|K_1^{(n)}(\bar{L})| \le M_{n+1}(P_{n+1} - 2) < P_n M_n < M_n(|K_2^{(n)}(\bar{L})| + 1) .$$

This implies that the value of $|K_1^{(n)}(\cdot)|$ is increasing in the next step of the iteration. Hence the above definition of the functions $\pi(\cdot)$, $b(k)$ and $j(k)$ is meaningful.

It is not difficult to check that the above defined function π is an isomorphism from $\mathbf{Z}^+$ to $\mathbf{Z}^+$. Hence we can use it as the permutation π in the definition of $\mathbf{S}$. We define the functions g_k needed in this definition in the following way: Let $g_k(u) = u$ if $k \le 0$ or $j(k) = 0$. If $j(k) = 1$ and $b(k) = n$, then let $g_k(u)$ be the interval exchange transformation described in formula (6) with the intervals $I_k^{(1)} = [0, 2^{-n})$ and $I_k^{(2)} = [2^{-n}, 2^{-n+1})$. (We defined the function $j(k)$ in order to decide whether we want to make an interval exchange transformation $g_k(u)$ in the k-th coordinate. The function $b(k)$ identifies in which interval $(B_{n-1}, B_n]$ the number $\pi(k)$ lies.)

Now we turn to the proof of relation (8). First we show that if $k \in (B_{n-1}, B_n]$, $n \ge 2$, then $\pi(k) \in (B_{n-2}, B_{n+1}]$. This follows from the following observations: If $k \in (R_n, S_n]$ then $k \in (B_{n-1}, B_n]$ and $\pi(k) \in (B_{n-1}, B_n]$, and if $k \in (S_n, R_{n+1}]$ then $k \in (B_{n-1}, B_{n+1}]$ and $\pi(k) \in J_1^{(n)} \cup J_2^{(n)} \subset (B_{n-1}, B_{n+1}]$. This relation implies that $\pi(k) > B_{n-1}$ and $\pi^{-1}(k) > B_{n-1}$ for $k > B_n$. Hence we get that if $m > B_L$ with some $L > 1$ then $\pi(k + \pi^{-1}(m)) > B_{L-2}$ for all $k \ge 0$, and

$$U_{m,n}(u) = 0 \quad \text{if } m > B_L \text{ and } 2^{-L+3} \le u \le 1 .$$

For $1 \le m \le B_L$ write

$$\begin{aligned} U_{m,n}(u) &= \frac{1}{n} \sum_{k=1}^{B_L} f_{\pi(k+\pi^{-1}(m))}\big(g_{\pi(k+\pi^{-1}(m))}(g_m(u))\big) \\ &\quad + \frac{1}{n} \sum_{k=B_L+1}^{n} f_{\pi(k+\pi^{-1}(m))}\big(g_{\pi(k+\pi^{-1}(m))}(g_m(u))\big) \\ &= \Sigma_L^1(m,n,u) + \Sigma_L^2(m,n,u) . \end{aligned}$$

Then

$$\Sigma_L^1(m,n,u) < \frac{C(L)}{n} ,$$

and

$$\Sigma_L^2(m,n,u) = 0 \quad \text{if } 2^{-L+3} \le g_m(u) \le 1 .$$

Hence

$$\sum_{m=B_L+1}^{\infty} U_{m,n}(x_m) = 0 \quad \text{on the set}$$

$$D_L = \{x = (x_j,\ j \in \mathbf{Z}),\ x_j \ge 2^{-(p+L)+3} \text{ for } B_{L+p} < j \le B_{L+p+1},\ \ p = 0, 1, \dots\} ,$$

$$\sum_{m=1}^{B_L} \Sigma_L^2(m,n,x_m) = 0 \quad \text{on the set}$$

$$E_L = \{x = (x_j,\ j \in \mathbf{Z}),\ g_j(x_j) \ge 2^{-L+3} \text{ for } \ 1 \le j \le B_L\}$$

and

$$\lim_{n\to\infty} \sum_{m=1}^{B_L} \Sigma_L^1(m,n,u) = 0\ .$$

Since

$$\mu(\mathcal{X} \setminus D_L) \le 2\sum_{p=0}^{\infty} A_{p+L} 2^{-(p+L)+3} < \text{const.}\ \frac{1}{L}\ ,$$

$$\mu(\mathcal{X} \setminus E_L) \le 2^{-L+3} B_L < \text{const.}\ \frac{1}{L^2}\ ,$$

and L can be chosen arbitrary large, the above relations imply (8).

Since $\pi(m) = m$ and $g_m(u) = u$ for $m \le 0$, the expression $U_{m,n}(u)$ is simpler in this case. We can write

$$U_{m,n}(u) = \frac{1}{n}\sum_{k=1}^{n} f_{\pi(k+m)}\big(g_{\pi(k+m)}(u)\big) = \frac{1}{n}\sum_{l=1}^{n+m} h_l(u) = \frac{H_{n+m}(u)}{n} \quad \text{for } 1-n \le m \le 0$$

with

$$h_l(u) = f_{\pi(l)}(g_{\pi(l)}(u)) \quad \text{and} \quad H_L(u) = \sum_{l=1}^{L} h_l(u)\ , \quad (H_L(u) = 0 \text{ for } L \le 0)\ .$$

Let us study the expression $H_L(u)$. It follows from the definition of the functions $\pi(k)$ and $g_k(u)$ that

$$H_L(u) = \begin{cases} L & \text{for } 0 \le u < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} \le u \le 1 \end{cases} \qquad \text{if } 1 \le L \le P_1\ ,$$

$$H_L(u) - H_{R_n}(u) = \begin{cases} n^3 & \text{if } 0 \le u < 2^{-n} \text{ and } L - R_n \text{ is odd} \\ 0 & \text{if } 2^{-n} \le u \le 1 \text{ or } L - R_n \text{ is even} \end{cases} \quad \text{if } R_n \le L \le S_n,\ \ n \ge 2\ .$$

In particular,

$$H_{S_n}(u) - H_{R_n}(u) = 0, \quad \text{for all } 0 \le u \le 1,\ n \ge 2\ .$$

We have

$$H_{R_{n+1}}(u) - H_{S_n}(u) = \begin{cases} \frac{1}{2}P_{n+1}(n+1)^3 - P_n n^3 & \text{if } 0 \le u < 2^{-n} \\ 0 & \text{if } 2^{-n} \le u \le 1 \end{cases}\ ,$$

and the functions

$$P_{n,L}(u) = H_L(u) - H_{S_n}(u)\ , \quad S_n < L \le R_{n+1}\ , \qquad n \ge 1\ ,$$

satisfy the identity

$$P_{n,L}(u) = \begin{cases} 0 & \text{if } 2^{-n} \le u \le 1 \\ \frac{1}{2}|K_1^{(n)}(L)|(n+1)^3 - |K_2^{(n)}(L)|n^3 & \text{if } 0 \le u < 2^{-n} \\ & \quad \text{and } |K_1^{(n)}(L)| \text{ is even} \\ \frac{1}{2}\left(|K_1^{(n)}(L)|+1\right)(n+1)^3 - |K_2^{(n)}(L)|n^3 & \text{if } 0 \le u < 2^{-(n+1)} \\ & \quad \text{and } |K_1^{(n)}(L)| \text{ is odd} \\ \frac{1}{2}\left(|K_1^{(n)}(L)|-1\right)(n+1)^3 - |K_2^{(n)}(L)|n^3 & \text{if } 2^{-(n+1)} \le u < 2^{-n} \\ & \quad \text{and } |K_1^{(n)}(L)| \text{ is odd} \end{cases},$$

where the functions $K_i^{(n)}(L)$, $i = 1, 2$, appeared in the definition of the function $\pi(k)$. To bound the function $P_{N,L}(u)$, first we prove the following relation for $S_n < L \le R_{n+1}$ by induction:

$$0 \le M_{n+1}|K_1^{(n)}(L)| - M_n|K_1^{(n)}(L)| < \begin{cases} M_n + M_{n+1} & \text{if } |K_1^{(n)}(L)| \text{ is odd} \\ M_n + 2M_{n+1} & \text{if } |K_1^{(n)}(L)| \text{ is even} \end{cases}. \tag{11}$$

Indeed, relation (11) holds for $L = S_n + 1$. Taking into consideration the inequality which decides which one of the sets $K_i^{(n)}(L)$, $i = 1, 2$, increases in the next step of the iteration we get the proof of relation (11) by separating the two cases by induction in the following way. If

$$M_n \le M_{n+1}|K_1^{(n)}(L)| - M_n|K_2^{(n)}(L)| < M_n + 2M_{n+1} ,$$

then $|K_1^{(n)}(L+1)| = |K_1^{(n)}(L)|$ and $|K_2^{(n)}(L+1)| = |K_2^{(n)}(L)| + 1$, hence relation (11) holds for $L + 1$. If

$$0 \le M_{n+1}|K_1^{(n)}(L)| - M_n|K_2^{(n)}(L)| < M_n \quad \text{and } |K_1^{(n)}(L)| \text{ is even} ,$$

then $|K_1^{(n)}(L+i)| = |K_1^{(n)}(L)| + i$ and $|K_2^{(n)}(L+i)| = |K_2^{(n)}(L)|$, $i = 1, 2$. Hence, in this case relation (11) holds for $L + 1$ and $L + 2$.

Since $n^3 = 2^n M_n$, relation (11) implies that

$$0 \le P_{n,L}(u) \le 3(n+1)^3 .$$

The above inequalities imply that for $B_p < L \le B_{p+1}$

$$0 \le H_L(u) \le \begin{cases} 0 & \text{if } \frac{1}{2} \le u \le 1 \\ \text{const.}\, s^4 & \text{for } 2^{-s-1} \le u < 2^{-s} \quad \text{for } 1 \le s \le p \\ \text{const.}\, p^4 & \text{for } 0 \le u < 2^{-p-1} \end{cases}.$$

Define the moments of the random variables $H_L(x_m)$

$$E_L^{(p)}(m) = E_L^{(p)} = \int H_L^p(x_m)\, d\mu(x) = \int_0^1 H_L^p(u)\, du\ , \qquad \text{for } L \geq 1 \text{ and } m \leq 0\ .$$

Because of the bound given for the functions $H_L(u)$

$$0 \leq E_L^{(p)} < C(p) < \infty \quad \text{for all } p = 1, 2, \ldots$$

with some appropriate constant $C(p)$. In particular, the first moment $E_L^{(1)}$ satisfies the relation

$$\lim_{L\to\infty} E_L^{(1)} = E = A_1 + \sum_{k=1}^{\infty}(P_{k+1}M_{k+1} - P_k M_k) > 0\ . \tag{12}$$

Also the relation $E = \lim\limits_{L\to\infty} E_L^{(1)} < \infty$ holds, since $E = A_1 + \lim\limits_{k\to\infty} P_k M_k$, and

$$P_{k+1}M_{k+1} = \left\lceil \frac{P_k}{2} + \frac{M_k}{M_{k+1}} \right\rceil M_{k+1} \leq P_k M_k + M_k\ ,$$

hence $\sup\limits_k P_k M_k \leq P_0 M_0 + \sum\limits_{k=1}^{\infty} M_k < \infty$.

Introduce the random variables $\xi_L(m) = H_L(x_{-m}) - E_L^{(1)}$, $m = 0, 1, 2, \ldots$, on the probability space $(\mathcal{X}, \mathcal{A}, \mu)$. The random variables $\xi_{L(m)}(m)$, $m = 1, 2, \ldots$, are independent for an arbitrary sequence $L(m)$, $E\xi_L(m) = 0$ and $E\xi_L^4(m) \leq C < \infty$ for all $L \geq 0$ and m. Hence

$$E\left(\frac{\xi_n(0) + \cdots + \xi_1(n-1)}{n}\right)^4 < \frac{\text{const.}}{n^2}\ ,$$

and

$$\mu\left(\left|\frac{\xi_n(0) + \cdots + \xi_1(n-1)}{n}\right| > \varepsilon\right) < \frac{\text{const.}}{n^2\varepsilon^4} \quad \text{for all } \varepsilon > 0\ .$$

Since the right-hand side of the last expression is summable in n for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\xi_n(0) + \cdots + \xi_1(n-1)}{n} = 0 \quad \text{a. s.}\ . \tag{13}$$

Relations (12) and (13) imply that

$$\lim_{n\to\infty} \sum_{m=-n+1}^{0} U_{m,n}(x_m) = \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \xi_{n-m}(m) + \lim_{n\to\infty} \frac{1}{n} \sum_{m=o}^{n-1} E_{n-m}^{(1)} = E > 0 \quad \text{a. s.}\ .$$

Relation (9), hence relation (2) is proved.

Reference:

[1] Buczolich, Z: Arithmetic averages of rotations of measurable functions. To appear in *Journal of Ergodic Theory and Dynamical Systems*