

ESTIMATION OF WIENER–ITÔ INTEGRALS AND OF POLYNOMIALS OF INDEPENDENT GAUSSIAN RANDOM VARIABLES.

Péter Major

*Alfréd Rényi Mathematical Institute
of the Hungarian Academy of Sciences*

In this paper I prove good estimates on the moments and tail distribution of k -fold Wiener–Itô integrals and also present their natural counterpart for polynomials of independent Gaussian random variables. The proof is based on the so-called diagram formula for Wiener–Itô integrals which yields a good representation for their products as a sum of such integrals. In a subsequent paper I shall show that this method also enables us to prove a natural counterpart of these estimates for U -statistics. The main result of this paper is a generalization of the estimates of Hanson and Wright about bilinear forms of independent standard normal random variables. On the other hand, it is a weaker estimate than the main result of a paper of Latała [5]. But that paper contains an error, and it is not clear whether its result is true. This question is also discussed here.

1. Introduction. Formulation of the main results.

The goal of this paper is to give good estimates on the tail-distribution and on high moments of Wiener–Itô integrals. This problem can be reformulated in an equivalent form to the estimation of polynomials of independent Gaussian random variables. The results obtained in such a way will be also proved.

This paper can be considered as a continuation of my investigation in paper [9] where good estimates were given for the tail-distribution and high moments of Wiener–Itô integrals and (degenerate) U -statistics with the help of their variance. The results of [9] were proved by means of the so-called diagram formula which yields a useful expression for the moments of Wiener–Itô integrals or degenerate U -statistics. In the present paper it is shown that this method can be applied also in cases when we have more information about a Wiener–Itô integral than its variance and want to exploit this. In a subsequent paper I shall show in a subsequent paper [10] that similar improvements can be proved about the moments and tail-distribution of degenerate U -statistics.

Previous papers in this field (see [3] or [5]) dealt with the estimation of polynomials of independent standard normal random variables and not of Wiener–Itô integrals. I shall show that the result of [3] about the estimation of bilinear forms of independent standard normal random variables is equivalent to the special case of the main result in this paper when only two-fold Wiener–Itô integrals are considered. On the other hand, paper [5] formulates sharper estimates about Gaussian polynomials of higher order than our results. But the proof in [5] contains an error, hence some problems arise with respect to this paper. I return to this question later.

First I introduce some notations needed to formulate the results of the present paper.

Let us have a σ -finite non-atomic measure μ on some measurable space (X, \mathcal{X}) together with a white noise μ_W with reference measure μ , i.e. a set of jointly Gaussian random variables $\mu_W(B)$ indexed by the sets $B \in \mathcal{X}$ such that $\mu(B) < \infty$, whose joint distribution is determined by the relations $E\mu_W(B) = 0$, $E\mu_W(A)\mu_W(B) = \mu(A \cap B)$ for all sets $A, B \in \mathcal{X}$ such that $\mu(A) < \infty$, $\mu(B) < \infty$. Let us also introduce the quantity

$$V_1^2(f) = \int f^2(x_1, \dots, x_k) \mu(dx_1) \dots \mu(dx_k) \quad (1.1)$$

for a function f of k variables on the space (X, \mathcal{X}) . The k -fold Wiener–Itô integral

$$I_k(f) = \frac{1}{k!} \int f(x_1, \dots, x_k) \mu_W(dx_1) \dots \mu_W(dx_k) \quad (1.2)$$

can be defined for all functions f such that $V_1(f) < \infty$. (See e.g. [4] or [7].) Here the knowledge of the definition of Wiener–Itô integrals is not assumed. We only need the so-called the diagram formula which enables us to calculate the moments of these random integrals. This result will be recalled in Section 3.

We are interested in good estimates on the probability $P(k!|I_k(f)| > x)$ for large numbers $x > 0$. This problem is closely related to the question about good moment estimates $E(k!I_k(f))^{2M}$ for large values M . The results of paper [9] yield a good estimate for these quantities with the help of $V_1(f)$. I recall them in the following Theorem A.

Theorem A. *Let a measurable space (X, \mathcal{X}) be given together with a non-atomic σ -finite measure on it. Let μ_W be a white-noise with reference measure μ , and let such a function $f(x_1, \dots, x_k)$ of k variables be given on the space (X, \mathcal{X}) for which $V_1^2(f) \leq \sigma^2$ with some $\sigma < \infty$, where the quantity $V_1(f)$ was defined in (1.1). Then the k -fold Wiener–Itô integral $I_k(f)$ introduced in (1.2) satisfies the inequalities*

$$E(k!I_k(f))^{2M} \leq C \left(\frac{2kM}{e} \right)^{kM} \sigma^{2M} \quad \text{for all } M = 1, 2, \dots \quad (1.3)$$

and

$$P(k!|I_k(f)| > x) \leq C \exp \left\{ -\frac{1}{2} \left(\frac{x}{\sigma} \right)^{2/k} \right\} \quad \text{for all } x > 0 \quad (1.4)$$

with some appropriate universal $C > 0$ depending only on the multiplicity k of the Wiener–Itô integral.

This estimate is sharp in the following sense. There are functions $f(x_1, \dots, x_k)$, (a function of the form $f(x_1, \dots, x_k) = g(x_1) \dots g(x_k)$, $\int g^2(x) \mu(dx) < \infty$, is an appropriate choice) for which the constant in the exponent of the probability estimate (1.4) cannot be increased, i.e. the inequality $P(k!|I_k(f)| > x) \leq C e^{-K(x/\sigma)^{2/k}}$ does not hold for $K > \frac{1}{2}$.

Let me remark that $EI_k^2(f) = \frac{1}{k!}V_1^2(f)$, and $EI_k(f) = 0$. So Theorem A gave a good estimate on the moments and tail-distribution of Wiener–Itô integrals with the help of their variance. If we have no more information about the kernel function f of the Wiener–Itô integral $I_k(f)$, than a bound on $V_1(f)$ then we cannot improve the estimates in Theorem A. On the other hand, one may introduce new quantities to get better estimates. In this paper such results will be proved. For this goal this first I introduce some new quantities.

Let us fix a function $f(x_1, \dots, x_k)$ on (X, \mathcal{X}) , introduce the set $K = \{1, \dots, k\}$ of the indices of its variables, and let $\mathcal{P} = \mathcal{P}(K)$ denote the set of partitions of the set K to non-empty sets. Given a $P = \{A_1, \dots, A_s\} \in \mathcal{P}(K)$ let us define the class of appropriate sequences of functions denoted by \mathcal{F}_P by the following formula:

$$\mathcal{F}_P = \left\{ g_r(x_j, j \in A_r), 1 \leq r \leq s: \int g_r^2(x_j, j \in A_r) \prod_{j \in A_r} \mu(dx_j) \leq 1 \text{ for all } 1 \leq r \leq s \right\} \quad (1.5)$$

if $P = \{A_1, \dots, A_s\} \in \mathcal{P}(K)$. This means that \mathcal{F}_P consists of a sequence of functions g_r on (X, \mathcal{X}) whose variables are indexed by the elements of corresponding sets A_r in the partition \mathcal{P} , and with L_2 -norm less than or equal to 1 with respect to the appropriate product of the copies of the measure μ .

Let us define for all sets $P = \{A_1, \dots, A_s\} \in \mathcal{P}$ the quantity

$$V_P(f) = \sup_{(g_1, \dots, g_s) \in \mathcal{F}_P} \int f(x_1, \dots, x_k) \prod_{1 \leq r \leq s} g_r(x_j, j \in A_r) \prod_{j \in K} \mu(dx_j). \quad (1.6)$$

Beside this, introduce the class $\mathcal{P}_s = \mathcal{P}_s(K) \subset \mathcal{P}(K)$ of partitions of K which consist of exactly s elements, and put

$$V_s(f) = \sup_{P \in \mathcal{P}_s} V_P(f). \quad (1.7)$$

Let me remark that the definition of $V_1(f)$ introduced in (1.1) agrees with the definition of $V_s(f)$ in (1.7) for $s = 1$. Beside this, the definition of the quantities \mathcal{F}_P , $V_P(f)$ and $V_s(f)$ were formulated in such a way that they are meaningful for any set $K = \{u_1, \dots, u_t\}$ and square integrable function $f(x_{u_1}, \dots, x_{u_t})$ with variables indexed by the set K . In the formulation of our results we can restrict our attention to the case $K = \{1, \dots, k\}$, but in later considerations we shall also work with these quantities in the case of more general sets K .

The main result of this paper, an estimate about the moments and tail distribution of a Wiener–Itô integral $I_k(f)$, can be formulated with the help of the quantities $V_s(f)$ introduced in formulas (1.5), (1.6) and (1.7).

Theorem about the tail-distribution of Wiener–Itô integrals. *Let us consider a k -fold Wiener–Itô integral $I_k(f)$, $k \geq 2$, defined in (1.1) by means of a white noise*

μ_W with a non-atomic σ -additive reference measure μ on a measurable space (X, \mathcal{X}) and a measurable function $f(x_1, \dots, x_k)$ of k variables on the space (X, \mathcal{X}) such that $V_1^2(f) < \infty$ for the quantity $V_1(f)$ defined in (1.1). Then there exist some universal constants $C > 0$, $C_1 > 0$ and $C_2 > 0$ depending only on the multiplicity k of the Wiener–Itô integral $I_k(f)$ such that

$$E(k!I_k(f))^{2M} \leq C^M \max_{1 \leq s \leq k} \left(V_s(f)^{2M} M^{[(s-1)(k-1)+1]M} \right), \quad \text{for all } M = 1, 2, \dots, \quad (1.8)$$

and

$$P(|k!I_k(f)| > x) \leq C_1 \exp \left\{ -C_2 \min_{1 \leq s \leq k} \left(\frac{x}{V_s(f)} \right)^{2/[(s-1)(k-1)+1]} \right\} \quad (1.9)$$

with the quantities $V_s(f)$, $1 \leq s \leq k$, defined in formulas (1.5), (1.6) and (1.7) with the set $K = \{1, \dots, k\}$.

It will be more convenient to prove first the following simpler version of this theorem and to deduce the result in the general case from it.

Simplified version of the theorem about the tail distribution of Wiener–Itô integrals. Let a k -fold Wiener–Itô integral $I_k(f)$, $k \geq 2$, be considered with respect to a white noise μ_W with reference measure μ whose kernel function f satisfies the inequalities

$$V_s(f) \leq M^{-(s-1)(k-1)/2} \quad \text{for all } 1 \leq s \leq k \quad (1.10)$$

with some positive integer M . Then the inequality

$$E(k!I_k(f))^{2M} \leq C^M M^M \quad (1.11)$$

holds with some universal constant C depending only on the multiplicity k of the Wiener–Itô integral.

Reduction of the theorem about the tail-distribution of Wiener–Itô integrals to its simplified version. Let us introduce with the help of the kernel function f of a Wiener–Itô integral $I_k(f)$ the number $B(M) = \max_{1 \leq s \leq k} V_s(f) M^{(k-1)(s-1)/2}$, and put $\bar{f} = \bar{f}(M) = \frac{1}{B(M)} f$. Then the function \bar{f} satisfies the conditions of the reduced theorem, hence it implies that relation (1.11) holds for the Wiener–Itô integral with kernel function \bar{f} . This relation implies formula (1.8). Formula (1.9) can be proved in the standard way by means of formula (1.8) and the Markov inequality $P(|k!I_k(f)| > x) \leq \frac{E(k!I_k(f))^{2M}}{x^{2M}}$ with a good choice of the parameter M . The choice of the closest integer to

$$\tilde{C} \min_{1 \leq s \leq k} \left(\frac{x}{V_s(f)} \right)^{2/[(s-1)(k-1)+1]}$$

for the parameter M with a sufficiently small $\tilde{C} > 0$ (if $x > 0$ is sufficiently large) supplies formula (1.9).

This paper contains the proof of the above result. But before turning to it I discuss what kind of estimates it yields for polynomials of independent standard normal random variables. This will be the subject of Section 2. Beside this, this section contains a comparison of these estimates with earlier results in this field. In particular, a result of Latała is discussed there together with the problem appearing in its proof. Section 3 contains a formula which helps to calculate the moments of a Wiener–Itô integral. This formula is a consequence of the diagram formula for the product of Wiener–Itô integrals, and it enables us to prove good moment estimates for Wiener–Itô integrals if we can find good bounds on some integrals defined with the help of some diagrams. A closer study showed that it is useful to consider a special class of the diagrams, the so-called connected diagrams, and it is enough to estimate the integrals related to them. A result about a good bound on such integrals, called the Basic Estimate is also formulated in Section 3, and it is shown that the proof of the main result of this paper can be reduced to that of the Basic Estimate. In Section 4 a result called the Main Inequality is proved, and the Basic Estimate is proved with its help. Finally in Section 5 Latała’s result about an improvement of our estimates is discussed in more detail, and the question is investigated what kind of result has to be solved to decide whether it is true.

2. Bounds on random polynomials of Gaussian random variables.

The natural counterpart of the estimation of k -fold Wiener–Itô integrals is the estimation of some special polynomials of order k of independent standard normal random variables defined with the help of Hermite polynomials. In more detail, polynomials of the following form will be considered. Let us take independent, standard normal random variables ξ_1, ξ_2, \dots , and introduce with their help the following random polynomials.

$$Z_k = \sum_{\substack{((j_1, l_1), \dots, (j_s, l_s)), \\ j_u \neq j_{u'} \text{ if } u \neq u', l_1 + \dots + l_s = k}} a((j_1, l_1), \dots, (j_u, l_u)) H_{l_1}(\xi_{j_1}) \cdots H_{l_s}(\xi_{j_s}). \quad (2.1)$$

Here the coefficients $a((j_1, l_1), \dots, (j_u, l_u))$ are some real numbers, and $H_l(x)$ denotes the Hermite polynomial of order l with leading coefficient 1. For the sake of simplicity let us assume that the sum in formula (2.1) contains only finitely many terms. Infinite sums could also be allowed in formula (2.1), but in such a case one should deal with some convergence problems.

It is more convenient to rewrite the random polynomials Z_k in formula (2.1) in a different form. This new notation introduced below means to work with Wick polynomials of Gaussian random variables, i.e. to apply a multivariate generalization of Hermite polynomials. To introduce the new notation put

$$H_{l_1}(\xi_{j_1}) \cdots H_{l_s}(\xi_{j_s}) =: \underbrace{\xi_{j_1}, \dots, \xi_{j_1}}_{l_1\text{-times}}, \dots, \underbrace{\xi_{j_s}, \dots, \xi_{j_s}}_{l_s\text{-times}} : \cdot$$

At the right-hand side of the last formula there is the product of k Gaussian random variables ξ_{j_s} , $1 \leq j_s \leq n$, $1 \leq s \leq k$, between the two signs $: \cdot$, and some of the terms

ξ_{j_s} may agree. For the sake of a more convenient notation let us slightly extend the definition of the above expression. Let us allow to write the terms in this product in an arbitrary order, i.e. put

$$:\xi_{j_{\pi(1)}}, \dots, \xi_{j_{\pi(k)}} :=: \xi_{j_1}, \dots, \xi_{j_k} :$$

for all permutations $\pi = (\pi(1), \dots, \pi(k))$ of the set $\{1, \dots, k\}$. With such a notation the random polynomials we are working with can be rewritten in the form

$$Z_k = \sum a(n_1, \dots, n_k) : \xi_{n_1} \xi_{n_2} \cdots \xi_{n_k} : \quad (2.2)$$

with some real coefficients $a(n_1, \dots, n_k)$ that can be calculated by means of the coefficients $a((j_1, l_1), \dots, (j_u, l_u))$ in formula (2.1). In the subsequent considerations I shall estimate random polynomials presented in the form (2.2).

Let us consider the unit interval $[0, 1]$ together with the Lebesgue measure on it denoted by μ , and let μ_W be a white noise with this reference measure μ . Take a complete orthonormal system of functions $\varphi_1(x), \varphi_2(x), \dots$ with respect to the Lebesgue measure on $[0, 1]$ and put $\xi_n = \int \varphi_n(x) \mu_W(dx)$, $n = 1, 2, \dots$. Then ξ_1, ξ_2, \dots is a sequence of independent, standard normal random variables, and we do not change the distribution of the random polynomial Z_k in (2.2) by choosing the above introduced standard normal random variables ξ_n in it. Thus we may assume that these standard normal random variables appear in the definition of the random polynomial Z_k , and I shall exploit this liberty. With such a choice Itô's formula for multiple Wiener–Itô integrals (see e.g. [4]) enables us to rewrite the random polynomial Z_k in the form of a k -fold Wiener–Itô integral. Such a representation of the random polynomial Z_k is useful, because it enables us to apply the results we know about Wiener–Itô integrals in our investigations.

To find the above indicated representation observe that by Itô's formula

$$:\xi_{n_1} \xi_{n_2} \cdots \xi_{n_k} := \int \varphi_{n_1}(x_1) \cdots \varphi_{n_k}(x_k) \mu_W(dx_1) \dots \mu_W(dx_k),$$

hence

$$Z_k = k! I_k(f) = \int f(x_1, \dots, x_k) \mu_W(dx_1) \dots \mu_W(dx_k) \quad (2.3)$$

with

$$f(x_1, \dots, x_k) = \sum a(n_1, \dots, n_k) \varphi_{n_1}(x_1) \cdots \varphi_{n_k}(x_k). \quad (2.4)$$

We get an estimate for the moments and tail-distribution of the random polynomial Z_k with the help of the results formulated in Section 1 if we can express the quantities $V_s(f)$, $1 \leq s \leq k$, for the function f introduced in (2.4) by means of the coefficients $a(n_1, \dots, n_k)$ in formula (2.2). We shall define some quantities \bar{V}_s , $1 \leq s \leq k$, with the help of the coefficients $a(n_1, \dots, n_k)$ and show that they are equal to $V_s(f)$, $1 \leq s \leq k$, with the function f defined in (2.4).

To define the desired quantities let us first introduce the set \mathcal{G}_P corresponding to \mathcal{F}_P defined in (1.5) for a partition $P = \{A_1, \dots, A_s\} \in \mathcal{P}(K)$ of the set $K = \{1, \dots, k\}$.

$$\mathcal{G}_P = \left\{ b_r(n_j, j \in A_r), 1 \leq n_j < \infty \text{ for all } j \in A_r, 1 \leq r \leq s: \right. \\ \left. \sum_{1 \leq n_j < \infty, 1 \leq j \leq r} b_r^2(n_j, j \in A_r) \leq 1 \text{ for all } 1 \leq r \leq s \right\}$$

if $P = \{A_1, \dots, A_s\} \in \mathcal{P}(K)$. With the help of this notion define similarly to the definition of V_P in (1.6) the quantity

$$\bar{V}_P(a(\cdot)) = \sup_{(b_1, \dots, b_s) \in \mathcal{G}_P} \sum a(n_1, \dots, n_k) \prod_{1 \leq r \leq s} b_r(n_j, j \in A_r) \quad (2.5)$$

for a partition $P = \{A_1, \dots, A_s\} \in \mathcal{P}(K)$ and a sequence $a(n_1, \dots, n_k)$, $0 \leq n_j < \infty$ for all $1 \leq j \leq k$ such that $\sum a^2(n_1, \dots, n_k) < \infty$. Finally, put, similarly to formula (1.7)

$$\bar{V}_s(a(\cdot)) = \sup_{P \in \mathcal{P}_s} \bar{V}_P(a(\cdot)). \quad (2.6)$$

I claim that

$$\bar{V}_P(a(\cdot)) = V_P(f) \quad (2.7)$$

for all partitions $P \in \mathcal{P}(K)$, and as a consequence

$$\bar{V}_s(a(\cdot)) = V_s(f) \quad (2.8)$$

with the function f defined in (2.4) and the sequence $a(n_1, \dots, n_k)$ appearing in formula (2.2).

To show that relation (2.7) holds fix some partition $P = \{A_1, \dots, A_s\} \in \mathcal{P}(K)$ with $A_r = (j_1^{(r)}, \dots, j_{l(r)}^{(r)})$, $1 \leq r \leq s$, take a set of sequences

$$(b_1(n_j, j \in A_1), \dots, b_s(n_j, j \in A_s)) \in \mathcal{G}_P,$$

and correspond to it the set of functions (g_1, \dots, g_r) defined by the formula

$$g_r(x_j, j \in A_r) = \sum_{n_j^{(u)}: 1 \leq n_{j_u^{(r)}} < \infty, 1 \leq u \leq l(r)} b_r(n_{j_1^{(r)}}, \dots, n_{j_{l(r)}^{(r)}}) \prod_{u=1}^{l(r)} \varphi_{n_{j_u^{(r)}}}(x_{j_u^{(r)}}), \quad 1 \leq r \leq s. \quad (2.9)$$

Then it follows from the Parseval formula that

$$\int g_r^2(x_j, j \in A_r) \prod_{j \in A_r} \mu(dx_j) = \sum_{n_j: 1 \leq n_j < \infty, j \in A_r} b_r^2(n_j, j \in A_r), \quad 1 \leq r \leq s,$$

$(g_1, \dots, g_r) \in \mathcal{F}_P$, and the mapping $b_r \rightarrow g_r$, $1 \leq r \leq s$, defined in formula (2.9) is a one-to-one map from \mathcal{G}_P to \mathcal{F}_P . Beside this, the Parseval formula also implies that

$$\sum a(n_1, \dots, n_k) \prod_{r=1}^s b_r(n_j, j \in A_r) = \int f(x_1, \dots, x_k) \prod_{r=1}^s g_r(x_j, j \in A_r) \prod_{j=1}^k \mu(dx_j) \quad (2.10)$$

with the sequences $a(n_1, \dots, n_k)$ in (2.2) and the function $f(x_1, \dots, x_k)$ in (2.4).

Then taking the supremum of both sides of formula (2.10) for all $(b_1, \dots, b_r) \in \mathcal{G}_P$, and exploiting that the two suprema equal we get relation (2.5). Indeed, the supremum of the left-hand side equals $\bar{V}_P(a(\cdot))$ by definition. The supremum of the right-hand side equals $V_P(f)$, because as the properties of the above defined one-to-one mapping imply that the supremum of the right-hand side equals the supremum of the same expressions if supremum is taken for all $(g_1, \dots, g_s) \in \mathcal{F}_P$. Finally, taking supremum in formula (2.5) for all $P \in \mathcal{P}_s$ we get formula (2.6).

The above considerations together with the theorem about the tail-distribution of Wiener–Itô integrals formulated in Section 1 supply the following result.

Theorem about the estimation of moments and tail distribution of polynomials of independent standard normal random variables. *Let us consider the random polynomial Z_k of standard normal random variables defined in formula (2.2) There exist some universal constants $C > 0$, $C_1 > 0$ and $C_2 > 0$ depending only on the order k of this random polynomial such that*

$$EZ_k^{2M} \leq C^M \max_{1 \leq s \leq k} \left(\bar{V}_s(a(\cdot))^{2M} M^{[(s-1)(k-1)+1]M} \right), \quad \text{for all } M = 1, 2, \dots,$$

and

$$P(|Z_k| > x) \leq C_1 \exp \left\{ -C_2 \min_{1 \leq s \leq k} \left(\frac{x}{\bar{V}_s(a(\cdot))} \right)^{2/[(s-1)(k-1)+1]} \right\} \quad (2.11)$$

with the quantities $\bar{V}_s(f)$, $1 \leq s \leq k$, defined in formulas (2.5) and (2.6).

The paper of Hanson and Wright [3] contains some estimates on the tail distribution of a bilinear form

$$S_n = \sum_{i,j=1}^n a(i,j)(\xi_i \xi_j - E\xi_i \xi_j), \quad (2.12)$$

where ξ_1, \dots, ξ_n are independent standard normal random variables, and $A = (a(i,j))$, $1 \leq i, j \leq n$, is an $n \times n$ symmetric matrix. Hanson and Wright introduced the Hilbert–Schmidt norm $\Lambda^2 = \sum_{i,j=1}^n a(i,j)^2$ and the usual norm $\|A\| = \sup_{|x|=1} |Ax|$ of the matrix

$A = (a(i,j))$, where $|x|$ denotes the usual Euclidean norm of the vector $x = (x_1, \dots, x_n)$. They proved the following estimate for the tail distribution of S_n .

$$P(S_n > x) \leq C_1 \exp \left\{ - \min \left\{ \frac{C_2 x}{\|A\|}, \frac{C_2 x^2}{\Lambda^2} \right\} \right\} \quad (2.13)$$

for all $x > 0$ with some universal constants $C_1 > 0$ and $C_2 > 0$.

This inequality agrees with the estimate (2.11) in the theorem about the estimation of moments and tail distribution of polynomials of independent standard normal random variables in the case $k = 2$. Indeed, the random polynomials Z_k defined in (2.2) agree with the polynomials S_n in (2.12), if they contain the same coefficients $a(i, j)$. Beside this, in this case $\bar{V}_1((a(\cdot))^2) = \Lambda^2$, and $\bar{V}_2(a(\cdot)) = \|A\|$ for the terms $\bar{V}_1(a(\cdot))$ and $\bar{V}_2(a(\cdot))$ defined in (2.5) and (2.6). The condition that the matrix $A = ((a(i, j)))$ must be symmetric does not mean a real restriction, because a general polynomial S_n can be replaced by its symmetrization, which does not change its distribution. With such a notation the estimate (2.13) agrees with the estimate (2.11) for $k = 2$.

Latała (see [5]) studied the estimation of polynomials of independent standard normal random variables which have the following special form.

$$Z_k = \sum_{1 \leq j_s \leq n, 1 \leq s \leq k} a(j_1, \dots, j_k) \xi_{j_1}^{(1)} \dots \xi_{j_k}^{(k)}, \quad (2.14)$$

where $\xi_1^{(s)}, \dots, \xi_n^{(s)}$, $1 \leq s \leq k$, are independent random sequences of independent standard normal random variables. In this case he formulated an estimate sharper than ours. But the proof of this result contains an error, and it is not clear whether Latała's estimate holds. Hence I present it as a conjecture. Its formulation presented here slightly is slightly different from that of paper [5], but they are equivalent. They have a similar relation to each other as the original and simplified versions of the theorem about the tail distribution of Wiener–Itô integrals in Section 1.

Latała's conjecture. *Let the coefficients of the random Gaussian polynomial Z_k of order k defined in (2.14) satisfy the inequality $\bar{V}_s(a(\cdot)) \leq M^{-(s-1)/2}$ with some positive integer M for all $1 \leq s \leq k$, with the quantities $\bar{V}_s(a(\cdot))$ defined in formulas (2.5) and (2.6). Then there exists some universal constant C depending only on the order k of the random polynomial Z_k such that*

$$EZ_k^{2M} \leq C^M M^M. \quad (2.15)$$

It follows from a result of de la Peña and Montgomery–Smith [1] that if Latała's conjecture holds for random polynomials of the form (2.14), then it also holds for polynomials of the form

$$Z_k = \sum_{1 \leq j_s \leq n, 1 \leq s \leq k, j_s \neq j_{s'} \text{ if } s \neq s'} a(j_1, \dots, j_k) \xi_{j_1} \dots \xi_{j_k},$$

where ξ_1, \dots, ξ_n are independent standard normal random variables. With the help of this observation and some additional work Latała's conjecture can be verified for general polynomials of the form (2.2), provided that it holds in its original form. I omit the details.

On the other hand, Latała's argument heavily exploited the special form of the random polynomials in (2.14). In the study of these polynomials it was important that the terms in the sum (2.14) contain such products $\xi_{j_1}^{(1)} \dots \xi_{j_k}^{(k)}$ whose terms are elements of independent copies of a random sequences. This fact enabled Latała to apply a conditioning argument, and he could reduce the original problem to the estimation of the supremum of an appropriately defined class of Gaussian random variables with its help. But the estimation of such a supremum is very hard, and at this point some problems arise about Latała's proof. A problem of the following type has to be considered.

Let us have a set of (jointly) Gaussian random variables, $\eta(x)$, $E\eta(x) = 0$, $E\eta^2(x) \leq 1$, $x \in X$, indexed by a parameter set X , and try to give a good estimate on the expression $E \left(\sup_{x \in X} \eta(x) \right)^{2M}$ for large positive integers M . To study this problem let us introduce the (pseudo)metric ρ_α defined by the formula $\rho_\alpha(x, y) = (E(\eta(x) - \eta(y))^2)^{1/2}$, $x, y \in X$, in the parameter space X . There is a natural way to give good estimates on the moments of the supremum we are interested in if we can give for all $\varepsilon > 0$ a good estimate on the minimal number $N(X, \rho_\alpha, \varepsilon)$ of balls of radius ε with respect to the distance ρ_α which cover the whole parameter set X .

The value $N(X, \rho_\alpha, \varepsilon)$ can be estimated in the following special case. If a probability measure μ_ε can be introduced in the parameter set X (or on an extension of the set X as it is done in Latała's paper) for all $\varepsilon > 0$ in such a way that the estimate $\mu_\varepsilon(y: \rho(x, y) > \varepsilon/2) \geq H(\varepsilon)$ is valid with some function $H(\cdot)$ for all $x \in X$, then the inequality $N(X, \rho_\alpha, \varepsilon) \leq \frac{1}{H(\varepsilon)}$ holds. Latała claimed that such a construction is possible in the problem he is investigating. He formulated two lemmas, Lemma 1 and Lemma 2 in his paper which supply a good estimate, presented in Corollary 2, on $N(X, \rho_\alpha, \varepsilon)$.

However, the proof of these Lemmas 1 and 2 is problematic. Lemma 1 contains a small inaccuracy (it states the upper bound $e^{-t^2/2}$ instead of $e^{-1/2t^2}$, and this wrong formula is written rather consequently), but this seems to be a corrigible error. The main problem is that Lemma 1 supplies a much weaker estimate than Lemma 2 which is not sufficient to get the results of the paper. At the explanation of this point I refer to the notation of the paper [5].

The right formula in the first line of the proof of Lemma 2 for $d = 1$ (page 2319 in paper [5]) would be

$$\begin{aligned} B_\alpha(\mathbf{x}, W_d^{[\mathbf{x}]}(\alpha, 4t)) &= \{y \in R^{n_1}: \alpha(x - y) \leq W_d^{[\mathbf{x}]}(\alpha, 4t)\} \\ &= \{y \in R^{n_1}: \alpha(x - y) \leq 4t\alpha(xG_{n_1})\}, \end{aligned}$$

where $xG_{n_1} = (x_j g_j, 1 \leq j \leq n_1)$, with a standard normal random vector $G_{n_1} = (g_1, \dots, g_{n_1})$. Hence in the proof of Lemma 2 with $d = 1$ Lemma 1 should be applied with the parameter \bar{t} determined by the equation $4t\alpha(G_{n_1}) = 4\bar{t}\alpha(xG_{n_1})$ (instead of the parameter t as it is done in [5]). This number \bar{t} can be very small, since we have to consider such vectors x for which $\sum_{j=1}^{n_1} x_j^2 \leq 1$. Hence Lemma 1 does not supply a good estimate in such a case. In particular, the estimate we get, depends on the dimension

n_1 of the space R^{n_1} , i.e. of the space where the random vector G_{n_1} takes its values. On the other hand, we need such estimates which do not depend on this dimension.

The question arises whether the proof can be saved despite of this error. I think that the hardest problem about Latała's proof is hiding behind this question. In Section 5 I return to it. For the sake of simpler notations I shall consider only the case $k = 3$. I show that Latała's conjecture is equivalent to a rather hard estimate on the expected value of the supremum of some random multilinear forms whose study demands new ideas.

3. The diagram formula for Wiener–Itô integrals.

In this section the diagram formula for products of Wiener–Itô integrals is formulated, and it is shown how the proof of the simplified version of the theorem about the tail distribution of Wiener–Itô integrals can be reduced with its help to an estimate that I call the Basic Estimate.

The diagram formula makes possible to rewrite the product of Wiener–Itô integrals as a sums of such integrals, and as a consequence, it supplies a formula about the moments of Wiener–Itô integrals. It was shown in [8] that this formula yields a good estimate on the moments of Wiener–Itô integrals. In this paper it will be shown that if the quantities $V_s(f)$, $1 \leq s \leq k$, defined in (1.6) and (1.7) are very small, then this method yields a better moment estimate. I recall this formula. It is the same result that I presented in paper [8], only I made some small changes in the notation. The indices of the arguments of the functions we are working with will be indexed in a different way, because this simplifies the discussion.

The following problem is considered: Let us have m such real-valued functions

$$f_j(x_{(j,1)}, \dots, x_{(j,k_j)}), \quad 1 \leq j \leq m$$

with k_j variables on a measure space (X, \mathcal{X}, μ) with some σ -finite non-atomic measure μ for which

$$\int f_j^2(x_{(j,1)}, \dots, x_{(j,k_j)}) \mu(dx_{(j,1)}) \dots \mu(dx_{(j,k_j)}) < \infty \quad \text{for all } 1 \leq j \leq m \quad (3.1)$$

together with a white noise μ_W with reference measure μ on the space (X, \mathcal{X}) . The Wiener–Itô integrals $k_j! I_{k_j}(f_j)$, $1 \leq j \leq m$, introduced in (1.3) can be defined with the above kernel functions f_j and white noise μ_W . We are interested in a good explicit

formula for the expectation $E \left(\prod_{j=1}^m I_{k_j}(f_j) \right)$. This formula, which is a simple consequence of the diagram formula for products of Wiener–Itô integrals (see [2] or [7]) will be presented below.

The expectation of the above product can be expressed by means of some (closed) diagrams introduced below. A class of (closed) diagrams denoted by $\bar{\Gamma} = \bar{\Gamma}(k_1, \dots, k_m)$ is defined in the following way. A diagram $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$ consists of vertices of the form (j, l) , $1 \leq j \leq m$, $1 \leq l \leq k_j$, and edges $((j, l), (j', l'))$, $1 \leq j, j' \leq m$, $1 \leq l \leq k_j$,

$1 \leq l' \leq k'_j$. The set of vertices (j, l) , $1 \leq l \leq k_j$, with a fixed number j is called the j -th row of the diagram. All edges $((j, l), (j', l'))$ of a diagram $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$ connect vertices from different rows, i.e. $j \neq j'$. It is also demanded that exactly one edge starts from all vertices of a (closed) diagram γ . The class $\bar{\Gamma}(k_1, \dots, k_m)$ of (closed) diagrams contains the diagrams γ with the above properties. Beside this, I introduce the set $U(\gamma)$ containing the edges of the diagram γ for all $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$ and enumerate their elements. An arbitrary enumeration is allowed, it is only demanded that different edges must get different labels. Let $N(\gamma)$ denote the set of indices attached to the elements of $U(\gamma)$. In this section I shall consider only closed diagrams in which every vertex is the end-point of some edge. In the next section we have to work with more general, not necessarily closed diagrams. I introduce their definition there.

Let us fix a diagram $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$. I introduce the following function $u_\gamma(\cdot)$ on the vertices (j, l) , $1 \leq j \leq m$, $1 \leq l \leq k_j$.

For each vertex of γ there is a unique edge $((j, l), (j', l')) \in U(\gamma)$, i.e. an edge in γ which connects (j, l) with some other vertex (j', l') of the diagram. If this edge has label $n \in N(\gamma)$, then we put $u_\gamma((j, l)) = n$.

Given a fixed diagram $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$ let us rewrite the functions f_j with reindexed variables as $f_j(x_{u_\gamma(j,1)}, \dots, x_{u_\gamma(j,k_j)})$, $1 \leq j \leq m$, with the help of the above defined function $u_\gamma(\cdot)$. (Two variables get the same index if the vertices related to them were connected by an edge of the diagram γ .) Define the product of these reindexed variables

$$\bar{F}_\gamma(x_n, n \in N(\gamma)) = \prod_{j=1}^m f_j(x_{u_\gamma(j,1)}, \dots, x_{u_\gamma(j,k_j)}) \quad (3.2)$$

together with the integral of these functions

$$F_\gamma = F_\gamma(f_1, \dots, f_m) = \int \bar{F}_\gamma(x_n, n \in N(\gamma)) \prod_{n \in N(\gamma)} \mu(dx_n) \quad (3.3)$$

for all $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$.

The expected value of the product of Wiener–Itô integrals $k_j! I_{k_j}(f_j)$, $1 \leq j \leq m$, can be expressed with the help of the above quantities F_γ in the following way.

Formula about the expected value of products of Wiener–Itô integrals. *Let us consider the Wiener–Itô integrals $k_j! I_{k_j}(f_j)$ of some functions f_j , $1 \leq j \leq m$, satisfying relation (3.1) with respect to a white noise μ_W with reference measure μ . The expected value of this product satisfies the identity*

$$E \left(\prod_{j=1}^m k_j! I_{k_j}(f_j) \right) = \sum_{\gamma \in \bar{\Gamma}(k_1, \dots, k_m)} F_\gamma \quad (3.4)$$

with the numbers F_γ defined in (3.2) and (3.3).

To get a good estimate on the expectation of the product of Wiener–Itô integrals by means of formula (3.4) we need a good bound on the quantities F_γ . For this goal

it is useful to rewrite them by means of an appropriate recursive formula. To present such a formula let us define the restrictions γ^r , $1 \leq r \leq m$, of all (closed) diagrams $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$ to its first r rows, $1 \leq r \leq m$.

More explicitly, the diagram γ^r contains the vertices (j, l) , $1 \leq j \leq r$, $1 \leq l \leq k_j$ and those edges of γ whose end-points are vertices in one of the first r rows in γ . I shall call γ^r also a diagram, although it may have vertices from which no edge starts. I return to this point in the next section. Let $U_1(\gamma^r)$ denote the set of edges and $U_2(\gamma^r)$ the set of those vertices of γ^r from which no edge starts in γ^r , i.e. which are connected with a vertex (j', l') with $j' > r$ in γ . Let all vertices of γ^r get the same enumeration that they got as a vertex of γ . Let $N_1(\gamma^r)$ denote the set of indices of the vertices which are end-points of an edge in $U_1(\gamma^r)$, and $N_2(\gamma^r)$ the set of indices of the vertices in $U_2(\gamma^r)$. Put $N(\gamma^r) = N_1(\gamma^r) \cup N_2(\gamma^r)$.

Let us define, similarly to the quantities \bar{F}_γ and F_γ , the functions

$$\bar{F}_{\gamma^r}(x_n, n \in N(\gamma^r)) = \prod_{j=1}^r f_j(x_{u_\gamma(j,1)}, \dots, x_{u_\gamma(j,k_j)}) \quad (3.5)$$

and

$$F_{\gamma^r}(x_n, n \in N_2(\gamma^r)) = \int \bar{F}_{\gamma^r}(x_n, n \in N(\gamma^r)) \prod_{n \in N_1(\gamma^r)} \mu(dx_n) \quad (3.6)$$

for all $1 \leq r \leq m$, and $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$. In the case $N_1(\gamma^r) = \emptyset$ the integral at the right-hand side equals $\bar{F}_{\gamma^r}(x_n, n \in N(\gamma^r))$. In general, the following convention is applied. If we integrate a function with respect to a product measure, then in the case when this product contains zero terms, then the integral equals the function itself.

It is not difficult to check that

$$F_{\gamma^1}(x_n, n \in N_2(\gamma^1)) = f_1(x_{u_\gamma(1,1)}, \dots, x_{u_\gamma(1,k_1)}), \quad (3.7)$$

and

$$\begin{aligned} & F_{\gamma^r}(x_n, n \in N_2(\gamma^r)) \\ &= \int F_{\gamma^{r-1}}(x_n, n \in N_2(\gamma^{r-1})) f_r(x_{u_\gamma(r,1)}, \dots, x_{u_\gamma(r,k_r)}) \prod_{n \in N_2(\gamma^r) \setminus N_2(\gamma^{r-1})} \mu(dx_n) \end{aligned} \quad (3.8)$$

for all $2 \leq r \leq m$. Beside this,

$$F_\gamma = F_{\gamma^m}(x_n, n \in N_2(\gamma^m)). \quad (3.9)$$

(Actually, $N_2(\gamma^m) = \emptyset$.) Relations (3.7), (3.8) and (3.9) yield a recursive formula for the quantity F_γ .

Next I introduce the notion of connected (closed) diagrams. They turned out to be a useful object, because the quantity F_γ can be better bounded for a connected than for a general diagram γ , and it is enough to bound them to prove our results.

Definition of connected (closed) diagrams. A (closed) diagram $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$ is connected if for all sets of rows $A \subset \{1, \dots, m\}$ of the diagram γ such that $1 \leq |A| \leq m - 1$ there is such an edge $((j_1, l_1), (j_2, l_2)) \in U(\gamma)$ of the diagram γ for which $j_1 \in A$ and $j_2 \notin A$.

The following result, called the Basic Estimate yields a bound on F_γ for connected diagrams γ . This estimate turned out to be sufficient for our purposes.

Basic Estimate. Let us consider a connected, closed diagram $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$, $m \geq 2$, and some functions f_j of k_j variables on a measure space (X, \mathcal{X}, μ) , $1 \leq j \leq m$, which satisfy the inequality $V_s(f_j) \leq R^{s-1}$ with some $0 \leq R \leq 1$, for all $1 \leq j \leq m$ and $1 \leq s \leq k_j$. (The quantities $V_s(f)$ were introduced in formulas (1.6) and (1.7).) The quantity $F_\gamma = F_\gamma(f_1, \dots, f_m)$, introduced in (3.3) satisfies the inequality

$$|F_\gamma| = |F_\gamma(f_1, \dots, f_m)| \leq R^{m-2}. \quad (3.10)$$

I show how the *simplified version of the theorem about the tail distribution of Wiener–Itô integrals* can be proved with the help of the Basic Lemma. To do this first I show that all diagrams $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$ can be decomposed to the union of disjoint connected diagrams in a unique way, and the quantity F_γ equals the product of the numbers $F_{\{\cdot\}}$ corresponding to these connected diagrams.

More explicitly, there is a unique partition $A_1 = A_1(\gamma), \dots, A_s = A_s(\gamma)$ of the set of rows $\{1, \dots, m\}$ of $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$ in such a way that the diagram γ equals the union of the diagrams γ_{A_u} , $1 \leq u \leq s$, where γ_{A_u} is the restriction of γ to the rows in A_u , i.e. it contains the rows of γ with indices in the set A_u together with the edges connecting vertices from these rows. Beside this, all diagrams γ_{A_u} of this decomposition must be connected. Also the identity

$$F_\gamma = \prod_{u=1}^s F_{\gamma_{A_u}} \quad (3.11)$$

holds.

In more detail, the restriction γ_{A_u} of γ to the rows with indices in A_u consists of vertices in the rows with index in the set of A_u , and those edges whose both end-points are among these vertices. The decomposition of γ to the diagrams γ_{A_u} , $1 \leq u \leq s$, means also that γ has no such edge which connects vertices from γ_{A_u} with vertices from $\gamma_{A_{u'}}$ with some $u \neq u'$. The connectedness of γ_{A_u} means that for all subsets $B \subset A_u$, $B \neq \emptyset$ and $B \neq A_u$ there is an edge of γ_{A_u} which connects a vertex in a row with index in B and a vertex in a row with index in $A_u \setminus B$. The quantities $F_{\gamma_{A_u}}$ can be defined similarly to F_γ , for instance by a natural adaptation of the recursive formulas (3.7), (3.8) and (3.9) in the calculation of F_γ to the calculation of $F_{\gamma_{A_u}}$. In this adaptation we can write similar recursive relations, only the row indices $1, \dots, m$ must be replaced by rows with indices $v_1, \dots, v_{|A_u|}$ if $A_u = \{v_1, \dots, v_{|A_u|}\}$ with $v_1 < v_2 < \dots < v_{|A_u|}$. It is not difficult to check that the Basic Estimate also implies that under the conditions of

this result the inequality $|F_{\gamma_A}| \leq R^{|A|-2}$ also holds for any connected (closed) diagram with rows in the set $A \subset \{1, \dots, m\}$.

To find the desired decomposition of a diagram $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$ let us define the graph $G(\gamma)$ with vertices $1, \dots, m$ in which two vertices j_1 and j_2 are connected with an edge if and only if the diagram γ contains an edge which connects two vertices from the j_1 -th and j_2 -th rows. Let A_1, \dots, A_s be the connected disjoint components of this graph. Then it is not difficult to see that $\gamma_{A_1}, \dots, \gamma_{A_s}$ supplies the desired decomposition of the diagram γ , and also relation (3.11) holds.

Given a set $A \subset \{1, \dots, m\}$ let $\bar{\Gamma}_c(k_j, j \in A)$ denote the class of those connected (closed) diagrams whose rows are the sequences $\{(j, 1), \dots, (j, k_j)\}$, $j \in A$. Put

$$K_c(A) = \sum_{\gamma \in \bar{\Gamma}_c(k_j, j \in A)} F_{\gamma_A}, \quad (3.12)$$

and let us also introduce for a partition $P = \{A_1, \dots, A_s\} \in \mathcal{P}(\{1, \dots, m\})$ of the set $\{1, \dots, m\}$ the class of diagrams $\bar{\Gamma}(A_1, \dots, A_s) = \bar{\Gamma}(A_1, \dots, A_s | k_1, \dots, k_m)$ containing those diagrams $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$ whose decomposition to connected components consists of diagrams with rows indexed by the sets A_1, \dots, A_s , i.e. of diagrams $\gamma_{A_r} \in \bar{\Gamma}_c(k_j, j \in A_r)$, $1 \leq r \leq s$. Define the quantity

$$K(A_1, \dots, A_s) = \sum_{\gamma \in \bar{\Gamma}(A_1, \dots, A_s)} F_{\gamma}$$

for all partitions $\{A_1, \dots, A_s\}$ of the set $\{1, \dots, m\}$.

It is not difficult to see with the help of relations (3.11) and (3.12) that

$$K(A_1, \dots, A_s) = K_c(A_1) \cdots K_c(A_s)$$

for all partitions $\{A_1, \dots, A_s\}$ of the set $\{1, \dots, m\}$. Summing up this identity for all classes $\bar{\Gamma}(A_1, \dots, A_r)$ we get with the help of the identity (3.4) and the fact that each diagram has a unique decomposition to connected diagrams that

$$E \left(\prod_{j=1}^m k_j! I_{k_j}(f_j) \right) = \sum_{\{A_1, \dots, A_s\} \in \mathcal{P}(\{1, \dots, m\})} K_c(A_1) \cdots K_c(A_s) \quad (3.13)$$

In the next calculations I shall restrict my attention to the case $m = 2M$, $f_j = f$, $k_j = k$ for all $1 \leq j \leq 2M$. I give a good upper bound on $E I_k(f)^{2M}$ with the help of relations (3.12), (3.13) and the Basic Estimate in the case when $V_s(f) \leq R^{s-1}$ for all $1 \leq s \leq k$ with $R = M^{-(k-1)/2}$. In these calculations I also exploit that a connected closed diagram has at least two rows, hence only such partitions $\{A_1, \dots, A_s\} \in \mathcal{P}(\{1, \dots, m\})$ have to be considered in the sum at the right-hand side of (3.13) for which $|A_r| \geq 2$ for all $1 \leq r \leq s$.

First I show that if $V_s(f) \leq R^{s-1}$ for all $1 \leq s \leq k$, then

$$|K_c(A)| \leq (k|A|)^{k|A|/2} R^{|A|-2} \quad \text{for all } A \subset \{1, \dots, m\} \quad (3.14)$$

for the quantity $K_c(A)$ defined in (3.12). Indeed, the Basic Estimate implies that $|F(\gamma_A)| \leq R^{|A|-2}$ for all terms in the sum at the right-hand side of (3.12), and clearly there are less than $(k|A|)^{k|A|/2}$ diagrams with $|A|$ rows and in each rows k vertices. (At this point we bounded the number of all (and not only the number of all connected) diagrams.)

To estimate the expression at the right-hand side of (3.13) let us introduce the class of those partitions $\mathcal{P}_{s,t_1,\dots,t_s}$, $t_1 + \dots + t_s = 2M$, $t_r \geq 2$, $1 \leq r \leq s$, of the set $\{1, \dots, 2M\}$ which consist of s sets A_1, \dots, A_s , and the set A_r has with t_r elements, $1 \leq r \leq s$. (The set $\mathcal{F}_{s,t_1,\dots,t_s}$ depends on the number s and the set $\{t_1, \dots, t_s\}$, but it does not depend on the order of the elements t_r in this set.) Let us first estimate the contribution of the partitions $\mathcal{P}_{s,t_1,\dots,t_s}$ to the sum in (3.13). I claim that

$$\begin{aligned} & \left| \sum_{\{A_1, \dots, A_s\} \in \mathcal{P}_{s,t_1,\dots,t_s}} K_c(A_1) \cdots K_c(A_s) \right| \\ & \leq |\mathcal{P}_{s,t_1,\dots,t_s}| \prod_{r=1}^s (kt_r)^{kt_r/2} R^{t_r-2} \leq C^M M^M \left[M^{(k-1)} R^2 \right]^{M-s} \end{aligned} \quad (3.15)$$

with a universal constant $C > 0$ depending only on the parameter k . In particular,

$$\left| \sum_{\{A_1, \dots, A_s\} \in \mathcal{P}_{s,t_1,\dots,t_s}} K_c(A_1) \cdots K_c(A_s) \right| \leq C^M M^M \quad (3.16)$$

if $V_s(f) \leq M^{-(k-1)(s-1)/2}$ for all $1 \leq s \leq k$, because (3.15) holds with $R = M^{-(k-1)/2}$ in this case.

The first inequality in (3.15) is a straight consequence of relation (3.14). To prove the second inequality we need a good bound on $|\mathcal{P}_{s,t_1,\dots,t_s}|$. To get such a bound let us first list the element A_1, \dots, A_s of a partition in $\mathcal{P}_{s,t_1,\dots,t_s}$ in the following way. Let A_1 be the set which contains the number 1, A_2 the set containing the smallest number not contained in A_1 , e.t.c.. Then let t_r denote the cardinality of the set A_r with such an indexation. Then the number $|\mathcal{P}_{s,t_1,\dots,t_s}|$ can be given explicitly, but the following estimate will be sufficient for our purposes.

$$|\mathcal{P}_{s,t_1,\dots,t_s}| \leq \prod_{r=1}^s \frac{(2M)^{t_r-1}}{(t_r-1)!} \leq C_1^M \frac{M^{2M-s}}{\prod_{r=1}^s t_r^{t_r}} \quad (3.17)$$

with some appropriate $C_1 > 0$.

The first inequality in relation (3.17) can be simply checked. To prove the second inequality let us first observe that $\prod_{r=1}^s t_r \leq \left(\frac{1}{s} \sum_{r=1}^s t_r \right)^s = \left(\frac{2M}{s} \right)^s \leq C_2^M$ with some universal constant C_2 . This inequality together with the Stirling formula imply that

$\prod_{r=1}^s (t_r - 1)! \geq C_2^{-M} \prod_{r=1}^s t_r! \geq C_3^{-M} \prod_{r=1}^s t_r^{t_r}$ with some appropriate values C_3 , hence relation (3.17) holds.

It follows from relation (3.17) that

$$|\mathcal{P}_{s,t_1,\dots,t_s}| \prod_{r=1}^s (kt_r)^{kt_r/2} R^{t_r-2} \leq C_4^M M^{2M-s} \left(\prod_{r=1}^s t_r^{(k-2)t_r/2} \right) R^{2M-2s}. \quad (3.18)$$

Let us consider the maximum of the right-hand side in the inequality (3.18) in the parameters t_1, \dots, t_s with a fixed value s . Since $k-2 \geq 0$, this expression takes its maximum if $t_1 = 2M - 2s + 2$ and $t_r = 2$, $2 \leq r \leq s$. Hence

$$\begin{aligned} |\mathcal{P}_{s,t_1,\dots,t_s}| \prod_{r=1}^s (kt_r)^{kt_r/2} R^{t_r-2} &\leq C_4^M M^{2M-s} (2M - 2s + 2)^{(k-2)(M-s+1)} 2^{(s-1)(k-1)} R^{2M-2s} \\ &\leq C^M M^{2M-s} M^{(k-2)(M-s)} R^{2M-2s} = C^M M^M [M^{k-1} R^2]^{M-s} \end{aligned}$$

with some $C > 0$ depending only on the parameter k . This inequality finishes the proof of relation (3.15).

Relation (3.16) is a simple consequence of (3.15). Finally, it follows from (3.4) (with $m = 2M$ and $f_j = f$ for all $1 \leq j \leq 2M$) and (3.16) that

$$E(k! I_k(f))^{2M} \leq 2^{2M} C^M M^M$$

if $V_s(f) \leq M^{-(k-1)(s-1)/2}$ for all $1 \leq s \leq k$, which means that the simplified version of the theorem about the tail distribution of Wiener–Itô integrals is proved with the help of the Basic Estimate.

To see that relations (3.4) and (3.16) really imply the last inequality it is enough to observe that the class of sets $\mathcal{P}(\{1, \dots, 2M\})$ is the union of the classes of sets $\mathcal{P}_{s,t_1,\dots,t_s}$, and there are less than 2^{2M} classes $\mathcal{P}_{s,t_1,\dots,t_s}$. (The number of such classes is bounded by the number of sets of positive integers $\{t_1, \dots, t_s\}$ such that $t_1 + \dots + t_s = 2M$.)

We could prove our result by means of a good estimate on the quantity F_γ for connected diagrams. The introduction of connected diagrams was very important in our considerations, because the Basic Estimate holds only for such diagrams. I show an example of non-connected (closed) diagrams which satisfies only a much weaker estimate. Let us consider a diagram $\gamma \in \Gamma(k_1, \dots, k_m)$ and a function $f_j = f$ with $m = 2M$, $k_j = k$ for all $1 \leq j \leq 2M$. Beside this, let the f be a function, symmetric in its variables and such that $V_1(f) = 1$. Let us consider such a diagram γ whose rows can be put into pairs in such a way that edges can connect only vertices from rows which are paired. For such a diagram $F_\gamma = V_1(f)^{2M} = 1$, and this value is much larger than the bound in the Basic Estimate. But there are relatively few such diagrams, their number equals $\frac{(2M)!}{2^M M!} (k!)^M$. Hence the relatively great value of F_γ for such diagrams causes no problem.

The estimation of the moments with the help of connected diagrams corresponds to a classical method of probability theory, to the estimation of moments by means of semi-invariants. The quantity $K_c(A)$ introduced in formula (3.12) is actually the semi-invariant of the random variables $k_j! I_{k_j}(f)$ for $j \in A$, and the identity (3.13) is a special case of the formula about the expression of the expectation of products of random variables by means of semi-invariants. Since I shall not work with semi-invariants in an explicit form in this paper, here I do not discuss this subject in more detail. Some useful information about semi-invariants can be found in the second chapter of the book [11], and I also return to this question in paper [10].

4. The proof of the Basic Estimate.

The Basic Estimate will be proved by means of an inductive proposition about the behaviour of the functions F_{γ^r} , $1 \leq r \leq m$, where γ^r is the restriction of the closed diagram γ to its first r rows. To formulate and prove this result the notion of diagrams, introduced in Section 3, will be generalized. Such diagrams will be defined which may have such vertices from which no edge starts. The definition of some other objects related to these new diagrams will be also presented together with some results needed in our further discussion.

A diagram with rows in a finite set $A = \{j_1, \dots, j_m\}$, $1 \leq j_1 < j_2 < \dots < j_m$ of the positive integers and with row length j_t , $1 \leq t \leq m$, is a graph whose vertices are the pairs (j_t, l) with $1 \leq t \leq m$ and $1 \leq l \leq k_{j_t}$. The set of points $\{(j_t, l): 1 \leq l \leq k_{j_t}\}$ is called the t -th row of the diagram. A diagram may contain such edges $((j_t, l), (j_{t'}, l'))$ for which $j_t, j_{t'} \in A$, $1 \leq l \leq k_{j_t}$, $1 \leq l' \leq k_{j_{t'}}$, and $j_t \neq j_{t'}$. In words this means that edges can connect only vertices from different rows. Beside this, it is required that from each vertex there starts either zero or one edge. Graphs satisfying all these properties will be called diagrams. The class of diagrams with rows indexed by a set A and with k_j element in the row indexed by $j \in A$ will be denoted by $\Gamma(k_j | j \in A)$. The main difference between diagrams and closed diagrams introduced in the previous section is that a general diagram may contain vertices from which no edge starts. Those vertices of a diagram γ from which no edge starts will be called open vertices. The notion of connected diagrams will be also introduced for this more general class of diagrams.

Definition of connected diagrams in the general case. *A diagram $\gamma \in \Gamma(k_j | j \in A)$ is connected if for all sets $B \subset A$ such that $B \neq \emptyset$ and $B \neq A$ there exists an edge $((j_t, l), (j_{t'}, l'))$ of γ such that $j_t \in B$ and $j_{t'} \in A \setminus B$.*

Similarly to closed diagrams, general diagrams $\gamma \in \Gamma(k_j | j \in A)$ also have a unique decomposition to connected diagrams. To formulate this statement precisely let us first introduce the reduction of a diagram to some of its rows. Given a diagram $\gamma \in \Gamma(k_j | j \in A)$, $A = \{j_1, \dots, j_m\}$ and a set $B \subset A$ let $\gamma_B \in \Gamma(k_j | j \in B)$ denote that diagram whose vertices are points of the form (j_t, l) , $j_t \in B$, $1 \leq l \leq k_{j_t}$, and two vertices in γ_B are connected by an edge if and only if they are connected by an edge in γ . With this notation we can state that for all diagrams $\gamma \in \Gamma(k_j | j \in A)$ there exists a unique partition A_1, \dots, A_s of the set of rows A in such a way that all diagrams γ_{A_u} , $1 \leq u \leq s$, are connected, and the diagram γ_A is the union of the diagrams γ_{A_u} , $1 \leq u \leq s$. This

can be proved similarly to the case of closed diagrams. One has to take the graph whose vertices are the points of the set A and draw an edge between two vertices $j_t \in A$, and $j_{t'} \in A$ if there is an edge in the diagram γ connecting some vertices from the j_t -th and the $j_{t'}$ -th row. By taking the decomposition of this graph to connected components we also get the decomposition of the diagram γ to connected components.

Given a diagram $\gamma \in \Gamma(k_j | j \in A)$ with $A = \{j_1, \dots, j_m\}$ together with some functions $f_t(x_{(j_t,1)}, \dots, x_{(j_t, k_{j_t})})$, $1 \leq t \leq m$, a natural generalization of the quantities \bar{F}_γ and F_γ defined in (3.2) and (3.3) can be introduced, and it can be shown that they have properties similar to them. To introduce these quantities let us enumerate first the edges and then the vertices of the diagram γ . A vertex from which an edge starts gets the same label as the edge starting from it. The remaining vertices, from which no edge starts get a new label. In this enumeration two vertices get the same index if and only if they are connected by an edge. Let $u_\gamma(j, l)$ denote the label of the vertex (j, l) in this enumeration of the vertices of a diagram γ . Let $N(\gamma)$ denote the set of labels of all vertices and $N_1(\gamma)$ the set of labels of all open vertices in γ . Then we define, similarly to formulas (3.2) and (3.3) the quantities

$$\bar{F}_\gamma(x_n, n \in N(\gamma)) = \prod_{t=1}^m f_t(x_{u_\gamma(j_t,1)}, \dots, x_{u_\gamma(j_t, k_{j_t})}) \quad (4.1)$$

and

$$\begin{aligned} F_\gamma(x_n, n \in N_1(\gamma)) &= F_\gamma(x_n, n \in N_1(\gamma) | f_1, \dots, f_m) \\ &= \int \bar{F}_\gamma(x_n, n \in N(\gamma)) \prod_{n \in N_2(\gamma)} \mu(dx_n) \end{aligned} \quad (4.2)$$

with $N_2(\gamma) = N(\gamma) \setminus N_1(\gamma)$.

Given a diagram $\gamma \in \Gamma(k_j | j \in A)$ with $A = \{j_1, \dots, j_m\}$ let γ^r denote its restriction to its first r rows, i.e. to rows in $A_r = \{j_1, \dots, j_r\}$, $1 \leq r \leq m$. The natural modification of the recursive relations (3.7), (3.8) and (3.9) remain valid also for general diagrams γ . The only difference we have to make is to rewrite the indices $u_\gamma(1, l)$ and $u_\gamma(r, l)$ by $u_\gamma(j_1, l)$ and $u_\gamma(j_r, l)$ in the new relations. Beside this, the following natural modification of formula (3.11) holds.

If the decomposition of a diagram $\gamma = \gamma_A \in \Gamma(k_j | j \in A)$ to connected components consists of the diagrams $\gamma_{A_1}, \dots, \gamma_{A_s}$, where A_1, \dots, A_s is the partition of the set A we have to apply to get the desired decomposition to connected components, then the above defined function $F_\gamma(x_n, n \in N_1(\gamma))$ satisfies the identity

$$F_{\gamma_A}(x_n, n \in N_1(\gamma_A)) = \prod_{t=1}^s F_{\gamma_{A_t}}(x_n, n \in N_1(\gamma_{A_t})), \quad (4.3)$$

where $N_1(\gamma_{A_t})$ denotes the set of labels of the open vertices in γ_{A_t} , and $F_{\gamma_{A_t}}$ is the function defined in (4.2) with the diagram $\gamma = \gamma_{A_t}$. Beside this, the sets of labels $N_1(\gamma_{A_t})$ are disjoint for different indices t . Here I make the convention that the label of a vertex in the restriction γ_B of a diagram γ_A , $B \subset A$, agrees with its original label in

the diagram γ_A . Formula (4.3) can be checked similarly to (3.8) with the help of (the modified version of) the recursive relations (3.8) and (3.9).

Given a closed diagram γ , its restriction γ^r to its first r rows may be not a closed diagram. But it is also a diagram, and the above mentioned results can be applied for it. In particular, it has a decomposition to connected components, and the function F_{γ^r} defined by means of some functions f_j , $1 \leq j \leq r$ can be factorized. The next result, called *A More Detailed Version of the Basic Estimate* contains an estimate for the terms in this factorization. The Basic Estimate is a part of this result for the parameter $r = m$.

A More Detailed Version of the Basic Estimate. *Let us consider a connected, closed diagram $\gamma \in \bar{\Gamma}(k_1, \dots, k_m)$, $m \geq 2$, and some functions f_j of k_j variables on a measure space (X, \mathcal{X}, μ) , $1 \leq j \leq m$, which satisfy the inequality $V_s(f_j) \leq R^{s-1}$ with some $0 \leq R \leq 1$ for all $1 \leq j \leq m$ and $1 \leq s \leq k_j$. Let γ^r denote the restriction of the diagram γ to its first r rows and let F_{γ^r} be the function defined in formula (4.2) with the help of the diagram $\gamma = \gamma^r$ and the functions f_s , $1 \leq s \leq r$. Take the decomposition of the diagram γ^r to the union of disjoint connected diagrams $\gamma_{A_t^r}$ defined with the help of an appropriate partition $A_1^r, \dots, A_{s(r)}^r$ of the set $\{1, \dots, r\}$, $1 \leq t \leq s(r)$, with some number $s(r)$. Consider the factorization (4.3) of the function F_{γ^r} to*

$$F_{\gamma^r}(x_n, n \in N_1(\gamma^r)) = \prod_{t=1}^{s(r)} F_{\gamma_{A_t^r}}(x_n, n \in N_1(\gamma_{A_t^r})).$$

For $1 \leq r \leq m-1$ all diagrams $\gamma_{A_t^r}$, $1 \leq t \leq s(r)$, contain at least one open vertex, i.e. $N_1(\gamma_{A_t^r}) \geq 1$, and

$$|V_s(F_{\gamma_{A_t^r}}(x_n, n \in N_1(\gamma_{A_t^r})))| \leq R^{|A_t^r|+s-2} \quad \text{for all } 1 \leq t \leq s(r) \text{ and } 1 \leq s \leq |N_1(\gamma_{A_t^r})|, \quad (4.4)$$

where $|A_t^r|$ is the cardinality of the set A_t^r , i.e. it equals the number of rows in the diagram A_t^r , and $N_1(\gamma_{A_t^r})$ denotes the set of indices of the free vertices in $\gamma_{A_t^r}$.

For $r = m$ the decomposition of $\gamma^m = \gamma$ to connected components consists of the (closed) diagram γ itself, and

$$|F_\gamma| = |F_{\gamma^m}| \leq R^{m-2}. \quad (4.5)$$

The proof of the *More Detailed Version of the Basic Estimate* is based on the following result called the Main Inequality.

The Main Inequality. *Let $f(x_1, \dots, x_n, v_{n+m+1}, \dots, v_{n+m+p})$ and*

$$g(x_{n+1}, \dots, x_{n+m}, v_{n+m+1}, \dots, v_{n+m+p})$$

be two square integrable functions with $m+p$ and $n+p$ variables on a measure space (X, \mathcal{X}, μ) , and define the function

$$F(x_1, \dots, x_{n+m}) = \int f(x_1, \dots, x_n, v_{n+m+1}, \dots, v_{n+m+p}) g(x_{n+1}, \dots, x_{n+m}, v_{n+m+1}, \dots, v_{n+m+p}) \prod_{u=n+m+1}^{n+m+p} \mu(dv_u). \quad (4.6)$$

Let $m+n \geq 1$, $p \geq 1$, and let the functions f and g satisfy the relation $V_s(f) \leq D_1 R^{s-2}$ and $V_s(g) \leq D_2 R^{s-2}$ with some $A > 0$, $B > 0$ and $0 \leq R \leq 1$ for all $1 \leq s \leq m+p$ and $1 \leq s \leq n+p$ respectively. Then the function F satisfies the inequality

$$V_s(F(x_1, \dots, x_{n+m})) \leq D_1 D_2 R^{s-2} \quad \text{for all } 1 \leq s \leq m+n. \quad (4.7)$$

First I prove the More Detailed Version of the Basic Estimate with the help of the Main Inequality.

Proof of the More Detailed Version of the Basic Estimate with the help of the Main Inequality. Let us first observe that all components $\gamma_{A_t^r}$, $1 \leq t \leq s(r)$, of the diagram γ^r contain at least one open vertex for $r \leq m-1$. Otherwise the diagram γ would not satisfy the condition of connectedness with $B = A_t^r$. Relation (4.4) clearly holds for $r = 1$. It will be proved by induction with respect to r that it holds for all $1 \leq r \leq m-1$.

Let us assume that relation (4.4) holds for $r-1$ and let us prove it for r . Some of the connected components $\gamma_{A_t^{r-1}}$ of γ^{r-1} may have a vertex connected with a vertex of the r -th row $\{(r, 1), \dots, (r, k_r)\}$ of the diagram γ . Let us make such an enumeration of the connected components of γ^{r-1} in which there is a constant $\bar{s}(r)$ such that the components $\gamma_{A_t^{r-1}}$ with $1 \leq t \leq \bar{s}(r)$ have a vertex connected with a vertex of the r -th row of γ , and the components $\gamma_{A_t^{r-1}}$, $\bar{s}(r) < t \leq s(r-1)$, have no such vertices. To simplify the following discussion let us introduce the notation $\gamma_{A_0^{r-1}} = \{(r, 1), \dots, (r, k_r)\}$ for the r -th row of the diagram γ . Then the connected components of γ^r are the diagrams $\gamma_{B^r} = \bigcup_{t=0}^{\bar{s}(r)} \gamma_{A_t^{r-1}}$ and $\gamma_{A_t^{r-1}}$, $\bar{s}(r) < t \leq s(r-1)$. The latter diagrams satisfy relation (4.4) by induction, so it is enough to show that

$$|V_s(F_{\gamma_{B^r}})| \leq R^{|B^r|+s-2} \quad \text{for all } 1 \leq s \leq |N_1(\gamma_{B^r})|, \quad (4.8)$$

where $|N_1(\gamma_{B^r})|$ denotes the number of open vertices in the diagram γ_{B^r} . It is possible that no vertex of the diagram γ^{r-1} is connected with a vertex from the r -th row of γ . In this case $\bar{s}(r) = 0$, $B^r = A_0^{r-1}$, and relation (4.8) clearly holds.

To prove relation (4.8) let us introduce the following notations. Put $B_j^r = \bigcup_{t=0}^j A_t^{r-1}$ for all $0 \leq j \leq \bar{s}(r)$. With such a notation $B_{\bar{s}(r)}^r = B^r$ for $j = \bar{s}(r)$. I shall show that

$$|V_s(F_{\gamma_{B_j^r}}(x_s, s \in N_1(\gamma_{B_j^r}))| \leq R^{|B_j^r|+s-2} \quad \text{for all } 1 \leq j \leq \bar{s}(r) \text{ and } 1 \leq s \leq |N_1(\gamma_{B_j^r})|. \quad (4.9)$$

Relation (4.9) for $j = \bar{s}(r)$ implies relation (4.8). Relation (4.9) holds for $j = 0$, and it will be proved for a general parameter j , $1 \leq j \leq \bar{s}(r)$, by induction.

For this goal I write the following recursive relation for the functions $F_{\gamma_{B_j^r}}$

$$F_{\gamma_{B_j^r}}(x_s, s \in N_1(\gamma_{B_j^r})) = \int F_{\gamma_{A_j^{r-1}}}(x_s, s \in N_1(\gamma_{A_j^{r-1}})) F_{\gamma_{B_{j-1}^r}}(x_s, s \in N_1(\gamma_{B_{j-1}^r})) \prod_{s \in N_1(\gamma_{A_j^{r-1}}) \cap N_1(\gamma_{B_{j-1}^r})} \mu(dx_s) \quad (4.10)$$

for all $1 \leq j \leq \bar{s}(r)$. I show that relation (4.9) follows from relation (4.10) and the Main Inequality. Indeed, let us apply the Main Inequality with the functions $f = F_{\gamma_{A_j^{r-1}}}(x_s, s \in N_1(\gamma_{A_j^{r-1}}))$ and $g = F_{\gamma_{B_{j-1}^r}}(x_s, s \in N_1(\gamma_{B_{j-1}^r}))$. (More precisely, we apply an equivalent version of the Main Inequality where the indices of the variables of the functions f and g may be different, and the variables by which we integrate and by which we do not integrate may be listed in an arbitrary order.) By our inductive hypothesis these functions f and g satisfy the inequalities $V_s(f) \leq D_1 R^{s-2}$ and $V_s(g) \leq D_2 R^{s-2}$ with $D_1 = R^{|\gamma_{A_j^{r-1}}|}$ and $D_2 = R^{|\gamma_{B_{j-1}^r}|}$. Beside this, $|N_1(\gamma_{A_j^{r-1}}) \cap N_1(\gamma_{B_{j-1}^r})| = |N_1(\gamma_{A_j^{r-1}}) \cap N_1(\gamma_{A_0^{r-1}})| \geq 1$, because there is an edge connecting a vertex of $\gamma_{A_j^{r-1}}$ with a vertex of the r -th row $\gamma_{A_0^{r-1}}$ of the diagram γ . This inequality corresponds to the condition $p \geq 1$ in the Main Inequality. This number p is the multiplicity of the integral in formula (4.6). Beside this, the diagram $\gamma_{A_j^{r-1}} \cup \gamma_{B_{j-1}^r}$ has an open vertex because of the connectedness of the diagram γ . This corresponds to the condition $m + n \geq 1$ in the Main Inequality.

The above considerations show that the Main Inequality can be applied in the present case. It yields that $|V_s(F_{\gamma_{B_j^r}})| \leq R^{|\gamma_{A_j^{r-1}}| + |\gamma_{B_{j-1}^r}|} R^{s-2} = R^{|\gamma_{B_j^r}| + s - 2}$, and this is what we had to prove.

The proof of relation (4.5) is similar. In the proof the above decomposition of γ^{r-1} is applied to the connected components for $r = m$. For this parameter $r = m$ all components $\gamma_{A_t^{m-1}}$, $1 \leq t \leq s(m-1)$ have a vertex which is connected with a vertex of the m -th row of the diagram γ . (The m -th row of γ will be sometimes denoted by γ_0^{m-1} .) Thus $\bar{s}(m) = s(m-1)$, the connected components $\gamma_{A_j^{m-1}}$, $1 \leq j \leq s(m-1)$, can be listed in an arbitrary order, and the diagrams $\gamma_{B_j^m}$, $1 \leq j \leq s(m-1)$ can be defined similarly to the case $r < m$. Beside this, relation (4.9) can be proved for $r = m$ and $j \leq s(m-1) - 1$, similarly as it was proved for $r < m$. The main difference between the cases $r < m$ and $r = m$ is that in the latter case the proof of relation (4.9) works only for $j \leq s(m-1) - 1$, but not for $j = s(m-1)$. In the case $j = s(m)$ the Main Inequality cannot be applied in the proof, because $\gamma_{B_{s(m)}^m} = \gamma$ is a diagram without open vertices, and this property does not allow the application of the Main Inequality in the case $r = m$, $j = s(m-1)$.

On the other hand the identity

$$F_\gamma = \int F_{\gamma_{B_{s(m-1)-1}^m}}(x_s, s \in N_1(\gamma_{A_{s(m-1)}^{m-1}}))$$

$$F_{\gamma_{A_{s(m-1)}^{m-1}}} (x_s, s \in N_1(\gamma_{A_{s(m-1)}^{m-1}})) \prod_{s \in N_1(\gamma_{A_{s(m-1)}^{m-1}})} \mu(dx_s) \quad (4.11)$$

holds, and the function integrated in (4.11) is the product of two terms whose L_2 -norm can be well bounded. Namely, since the L_2 -norm of a function f of several variables (defined on the measure space (X, \mathcal{X}, μ)) equals $V_1(f)$, relation (4.9) with the choice $r = m$ and $j = s(m-1) - 1$ together with formula (4.4) for $r = m-1$ yield, with the parameter $s = 1$, the bound $R^{|B_{s(m-1)-1}^m| - 1}$ and $R^{|A_{s(m-1)}^{m-1}| - 1}$ for the L_2 -norm of these terms. Hence relation (4.11) and the Schwarz inequality imply that

$$|F_\gamma| \leq R^{|B_{s(m-1)-1}^m| + |A_{s(m-1)}^{m-1}| - 2} = R^{m-2},$$

as it was stated.

It remained to prove the Main Inequality.

Proof of the Main Inequality. Let us fix a partition $A_j = A_j^{(1)} \cup A_j^{(2)}$, $1 \leq j \leq s_1$, B_k , $1 \leq k \leq s_2$, and C_l , $1 \leq l \leq s_3$, $s_1 + s_2 + s_3 = s$ of the set $\{1, \dots, n+m\}$ such that all sets $A_j^{(1)}$, $A_j^{(2)}$, B_k and C_l are non-empty, and $A_j^{(1)}, B_k \subset \{1, \dots, n\}$, $A_j^{(2)}, C_l \subset \{n+1, \dots, n+m\}$. Define some functions $u_j(x_q, q \in A_j)$, $1 \leq j \leq s_1$, $v_k(x_r, r \in B_k)$, $1 \leq k \leq s_2$ and $w_l(x_t, t \in C_l)$, $1 \leq l \leq s_3$ such that $\int u_j^2(x_q, q \in A_j) \prod_{q \in A_j} \mu(dx_q) \leq 1$, $\int v_k^2(x_r, r \in B_k) \prod_{r \in B_k} \mu(dx_r) \leq 1$, and $\int w_l^2(x_t, t \in C_l) \prod_{t \in C_l} \mu(dx_t) \leq 1$. It has to be shown that for all such partitions and the functions f and g the inequality

$$\begin{aligned} & \int f(x_1, \dots, x_n, v_{n+m+1}, \dots, v_{n+m+p}) g(x_{n+1}, \dots, x_{n+m}, v_{n+m+1}, \dots, v_{n+m+p}) \\ & \prod_{j=1}^{s_1} u_j(x_q, q \in A_j) \prod_{k=1}^{s_2} v_k(x_r, r \in B_k) \prod_{l=1}^{s_3} w_l(x_t, t \in C_l) \\ & \prod_{i=1}^{n+m} \mu(dx_i) \prod_{u=n+m+1}^{n+m+p} \mu(dv_u) \leq D_1 D_2 R^{s-2} \end{aligned} \quad (4.12)$$

holds.

Let us first show that this inequality can be reduced to the case when $s_2 = s_3 = 0$, that is when the partition of the set $\{1, \dots, n+m\}$ consists only of such sets A_j which have a non-empty intersection with both sets $\{1, \dots, n\}$ and $\{n+1, \dots, n+m\}$. To carry out this reduction let us define the functions

$$\begin{aligned} & \bar{f}(x_j, j \in \{1, \dots, n\} \setminus B, v_{n+m+1}, \dots, v_{n+m+p}) \\ & = \int f(x_1, \dots, x_n, v_{n+m+1}, \dots, v_{n+m+p}) \prod_{k=1}^{s_2} v_k(x_r, r \in B_k) \prod_{r \in B} \mu(dx_r), \end{aligned} \quad (4.13)$$

$$\begin{aligned}
& \bar{g}(x_j, j \in \{n+1, \dots, n+m\} \setminus C, v_{n+m+1}, \dots, v_{n+m+p}) \\
&= \int g(x_{n+1}, \dots, x_{n+m}, v_{n+m+1}, \dots, v_{n+m+p}) \prod_{l=1}^{s_3} w_l(x_t, t \in C_l) \prod_{t \in C} \mu(dx_t)
\end{aligned} \tag{4.14}$$

with $B = \bigcup_{k=1}^{s_2} B_k$ and $C = \bigcup_{l=1}^{s_2} C_l$ together with the function

$$\begin{aligned}
& \bar{F}(x_j, j \in \{1, \dots, n+m\} \setminus (B \cup C)) \\
&= \int \bar{f}(x_j, j \in \{1, \dots, n\} \setminus B, v_{n+m+1}, \dots, v_{n+m+p}) \\
& \quad \bar{g}(x_j, j \in \{n+1, \dots, n+m\} \setminus C, v_{n+m+1}, \dots, v_{n+m+p}) \prod_{u=n+m+1}^{n+m+p} \mu(dv_u).
\end{aligned} \tag{4.15}$$

With this notation inequality (4.12) can be rewritten as

$$\begin{aligned}
& \int \bar{F}(x_i, i \in \{1, \dots, n+m\} \setminus (B \cup C)) \prod_{j=1}^{s_1} u_j(x_q, q \in A_j) \\
& \quad \prod_{i \in \{1, \dots, n+m\} \setminus (B \cup C)} \mu(dx_i) \leq D_1 D_2 R^{s_1 + s_2 + s_3 - 2}.
\end{aligned} \tag{4.16}$$

Beside this, it is not difficult to check that the functions \bar{f} and \bar{g} satisfy the inequalities $V_s(\bar{f}) \leq \bar{D}_1 R^{s-2}$ for all $1 \leq s \leq n+p-|B|$ and $V_s(\bar{g}) \leq \bar{D}_2 R^{s-2}$ for all $1 \leq s \leq m+p-|C|$ with $\bar{D}_1 = D_1 R^{s_2}$ and $\bar{D}_2 = D_2 R^{s_3}$ respectively. For the parameter $s = 1$ these inequalities yield the bounds $\bar{D}_1 R^{-1}$ and $\bar{D}_2 R^{-1}$ for the L_2 -norm of the functions \bar{f} and \bar{g} respectively. They imply together with relation (4.15) and the Schwarz inequality that relation (4.16) holds in the special case $s_1 = 0$, i.e. when there is no set of the type A_j in the partition we consider. (In this case \bar{F} is a function of zero variables, i.e. it is a constant.)

This means that it is enough to prove relation (4.16) in the case $s_1 \geq 1$. This enables us to reduce the proof of relation (4.12) to the case $s_2 = s_3 = 0$, i.e. to the case when all elements of the partition is such a set as the sets A_j . To get this reduction it is enough to work with the functions \bar{f} , \bar{g} and \bar{F} defined in formulas (4.13), (4.14) and (4.15) instead of the original functions f , g and F in the proof of the Main Inequality, and observe that these functions also satisfy its conditions (with an appropriate reindexation of the variables in these functions). Hence the Main inequality follows from the following reduced form of relation (4.12):

$$\begin{aligned}
I &= \int f(x_1, \dots, x_n, v_{n+m+1}, \dots, v_{n+m+p}) g(x_{n+1}, \dots, x_{n+m}, v_{n+m+1}, \dots, v_{n+m+p}) \\
& \quad \prod_{j=1}^s u_j(x_q, q \in A_j) \prod_{i=1}^{n+m} \mu(dx_i) \prod_{u=n+m+1}^{n+m+p} \mu(dv_u) \leq D_1 D_2 R^{s-2}
\end{aligned} \tag{4.17}$$

for a partition $A_j = A_j^{(1)} \cup A_j^{(2)}$, $1 \leq j \leq s$, of the set $\{1, \dots, n+m\}$. To prove this inequality let us introduce the functions

$$U_j(x_q, q \in A_j^{(2)}) = \left[\int u_j^2(x_q, q \in A_j) \prod_{r \in A_j^{(1)}} \mu(dx_r) \right]^{1/2}$$

and

$$\bar{u}_j(x_q, q \in A_j^{(1)} | x_q, q \in A_j^{(2)}) = \frac{u_j(x_q, q \in A_j)}{U_j(x_q, q \in A_j^{(2)})}$$

for all $1 \leq j \leq s$ together with the functions

$$\begin{aligned} & G(x_{n+1}, \dots, x_{n+m}) \\ &= \left[\int g^2(x_{n+1}, \dots, x_{n+m}, v_{n+m+1}, \dots, v_{n+m+p}) \prod_{u=n+m+1}^{n+m+p} \mu(dv_u) \right]^{1/2} \end{aligned}$$

and

$$\bar{g}(v_{n+m+1}, \dots, v_{n+m+p} | x_{n+1}, \dots, x_{n+m}) = \frac{g(x_{n+1}, \dots, x_{n+m}, v_{n+m+1}, \dots, v_{n+m+p})}{G(x_{n+1}, \dots, x_{n+m})}.$$

Observe that

$$\int U_j^2(x_q, q \in A_j^{(2)}) \prod_{q \in A_j^{(2)}} \mu(dx_q) \leq 1, \quad (4.18)$$

$$\int \bar{u}_j^2(x_q, q \in A_j^{(1)} | x_q, q \in A_j^{(2)}) \prod_{q \in A_j^{(1)}} \mu(dx_q) = 1 \quad \text{for all } x = \{x_q, q \in A_j^{(2)}\} \quad (4.19)$$

for all $1 \leq j \leq s$, because the L_2 -norm of the functions $u_j(\cdot)$ are less than 1. Similarly,

$$\int G^2(x_{n+1}, \dots, x_{n+m}) \prod_{j=n+1}^{n+m} \mu(dx_j) \leq D_2^2 R^{-2}, \quad (4.20)$$

since the L_2 -norm of the function G equals the L_2 -norm of g which is $U_1(g)$, and it is bounded by $D_2 R^{-1}$. Beside this,

$$\begin{aligned} & \int \bar{g}^2(v_{n+m+1}, \dots, v_{n+m+p} | x_{n+1}, \dots, x_{n+m}) \prod_{u=n+m+1}^{n+m+p} \mu(dv_u) = 1 \\ & \text{for all } x = (x_{n+1}, \dots, x_{n+m}) \end{aligned} \quad (4.21)$$

The function

$$Z(x_r, n+1 \leq r \leq n+m) = \int f(x_1, \dots, x_n, v_{n+m+1}, \dots, v_{n+m+p}) \prod_{j=1}^s \bar{u}_j(x_q, q \in A_j^{(1)} | x_q, q \in A_j^{(2)}) \bar{g}(v_{n+m+1}, \dots, v_{n+m+p} | x_{n+1}, \dots, x_{n+m}) \prod_{i=1}^n \mu(dx_i) \prod_{u=n+m+1}^{n+m+p} \mu(dv_u)$$

satisfies the inequality

$$|Z(x_r, n+1 \leq r \leq n+m)| \leq D_1 R^{s-1} \quad \text{for all } x = (x_r, n+1 \leq r \leq n+m), \quad (4.22)$$

because of the inequality $V_{s+1}(f) \leq D_1 R^{s-1}$ (we consider the partition of the index set consisting of the sets $A^{(1)}$, $1 \leq j \leq s$. and the index set of the variables v_j) and the bounds (4.19) and (4.21) about the L_2 -norm of the functions $\bar{u}_j(\cdot|\cdot)$ and $\bar{g}(\cdot|\cdot)$.

By rewriting the integrand in formula (4.17) with the help of the identities $u_j(\cdot) = \bar{u}_j(\cdot|\cdot)U_j(\cdot)$, $1 \leq j \leq s$ and $g(\cdot) = \bar{g}(\cdot|\cdot)G(\cdot)$ we can rewrite the expression I in formula (4.17) with the help of the above introduced quantities as

$$I = \int Z(x_r, n+1 \leq r \leq n+m) \prod_{j=1}^s U_j(x_q, q \in A_j^{(2)}) G(x_{n+1}, \dots, x_{n+m}) \prod_{i=n+1}^{n+m} \mu(dx_i).$$

Beside this, the L_2 -norms of the functions

$$\prod_{j=1}^s U_j(x_q, q \in A_j^{(2)}) \quad \text{and} \quad G(x_{n+1}, \dots, x_{n+m})$$

are bounded by 1 and $D_2 R^{-1}$ respectively by relations (4.18) and (4.20), and the function $Z(\cdot)$ satisfies inequality (4.22). These inequalities together with the Schwarz inequality imply that $I \leq D_1 D_2 R^{s-2}$, i.e. relation (4.17) holds. The proof of the Main Inequality is completed.

5. On Latała's conjecture.

In this section I discuss Latała's conjecture. I show, by working out the details of the arguments leading to this conjecture that it is equivalent to an estimate about the expected value of the supremum of certain random multilinear forms. The original form of this conjecture contains an estimate about the moments of certain Gaussian polynomials. The would-be proof applies an inductive argument with respect to the order of the polynomials we consider. It is known that the conjecture holds for polynomials of order 2. I shall consider polynomials of order 3 whose study also reveals very much about the general situation. First I formulate Latała's conjecture for polynomials of order three in an explicit form.

Latała's conjecture for Gaussian random polynomials of order 3. *Let us fix a large positive integer M and consider a random polynomial of the form*

$$Z = \sum_{i,j,k} a(i,j,k) \xi_i \eta_j \zeta_k, \quad (5.1)$$

where all random variables ξ_i , η_j and ζ_k have standard normal distribution, and they are independent of each other. Let the coefficients $a(i,j,k)$ of the polynomial Z in formula (5.1) satisfy the following inequalities depending on the fixed parameter M :

$$\sum_{i,j,k} a(i,j,k) u(i,j,k) \leq 1 \quad \text{if} \quad \sum_{i,j,k} u^2(i,j,k) \leq 1, \quad (5.2)$$

$$\begin{aligned} \sum_{i,j,k} a(i,j,k) u(i,j) v(k) &\leq M^{-1/2} \quad \text{if} \quad \sum_{i,j} u^2(i,j) \leq 1 \quad \text{and} \quad \sum_k v^2(k) \leq 1, \\ \sum_{i,j,k} a(i,j,k) u(i,k) v(j) &\leq M^{-1/2} \quad \text{if} \quad \sum_{i,k} u^2(i,k) \leq 1 \quad \text{and} \quad \sum_j v^2(j) \leq 1, \\ \sum_{i,j,k} a(i,j,k) u(j,k) v(i) &\leq M^{-1/2} \quad \text{if} \quad \sum_{j,k} u^2(j,k) \leq 1 \quad \text{and} \quad \sum_i v^2(i) \leq 1, \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} \sum_{i,j,k} a(i,j,k) u(i) v(j) w(k) &\leq M^{-1} \\ \text{if} \quad \sum_i u^2(i) &\leq 1, \quad \sum_j v^2(j) \leq 1 \quad \text{and} \quad \sum_k w^2(k) \leq 1. \end{aligned} \quad (5.4)$$

Then the random polynomial Z satisfies the inequality

$$EZ^{2M} \leq C^M M^M \quad (5.5)$$

with some universal constant $C > 0$.

In the calculation of EZ^{2M} it is useful to consider first its conditional expectation under the condition that the value of all random variables ξ_i are prescribed. This

conditional expectation has a simple form which can be well bounded because of the independence of the variables ξ_i , η_j and ζ_k . We get

$$E(Z^{2M} | \xi_i = x_i) = E \left(\sum_{i,j,k} a(i,j,k) x_i \eta_j \zeta_k \right)^{2M} = E \left(\sum_{j,k} A(j,k|x) \eta_j \zeta_k \right)^{2M}, \quad (5.6)$$

where

$$A_i(j,k|x) = A_i(j,k|x_1, x_2, \dots) = \sum_i a(i,j,k) x_i.$$

The moment estimates known for Gaussian polynomials of order two enables us to bound the expression in formula (5.6). These estimates depend on the Hilbert–Schmidt norm $D_1(x)$ and usual norm $D_2(x)$ of the matrix $A(j,k|x)$ appearing in formula (5.6). To get a formula more appropriate for our investigations I give the value of these quantities by means of the the following variational principle.

$$D_1(x) = \sup_{v(j,k): \sum v^2(j,k) \leq 1} \sum_{j,k} A(j,k|x) v(j,k),$$

and

$$D_2(x) = \sup_{(v(j), w(k)): \sum v^2(j) \leq 1, \sum w^2(k) \leq 1} \sum_{j,k} A(j,k|x) v(j) w(k).$$

In such a way we get the following estimate.

$$\begin{aligned} E(Z^{2M} | \xi_i = x_i) &= E \left(\sum_{j,k} A(j,k|x) \eta_j \zeta_k \right)^{2M} \\ &\leq C^M M^M \left(\sup_{\substack{v(j,k) \\ \sum v^2(j,k) \leq 1}} \sum_{j,k} A(j,k|x) v(j,k) \right)^{2M} \\ &\quad + C^M M^{2M} \left(\sup_{\substack{v(j), w(k) \\ \sum v^2(j) \leq 1, \sum w^2(k) \leq 1}} \sum_{j,k} A(j,k|x) v(j) w(k) \right)^{2M}. \end{aligned} \quad (5.7)$$

Taking expectation in inequality (5.7) we get that

$$\begin{aligned} E(Z^{2M}) &\leq C^M M^M E \left(\sup_{\substack{u(j,k) \\ \sum u^2(j,k) \leq 1}} \sum_{j,k} A(j,k|\xi_i) u(j,k) \right)^{2M} \\ &\quad + C^M M^{2M} E \left(\sup_{\substack{v(j), w(k) \\ \sum v^2(j) \leq 1, \sum w^2(k) \leq 1}} \sum_{j,k} A(j,k|\xi_i) v(j) w(k) \right)^{2M}. \end{aligned}$$

The last inequality can be rewritten in the form

$$E(Z^{2M}) \leq C^M M^M E \left(\sup_{\substack{u(j,k) \\ \sum u^2(j,k) \leq 1}} \sum_i B_{i,1}(u(j,k)) \xi_i \right)^{2M} \\ + C^M M^{2M} E \left(\sup_{\substack{v(j), w(k) \\ \sum v^2(j) \leq 1, \sum w^2(k) \leq 1}} \sum_i B_{i,2}(v(j), w(k)) \xi_i \right)^{2M} .$$

with

$$B_{i,1}(u(j,k), j, k = 1, 2, \dots) = \sum_{j,k} a(i, j, k) u(j, k),$$

and

$$B_{i,2}(v(j), w(k), j, k = 1, 2, \dots) = \sum_{j,k} a(i, j, k) v(j) w(k),$$

or by introducing the notations $u = (u(j, k), j, k = 1, 2, \dots)$, $v = (v(j), j = 1, 2, \dots)$ and $w = (w(k), k = 1, 2, \dots)$ together with the Gaussian random variables

$$X(u) = \sum_{i,j,k} a(i, j, k) u(j, k) \xi_i = \sum_i B_{i,1}(u(j, k), j, k = 1, 2, \dots) \xi_i, \\ Y(v, w) = \sum_{i,j,k} a(i, j, k) v(j) w(k) \xi_i = \sum_i B_{i,2}(v(j), w(k), j, k = 1, 2, \dots) \xi_i, \quad (5.8)$$

this inequality can be written in the form

$$E(Z^{2M}) \leq C^M M^M E \left(\sup_{\substack{u(j,k) \\ \sum u^2(j,k) \leq 1}} X(u) \right)^{2M} \\ + C^M M^{2M} E \left(\sup_{\substack{v(j), w(k) \\ \sum v^2(j) \leq 1, \sum w^2(k) \leq 1}} Y(v, w) \right)^{2M} . \quad (5.9)$$

The right-hand side of (5.9) can be bounded by means of some concentration theorem type inequalities about the supremum of a Gaussian process. Ledoux has a result about the supremum of Gaussian processes (Theorem 7.1 in the book [6] *The Concentration of Measure Phenomenon*) which states that the supremum of a Gaussian process $U(t)$, $EU(t) = 0$, $t \in T$, takes a value larger than $E \sup_{t \in T} U(t)$ with relatively small probability.

More explicitly, it states that $P \left(\sup_{t \in T} U(t) \geq E \sup_{t \in T} U(t) + x \right) \leq C_1 e^{-C_2 x^2 / \lambda}$,

where $\lambda = \sup_{t \in T} EU^2(t)$. Some calculation with the help of this inequality yields the estimates

$$\begin{aligned}
& E \left(\sup_{\substack{u(j,k) \\ \sum u^2(j,k) \leq 1}} X(u) \right)^{2M} \leq D^M \left(E \left| \sup_{\substack{u(j,k) \\ \sum u^2(j,k) \leq 1}} X(u) \right| \right)^{2M} \\
& \quad + D^M \sup_{\substack{u(j,k) \\ \sum u^2(j,k) \leq 1}} EX(u)^{2M} \\
& \leq D^M \left(E \left| \sup_{\substack{u(j,k) \\ \sum u^2(j,k) \leq 1}} X(u) \right| \right)^{2M} + D^M M^M \sup_{\substack{u(j,k) \\ \sum u^2(j,k) \leq 1}} (EX(u)^2)^M
\end{aligned}$$

and

$$\begin{aligned}
& E \left(\sup_{\substack{v(j), w(k) \\ \sum v^2(j) \leq 1, \sum w^2(k) \leq 1}} Y(v, w) \right)^{2M} \\
& \leq D^M \left(E \left| \sup_{\substack{v(j), w(k) \\ \sum v^2(j) \leq 1, \sum w^2(k) \leq 1}} Y(v, w) \right| \right)^{2M} \\
& \quad + D^M \sup_{\substack{v(j), w(k) \\ \sum v^2(j) \leq 1, w^2(k) \leq 1}} EY(v, w)^{2M} \\
& \leq D^M \left(E \left| \sup_{\substack{v(j), w(k) \\ \sum v^2(j) \leq 1, \sum w(k)^2 \leq 1}} Y(v, w) \right| \right)^{2M} \\
& \quad + D^M M^M \sup_{\substack{v(j), w(k) \\ \sum v^2(j) \leq 1, \sum w^2(k) \leq 1}} (EY(v, w)^2)^M.
\end{aligned}$$

The content of the above inequalities is that to get a good estimate on the high moments of the supremum of a Gaussian process with expectation zero it is enough to have a good estimate on the expectation of the absolute value of this supremum and on the moments of the single random variables in this stochastic process. The latter terms can be expressed by means of the variance of these random variables.

A relatively simple calculation by means of the Schwarz inequality shows that under conditions (5.3) and (5.4) the inequalities $EX(u)^2 \leq M^{-1}$, and $EY(v, w)^2 \leq M^{-2}$ hold

if $\sum u(j,k)^2 \leq 1$, $\sum v^2(j) \leq 1$, and $\sum w^2(k) \leq 1$. Some calculation also shows that under the condition (5.2) the inequality

$$E \left| \sup_{\substack{u(j,k) \\ \sum u^2(j,k) \leq 1}} X(u) \right| \leq \left(E \left(\sup_{\substack{u(j,k) \\ \sum u^2(j,k) \leq 1}} X(u) \right)^2 \right)^{1/2} \leq C \quad (5.10)$$

holds. The second term in (5.10) can be well estimated, because the supremum of the random variables $X(u)$ can be explicitly calculated for all fixed random vectors ξ_i , $i = 1, 2, \dots$, and after this, the middle term in (5.10) can be well bounded with the help of relation (5.2) because of the orthogonality of the random variables ξ_i .

Because of the above inequalities to show that relation (5.5) holds under conditions (5.2), (5.3) and (5.4) it would be sufficient to prove the inequality

$$E \left| \sup_{\substack{v(j), w(k) \\ \sum v^2(j) \leq 1, \sum w(k)^2 \leq 1}} Y(v, w) \right| \leq \frac{C}{\sqrt{M}}. \quad (5.11)$$

under the above conditions with the random variables $Y(v, w)$ introduced in (5.8) and some universal constant $C < \infty$. This would mean that Latała's conjecture holds for Gaussian polynomials of order 3.

A more careful analysis would even show that the validity of relation (5.11) under conditions (5.2), (5.3) and (5.4) is equivalent to Latała's conjecture. This argument can be adapted to the case of general parameter k , and it yields that Latała's conjecture is equivalent to Theorem 2 of his paper [5].

More generally, it can be said that even if we cannot prove inequality (5.11), the bound we can give for the high moments of a random polynomial Z defined in (5.1) depends on what kind of estimate we can prove for the expression in (5.11). But the estimation of such an expression is a hard problem. The analogous problem in formula (5.10) was much simpler.

The estimation of the left-hand side of (5.11) is much harder than that of formula (5.10), because in this case the supremum of random trilinear forms had to be considered (and not the supremum of random bilinear forms as in formula (5.10)). Latała tried to get a good estimate for such an expression by means of a good bound on a quantity $N(X, \rho_\alpha, \varepsilon)$ which was introduced also in Section 2 of this paper. But his proof of the estimate for $N(\cdot)$ contained an error which seems to be serious. I do not know whether the estimation of the quantity $N(X, \rho_\alpha, \varepsilon)$ is the right way to give a good bound on the expression in formula (5.11). But the proof of such an estimate demands a deeper analysis than the method of paper [5]. Such an analysis should exploit the finer structure of the model we consider.

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