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A NOTE ON KOLMOGOROV'S LAW OF ITERATED LOGARITHM

by

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Summary: Let the sequence of independent random variables $X_1, X_2, \dots$ satisfy the conditions of Kolmogorov's law of iterated logarithm. Then the partial sums $S_n=\sum_{i=1}^{n} X_i$, $n=1, 2, \dots$ can be approximated by an appropriate Wiener process. This implies a Strassen type law of iterated logarithm.

Introduction

Kolmogorov's law of iterated logarithm (see e.g. [1]) states the following result:

Let $X_1, X_2, \dots$ be independent random variables $EX_i=0$, $EX_i^2=\sigma_i^2$, $i=1, 2, \dots$, $S_n=\sum_{i=1}^{n} X_i$, $B_n=\sum_{i=1}^{n} \sigma_i^2$, $n=1, 2, \dots$. Let $B_n \to \infty$. Assume the existence of a numerical sequence M_n, $n=1, 2, \dots$, such that

$$M_n = o\left(\sqrt{\frac{B_n}{\log\log B_n}}\right)$$

and

$$P(|X_n| \leqq M_n) = 1.$$

Then the relation

$$\limsup \frac{S_n}{\sqrt{2B_n \log\log B_n}} = 1 \quad \textit{with pr. 1}$$

holds true.

Let us define the process $S(t)$, $t \geqq 0$ in the following way: $S(B_n)=S_n$, $(B_0=0,\ S(0)=0)$ and $S(t)=S_n\dfrac{B_{n+1}-t}{B_{n+1}-B_n}+S_{n+1}\dfrac{t-B_n}{B_{n+1}-B_n}$ if $B_n<t<B_{n+1}$. We prove the following

THEOREM. *Let the sequence of independent random variables $X_1, X_2, \dots$ satisfy the conditions of Kolmogorov's law of iterated logarithm. If the probability space where the X_i-s are given is sufficiently rich, one can construct a standard Wiener process $W(t)$ such that*

$$\lim_{t\to\infty} \frac{|W(t)-S(t)|}{\sqrt{t \log\log t}} = 0 \quad \textit{with probability } 1.$$

Since $B_{n+1}-B_n \leqq M_n^2$, and $M_n^2=o(B_n)$ Strassen's law of iterated logarithm for Wiener process (see [2]) yields the following

COROLLARY. *Define*

$$S_n(t) = \frac{S(B_n t)}{\sqrt{2B_n \log\log B_n}}, \quad n = 1, 2, \dots, 0 \leqq t \leqq 1.$$

This sequence of functions is relatively compact in the Banach space $C[0, 1]$, *and its limit points agree with the set* K, $K=\{f(t),\ 0\leqq t\leqq 1;\ f(0)=0,\ f(t)$ *is absolute continuous,* $\int_0^1 \dot f^2(t)\,dt \leqq 1\}$ *with probability* 1.

This corollary contains Kolmogorov's law of iterated logarithm as a special case.

PROOF OF THE THEOREM. We need an estimate of $P(S_n - S_m > x)$. Though this estimate is very similar to those needed in the proof of Kolmogorov's law of iterated logarithm, for the sake of completeness we prove it. Our estimation is based on an idea of FELLER (see [3]).

LEMMA. *Let* ε, δ, L *be arbitrary positive numbers. Under the conditions of the Theorem we have for every large* n

$$c \exp\left[-(1+\delta)\frac{x^2}{2(B_n - B_m)}\right] \leqq P(S_n - S_m > x) \leqq \exp\left[-(1-\delta)\frac{x^2}{2(B_n - B_m)}\right]$$

if $n>m$, $B_n - B_m > \varepsilon B_n$, $0 \leqq x \leqq L\sqrt{B_n \log\log B_n}$. *($c$ is a universal constant.)*

PROOF. First we estimate the moment generating function of $S_n - S_m$ from below and from above. Let $0 \leqq t \leqq t_0 = K\sqrt{\dfrac{\log\log B_n}{B_n}}$ where $K = 2L/\varepsilon$.

We have for $j \leqq n$ $tM_j < \delta/3$ if n is sufficiently large. Thus

$$E \exp tX_j = 1 + \sum_{k=2}^{\infty} \frac{t^k}{k!} EX_j^k \leqq 1 + \sigma_j^2 \frac{t^2}{2}\left(1 + \frac{t}{3} M_j + \frac{t^2}{12} M_j^2 + \dots\right) \leqq$$

$$\leqq 1 + \frac{t^2}{2}\sigma_j^2\left(1 + \frac{t}{2} M_j\right) \leqq \exp\left[\left(1+\frac{\delta}{2}\right)\sigma_j^2 \frac{t^2}{2}\right],$$

and

$$E \exp tX_j \geqq 1 + \frac{t^2}{2}\sigma_j^2\left(1 - \frac{t}{3} M_j - \frac{t^2}{12} M_j^2 - \dots\right) \geqq \exp\left[\left(1-\frac{\delta}{2}\right)\sigma_j^2 \frac{t^2}{2}\right].$$

These estimations imply that

$$\exp\left[\left(1-\frac{\delta}{2}\right)\frac{t^2}{2}(B_n - B_m)\right] \leqq E \exp[t(S_n - S_m)] \leqq \exp\left[\left(1+\frac{\delta}{2}\right)\frac{t^2}{2}(B_n - B_m)\right].$$

Define the probability distributions

$$F_j^t(dx) = \frac{\exp(tx)F_j(dx)}{\int \exp(tx)F_j(dx)}, \quad j = 1, 2, \dots$$

and

$$G_{n,m}^t(dx) = (F_m^t * \dots * F_n^t)(dx),$$

where $F_j(x)$ is the distribution function of X_j, and $*$ means convolution.

For any Borel set H the equation

$$P(S_n - S_m \in H) = E[\exp t(S_n - S_m)] \int_H \exp(-tx) G_{n,m}^t(dx)$$

holds. (These formulae follow from the basic properties of conjugated distributions, see e.g. [4] Chapter XVI. 6.)

Choose

$$t = \frac{x}{(B_n - B_m)\left(1 + \frac{\delta}{2}\right)}.$$

Since

$$t \leqq \frac{L\sqrt{B_n \log\log B_n}}{\varepsilon B_n} \leqq t_0$$

the estimation

$$P(S_n - S_m > x) \leqq E\exp[t(S_n - S_m)]\exp(-tx) \leqq \exp\left[-\frac{(1-\delta)x^2}{2(B_n - B_m)}\right]$$

holds true.

Set

$$E_j^t = \int x F_j^t(dx), \quad (D_j^t)^2 = \int x^2 F_j^t(dx) - (E_j^t)^2$$

and

$$E_{m,n}^t = \sum_{j=m}^n E_j^t, \quad (D_{m,n}^t)^2 = \sum_{j=m}^n (D_j^t)^2.$$

In order to get an estimate from below first we show that

$$(1) \qquad \left(1 - \frac{\delta}{2}\right)(B_n - B_m) < (D_{m,n}^t)^2 < \left(1 + \frac{\delta}{2}\right)(B_n - B_m)$$

and

$$(2) \qquad \left(1 - \frac{\delta}{2}\right) t(B_n - B_m) < E_{m,n}^t < \left(1 + \frac{\delta}{2}\right) t(B_n - B_m)$$

if $t < t_0$.

$$(3) \qquad (D_j^t)^2 \leqq \int_{-M_j}^{M_j} \exp(tx) x^2 F_j(dx) \leqq \left(1 + \frac{\delta}{2}\right) \int x^2 F_j(dx) = \left(1 + \frac{\delta}{2}\right) \sigma_j^2$$

if $j \leqq n$ since $tM_n \leqq \delta/4$.

Similarly, exploiting that $tM_n \to 0$, as $n \to \infty$, we obtain that

$$(4) \qquad \int x^2 F_j^t(dx) \geqq \left(1 - \frac{\delta}{4}\right) \sigma_j^2$$

and

$$\left|\int_0^{M_j} x F_j^t(dx) - \int_0^{M_j} x F_j(dx)\right| \leqq \frac{\delta}{4} \int_0^{M_j} x F_j(dx),$$

$$\left|\int_{-M_j}^0 x F_j^t(dx) - \int_{-M_j}^0 x F_j(dx)\right| \leqq \frac{\delta}{4} \int_{-M_j}^0 |x| F_j(dx).$$

The last two inequalities imply

(5) $$(E_j^t)^2 \leqq \left[\frac{\delta}{4}\int |x| F_j(dx)\right]^2 \leqq \frac{\delta^2}{16}\sigma_j^2.$$

(3), (4) and (5) give that

$$\left(1-\frac{\delta}{2}\right)\sigma_j^2 < (D_j^t)^2 < \left(1+\frac{\delta}{2}\right)\sigma_j^2 \quad \text{if} \quad j \leqq n.$$

Summing up this inequality from m to n we obtain (1).

Since

$$\frac{d}{dt} E_{m,n}^t = (D_{m,n}^t)^2$$

relation (1) implies (2).

Let us choose $\bar{t}$ as the solution of the equation $E_{m,n}^t = x + 2\sqrt{B_n - B_m}$. ($E_{m,n}^t$ is monotonically increasing in t therefore the equation has a unique solution.)

$$\frac{\left(1-\frac{\delta}{2}\right)(x+2\sqrt{B_n-B_m})}{B_n-B_m} < \bar{t} < \frac{\left(1+\frac{\delta}{2}\right)(x+2\sqrt{B_n-B_m})}{B_n-B_m}$$

because of (2).

Relation (1) and the Chebyshev inequality yield that

$$\int_x^{x+4\sqrt{B_n-B_m}} \exp(-\bar{t}x)\, G_{n,m}^{\bar{t}}(dx) \geqq$$

$$\geqq \exp\left[-\bar{t}\,(x+4\sqrt{B_n-B_m})\right]\left[G_{n,m}^{\bar{t}}(x+4\sqrt{B_n-B_m}) - G_{n,m}^{\bar{t}}(x)\right] \geqq$$

$$\geqq \frac{2}{3}\exp\left(-\bar{t}x - 4\bar{t}\sqrt{B_n-B_m}\right) \geqq \frac{1}{100}\exp(-\bar{t}x).$$

Thus we can make the following estimation:

$$P(S_n - S_m > x) \geqq P\left(|S_n - S_m - x - 2\sqrt{B_n-B_m}| < 2\sqrt{B_n-B_m}\right) =$$

$$= E\exp\left[E(S_n - S_m)\right] \int_x^{x+4\sqrt{B_n-B_m}} \exp(-\bar{t}x)\, G_{n,m}^{\bar{t}}(dx) \geqq$$

$$\geqq \frac{1}{100}\exp\left[\left(1-\frac{\delta}{2}\right)\frac{\bar{t}^2}{2}(B_n - B_m - \bar{t}x\right] \geqq c\exp\left[-\frac{(1-\delta)x^2}{2(B_n-B_m)}\right].$$

This estimation completes the Proof of the lemma.

Similar inequality holds in the case $0 \geqq x \geqq -L\sqrt{B_n \log\log B_n}$.

Set $G_{n,m}(x) = P(S_n - S_m > x)$, and let α be a uniformly distributed random variable in [0, 1] independent of the S_i-s.

Define

$$\eta = \tilde{G}_{n,m}(S_n - S_m)$$

and

$$T_n - T_m = \Phi^{-1}\left(\frac{\eta}{\sqrt{B_n - B_m}}\right),$$

where $\tilde{G}_{n,m}(x) = G_{n,m}(x) + \alpha(G_{n,m}(x+0) - G_{n,m}(x))$, and $\Phi(x)$ is the standard normal distribution function. η is uniformly distributed in [0, 1] and $T_n - T_m$ has a normal distribution functions with expectation 0 and variance $B_n - B_m$.

Our lemma has the following

COROLLARY. *Let ε, η and L be arbitrary positive numbers. Let $B_n - B_m > \varepsilon B_n$. We have for every sufficiently large n*

$$|(S_n - S_m) - (T_n - T_m)| < \eta \sqrt{B_n \log\log B_n}$$

on the set $|S_n - S_m| < L\sqrt{B_n \log\log B_n}$.

PROOF. Since the Lemma holds for every $\delta > 0$ we have

$$\Phi\left(\frac{x - \eta\sqrt{B_n \log\log B_n}}{\sqrt{B_n - B_m}}\right) \leqq G_{n,m}(x) \leqq \Phi\left(\frac{x + \eta\sqrt{B_n \log\log B_n}}{\sqrt{B_n - B_m}}\right),$$

if $|x| < L\sqrt{B_n \log\log B_n}$. This relation implies the corollary.

PROOF OF THE THEOREM. Given any $\varepsilon > 0$ we show that for $n > n(\varepsilon)$ there exists a Wiener process such that

$$P\left(\sup_{t \geqq B_n} \frac{|S(t) - S(B_n) - W(t - B_n)|}{\sqrt{t \log\log t}} > \varepsilon\right) < \varepsilon. \tag{6}$$

Define a sequence of integers $n_0, n_1, \dots$ in such a way that $n_0 = n$ and $\left(1 + \frac{\varepsilon}{20}\right) B_{n_k} < B_{n_{k+1}} < \left(1 + \frac{\varepsilon}{10}\right) B_{n_k}$. Let us construct a sequence of random variables $T_{n_k} - T_{n_{k-1}}$, $k = 1, 2, \dots$ as it was done in the corollary. We may assume that the random variables $T_{n_k} - T_{n_{k-1}}$, $k = 1, 2, \dots$ are independent. If the probability space is rich enough, a Wiener process $W(t)$, $t \geqq 0$ can be constructed in such a way that $W(B_{n_k} - B_n) = T_{n_k} - T_n$ for any $k \geqq 0$. We claim that this Wiener process satisfies relation (6).

First we show that

$$P\left(\sup_k \frac{|S(B_{n_k}) - S(B_n) - W(B_{n_k} - B_n)|}{\sqrt{B_{n_k} \log\log B_{n_k}}} > \frac{\varepsilon}{2}\right) < \frac{\varepsilon}{2}. \tag{7}$$

Indeed,

$$|S(B_{n_k}) - S(B_n) - W(B_{n_k} - B_n)| = \left|\sum_{j=1}^{k} (S_{n_j} - S_{n_{j-1}}) - (T_{n_j} - T_{n_{j-1}})\right|.$$

But the last sum is less than

$$\eta \sum_{j=1}^{k} \sqrt{B_{n_j} \log\log B_{n_j}} < \eta \cdot \frac{20}{\varepsilon} \sqrt{B_{n_k} \log\log B_{n_k}}$$

if $|S_{n_j} - S_{n_{j-1}}| < L\sqrt{B_{n_j} \log\log B_{n_j}}$ for every $j=1, 2, \ldots$.

But choosing L sufficiently large the Lemma implies that the exceptional set has very little probability. Thus choosing $\eta = \varepsilon^2/40$ we obtain relation (7).

To finish the proof of relation (6) it is enough to show that

$$\sum_{k=0}^{\infty} P\left(\sup_{n_k < t < n_{k+1}} \frac{|S(t) - S(B_{n_k})|}{\sqrt{B_{n_k} \log\log B_{n_k}}} > \frac{\varepsilon}{4}\right) < \frac{\varepsilon}{4},$$

and

$$\sum_{k=0}^{\infty} P\left(\sup_{n_k < t < n_{k+1}} \frac{|W(t-B_n) - W(B_{n_k} - B_n)|}{\sqrt{B_{n_k} \log\log B_{n_k}}} > \frac{\varepsilon}{4}\right) < \frac{\varepsilon}{4}.$$

Using a well-known estimation about the supremum of independent random variables one gets that the first sum is less than

$$2 \sum_{k=0}^{\infty} P\left(|S(B_{n_{k+1}}) - S(B_{n_k})| > \frac{\varepsilon}{5} \sqrt{B_{n_k} \log\log B_{n_k}}\right) \leqq$$

$$\leqq 2 \sum_{k=0}^{\infty} P\left(|S_{n_{k+1}} - S_{n_k}| > \frac{\varepsilon}{5} \cdot \frac{10}{\varepsilon} (\log k + c(n))\right),$$

where $C(n) \to \infty$ as $n \to \infty$. If n is sufficiently large then this sum is less than $\varepsilon/4$ because of the Lemma. The second relation may be proved similarly.

One can choose a monotone sequence r_k, $k=1, 2, \ldots$ in such a way that relation (6) is satisfied with $\varepsilon = \frac{1}{k^2}$ for $n \geqq r_k$, and the relations

$$P\left(|S_{r_k}| > \sqrt{B_{r_k} \log\log\log B_{r_k}}\right) < \frac{1}{k^2}, \tag{8}$$

$$1 - \Phi\left(\sqrt{\log\log\log B_{r_k}}\right) < \frac{1}{k^2} \tag{8'}$$

also hold. (Here again we apply the Lemma.) One can construct a sequence of Wiener processes

$$W^k(t), \quad 0 \leqq t \leqq B_{r_{k+1}} - B_{r_k}, \quad k = 1, 2, \ldots$$

which satisfy

$$P\left(\sup_{B_{r_k} < t < B_{r_{k+1}}} \frac{|S(t) - S(B_{r_k})| - W(t - B_{r_k})}{\sqrt{t \log\log t}} > \frac{1}{k^2}\right) < \frac{1}{k^2}. \tag{6'}$$

We may assume that the processes $W^k(t)$, $k=1, 2, \ldots$ are independent. Let $W^0(t)$, $0<t<B_{r_1}$ be an arbitrary Wiener process, independent of the $W^k(t)$-s, $k=1, 2, \ldots$. Define $W(t)$, $t\geqq 0$ as

$$W(t) = \sum_{j=0}^{k-1} W^j(B_{r_{j+1}}-B_{r_j})+W^k(t-B_{r_k}) \quad \text{if} \quad B_{r_k} \leqq t < B_{r_{k+1}}.$$

We claim that this $W(t)$ satisfies the Theorem. Let us first observe that

$$\frac{S(B_{r_k})}{\sqrt{B_{r_k}\log\log B_{r_k}}} \to 0, \quad \frac{W(B_{r_k})}{\sqrt{B_{r_k}\log\log B_{r_k}}} \to 0 \quad \text{with probability 1}$$

because of (8), (8') and the Borel—Cantelli lemma. Thus

$$\frac{S(B_{r_k})-W(B_{r_k})}{\sqrt{B_{r_k}\log\log B_{r_k}}} \to 0 \quad \text{with probability 1.}$$

On the other hand using again the Borel—Cantelli lemma and relation (6') one gets that

$$\lim_{k\to\infty} \sup_{B_{r_k}\leqq t<B_{r_{k+1}}} \frac{|S(t)-S(B_{r_k})-W(t)-W(B_{r_k})|}{\sqrt{t\log\log t}} = 0 \quad \text{with probability 1.}$$

These last two relations imply the theorem.

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