ON THE NUMBER OF LATTICE POINTS BETWEEN TWO ENLARGED AND RANDOMLY SHIFTED COPIES OF AN OVAL

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Summary: Let $\mathbf{A}$ be an oval with a nice boundary in $\mathbb{R}^2$, R a large positive number, $c > 0$ some fixed number and α a uniformly distributed random vector in the unit square $[0,1]^2$. We are interested in the number of lattice points in the shifted annular region consisting of the difference of the sets $\left\{\left(R+\dfrac{c}{R}\right)\mathbf{A}-\alpha\right\}$ and $\left\{\left(R-\dfrac{c}{R}\right)A-\alpha\right\}$. We prove that when R tends to infinity, the expectation and the variance of this random variable tend to $4c$ times the area of the set $\mathbf{A}$, i.e. to the area of the domain where we are counting the number of lattice points. This is consistent with computer studies in the case of a circle or an ellipse which indicate that the distribution of this random variable tends to the Poisson law. We also make some comments about possible generalizations.

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1. Introduction

Using computer simulation Cheng and Lebowitz [4] studied the distribution of the number of lattice points in the domain between two concentric circles of radii $R+\dfrac{c}{R}$ and $R-\dfrac{c}{R}$ whose center is uniformly distributed on the unit square $[0,1]^2$. (By lattice points we mean points from $\mathbb{Z}^2$, i.e. from the set of points in $\mathbb{R}^2$ with integer coordinates.) This computer study, motivated by works of Sinai [9], [10], Bleher [2] and Major [8], suggested that for large R this distribution is asymptotically Poissonian with parameter $4\pi c$, i.e. with the area of the domain where we are counting the number of lattice points. A first step to check the correctness of this statement is to investigate whether the variance of this distribution is asymptotically $4\pi c$, i.e. whether the variance behaves as the Poissonian limit suggests. We answer this question in the affirmative. A similar statement holds for a class of ovals defined as follows.

Definition of an oval. *A closed bounded convex set* $\mathbf{A}$ *is an oval in* $\mathbb{R}^2$ *if it contains the origin in its interior, and its boundary is a smooth four times differentiable Jordan curve whose curvature is positive at all points.*

We also introduce the following notations. Let $|\mathbf{A}|$ denote the area of a measurable set $\mathbf{A}$ in $\mathbb{R}^2$. Given some set $\mathbf{A}\subset\mathbb{R}^2$, $\alpha\in\mathbb{R}^2$ and number $R\in\mathbb{R}^+$ define the set $R\mathbf{A}-\alpha$ as

$$R\mathbf{A}-\alpha=\{\mathbf{u}\in\mathbb{R}^2;\quad \mathbf{u}=R\mathbf{v}-\alpha,\ \mathbf{v}\in\mathbf{A}\}$$

and for $c>0$, $R^2>c$ introduce the difference set

$$\mathbb{O}_R(c,\alpha)=\left[\left(R+\frac{c}{R}\right)\mathbf{A}-\alpha\right]\setminus\left[\left(R-\frac{c}{R}\right)\mathbf{A}-\alpha\right]. \tag{1.1}$$

Clearly, $|\mathbb{O}_R(c,\alpha)|=4c|\mathbf{A}|$ for all α. The following Theorem is the main result of this paper.

Theorem. *Let* $\mathbf{A}$ *be an oval,* $c>0$ *some fixed positive number and* α *a uniformly distributed random variable on* $[0,1]^2$*. For* $R>\sqrt{c}$ *let* $\xi_R=\xi_R(\alpha)$ *denote the number of lattice points in the set* $\mathbb{O}_R(c,\alpha)$ *defined in (1.1). Then the relations*

$$E\xi_R=|\mathbb{O}_R|=4c|\mathbf{A}| \tag{1.2}$$

$$\lim_{R\to\infty}\operatorname{Var}\xi_R=|\mathbb{O}_R|=4c|\mathbf{A}| \tag{1.3}$$

hold. Here $\mathbb{O}_R$ *denotes* $\mathbb{O}_R(c,\alpha)$ *with* $\alpha=0$.

The investigation of the number of lattice points in a domain is a popular subject in number theory. See e.g. [6] for a recent treatment or [7]. This problem also has physical motivations, relating to the investigation of the statistics of eigenvalues in a quantum system with an integrable classical Hamiltonian. For example, if $\mathbf{A}$ is a circle, the lattice points $\mathbf{n}$ label energy levels $E(\mathbf{n};\alpha)=|\mathbf{n}-\alpha|^2$ of the Laplacian $-(\nabla-\alpha)^2$ on the unit torus. These energies can be thought of as points on the real line and their statistics can be studied. This problem seems to be very hard. An easier problem is to consider not a fixed α but a random one distributed uniformly on the unit square $[0,1]^2$, and this is what we have done. A widely accepted conjecture in the physics community is that the distribution of levels is, for typical systems in this class, locally Poissonian [1], i.e. the statistics of the energy levels in the interval $[E, E+L]$, L is fixed and E is uniformly distributed in an interval $[0,T]$ with $T \to \infty$, behave like Poisson distributed points with density π. In our context the conjecture is the following:

Let $P(n;R)$ be the probability that there are exactly n lattice points in $\mathbb{O}_R(c,\alpha)$. Then

$$\lim_{R\to\infty} P(n,R) = p(n) \quad \text{with } p(n) = \frac{(4c\,|\mathbf{A}|)^n}{n!} e^{-4c|\mathbf{A}|n} \ . \tag{1.4}$$

Such a result was proved by Sinai [8] and Major [9] for the number of lattice points in scaled annuli domains bounded by *very* random curves. Here all the randomness comes from α, so the proof of (1.4), if indeed it is true, is far from trivial. Our theorem proves that the limit of the first and second moment of $P(n;R)$ has the right behavior when $R \to \infty$. In fact, our result also shows that the covariance of pairs of distinct random variables $\eta_R^{(j)} = \xi_{R+jc} - |\mathbb{O}_R|$ vanishes as $R \to \infty$. This suggests that $\lim\limits_{c\to\infty} \lim\limits_{R\to\infty} \dfrac{\xi_R - |\mathbb{O}_R|}{\sqrt{|\mathbb{O}_R|}}$ should be a Gaussian random variable. This is consistent with taking the large parameter limit of the Poissonian distribution, but may be valid more generally.

2. Proof of the Theorem

The proof of relation (1.2) is simple. We can write $E\xi_R(\alpha)$ as the the sum of the probabilities

$$E\xi_R(\alpha) = \sum_{\mathbf{m}\in\mathbb{Z}^2} P(\mathbf{m} \in \mathbb{O}_R(c,\alpha)) = \sum_{\mathbf{m}\in\mathbb{Z}^2} \left|\mathbb{O}_R \cap ([0,1]^2 - \mathbf{m})\right| = |\mathbb{O}_R| \ .$$

Because of (1.2) formula (1.3) is equivalent to the relation

$$\lim_{R\to\infty} E\xi_R(\alpha)(\xi_R(\alpha)-1) = |\mathbb{O}_R|^2 \ . \tag{2.1}$$

We claim that

$$E\xi_R(\alpha)(\xi_R(\alpha)-1) = \sum_{\mathbf{m}\in\mathbb{Z}^2\setminus\{0\}} |\mathbb{O}_R \cap (\mathbb{O}_R - \mathbf{m})| \ . \tag{2.2}$$

Indeed,

$$\begin{aligned} E\xi_R(\alpha)(\xi_R(\alpha)-1) &= \sum_{\mathbf{m}\in\mathbb{Z}^2} \sum_{\mathbf{m}_1\in\mathbb{Z}^2\setminus\{\mathbf{m}\}} P\left(\mathbf{m} \in \mathbb{O}_R(c,\alpha),\ \mathbf{m}_1 \in \mathbb{O}_R(c,\alpha)\right) \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^2} \sum_{\mathbf{m}_1\in\mathbb{Z}^2\setminus\{\mathbf{m}\}} \left|[0,1]^2 \cap (\mathbb{O}_R - \mathbf{m}) \cap (\mathbb{O}_R - \mathbf{m}_1)\right| \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^2} \sum_{\mathbf{m}_1\in\mathbb{Z}^2\setminus\{0\}} \left|([0,1]^2 + \mathbf{m}) \cap \mathbb{O}_R \cap (\mathbb{O}_R - \mathbf{m}_1)\right| \\ &= \sum_{\mathbf{m}\in\mathbb{Z}^2\setminus\{0\}} |\mathbb{O}_R \cap (\mathbb{O}_R - \mathbf{m})| \ . \end{aligned}$$

Hence to prove the Theorem it is enough to prove relation (2.1) with the help of relation (2.2). This requires a good estimate on the area of $\mathbb{O}_R \cap (\mathbb{O}_R - \mathbf{m})$. First we introduce some notations.

Let us denote the boundary of the set $t\mathbf{A}$ for $t > 0$, by γ_t, and let $\gamma = \gamma_1$. For some $\mathbf{x} \in \gamma_t$ let $\varphi(\mathbf{x})$ denote the angle of the vector $\mathbf{x}$ and $\psi(\mathbf{x})$ the angle of the normal of the curve γ_t at $\mathbf{x}$ (pointing outside of the domain $t\mathbf{A}$) with the vector $\mathbf{e}_1 = (1, 0)$. Given some $\mathbf{z} \in \mathbb{R}^2 \setminus \{0\}$ let $d(\mathbf{z}, t)$ denote the diameter of the set $t\mathbf{A}$ in the direction $\mathbf{z}$, i.e.

$$d(\mathbf{z}, t) = \max \{|\mathbf{z}_1 - \mathbf{z}_2|; \quad \mathbf{z}_1 \in \gamma_t,\ \mathbf{z}_2 \in \gamma_t,\ \mathbf{z}_1 - \mathbf{z}_2 = \lambda \mathbf{z},\ \text{with some } \lambda > 0\} \ .$$

Let $\mathbf{z}_{d,r}(t)$ and $\mathbf{z}_{d,l}(t)$ be the end points of this maximal vector, i.e. $\mathbf{z}_{d,r}(t) \in \gamma_t$, $\mathbf{z}_{d,l}(t) \in \gamma_t$, $\mathbf{z}_{d,r}(t) - \mathbf{z}_{d,l}(t) = \lambda \mathbf{z}$ with $\lambda > 0$ and $|\mathbf{z}_{d,r}(t) - \mathbf{z}_{d,r}(t)| = d(\mathbf{z}, t)$. For $\mathbf{z} \in \mathbb{R}^2$ let $\mathbf{z}^\perp$ denote the vector $\mathbf{z}$ rotated by $+\pi/2$, and define $K_t^+(\mathbf{z})$ and $K_t^-(\mathbf{z})$ as the half planes whose boundary is the line going through the points $\mathbf{z}_{d,r}(t)$ and $\mathbf{z}_{d,l}(t)$ and which are in the direction $\mathbf{z}^\perp$ and $-\mathbf{z}^\perp$ of this line respectively. For $\mathbf{z} \in \mathbb{R}^2$ and $0 < |\mathbf{z}| \le d(\mathbf{z}, t)$ define the (unique) points $\mathbf{z}_{t,r}^+$, $\mathbf{z}_{t,l}^+ \in K_t^+(\mathbf{z}) \cap \gamma_t$ and $\mathbf{z}_{t,r}^-$, $\mathbf{z}_{t,l}^- \in K_t^-(\mathbf{z}) \cap \gamma_t$ such that $\mathbf{z}_{t,r}^+ - \mathbf{z}_{t,l}^+ = \mathbf{z}$ and $\mathbf{z}_{t,r}^- - \mathbf{z}_{t,l}^- = \mathbf{z}$.

For $\mathbf{z}_1$, $\mathbf{z}_2 \in \gamma_R$ define the function

$$F_R(\mathbf{z}_1, \mathbf{z}_2) = \frac{4c^2}{R^4} \frac{|\mathbf{z}_1||\mathbf{z}_2| \cos(\varphi(\mathbf{z}_1) - \psi(\mathbf{z}_1)) \cos(\varphi(\mathbf{z}_2) - \psi(\mathbf{z}_2))}{|\sin(\psi(\mathbf{z}_1) - \psi(\mathbf{z}_2))|}$$

and for some $\mathbf{z} \in \mathbb{R}^2$, $0 < |\mathbf{z}| \le d(\mathbf{z}, R)$ the functions

$$f_R^+(\mathbf{z}) = F_R(\mathbf{z}_{R,l}^+, \mathbf{z}_{R,r}^+) \ , \qquad f_R^-(\mathbf{z}) = F_R(\mathbf{z}_{R,l}^-, \mathbf{z}_{R,r}^-) \ .$$

For $\mathbf{A} \subset \mathbb{Z}^2$ and $\mathbf{B} \subset \mathbb{Z}^2$ define their sum

$$\mathbf{A} + \mathbf{B} = \{\mathbf{x} + \mathbf{y}, \quad \mathbf{x} \in \mathbf{A},\ \mathbf{y} \in \mathbf{B}\}$$

and

$$\mathbf{A}^{(2)}(R) = R\mathbf{A} + (-R\mathbf{A}) \ .$$

We claim that

$$\int_{\mathbf{A}^{(2)}(R)} \left[f_R^+(\mathbf{z}) + f_R^-(\mathbf{z}) \right] \, d\mathbf{z} = 16c^2 \, |\mathbf{A}|^2 \ . \tag{2.3}$$

Put

$$h_R(\mathbf{z}) = |\mathbb{O}_R \cap (\mathbb{O}_R - \mathbf{z})| \ , \quad \mathbf{z} \in \mathbb{R}^2 \ . \tag{2.4}$$

We then also claim that

$$\lim_{R \to \infty} \left\{ \sum_{\mathbf{m} \in \mathbb{Z}^2 \setminus \{0\}} h_R(\mathbf{m}) - \int_{\mathbf{A}^{(2)}(R)} \left[f_R^+(\mathbf{z}) + f_R^-(\mathbf{z}) \right] \, dz \right\} = 0 \ . \tag{2.5}$$

Relations (2.2), (2.3) and (2.5) together imply (2.1) hence also the Theorem. To prove (2.3) we introduce the maps

$$\begin{aligned} &G^\pm \colon \mathbf{A}^{(2)}(R) \setminus \{0\} \mapsto \gamma_R \times \gamma_R \ , \\ &G^+(\mathbf{z}) = (\mathbf{z}_{R,l}^+, \mathbf{z}_{R,r}^+), \quad G^-(\mathbf{z}) = (\mathbf{z}_{R,l}^-, \mathbf{z}_{R,r}^-) \end{aligned}$$

Observe that the set $\operatorname{Int} G^+(\mathbf{A}^{(2)}(R)\setminus\{0\})\cap\operatorname{Int} G^-(\mathbf{A}^{(2)}(R)\setminus\{0\})$ is empty, the maps $G^\pm$ are diffeomorphisms on $\operatorname{Int}\mathbf{A}^{(2)}(R)\setminus\{0\}$, and $G^+(\mathbf{A}^{(2)}(R)\setminus\{0\})\cup G^-(\mathbf{A}^{(2)}(R)\setminus\{0\})$ is $\gamma_R\times\gamma_R\setminus\{(\mathbf{z},\mathbf{z}),\ \mathbf{z}\in\gamma_r\}$.

The inverses of the maps $G^\pm(\mathbf{z})$ have Jacobians $\left|\sin\left(\psi(\mathbf{z}^\pm_{R,r})-\psi(\mathbf{z}^\pm_{R,l})\right)\right|$. To see this we make the following observation: Let $[\mathbf{z}_1,\mathbf{z}_1+d\mathbf{z}_1]$ and $[\mathbf{z}_2,\mathbf{z}_2+d\mathbf{z}_2]$ be two small curves on γ_R starting from some points $\mathbf{z}_1$ and $\mathbf{z}_2$ respectively. Then the inverse of the map $G^\pm$ maps the set $[\mathbf{z}_1,\mathbf{z}_1+d\mathbf{z}_1]\times[\mathbf{z}_2,\mathbf{z}_2+d\mathbf{z}_2]$ approximately to $\mathbf{z}_2-\mathbf{z}_1+\Delta(d\mathbf{z}_1,d\mathbf{z}_2)$, where $\Delta(d\mathbf{z}_1,d\mathbf{z}_2)$ is a parallelogram with one vertex at the origin, whose sides are the vectors $d\mathbf{z}_1$ and $d\mathbf{z}_2$. The area of this parallelogram is $|d\mathbf{z}_1||d\mathbf{z}_2||\sin(\psi(\mathbf{z}_1)-\psi(\mathbf{z}_2))|$. We can approximate the area of the image of the above domain by the inverse map $(G^\pm)^{-1}$ with the area of this parallelogram. Since this approximation gives only an error of order $o(|d\mathbf{z}_1||d\mathbf{z}_2|)$ the Jacobian has the form we have stated.

The above relations imply that

$$\begin{aligned}
&\int_{\mathbf{A}^{(2)}(R)}\left[f_R^+(\mathbf{z})+f_R^-(\mathbf{z})\right]\,d\mathbf{z}\\
&\quad=\int_{\gamma_R\times\gamma_R}\frac{4c^2}{R^4}|\mathbf{z}_1||\mathbf{z}_2|\cos(\psi(\mathbf{z}_1)-\varphi(\mathbf{z}_1))\cos(\psi(\mathbf{z}_2)-\varphi(\mathbf{z}_2))\,d\mathbf{z}_1\,d\mathbf{z}_2\\
&\quad=\left[\frac{2c}{R^2}\int_{\gamma_R}|\mathbf{z}|\cos(\psi(\mathbf{z})-\varphi(\mathbf{z}))\,d\mathbf{z}\right]^2\\
&\quad=\left[2c\int_{\gamma_1}|\mathbf{z}|\cos(\psi(\mathbf{z})-\varphi(\mathbf{z}))\,d\mathbf{z}\right]^2=16c^2\,|\mathbf{A}|^2\ ,
\end{aligned}$$

hence relation (2.3) holds. To prove relation (2.5) we need some geometrical facts formulated in relations (2.9) and (2.10) and a lemma about the value of $h_R(\mathbf{z})$. They will be proved in the next Section.

Lemma 1. *There is some $\varepsilon>0$ such that the function $h_R(\mathbf{z})$ defined in (2.4) satisfies the following estimates:*

a) For $1\le|\mathbf{z}|<\varepsilon R,\quad h_R(\mathbf{z})<\dfrac{\text{const.}}{R\,|\mathbf{z}|}$.

b) For all $0<\eta\le\varepsilon$ *and* $\eta R\le|\mathbf{z}|\le(1-\eta)d(\mathbf{z},R)$

$$R^2\left\{h_R(\mathbf{z})-\left[f_R^+(\mathbf{z})+f_R^-(\mathbf{z})\right]\right\}\to0\quad\textit{as } R\to\infty\ ,$$

and the convergence is uniform in $\mathbf{z}$.

c) Let us fix some positive constant $B>0$. Then the following inequalities hold:

c1) $h_R(\mathbf{z})<\dfrac{\text{const.}}{R^{3/2}\sqrt{d(\mathbf{z},R)-|\mathbf{z}|}}\qquad$ *if* $(1-\varepsilon)d(\mathbf{z},R)\le|\mathbf{z}|\le d(\mathbf{z},R)-\dfrac{B}{R}$.

c2) $h_R(\mathbf{z})<\dfrac{\text{const.}}{R}\qquad$ *if* $d(\mathbf{z},R)-\dfrac{B}{R}\le|\mathbf{z}|\le d(\mathbf{z},R)+\dfrac{B}{R}$.

d) $h_R(\mathbf{z})=0$ *if* $|\mathbf{z}|>d(\mathbf{z},R)+\dfrac{B}{R}$*, and B is larger than c times the diameter of the set* $\mathbf{A}$.

To prove relation (2.5) let us introduce the sets

$$
\begin{aligned}
D_1^\varepsilon(R) &= \left\{\mathbf{z} \in \mathbb{R}^2,\ 0 < |\mathbf{z}| < \varepsilon R\right\} \\
D_2^\varepsilon(R) &= (1-\varepsilon)\mathbf{A}^{(2)}(R) \setminus D_1^\varepsilon(R) \\
D_3^\varepsilon(R) &= \left(1 - \frac{B}{R^2}\right)\mathbf{A}^{(2)}(R) \setminus (1-\varepsilon)\mathbf{A}^{(2)}(R) \\
D_4^\varepsilon(R) &= \left(1 + \frac{B}{R^2}\right)\mathbf{A}^{(2)}(R) \setminus \left(1 - \frac{B}{R^2}\right)\mathbf{A}^{(2)}(R)\ .
\end{aligned}
$$

Define the discrete measure μ on the positive half-line $[0,\infty]$,

$$\mu([0,x]) = \left\{\text{the number of lattice points in the set } x\mathbf{A}^{(2)}\right\}\ ,$$

where $\mathbf{A}^{(2)}$ denotes $\mathbf{A}^{(2)}(R)$ with $R = 1$. Define also the signed measure

$$\nu([0,x]) = \mu([0,x]) - x^2 \text{Area}\,(\mathbf{A}^{(2)})\ .$$

If $\mathbf{A}$ is an oval, then $\mathbf{A}^{(2)}$ is again an oval. (See [3]). (This means that the boundary of $\mathbf{A}^{(2)}$ is again strictly convex, smooth, and has positive curvature at all points.) Hence the results known for ovals can be applied to $\mathbf{A}^{(2)}$. In particular, we can state because of a result of Colin de Verdière [5] that

$$|\nu([0,x])| < \text{const.}\, x^{2/3} \quad \text{for all } x > 1\ . \tag{2.6}$$

Let us also remark that the normals of γ_R in the points $\mathbf{z}_{d,r}(R)$ and $\mathbf{z}_{d,l}(R)$ satisfy the relation

$$\psi(\mathbf{z}_{d,r}(R)) = \psi(\mathbf{z}_{d,l}(R)) + \pi \tag{2.7}$$

for all $\mathbf{z} \in \mathbb{R}^2 \setminus \{0\}$, i.e. the normals in the points $\mathbf{z}_{d,r}(R)$ and $\mathbf{z}_{d,l}(R)$ point in opposite directions. The half-line $\lambda\mathbf{z}$, $\lambda > 0$, intersects the boundary of $\mathbf{A}^{(2)}(R)$ at distance $d(\mathbf{z}, R)$ from the origin. Hence Part c1) of Lemma 1 bounds the value of $h_R(\mathbf{z})$ for $\mathbf{z} \in D_3^\varepsilon(R)$ and Part c2) bounds the value of $h_R(\mathbf{z})$ for $\mathbf{z} \in D_4^\varepsilon(R)$.

By Part a) of Lemma 1

$$\sum_{\mathbf{m} \in \mathbb{Z}^2 \cap D_1^\varepsilon(R) \setminus \{0\}} h_R(\mathbf{m}) < \text{const.}\, \frac{1}{R} \sum_{\substack{\mathbf{m} \in \mathbb{Z}^2 \\ 0 < |\mathbf{m}| < \varepsilon R}} \frac{1}{|\mathbf{m}|} < \text{const.}\, \varepsilon.$$

Since $R^2\left(f_R^+(R\mathbf{z}) + f_R^-(R\mathbf{z})\right)$ is uniformly continuous in the set $\dfrac{1}{R} D_2^\varepsilon(R)$, hence

$$\sum_{\mathbf{m} \in \mathbb{Z}^2 \cap D_2^\varepsilon(R)} h_R(\mathbf{m}) - \int_{D_2^\varepsilon(R)} \left[f_R^+(\mathbf{z}) + f_R^-(\mathbf{z})\right] d\mathbf{z} \to 0 \quad \text{as } R \to \infty$$

by Part b) of Lemma 1. Put

$$H_R(t) = \sup_{\mathbf{z} \in t\partial\mathbf{A}^{(2)}} h_R(\mathbf{z})\ ,$$

where $\partial\mathbf{A}^{(2)}$ denotes the boundary of $\mathbf{A}^{(2)}$. By using Part c1) of Lemma 1, integrating by parts and applying (2.6) we can write

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathbb{Z}^2\cap D_3^\varepsilon(R)} h_R(\mathbf{m}) &\le \int_{(1-\varepsilon)R}^{\left(R-\frac{B}{R}\right)} H_R(x)\,\mu(dx) \\
&< \frac{\text{const.}}{R^{3/2}}\int_{(1-\varepsilon)R}^{\left(R-\frac{B}{R}\right)} \frac{x}{\sqrt{R-x}}\,dx + \frac{\text{const.}}{R^{3/2}}\int_{(1-\varepsilon)R}^{\left(R-\frac{B}{R}\right)} \frac{\nu(\,dx)}{\sqrt{R-x}} \\
&< \text{const.}\sqrt{\varepsilon} + \frac{\text{const.}}{R^{3/2}}\left[\frac{\nu([0,x])}{\sqrt{R-x}}\right]_{(1-\varepsilon)R}^{\left(R-\frac{B}{R}\right)} \\
&\qquad + \text{const.}\int_{(1-\varepsilon)R}^{\left(R-\frac{B}{R}\right)} \frac{\nu([0,x])}{R^{3/2}\,(R-x)^{3/2}}\,dx \\
&< \text{const.}\left(\sqrt{\varepsilon} + R^{-1/3} + R^{-5/6}\int_{(1-\varepsilon)R}^{\left(R-\frac{B}{R}\right)} \frac{1}{(R-x)^{3/2}}\,dx\right) \\
&< \text{const.}\left[R^{-1/3} + \sqrt{\varepsilon}\right].
\end{aligned}$$

Similarly, by Part c2) of Lemma 1

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathbb{Z}^2\cap D_4^\varepsilon(R)} h_R(\mathbf{m}) &\le \text{const.}\int_{\left(R-\frac{B}{R}\right)}^{\left(R+\frac{B}{R}\right)} \frac{1}{R}\,\mu(\,dx) \\
&\le \text{const.}\int_{\left(R-\frac{B}{R}\right)}^{\left(R+\frac{B}{R}\right)} \frac{x}{R}\,dx + \frac{\text{const.}}{R}\nu\left(\left[R-\frac{B}{R}, R+\frac{B}{R}\right]\right) \\
&< \text{const.}\left[\frac{1}{R} + R^{-1/3}\right] < \text{const.}\,R^{-1/3}\,.
\end{aligned}$$

The above relations together with Part d) of Lemma 1 imply that

$$\sum_{\mathbf{m}\in\mathbb{Z}^2\setminus\{0\}} h_R(\mathbf{m}) - \int_{D_2^\varepsilon(R)} \left[f_R^+(\mathbf{z}) + f_R^-(\mathbf{z})\right]\,d\mathbf{z} = O\left(\sqrt{\varepsilon} + R^{-1/3}\right)\,. \tag{2.8}$$

We claim that

$$I_1 = \int_{D_1^\varepsilon(R)} \left[f_R^+(\mathbf{z}) + f_R^-(\mathbf{z})\right]\,d\mathbf{z} = O(\varepsilon) \tag{2.8$'$}$$

and

$$I_2 = \int_{\mathbf{A}^{(2)}(R)\setminus(1-\varepsilon)\mathbf{A}^{(2)}(R)} \left[f_R^+(\mathbf{z}) + f_R^-(\mathbf{z})\right]\,d\mathbf{z} = O\left(\sqrt{\varepsilon}\right)\,. \tag{2.8$''$}$$

Since $\varepsilon > 0$ can be chosen arbitrary small the above relations imply (2.5).

In Section 3 we shall prove the following statements. There is some $\varepsilon > 0$ such that if $|\mathbf{z}| = uR$ with some $0 < u < \varepsilon$, then the normals of γ_R satisfy the inequality

$$\left|\psi(\mathbf{z}_{R,r}^{\pm}) - \psi(\mathbf{z}_{R,l}^{\pm})\right| > \text{const.}\,u\,, \tag{2.9}$$

and if $|\mathbf{z}| = (1-u)d(\mathbf{z}, R)$ with some $0 < u < \varepsilon$, then

$$\left|\psi(\mathbf{z}^{\pm}_{R,r}) - \psi(\mathbf{z}^{\pm}_{R,l}) - \pi\right| > \text{const.}\,\sqrt{u}\ . \tag{2.10}$$

In the first case we get that

$$|f^{+}_{R}(\mathbf{z})| + |f^{-}_{R}(\mathbf{z})| < \text{const.}\,\frac{1}{R^2 u} \tag{2.11}$$

and in the second case

$$|f^{+}_{R}(\mathbf{z})| + |f^{-}_{R}(\mathbf{z})| < \text{const.}\,\frac{1}{R^2 \sqrt{u}}\ . \tag{2.12}$$

Thus we get, by integrating in a polar coordinate system and applying the estimate (2.11), that

$$I_1 < \text{const.}\,\frac{1}{R^2}\int_0^{\varepsilon} R^2 u \frac{1}{u}\, du < \text{const.}\,\varepsilon\ .$$

Relation (2.12) implies that for $(1-\varepsilon)R \le t \le R$

$$\begin{aligned}\oint_{t\partial\mathbf{A}^{(2)}} \left[f^{+}_{R}(\mathbf{z}) + f^{-}_{R}(\mathbf{z})\right]\, d\mathbf{z} &\le \text{const.}\,\oint_{t\partial\mathbf{A}^{(2)}} (1-t)^{-1/2} R^{-3/2}\, d\mathbf{z} \\ &\le \text{const.}\,\frac{1}{\sqrt{(1-t)R}}\ .\end{aligned}$$

Integrating first on the curves γ_t, $(1-\varepsilon)R < t < R$, we get that

$$I_2 < \text{const.}\,\int_0^{\varepsilon R} R^{-1/2} u^{-1/2}\, du < \text{const.}\,\sqrt{\varepsilon}\ .$$

Hence we proved the Theorem with the help of Lemma 1 and formulas (2.9) and (2.10).

3. Proof of Lemma 1

We shall need a result about convex sets in the proof. To formulate it we introduce some notations. Let us fix some vector $\mathbf{z} \in \mathbb{R}^2$, $\mathbf{z} \neq 0$. Introduce the coordinate system whose coordinate axis x is in the direction $\dfrac{\mathbf{z}}{|\mathbf{z}|}$ and the coordinate axis y is its rotation with $+\frac{\pi}{2}$, in the direction $\dfrac{\mathbf{z}^\perp}{|\mathbf{z}^\perp|}$. In this new coordinate system let $(x_0(t), y_0(t))$ and $(x_1(t), y_1(t))$ be the points of tangency of the curve γ_t with the line parallel to the vector $\mathbf{z}$ in the half-spaces $K_t^+(\mathbf{z})$ and $K_t^-(\mathbf{z})$ respectively. For $y_1(t) < u < y_0(t)$ the line $y = u$ intersects the curve γ_t in the points $x_{l,t}(u)$ and $x_{r,t}(u)$, $x_{l,t}(u) < x_{r,t}(u)$. We shall prove the following

Lemma 2. *There are some positive constants $A > 0$, $0 < B_1 < B_2$ depending only on the curve γ such that*

$$
\begin{gathered}
B_1\sqrt{t(y_0(t)-u)} < x_{r,t}(u) - x_0(t) < B_2\sqrt{t(y_0(t)-u)}\,,\\
B_1\sqrt{t(y_0(t)-u)} < x_0(t) - x_{l,t}(u) < B_2\sqrt{t(y_0(t)-u)}\\
\frac{B_1\sqrt{t}}{\sqrt{y_0(t)-u}} < -\frac{d}{du}x_{r,t}(u) < \frac{B_2\sqrt{t}}{\sqrt{y_0(t)-u}}\,,\\
\frac{B_1\sqrt{t}}{\sqrt{y_0(t)-u}} < \frac{d}{du}x_{l,t}(u) < \frac{B_2\sqrt{t}}{\sqrt{y_0(t)-u}}\,,
\end{gathered}
$$

if $y_0(t) - tA < u < y_0(t)$, and

$$
\begin{gathered}
B_1\sqrt{t(u-y_1(t))} < x_{r,t}(u) - x_1(t) < B_2\sqrt{t(u-y_1(t))}\,,\\
B_1\sqrt{t(u-y_1(t))} < x_1(t) - x_{l,t}(u) < B_2\sqrt{t(u-y_1(t))}\\
\frac{B_1\sqrt{t}}{\sqrt{u-y_1(t)}} < \frac{d}{du}x_{r,t}(u) < \frac{B_2\sqrt{t}}{\sqrt{u-y_1(t)}}\,,\\
\frac{B_1\sqrt{t}}{\sqrt{u-y_1(t)}} < -\frac{d}{du}x_{l,t}(u) < \frac{B_2\sqrt{t}}{\sqrt{u-y_1(t)}}\,,
\end{gathered}
$$

if $y_1(t) < u < y_1(t) + tA$.

Proof of Lemma 2. Let us first restrict our attention to the case $t = 1$. It is more convenient to work with the inverse of the function $x_{r,1}(u)$. Let $(x, g(x))$ be a small part of the curve γ in the neighborhood of the point $(x_0(1), y_0(1))$, and let $\rho(x)$ be the curvature of the curve γ in the point $(x, g(x))$. We can write in an interval $[x_0(1), x_0(1)+\eta]$, $\eta > 0$,

$$
\rho(x) = -\frac{\left[1 + g'(x)^2\right]^{3/2}}{g''(x)}\,.
$$

Put $z(x) = g'(x)$. Since $z(x_0(1)) = 0$, the last relation implies that

$$
\int_0^{z(x)} \frac{dt}{(1+t^2)^{3/2}} = -\int_{x_0(1)}^{x} \frac{1}{\rho(t)}\,dt
$$

or

$$\frac{z(x)}{\sqrt{1+z^2(x)}} = -P(x)$$

with $P(x) = \int_{x_0(1)}^{x} \frac{1}{\rho(t)}\,dt$. Since the curvature of γ is separated both from zero and infinity, there are some constants $K_2 > K_1 > 0$ such that

$$K_1(x - x_0(1)) < P(x) < K_2(x - x_0(1))$$

Since $z(x) = \frac{-P(x)}{\sqrt{1-P^2(x)}}$ the last relation implies that

$$C_1(x - x_0(1)) < -g'(x) < C_2(x - x_0(1))$$

with some $C_2 > C_1 > 0$ in an interval $x \in [x_0(1), x_0(1) + \eta)$. Since $g(x_0(1)) = y_0(1)$ we get by integrating that

$$-\frac{C_2}{2}(x - x_0(1))^2 < g(x) - y_0(1) < -\frac{C_1}{2}(x - x_0(1))^2.$$

These formulas imply that $x_{r,1}(u)$, the inverse of $g(x)$, satisfies Lemma 1 for $t = 1$ in an interval $[y_0(1) - A, y_0(1)]$. The remaining statements of Lemma 1 for $t = 1$ can be proved in the same way. The case of general $t > 0$ follows from the case $t = 1$ because of the homogeneity properties of γ_t. $\square$

Now we turn to the proof of relations (2.9) and (2.10). Here again we can restrict our attention to the case $R = 1$. Let us recall that the curvature $\rho(\mathbf{x})$ of γ in a point $\mathbf{x} \in \gamma$ and the angle of the normal $\psi(\mathbf{x})$ in this point satisfy the relation

$$\frac{d\psi(\mathbf{x})}{ds(\mathbf{x})} = \rho(\mathbf{x}) \ , \tag{3.1}$$

where $s(\mathbf{x})$ is the length of the arc $(\mathbf{x}_0, \mathbf{x})$ of γ with some fixed $\mathbf{x}_0 \in \gamma$.

Under the conditions imposed for formula (2.9) the length of the arc $(\mathbf{z}_{1,l}^{\pm}, \mathbf{z}_{1,r}^{\pm})$ is greater than u. Since $\rho(\mathbf{x})$ is bounded from below by a positive constant, we get formula (2.9) by integrating (3.1).

The proof of (2.10) is similar. Here we can apply formula (2.7). Because of this formula it is enough to show that

$$\begin{aligned} |\psi(\mathbf{z}_{1,l}^{\pm}) - \psi(\mathbf{z}_{d,l}(1))| &> \text{const.} \sqrt{u} \\ |\psi(\mathbf{z}_{1,r}^{\pm}) - \psi(\mathbf{z}_{d,r}(1))| &> \text{const.} \sqrt{u} \end{aligned} \tag{3.2}$$

under the conditions imposed for (2.10).

Given two vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ let $\angle(\mathbf{z}_1, \mathbf{z}_2)$ denote the angle between them. For $\mathbf{z} \in \gamma_1$ let $n(\mathbf{z})$ denote the normal vector to the curve γ_1 at $\mathbf{z}$. We make the following observation: For any $\mathbf{z} \in \mathbb{R}^2 \setminus \{0\}$ consider the end points $\mathbf{z}_{d,l} = \mathbf{z}_{d,l}(1)$ and $\mathbf{z}_{d,r} = \mathbf{z}_{d,r}(1)$

of the interval with maximal length in $\mathbf{A}$ in the direction of $\mathbf{z}$. The normal of γ_1 in these points cannot be almost orthogonal to $\mathbf{z}$. More explicitly, there is some $\eta > 0$ such that

$$-\frac{\pi}{2} + \eta < \angle(-\mathbf{z}, n(\mathbf{z}_{d,l})) < \frac{\pi}{2} - \eta \,. \tag{3.3}$$

(This statement is equivalent to the following one: If $\mathbf{z}$ is a boundary point of the oval $\mathbf{A}^{(2)}$, then the vector $\mathbf{z}$ cannot be almost orthogonal to the normal of the boundary of $\mathbf{A}^{(2)}$ in this point. The equivalence of these two statements follows from the following argument. The vector $\mathbf{z}_{d,l} - \mathbf{z}_{d,r}$ is on the boundary of $\mathbf{A}^{(2)}$, and it is parallel to $\mathbf{z}$. The normal of $\mathbf{A}^{(2)}$ in this point is parallel to $n(\mathbf{z}_{d,l})$. The proof of the second statement is simpler.)

We claim that under the conditions imposed for formula (2.10) the distance of the parallel lines going through the points $\mathbf{z}^{\pm}_{1,l}$ and $\mathbf{z}^{\pm}_{1,r}$ and through the points $\mathbf{z}_{d,l}(1)$ and $\mathbf{z}_{d,r}(1)$ is bigger than const. $\sqrt{u}$. Put $m_l = m_l(\mathbf{z}) = \dfrac{1}{\cos\angle(\mathbf{z}, n(\mathbf{z}_{d,l}))}$ and $m_r = m_r(\mathbf{z}) = \dfrac{1}{\cos\angle(\mathbf{z}, n(\mathbf{z}_{d,r}))}$. Then $m_r = -m_l$ by (2.7), and by (3.3) there is some $\infty > K > 0$ such that $-K < m_r(\mathbf{z}) < K$ for all $\mathbf{z} \in \mathbb{R}^2 \setminus \{0\}$.

Let us fix a new coordinate system with the origin in a point of the line going through the points $\mathbf{z}_{l,r}$ and $\mathbf{z}_{d,r}$, with the x axis in the direction of the vector $\mathbf{z}$ and y axis in the direction $\mathbf{z}^{\perp}$, and let us work in it. For $\varepsilon > v > 0$ let $\mathbf{z}^{+}_{l}(v) = (z^{+}_{l}(v), v)$ and $\mathbf{z}^{+}_{r}(v) = (z^{+}_{r}(v), v)$, $z^{+}_{l}(v) < z^{+}_{r}(v)$, be the two intersections of γ and the line $y = v$, and put $\mathbf{z}(v) = \mathbf{z}^{+}_{l}(v) - \mathbf{z}^{+}_{r}(v)$. It follows from Lemma 2 (with its application in the coordinate system with coordinate axes parallel to the normal and to the tangent vector of the curve γ_1 in the points $\mathbf{z}_{d,l}(1)$ and $\mathbf{z}_{d,r}(1)$ respectively) that there is some $C > 0$ such that $z^{+}_{l}(v) < z_{d,l}(1) + vm_l + Cv^2$ and $z^{+}_{r}(v) > z_{d,r}(1) + vm_l - Cv^2$. The above relations imply that $|\mathbf{z}(v)| = z^{+}_{r}(v) - z^{+}_{l}(v) > d(\mathbf{z}, 1) - 2Cv^2$. Since we imposed the condition $|\mathbf{z}| = d(\mathbf{z}, 1) - ud(\mathbf{z}, 1)$, this relation implies that the distance between the parallel lines going through the points $\mathbf{z}^{+}_{1,l}$ and $\mathbf{z}^{+}_{1,r}$ and through the points $\mathbf{z}_{d,l}(1)$ and $\mathbf{z}_{d,r}(1)$ is greater than const. $\sqrt{u}$. The same statement holds if $\mathbf{z}^{+}_{1,l}$ and $\mathbf{z}^{+}_{1,r}$ are replaced by $\mathbf{z}^{-}_{1,l}$ and $\mathbf{z}^{-}_{1,r}$. Hence the arcs $\mathbf{z}^{\pm}_{1,l}, \mathbf{z}_{d,l}$ and the arcs $\mathbf{z}^{\pm}_{1,r}, \mathbf{z}_{d,r}$ are longer than const. $\sqrt{u}$. Hence relation (3.1) and the strict positivity of $\rho(\mathbf{x})$ imply formula (3.2) and hence formula (2.10) too.

Now we turn to the proof of Lemma 1. We introduce the abbreviation R^+ instead of $\left(R + \dfrac{c}{R}\right)$ and R^- instead of $\left(R - \dfrac{c}{R}\right)$.

Proof of Part a). Let us introduce the coordinate system with x axis parallel to $\mathbf{z}$ and y axis parallel to $\mathbf{z}^{\perp}$. We shall estimate, by means of Lemma 2, the length of the intersection of the set $\mathbb{O}_R \cap (\mathbb{O}_R - \mathbf{z})$ with the lines $y = u$ for different u-s.

Define $u_{R^+}(\mathbf{z})$ as the (unique) solution of the equation

$$x_{r,R^+}(u) - x_{l,R^+}(u) = |\mathbf{z}|, \quad u > u_{d,R^+} \,,$$

where $u_{d,R^\pm}$ is defined by the formula

$$x_{r,R^\pm}\left(u_{d,R^\pm}\right) - x_{l,R^\pm}\left(u_{l,R^\pm}\right) = \max_u \left[x_{r,R^\pm}(u) - x_{l,R^\pm}(u)\right] \,,$$

that is, it is the level u at which the horizontal line $y=u$ has the longest intersection with the set $R^{\pm}\mathbf{A}$.

We claim that there is some $K>0$ such that

$$x_{r,R^+}(u)-x_{l,R^+}(u)<|\mathbf{z}|+\frac{\text{const.}}{|\mathbf{z}|} \quad \text{if } u_{R^+}(\mathbf{z})-\frac{K}{R}\le u\le u_{R^+}(\mathbf{z})\ , \tag{3.4}$$

$$x_{r,R^-}(u)-x_{l,R^-}(u)>|\mathbf{z}| \quad \text{if } u_{d,R^-}\le u\le u_{R^+}(\mathbf{z})-\frac{K}{R}\ , \tag{3.4$'$}$$

and

$$\left.\begin{aligned} x_{r,R^+}(u)-x_{r,R^-}(u)&<\frac{1}{2}\\ x_{l,R^+}(u)-x_{l,R^-}(u)&<\frac{1}{2}\end{aligned}\right\} \quad \text{if } u_{d,R^-}\le u\le u_{R^+}(\mathbf{z})-\frac{K}{R}\ . \tag{3.4$''$}$$

First we show that relations (3.4)—(3.4$''$) imply that

$$\left|\mathbb{O}_R\cap(\mathbb{O}_R-\mathbf{z})\cap K^+_{R^-}(\mathbf{z})\right|<\frac{\text{const.}}{R\,|\mathbf{z}|}\ . \tag{3.5}$$

To see this we show that the intersection of $\mathbb{O}_R\cap(\mathbb{O}_R-\mathbf{z})$ with the line $y=u$ has a length smaller than $\dfrac{\text{const.}}{|\mathbf{z}|}$ if $u_{R^+}(\mathbf{z})-\dfrac{K}{R}\le u\le u_{R^+}(\mathbf{z})$, and it is empty if $u>u_{R^+}(\mathbf{z})$ or $u_{d,R^-}\le u\le u_{R^+}(\mathbf{z})-\dfrac{K}{R}$. The above relations imply (3.5).

The above intersections are contained in the interval $[x_{l,R^+}(u),x_{r,R^+}(u)-|\mathbf{z}|]$ whose length is less than $\dfrac{\text{const.}}{|\mathbf{z}|}$ if $u_{R^+}(\mathbf{z})-\dfrac{K}{R}\le u\le u_{R^+}(\mathbf{z})$ by (3.4). The distance $x_{r,R^+}(u)-x_{l,R^+}(u)$ is less than $|\mathbf{z}|$ for $u>u_{R^+}(\mathbf{z})$, because it is a (convex) monotone decreasing function of u in the interval $\left[u_{d,R^+},y_0(R^+)\right]$ and it equals $|\mathbf{z}|$ for $u_{R^+}(\mathbf{z})$. This implies that the intersection $\mathbb{O}_R\cap(\mathbb{O}_R-\mathbf{z})\cap\{(x,u),\ x\in\mathbb{R}^1\}$ is empty for $u>u_{R^+}(\mathbf{z})$. To see that it is empty for $u_{d,R^-}\le u\le u_{R^+}(\mathbf{z})-\dfrac{K}{R}$ too, observe first that the intersection of $\mathbb{O}_R$ with the line $y=u$ consists of two intervals, $\left[x_{l,R^+}(u),x_{l,R^-}(u)\right]$ and $\left[x_{r,R^-}(u),x_{r,R^+}(u)\right]$. The intersections

$$\left[x_{l,R^+}(u),x_{l,R^-}(u)\right]\cap\left(\left[x_{l,R^+}(u),x_{l,R^-}(u)\right]-|\mathbf{z}|\right)$$

and

$$\left[x_{r,R^-}(u),x_{l,R^+}(u)\right]\cap\left(\left[x_{r,R^-}(u),x_{l,R^+}(u)\right]-|\mathbf{z}|\right)$$

are empty because of (3.4$''$) and the condition $|\mathbf{z}|>1$. The intersection

$$\left[x_{l,R^+}(u),x_{l,R^-}(u)\right]\cap\left(\left[x_{r,R^-}(u),x_{l,R^+}(u)\right]-|\mathbf{z}|\right)$$

is empty because of (3.4$'$), and the intersection

$$\left[x_{r,R^-}(u),x_{l,R^+}(u)\right]\cap\left(\left[x_{l,R^+}(u),x_{l,R^-}(u)\right]-|\mathbf{z}|\right)$$

is always empty. In such a way we deduced (3.5) from (3.4)—(3.4$''$).

Let us recall the following notation. The point $(x_0(R^+), y_R^+)$ is the point of tangency of the curve γ_{R^+} with the line parallel to $\mathbf{z}$ in the half-space $K_{R^+}^+(\mathbf{z})$. To prove relations (3.4)—(3.4″) let us first observe that because of the first two relations in Lemma 2

$$B_1 R(y_0(R^+0) - u_{R^+}(\mathbf{z}) < \mathbf{z}^2 < B_2 R(y_0(R^+0) - u_{R^+}(\mathbf{z}) \, .$$

In particular, for $|\mathbf{z}| \geq 1$ $\;y_0(R^+) - u_{R^+}(\mathbf{z}) > \dfrac{\text{const.}}{R}$. Hence, by the third and fourth relations in Lemma 2

$$C_1 \frac{R}{|\mathbf{z}|} < -\frac{d}{du}\left(x_{r,R^+}(u) - x_{l,R^+}(u)\right) < C_2 \frac{R}{|\mathbf{z}|} \quad \text{for } u_{R^+}(\mathbf{z}) - \frac{K}{R} < u < u_{R^+}(\mathbf{z}) \tag{3.6}$$

with some $C_2 > C_1 > 0$. Since $x_{r,R^+}(u_{R^+}(\mathbf{z})) - x_{l,R^+}(u_{R^+}(\mathbf{z})) = |\mathbf{z}|$ we get relation (3.4) by integrating the right-hand side of (3.6) in the interval $[u, u_{R^+}(\mathbf{z})]$.

Because of the left-hand side of (3.6) we can choose for any $D > 0$ a number $C = C(D) > 0$ such that for $u_0 = u_{R^+}(\mathbf{z}) - \dfrac{C}{R}$ $x_{r,R^+}(u_0) - x_{l,R^+}(u_0) > |\mathbf{z}| + \frac{D}{|\mathbf{z}|}$. We rewrite this relation by turning from γ_{R^+} to γ_{R^-}. In this calculation we exploit that $|\mathbf{z}| < \text{const.}\, R$ if $\mathbf{z} \in \gamma_{R^+}$. Putting $u_1 = (1-\eta)u_0 = \left(1 - \dfrac{c}{R^2}\right)\left(1 + \dfrac{c}{R^2}\right)^{-1} u_0$ we have $u_1 > u_{R^+}(\mathbf{z}) - \dfrac{K}{R}$ with an appropriate $K > 0$, $\eta = O(R^{-2})$ and

$$\begin{aligned} x_{r,R^-}(u_1) - x_{l,R^-}(u_1) &= (1-\eta)\left(x_{r,R^+}(u_0) - x_{l,R^+}(u_0)\right) \\ &> (1-\eta)\left(|\mathbf{z}| + \frac{D}{|\mathbf{z}|}\right) > |\mathbf{z}| \end{aligned}$$

if $D > 0$ is sufficiently large. This means that relation (3.4′) holds for $u_1 > u_{R^+}(\mathbf{z}) - \dfrac{K}{R}$. Because of the monotonicity of $x_{r,R^-}(u_0) - x_{l,R^-}(u_0)$ for $u_{d,R^-} < u < y_0(R^-)$ relation (3.4′) holds.

To prove relation (3.4″) first we observe that $x_{r,R^+}(\bar{u}) = (1+\beta)x_{r,R^-}(u)$ with $1+\beta = \left(1 + \dfrac{c}{R^2}\right)\left(1 - \dfrac{c}{R^2}\right)^{-1}$ and $\bar{u} = (1+\beta)u$. Hence $x_{r,R^+}(\bar{u}) = x_{r,R^-}(u) + O\left(\dfrac{1}{R}\right)$ and $|u - \bar{u}| < \dfrac{L}{R}$ with some $L > 0$. The derivative $\dfrac{d}{du} x_{r,R^+}(u)$ is a monotone decreasing function of u, hence it follows from the third relation in Lemma 2 that

$$\left|\frac{d}{du} x_{r,R^+}(u)\right| < \frac{R}{3L}$$

if $y_1(R^+) - \dfrac{K}{R} \leq u \leq y_0(R^+) - \dfrac{K}{R}$, and $K > 0$ is chosen sufficiently large. Hence

$$\left|x_{r,R^+}(u) - x_{r,R^+}(\bar{u})\right| < \frac{R}{3L}|u - \bar{u}| + O\left(\frac{1}{R}\right) < \frac{1}{2} \, .$$

The first relation of (3.4″) is proved, and the second one can be proved in the same way.

We have proved relation (3.5). It can be proved in the same way if $K^+_{R^+}(\mathbf{z})$ is replaced by $K^-_{R^+}(\mathbf{z})$. These two relations together imply Part a).

Proof of Part b). We define two parallelograms $P^+(\mathbf{z})$ and $P^-(\mathbf{z})$. The parallelogram $P^+(\mathbf{z})$ is bounded by two pairs of parallel lines, the lines of the first pair are going through the points $\left(1-\frac{c}{R^2}\right)\mathbf{z}^+_{R,l}$ and $\left(1+\frac{c}{R^2}\right)\mathbf{z}^+_{R,l}$ and they have normal $\psi(\mathbf{z}^+_{R,l})$, the lines of the second pair are going through the points

$$\left(1-\frac{c}{R^2}\right)\mathbf{z}^+_{R,r}-\mathbf{z}\quad\text{and}\quad\left(1+\frac{c}{R^2}\right)\mathbf{z}^+_{R,r}-\mathbf{z}\,,$$

and they have normal $\psi(\mathbf{z}^+_{R,r})$. The parallel pairs of lines bounding $P^-(\mathbf{z})$ are going through the points $\left(1-\frac{c}{R^2}\right)\mathbf{z}^-_{R,l}$ and $\left(1+\frac{c}{R^2}\right)\mathbf{z}^-_{R,l}$ with normal $\mathbf{z}^-_{R,l}(\psi)$, and through the points $\left(1-\frac{c}{R^2}\right)\mathbf{z}^-_{R,r}-\mathbf{z}$ and $\left(1+\frac{c}{R^2}\right)\mathbf{z}^-_{R,r}-\mathbf{z}$ with normal $\mathbf{z}^-_{R,r}(\psi)$. The parallelograms $P^+(\mathbf{z})$ and $P^-(\mathbf{z})$ have area $f^+_R(\mathbf{z})$ and $f^-_R(\mathbf{z})$ respectively, and they are disjoint if $|\mathbf{z}|>\eta R$. Since the difference of these parallelograms and the domains $\mathbb{O}_R\cap(\mathbb{O}_R-\mathbf{z})\cap K^+_R(\mathbf{z})$ and $\mathbb{O}_R\cap(\mathbb{O}_R-\mathbf{z})\cap K^-_R(\mathbf{z})$ have an area of order $o\left(R^{-2}\right)$, the above relations imply Part b) of Lemma 1.

Proof of Part c). Let us work in the coordinate system with origin $\mathbf{z}_{d,l}(R^+)$, with x-coordinate axis in the direction $-n(\mathbf{z}_{d,l}(R^+))$, the normal of γ_{R^+} in the point $\mathbf{z}_{d,l}(R^+)$ showing inside the domain $\mathbf{A}_{R^+}$, and with y-axis in the direction $\mathbf{z}_e=-n(\mathbf{z}_{d,l}(R^+))^\perp$, the tangent of γ_{R^+} in this point which is obtained when the x axis is rotated with angle $+\pi/2$. Let $y^\pm_{R^+}(u)$ be the y coordinate of the intersection of the set $\gamma_{R^+}\cap K^\pm_{R^+}(\mathbf{z})$ with the line $x=u$ and $y^\pm_{R^-}(u)$ the y coordinate of the intersection of the set $\gamma_{R^-}\cap K^\pm_{R^-}(\mathbf{z})$ with this line. We shall estimate the length of the intersection of $\mathbb{O}_R$ with the line $x=u$. We shall prove that the Lebesgue measure of this intersection satisfies the inequality

$$\lambda\left(\mathbb{O}_R\cap\{(u,y),\ y\in\mathbb{R}^{(1)}\}\right)<\text{const.}\max\left(\frac{1}{\sqrt{Ru}},1\right)\quad\text{if } 0<u<\eta R \tag{3.7}$$

with some $\eta>0$.

By Lemma 2

$$\left|y^\pm_{R^+}(u)\right|<\text{const.}\sqrt{Ru}\quad\text{if } 0<u<AR\,,$$

and this inequality implies (3.7) in the case $u<\frac{K}{R}$ with some $K>0$. We shall show, using again Lemma 2, arguing similarly as in the proof of relation (3.4) in the proof of Part a) that

$$\left|y^\pm_{R^+}(u)-y^\pm_{R^-}(u)\right|<\frac{\text{const.}}{\sqrt{Ru}}\,,\quad\text{if } \frac{K}{R}<u<\eta R\,, \tag{3.8}$$

which relation completes the proof of (3.7). To prove (3.8) we have to express $y^\pm_{R^-}(\cdot)$ by $y^\pm_{R^+}(\cdot)$ and to exploit that by Lemma 2 $\left|\frac{d}{dv}y^\pm_{R^-}(v)\right|<\text{const.}\frac{\sqrt{R}}{\sqrt{u}}$ if $\frac{K}{R}<u<v<u+\frac{K}{R}$.

Let $\mathbf{v}^+_{R^+}=(v^+_{R^+,1},v^+_{R^+,2})$ and $\mathbf{v}^-_{R^+}=(v^-_{R^+,1},v^-_{R^+,2})$ be the points of intersection of γ_{R^+} and $\gamma_{R^+}-\mathbf{z}$ in the half planes $K^+_{R^+}(\mathbf{z})$ and $K^-_{R^+}(\mathbf{z})$ respectively. We shall prove that there is some constant $K>0$ such that

$$\mathbb{O}_R\cap(\mathbb{O}_R-\mathbf{z})\cap K^+_{R^+}(\mathbf{z})\cap\{(v,y),\ y\in\mathbb{R}^1\}=\emptyset\quad\text{if } v\notin\left[v^+_{R^+,1}-\frac{K}{R},v^+_{R^+,1}+\frac{K}{R}\right] \tag{3.9}$$

and

$$\mathbb{O}_R \cap (\mathbb{O}_R - \mathbf{z}) \cap K^-_{R^+}(\mathbf{z}) \cap \{(v,y),\ y \in \mathbb{R}^1\} = \emptyset \quad \text{if } v \notin \left[v^-_{R^+,1} - \frac{K}{R}, v^-_{R^+,1} + \frac{K}{R} \right] . \tag{3.9$'$}$$

We also claim that under the conditions of Part c1) or Part c2) of Lemma 1

$$v^{\pm}_{R^+,1} > \text{const.}\,(d(\mathbf{z},R) - |\mathbf{z}|) . \tag{3.10}$$

Relations (3.7), (3.9), (3.9′) and (3.10) together imply Part c) of Lemma 1. To prove relation (3.10) let us consider the projection of the vectors $\mathbf{v}^{\pm}_{R^+}$ and $(d(\mathbf{z},R) - |\mathbf{z}|)\dfrac{\mathbf{z}}{|\mathbf{z}|} - \mathbf{v}^{\pm}_{R^+}$ to the direction of the vector $-n(\mathbf{z}_{d,l}(R^+))$. The sum of these two vectors, which is the projection of $d(\mathbf{z},R) - |\mathbf{z}|$ to $-n(\mathbf{z}_{d,l}(R^+))$ is longer than const. $(d(\mathbf{z},R) - |\mathbf{z}|)$ because of relation (3.3). On the other hand, the proportion of the length of these two vectors is separated both from zero and infinity because of relation (3.3) which implies this relation if the projection is done in the orthogonal direction $(d(\mathbf{z},R) - |\mathbf{z}|)^{\perp}$ and Lemma 2.

To prove relations (3.9) and (3.9′) we introduce the following notation. Let $s^{\pm}_{R^{\pm}}(u)$ be the y coordinate of the intersection of the set $(\gamma_{R^{\pm}} - \mathbf{z}) \cap K^{\pm}_{R^{\pm}}(\mathbf{z})$ with the line $x = u$. Since $s^+_{R^+}(v^+_{R^+,2}) = y^+_{R^+}(v^+_{R^+,2})$ we get by expressing $s^+_{R^-}(\cdot)$ through $s^+_{R^+}(\cdot)$, exploiting the lower bound on the derivative of the function $s^+_{R^+}(\cdot)$ given by Lemma 2 and arguing similarly to the proof of relation (3.7) that $s^+_{R^-}(v) > x^+_{R^+}(v)$ or $v < 0$ if $v < v^{\pm}_{R^+,1} - \dfrac{K}{R}$ with some sufficiently large $K > 0$. If $v < 0$, then the set $\mathbb{O}_R \cap \{(v,y),\ y \in \mathbb{R}^1\}$ is empty. Hence

$$\mathbb{O}_R \cap \{(v,y),\ y \in \mathbb{R}^1\} = \emptyset \quad \text{if } v < v^+_{R^+,1} - \frac{K}{R} .$$

By changing the role of γ_R and $\gamma_R - \mathbf{z}$ we get that

$$\mathbb{O}_R \cap \{(v,y),\ y \in \mathbb{R}^1\} = \emptyset \quad \text{if } v > v^+_{R^+,1} + \frac{K}{R} .$$

The last two relations together imply (3.9). The proof of (3.9′) is similar.

In such a way we have proved Part c) of Lemma 1. The proof of Part d) is trivial, since in this case even the set $R^+\mathbf{A} \cap (R^+\mathbf{A} - \mathbf{z})$ is empty. $\square$

4. Some concluding remarks

In this Section we discuss the conjecture about the Poissonian distribution of a randomly placed circle suggested by the computer study of Cheng and Lebowitz [4] and briefly explain what kind of approach is suggested by the present paper.

It is relatively easy to show that the Poissonian limit for the number of lattice points in $\mathbb{O}_R(c,\alpha)$ would follow from the following generalization of formula (2.1):

$$\lim_{R\to\infty} E\xi_R(\alpha)(\xi_R(\alpha)-1)\cdots(\xi_R(\alpha)-k+1) = |\mathbb{O}_R|^k = [4c\,|\mathbf{A}|]^k \quad \text{for all } k \ge 1\ . \tag{4.1}$$

Actually relation (4.1) is equivalent to the statement that all moments of the random variable $\xi_R(a)$ converge to the corresponding moments of a Poissonian random variable with parameter $|\mathbb{O}_R| = 4c\,|\mathbf{A}|$. Some modification of the argument leading to the proof of formula (2.2) gives that

$$\begin{aligned} &E\xi_R(\alpha)(\xi_R(\alpha)-1)\cdots(\xi_R(\alpha)-k+1) \\ &\qquad = \sum_{\substack{\mathbf{m}_1\in\mathbb{Z}^2\setminus\{0\},\dots,\mathbf{m}_{k-1}\in\mathbb{Z}^2\setminus\{0\} \\ \text{the points } \mathbf{m}_1,\dots,\mathbf{m}_{k-1} \text{ are different.}}} |\mathbb{O}_R\cap(\mathbb{O}_R-\mathbf{m}_1)\cap\cdots\cap(\mathbb{O}_R-\mathbf{m}_{k-1})|\ . \end{aligned} \tag{4.2}$$

It is relatively simple to prove the following identity:

$$\int |\mathbb{O}_R\cap(\mathbb{O}_R-\mathbf{z}_1)\cap\cdots\cap(\mathbb{O}_R-\mathbf{z}_{k-1})|\, d\mathbf{z}_1\dots d\mathbf{z}_{k-1} = [4c\,|\mathbf{A}|]^k\ . \tag{4.3}$$

Hence to prove the Poissonian limit it would be enough to show that for large R the replacement of the sum in (4.2) by the integral in (4.3) causes a negligible error for all $k=1,\,2,\,\dots$. Actually, this fact was proved for $k=1$ and 2 in this paper. But the proof for larger k is much harder. In our proof we exploited the independence caused by the random shift $\alpha\in[0,1]^2$. But this independence is not sufficient for the proof of (4.3) if $k\ge 3$. To prove formula (4.3) in this case some deep number theoretical statement would be needed which states that certain functions of the k-tuples of lattice points $(\mathbf{m}_1,\dots,\mathbf{m}_k)$ are almost uniformly distributed. We could give an explicit formulation of this statement, but since this would require complicated notations and would lead to a problem that we cannot handle we omit it.

We also discuss briefly the higher dimensional version of the problem in this paper.

Let $\mathbf{A}$ be a convex set with a nice boundary, $\alpha\in\mathbb{R}^d$ a vector, uniformly distributed in the d-dimensional unit cube, $c>0$ a fixed number and

$$\mathbb{O}_R(c,\alpha) = \left[\left(R+\frac{c}{R^{d-1}}\right)\mathbf{A}-\alpha\right] \setminus \left[\left(R-\frac{c}{R^{d-1}}\right)\mathbf{A}-\alpha\right]\ .$$

The volume of the set $\mathbb{O}_R(c,\alpha)$ is $2cd\,\mathrm{Volume}\,(\mathbf{A})+O\left(\frac{1}{R}\right)$. We are interested in the number of lattice points $\xi_R(\alpha)$ in the randomly shifted set $\mathbb{O}_R(c,\alpha)$. It can be proved that the first two moments of ξ_R tend to the first two moments of a Poissonian random variable with parameter $2d\,\mathrm{Volume}\,(\mathbf{A})$. This can be proved by methods similar to those of the present paper. Moreover, in this case a stronger result can be proved. It can be shown that in the d-dimensional case

$$\lim_{R\to\infty} E\xi_R(\alpha)(\xi_R(\alpha)-1)\cdots(\xi_R(\alpha)-k+1) = [2cd\,\text{Volume}\,(\mathbf{A})]^k \quad \text{for } 1 \le k \le d\,. \tag{4.4}$$

This means that the first d moments of the random variable ξ_R tend to that of a Poissonian random variable with parameter $2cd\,\text{Volume}\,(\mathbf{A})$ as $R \to \infty$. To explain why relation (4.5) holds let us remark that relations (4.2) and (4.3) remain valid for all $d \ge 2$ if the area is replaced by Volume, $\mathbb{Z}^2 \setminus \{0\}$ by $\mathbb{Z}^d \setminus \{0\}$ and $4c\,|\mathbf{A}|$ by $2cd\,\text{Volume}\,(\mathbf{A})$. We have to show that by replacing the sum in (4.2) by the integral in (4.3) we commit a negligible error. Let us also observe that for $k \le d$ the expression

$$h(\mathbf{z}_1,\dots,\mathbf{z}_{k-1}) = \text{Volume}\,(\mathbb{O}_R \cap (\mathbb{O}_R - \mathbf{z}_1) \cap \dots \cap (\mathbb{O}_R - \mathbf{z}_{k-1}))$$

changes very little if the arguments in this expression are changing with an order of constant. Hence the same technique works for $k \le d$, $d > 2$, as for the case $d = 2$ in the second Section of the present paper. Actually some technical difficulties have to be overcome if we want to carry out this program. We do not go into the details.

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