MULTIPLE WIENER–ITÔ INTEGRALS

with applications to limit theorems

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Introduction.

One of the most important problems in probability theory is the investigation of the limit distribution of partial sums of appropriately normalized random variables. The case where the random variables are independent is fairly well understood. Many results are known also in the case where independence is replaced by an appropriate mixing condition or some other "almost independence" property. Much less is known about the limit behaviour of partial sums of really dependent random variables. On the other hand, this case is becoming more and more important, not only in probability theory, but also in some applications in statistical physics.

The problem about the asymptotic behaviour of partial sums of dependent random variables leads to the investigation of some very complicated transformations of probability measures. The classical methods of probability theory do not seem to work for this problem. On the other hand, although we are still very far from a satisfactory solution of this problem, we can already present some nontrivial results.

The so-called multiple Wiener–Itô integrals have proved to be a very useful tool in the investigation of this problem. The proofs of almost all rigorous results in this field are closely related to this technique. The notion of multiple Wiener–Itô integrals was worked out for the investigation of non-linear functionals over Gaussian fields. It is closely related to the so-called Wick polynomials which can be considered as the multi-dimensional generalization of Hermite polynomials. The notion of Wick polynomials and multiple Wiener–Itô integrals were worked out at the same time and independently of each other. Actually, we discuss a modified version of the multiple Wiener–Itô integrals in greatest detail. The technical changes needed in the definition of these modified integrals are not essential. On the other hand, these modified integrals are more appropriate for certain investigations, since they enable us to describe the action of shift transformations and to apply some sort of random Fourier analysis. There is also some connection between multiple Wiener–Itô integrals and the classical stochastic Itô integrals. The main difference between them is that in the first case deterministic functions are integrated, and in the second case so-called non-anticipating functionals. The consequence of this difference is that no technical difficulty arises when we want to define multiple Wiener–Itô integrals in the multi-dimensional time case. On the other hand, a large class of nonlinear functionals over Gaussian fields can be represented by means $I_{n,k}(f)$ of multiple Wiener–Itô integrals.

In this work we are interested in limit problems for sums of dependent random variables. It is useful to consider this problem together with its continuous time version. The natural formulation of the continuous time version of this problem can be given by means of generalized fields. Consequently we also have to discuss some questions about generalized fields.

I have not tried to formulate all the results in the most general form. My main goal was to work out the most important techniques needed in the investigation of such problems. This is the reason why the greatest part of this work deals with multiple Wiener–Itô integrals. I have tried to give a self-contained exposition of this subject and also to explain the motivation behind the results.

I had the opportunity to participate in the Dobrushin–Sinai seminar in Moscow. What I learned there was very useful also for the preparation of this Lecture Note. Therefore I would like to thank the members of this seminar for what I could learn from them, especially P. M. Bleher, R. L. Dobrushin and Ya. G. Sinai.

Some remarks to this text.

This text is a slightly modified version of my Lecture Note *Multiple Wiener–Itô integrals with applications to limit theorems* published in the *Lecture Notes in Mathematics* series (number 849) of the Springer Verlag in 1981. I decided to make a special lecture on the basis of this work in the first semester of the university course in 2011–2012 at the University of Szeged. Preparing for it I observed how difficult the reading of formulas in this Lecture Note is. These difficulties arose because this Lecture Note was written at the time when the $\mathcal{T}_{\mathrm{E}}X$ program still did not exist, and the highest technical level of typing was writing on an IBM machine that enabled one to type beside the usual text also mathematical formulas. But the texts written in such a way are very hard to read. To make my text more readable I decided to retype it by means of the $\mathcal{T}_{\mathrm{E}}X$ program. This demanded some changes. It implied e.g. to follow such partly typographical partly linguistic rules by which one does not start a sentence with a formula. Beside this, it suggested to formulate the basic definitions in a (typographically) more explicit form and not as an explanation inside the text. When typing this work I also tried to rethink what I had written, to correct the errors and to make the proofs more understandable. It was surprising and a little bit shocking to meet my old personality by studying my old Lecture Note and to recognize how much I have changed. Now I would expose many details in a different way. Naturally I would also make many changes by taking into account the results proved since the time I wrote this note. Nevertheless I decided to make no essential changes in the text, to restrict myself to the correction of the errors I found, and to give a more detailed explanation of the proofs where I felt that it is useful. (There were many such places.) In doing so I was influenced by a Russian proverb which says: 'Luchshe vrag khoroshego'. I tried to follow the advice of this proverb. (I do not know of an English counterpart of it, but it has a French version: 'Le mieux est l'ennemi du bien'.)

I made only one exception. I decided to explain those basic notions and results in the theory of generalized functions which were applied in this work in an implicit way. In particular, I tried to explain how one gets with the help of this theory those results about the so-called spectral representation of the covariance function of stationary random fields that I have formulated under the name *Bochner's theorem* and *Bochner–Schwartz theorem.* This extension of the text is contained in the attachments to Sections 1 and 3. In the original version I only referred to a work where these notions and results can be found. But now I found such an approach not satisfactory, because these notions and results play an important role in some arguments of this work. Hence I felt that to make a self-contained presentation of the subject I have to explain them in more detail.

Budapest, 15 August 2011

Péter Major

1. On a limit problem.

We begin with the formulation of a problem which is important both for probability theory and statistical physics. The multiple Wiener–Itô integral proved to be a very useful tool at the investigation of this problem.

We shall consider a set of random variables ξ_n, $n \in \mathbb{Z}_\nu$, where $\mathbb{Z}_\nu$ denotes the ν-dimensional integer lattice, and we shall study their properties. Such a set of random variables will be called a discrete (ν-dimensional) random field. We shall be mainly interested in so-called stationary random fields. Let us recall their definition.

Definition of discrete (strictly) stationary random fields. *A set of random variables ξ_n, $n \in \mathbb{Z}_\nu$, is called a (strictly) stationary discrete random field if $(\xi_{n_1}, \dots, \xi_{n_k}) \stackrel{\Delta}{=} (\xi_{n_1+m}, \dots, \xi_{n_k+m})$ for all $k = 1, 2, \dots$ and $n_1, \dots, n_k$, $m \in \mathbb{Z}_\nu$, where $\stackrel{\Delta}{=}$ denotes equality in distribution.*

Let us also recall that a discrete random field ξ_n, $n \in \mathbb{Z}_\nu$, is called Gaussian if for every finite subset $\{n_1, \dots, n_k\} \subset \mathbb{Z}_\nu$ the random vector $(\xi_{n_1}, \dots, \xi_{n_k})$ is normally distributed.

Given a discrete random field ξ_n, $n \in \mathbb{Z}_\nu$, we define for all $N = 1, 2, \dots$ the new random fields

$$Z_n^N = A_N^{-1} \sum_{j \in B_n^N} \xi_j, \qquad N = 1, 2, \dots, \quad n \in \mathbb{Z}_\nu, \tag{1.1}$$

where

$$B_n^N = \{j\colon\ j \in \mathbb{Z}_\nu, \quad n^{(i)}N \le j^{(i)} < (n^{(i)} + 1)N,\ i = 1, 2, \dots, \nu\},$$

and A_N, $A_N > 0$, is an appropriate norming constant. The superscript i denotes the i-th coordinate of a vector in this formula. We are interested in the question when the finite dimensional distribution of the random fields Z_n^N defined in (1.1) have a limit as $N \to \infty$. In particular, we would like to describe those random fields Z_n^*, $n \in \mathbb{Z}_\nu$, which appear as the limit of such random fields Z_n^N. This problem led to the introduction of the following notion.

Definition of self-similar (discrete) random fields. *A (discrete) random field ξ_n, $n \in \mathbb{Z}_\nu$, is called self-similar with self-similarity parameter α if the random fields Z_n^N defined in (1.1) with their help and the choice $A_N = N^\alpha$ satisfy the relation*

$$(\xi_{n_1}, \dots, \xi_{n_k}) \stackrel{\Delta}{=} (Z_{n_1}^N, \dots, Z_{n_k}^N) \tag{1.2}$$

for all $N = 1, 2, \dots$ and $n_1, \dots, n_k \in \mathbb{Z}_\nu$.

We are interested in the choice $A_N = N^\alpha$ with some $\alpha > 0$ in the definition of the random variables Z_n^N in (1.2), because under slight restrictions, relation (1.2) can be satisfied only with such norming constants A_N. A central problem both in statistical physics and in probability theory is the description of self-similar fields. We are interested in self-similar fields whose random variables have a finite second moment.

This excludes the fields consisting of i.i.d. random variables with a non–Gaussian stable law.

The Gaussian self-similar fields and their Gaussian range of attraction are fairly well known. Much less is known about the non-Gaussian case. The problem is hard, because the transformations of measures over $R^{\mathbb{Z}_\nu}$ induced by formula (1.1) have a very complicated structure. We shall define the so-called subordinated fields below. (More precisely the fields subordinated to a stationary Gaussian field.) In case of subordinated fields the Wiener–Itô integral is a very useful tool for investigating the transformation defined in (1.1). In particular, it enables us to construct non–Gaussian self-similar fields and to prove non-trivial limit theorems. All known results are closely related to this technique.

Let X_n, $n \in \mathbb{Z}_\nu$, be a stationary Gaussian field. We define the shift transformations T_m, $m \in \mathbb{Z}_\nu$, over this field by the formula $T_m X_n = X_{n+m}$ for all n, $m \in \mathbb{Z}_\nu$. Let $\mathcal{H}$ denote the *real* Hilbert space consisting of the square integrable random variables measurable with respect to the σ-algebra $\mathcal{B} = \mathcal{B}(X_n,\ n \in \mathbb{Z}_\nu)$. The scalar product in $\mathcal{H}$ is defined as $(\xi, \eta) = E\xi\eta$, ξ, $\eta \in \mathcal{H}$. The shift transformations T_m, $m \in \mathbb{Z}_\nu$, can be extended to a group of unitary shift transformations over $\mathcal{H}$ in a natural way. Namely, if $\xi = f(X_{n_1}, \dots, X_{n_k})$ then we define $T_m\xi = f(X_{n_1+m}, \dots, X_{n_k+m})$. It can be seen that $\|\xi\| = \|T_m\xi\|$, and the above considered random variables ξ are dense in $\mathcal{H}$. (A more detailed discussion about the definition of shift operators and their properties will be given in Section 2 in a *Remark* after the formulation of Theorem 2C. Here we shall define the shift $T_m\xi$, $m \in \mathbb{Z}_\nu$, of all random variables ξ which are measurable with respect to the σ-algebra $\mathcal{B}(X_n,\ n \in Z_\nu)$, i.e. ξ does not have to be square integrable.) Hence $\|T_m\|$ can be extended to the whole space $\mathcal{H}$ by L_2 continuity. It can be proved that the norm preserving transformations T_m, $m \in \mathbb{Z}_\nu$, constitute a unitary group in $\mathcal{H}$, i.e. $T_{n+m} = T_n T_m$ for all n, $m \in \mathbb{Z}_\nu$, and $T_0 = \mathrm{Id}$. Now we introduce the following

Definition of subordinate fields. *Given a stationary Gaussian field X_n, $n \in \mathbb{Z}_\nu$, we define the Hilbert spaces $\mathcal{H}$ and the shift transformations T_m, $m \in \mathbb{Z}_\nu$, over $\mathcal{H}$ as before. A discrete stationary field ξ_n is called a random field subordinated to X_n if $\xi_n \in \mathcal{H}$, and $T_n\xi_m = \xi_{n+m}$ for all n, $m \in \mathbb{Z}_\nu$.*

We remark that ξ_0 determines the subordinated fields ξ_n completely, since $\xi_n = T_n\xi_0$. Later we give a more adequate description of subordinates fields by means of Wiener–Itô integrals. Before working out the details we formulate the continuous time version of the above notions and problems. In the continuous time case it is more natural to consider generalized random fields. To explain the idea behind such an approach we shortly explain a different but equivalent description of discrete random fields. We present them as an appropriate set of random variables indexed by the elements of a linear space. This shows some similarity with generalized random fields.

Let $\varphi_n(x)$, $n \in \mathbb{Z}_\nu$, $n = (n_1, \dots, n_\nu)$, denote the indicator function of the cube $[n_1 - \frac{1}{2}, n_1 + \frac{1}{2}) \times \cdots \times [n_\nu - \frac{1}{2}, n_\nu + \frac{1}{2})$, with center $n = (n_1, \dots, n_\nu)$ and with edges of length 1, ie. let $\varphi_n(x) = 1$, $x = (x_1, \dots, x_\nu) \in R^\nu$, if $n_j - \frac{1}{2} \le x_j < n_j + \frac{1}{2}$ for all $1 \le j \le \nu$, and let $\varphi_n(x) = 0$ otherwise. Define the linear space Φ of functions on R^ν consisting of all finite linear combinations of the form $\sum c_j \varphi_{n_j}(x)$, $n_j \in \mathbb{Z}_\nu$, with the

above defined functions $\varphi_n(x)$ and real coefficients c_j. Given a discrete random field ξ_n, $n \in \mathbb{Z}_\nu$, define the random variables $\xi(\varphi)$ for all $\varphi \in \Phi$ in the following way. Put $\xi(\varphi) = \sum c_j \xi_{n_j}$ if $\varphi(x) = \sum c_j \varphi_{n_j}(x)$. In particular, $\xi(\varphi_n) = \xi_n$ for all $n \in \mathbb{Z}_\nu$. The identity $\xi(c_1\varphi + c_2\psi) = c_1\xi(\varphi) + c_2\xi(\psi)$ also holds for all $\varphi, \psi \in \Phi$ and real numbers c_1 and c_2.

Let us also define the function $\varphi^{(N,A_N)}(x) = \frac{1}{A(N)}\varphi(\frac{x}{N})$ for all functions $\varphi \in \Phi$ and positive integers $N = 1, 2, \dots$, with some appropriately chosen constants $A_N > 0$. Observe that $\xi(\varphi_n^{(N,A_N)}) = Z_n^N$ with the random variable Z_n^N defined in (1.1). All previously introduced notions related to discrete random fields can be reformulated with the help of the set of random variables $\xi(\varphi)$, $\varphi \in \Phi$. Thus for instance the random field ξ_n, $n \in \mathbb{Z}_\nu$ is self-similar with self-similarity parameter α if and only if $\xi(\varphi^{(N,N^\alpha)}) \stackrel{\Delta}{=} \xi(\varphi)$ for all $\varphi \in \Phi$ and $N = 1, 2, \dots$. (To see why this statement holds observe that the distributions of two random vectors agree if and only if every linear combination of their coordinates have the same distribution. This follows from the fact that the characteristic function of a random vector determines its distribution.)

It will be more useful to define the continuous time version of discrete random fields as generalized random fields. The generalized random fields will be defined as a set of random variables indexed by the elements of a linear space of functions. They show some similarity to the class of random variables $\xi(\varphi)$, $\varphi \in \Phi$, defined above. The main difference is that instead of the space Φ a different linear space is chosen for the parameter set of the random field. We shall choose the so-called Schwartz space for this role.

Let $\mathcal{S} = \mathcal{S}_\nu$ be the Schwartz space of (real valued) rapidly decreasing, smooth functions on R^ν. (See e.g. [15] for the definition of $\mathcal{S}_\nu$. I shall present a more detailed discussion about the definition of the space $\mathcal{S}$ in the adjustment to Section 1.) Generally one takes the space of complex valued, rapidly decreasing, smooth functions as the space $\mathcal{S}$, but we shall denote the space of *real valued*, rapidly decreasing, smooth functions by $\mathcal{S}$ if we do not say this otherwise. We shall omit the subscript ν if it leads to no ambiguity. Now we introduce the notion of generalized random fields.

Definition of generalized random fields. *We say that the set of random variables $X(\varphi)$, $\varphi \in \mathcal{S}$, is a generalized random field over the Schwartz space $\mathcal{S}$ of rapidly decreasing, smooth functions if:*

a) $X(a_1\varphi_1 + a_2\varphi_2) = a_1X(\varphi_1) + a_2X(\varphi_2)$ *with probability 1 for all real numbers a_1 and a_2 and $\varphi_1 \in \mathcal{S}$, $\varphi_2 \in \mathcal{S}$. (The exceptional set of probability 0 where this identity does not hold may depend on a_1, a_2, φ_1 and φ_2.)*

b) $X(\varphi_n) \Rightarrow X(\varphi)$ *stochastically if $\varphi_n \to \varphi$ in the topology of $\mathcal{S}$.*

We also introduce the following definitions.

Definition of stationarity and Gaussian property of a generalized random field. On the notion of convergence of generalized random fields in distribution. *The generalized random field $X = \{X(\varphi), \varphi \in \mathcal{S}\}$ is stationary if $X(\varphi) \stackrel{\Delta}{=} X(T_t\varphi)$ for all $\varphi \in \mathcal{S}$ and $t \in R^\nu$, where $T_t\varphi(x) = \varphi(x-t)$. It is Gaussian if $X(\varphi)$ is a Gaussian*

random variable for all $\varphi \in \mathcal{S}$. *The relation* $X_n \xrightarrow{\mathcal{D}} X_0$ *as* $n \to \infty$ *holds for a sequence of generalized random fields* X_n, $n = 0, 1, 2, \dots$, *if* $X_n(\varphi) \xrightarrow{\mathcal{D}} X_0(\varphi)$ *for all* $\varphi \in \mathcal{S}$, *where* $\xrightarrow{\mathcal{D}}$ *denotes convergence in distribution.*

Given a stationary generalized random field X and a function $A(t) > 0$, $t > 0$, on the set of positive real numbers we define the (stationary) random fields X_t^A for all $t > 0$ by the formula

$$X_t^A(\varphi) = X(\varphi_t^A), \quad \varphi \in \mathcal{S}, \qquad \text{where } \varphi_t^A(x) = A(t)^{-1}\varphi\left(\frac{x}{t}\right). \tag{1.3}$$

We are interested in the following

Question. *When does a generalized random field* X^* *exist such that* $X_t^A \xrightarrow{\mathcal{D}} X^*$ *as* $t \to \infty$ *(or as* $t \to 0$*)?*

In relation to this question we introduce the following

Definition of self-similarity. *The stationary generalized random field* X *is self-similar with self-similarity parameter* α *if* $X_t^A(\varphi) \stackrel{\Delta}{=} X(\varphi)$ *for all* $\varphi \in \mathcal{S}$ *and* $t > 0$ *with the function* $A(t) = t^\alpha$.

To answer the above question one should first describe the generalized self-similar random fields.

We try to explain the motivation behind the above definitions. Given an ordinary random field $X(t)$, $t \in R^\nu$, and a topological space $\mathcal{E}$ consisting of functions over R^ν one can define the random variables $X(\varphi) = \int_{R^\nu} \varphi(t)X(t)\,dt$, $\varphi \in \mathcal{E}$. Some difficulty may arise when defining this integral, but it can be overcome in all interesting cases. If the space $\mathcal{E}$ is rich enough, and this is the case if $\mathcal{E} = \mathcal{S}$, then the integrals $X(\varphi)$, $\varphi \in \mathcal{E}$, determine the random process $X(t)$. The set of random variables $X(\varphi)$, $\varphi \in \mathcal{S}$, is a generalized random field in all nice cases. On the other hand, there are generalized random fields which cannot be obtained by integrating ordinary random fields. In particular, the generalized self-similar random fields we shall construct later cannot be interpreted through ordinary fields. The above definitions of various properties of generalized fields are fairly natural, considering what these definitions mean for generalized random fields obtained by integrating ordinary fields.

The investigation of generalized random fields is simpler than that of ordinary discrete random fields, because in the continuous case more symmetry is available. Moreover, in the study or construction of discrete random fields generalized random fields may play a useful role. To understand this let us remark that if we have a generalized random field $X(\varphi)$, $\varphi \in \mathcal{S}$, and we can extend the space $\mathcal{S}$ containing the test function φ to such a larger linear space $\mathcal{T}$ for which $\Phi \subset \mathcal{T}$ with the above introduced linear space Φ, then we can define the discrete random field $X(\varphi)$, $\varphi \in \Phi$, by a restriction of the space of test functions of the generalized random field $X(\varphi)$, $\varphi \in \mathcal{T}$. This random field can be considered as the discretization of the original generalized random field $X(\varphi)$, $\varphi \in \mathcal{S}$.

We finish this section by defining the generalized subordinated random fields. Let $X(\varphi)$, $\varphi \in \mathcal{S}$, be a generalized stationary Gaussian random field. The formula $T_t X(\varphi)) = X(T_t\varphi)$, $T_t\varphi(x) = \varphi(x-t)$, defines the shift transformation for all $t \in R^\nu$. Let $\mathcal{H}$ denote the real Hilbert space consisting of the $\mathcal{B} = \mathcal{B}(X(\varphi),\ \varphi \in \mathcal{S})$ measurable random variables with finite second moment. The shift transformation can be extended to a group of unitary transformations over $\mathcal{H}$ similarly to the discrete case.

Definition of generalized random fields subordinated to a generalized stationary Gaussian random field. *Given a generalized stationary Gaussian random field $X(\varphi)$, $\varphi \in \mathcal{S}$, we define the Hilbert space $\mathcal{H}$ and the shift transformations T_t, $t \in R^\nu$, over $\mathcal{H}$ as above. A generalized stationary random field $\xi(\varphi)$, $\varphi \in \mathcal{S}$, is subordinated to the field $X(\varphi)$, $\varphi \in \mathcal{S}$, if $\xi(\varphi) \in \mathcal{H}$ and $T_t\xi(\varphi) = \xi(T_t\varphi)$ for all $\varphi \in \mathcal{S}$ and $t \in R^\nu$, and $E[\xi\varphi_n) - \xi(\varphi)]^2 \to 0$ if $\varphi_n \to \varphi$ in the topology of $\mathcal{S}$.*

Attachment to Section 1. *A brief overview about some results on generalized functions.*

Let us first describe the Schwartz spaces $\mathcal{S}$ and $\mathcal{S}^c$ in more detail. The space $\mathcal{S}^c = (\mathcal{S}_\nu)^c$ consists of those complex valued functions of ν variables which decrease at infinity, together with their derivatives, faster than any polynomial degree. More explicitly, $\varphi \in \mathcal{S}^c$ for a complex valued function φ of ν variables if

$$\left| x_1^{k_1} \cdots x_\nu^{k_\nu} \frac{\partial^{q_1+\cdots+q_\nu}}{\partial x_1^{q_1} \dots \partial x_\nu^{q_\nu}} \varphi(x_1, \dots, x_\nu) \right| \le C(k_1, \dots, k_\nu, q_1, \dots, q_\nu)$$

for all point $x = (x_1, \dots, x_\nu) \in R^\nu$ and vectors $(k_1, \dots, k_\nu)$, $(q_1, \dots, q_\nu)$ with non-negative integer coordinates with some constant $C(k_1, \dots, k_\nu, q_1, \dots, q_\nu)$ which may depend on the function φ. This formula can be written in a more concise form as

$$|x^k D^q \varphi(x)| \le C(k, q) \quad \text{with } k = (k_1, \dots, k_\nu) \text{ and } q = (q_1, \dots, q_\nu),$$

where $x = (x_1, \dots, x_\nu)$, $x^k = x_1^{k_1} \cdots x_\nu^{k_\nu}$ and $D^q = \frac{\partial^{q_1+\cdots+q_\nu}}{\partial x_1^{q_1} \dots \partial x_\nu^{q_\nu}}$. The elements of the space $\mathcal{S}$ are defined similarly, with the only difference that they are real valued functions.

To define the spaces $\mathcal{S}$ and $\mathcal{S}^c$ we still have to define the convergence in them. We say that a sequence of functions $\varphi_n \in \mathcal{S}^c$ (or $\varphi_n \in \mathcal{S}$) converges to a function φ if

$$\lim_{n\to\infty} \sup_{x \in R^\nu} (1 + |x|^2)^k |D^q \varphi_n(x) - D^q \varphi(x)| = 0.$$

for all $k = 1, 2, \dots$ and $q = (q_1, \dots, q_\nu)$. It can be seen that the limit function φ is also in the space $\mathcal{S}^c$ (or in the space $\mathcal{S}$).

A nice topology can be introduced in the space $\mathcal{S}^c$ (or $\mathcal{S}$) which induces the above convergence. The following topology is an appropriate choice. Let a basis of neighbourhoods of the origin consist of the sets

$$U(k, q, \varepsilon) = \left\{ \varphi\colon \max_x (1 + |x|^2)^k |D^q \varphi(x)| < \varepsilon \right\}$$

with $k = 0, 1, 2, \dots$, $q = (q_1, \dots, q_\nu)$ with non-negative integer coordinates and $\varepsilon > 0$, where $|x|^2 = x_1^2 + \cdots + x_\nu^2$. A basis of neighbourhoods of an arbitrary function $\varphi \in \mathcal{S}^c$ (or $\varphi \in \mathcal{S}$) consists of sets of the form $\varphi + U(k, q, \varepsilon)$, where the class of sets $U(k, q, \varepsilon)$ is a basis of neighbourhood of the origin. The fact that the convergence in $\mathcal{S}$ has such a representation, (and a similar result holds in some other spaces studied in the theory of generalized functions) has a great importance in the theory of generalized functions. We also have exploited this fact in Section 6 of this Lecture Note. Topological spaces with such a topology are called countably normed spaces.

The space of generalized functions $\mathcal{S}'$ consists of the *continuous* linear maps $F\colon \mathcal{S} \to C$ or $F\colon \mathcal{S}^c \to C$, where C denotes the linear space of complex numbers. (In the study of the space $\mathcal{S}'$ we omit the upper index c, i.e. we do not indicate whether we are working in real or complex space when this causes no problem.) We shall write the map $F(\varphi)$, $F \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$ (or $\varphi \in \mathcal{S}^c$) in the form (F, φ).

We can define generalized functions $F \in \mathcal{S}'$ by the formula $(F, \varphi) = \int \overline{f(x)}\varphi(x)\,dx$ for all $\varphi \in \mathcal{S}$ or $\varphi \in \mathcal{S}^c$ with a function f such that $\int (1+|x|^2)^{-p}|f(x)|\,dx < \infty$ with some $p \geq 0$. (The upper script $^-$ denotes complex conjugate in the sequel.) Such functionals are called regular. There are also non-regular functionals in the space $\mathcal{S}'$. An example for them is the δ-function defined by the formula $(\delta, \varphi) = \varphi(0)$. There is a rather good description of the generalized functions $F \in \mathcal{S}'$, (see the book I. M. Gelfand and G. E. Shilov: Generalized functions, Volume 2, Chapter 2, Section 4), but we do not need this result, hence we do not discuss it here. Another important question in this field that we omit is about the interpretation of a usual function as a generalized function in the case when it does not define a regular functional because of its strong singularity in some points. In such cases some regularization can be applied. It is an important problem to find the appropriate generalized functions in such cases, but it does not appear in the study of the problems of this work.

The derivative and the Fourier transform of generalized functions are also defined, and they play an important role in some investigations. In the definition of these notions for generalized functions we want to preserve the old definition if nice regular functionals are considered for which these notions were already defined in classical analysis. Such considerations lead to the definition $(\frac{\partial_j}{\partial x_j} F, \varphi) = -(F, \frac{\partial \varphi}{\partial x_j})$ of the derivative of generalized functions. We do not discuss this definition in more detail, because here we do not work with the derivatives of generalized functions.

The Fourier transform of generalized functions in S' appears in our discussion, although only in an implicit form. The Bochner-Schwartz theorem discussed in Section 3 actually deals with the Fourier transform of generalized functions. Hence the definition of Fourier transform will be given in more detail.

We shall define the Fourier transform of a generalized function by means of a natural extension of the Parseval formula, more explicitly of a simplified version of it, where the same identity

$$\int_{R^\nu} \overline{f(x)} g(x)\,dx = \frac{1}{(2\pi)^\nu} \int_{R^\nu} \overline{\tilde{f}(u)} \tilde{g}(u)\,du$$

is formulated with $\tilde{f}(u) = \int_{R^\nu} e^{i(u,x)} f(x)\,dx$ and $\tilde{g}(u) = \int_{R^\nu} e^{i(u,x)} g(x)\,dx$. But now we

consider a pair of functions (f, g) with different properties. We demand that f should be an integrable function, and $g \in \mathcal{S}^c$. (In the original version of the Parseval formula both f and g are L_2 functions.)

The proof of this identity is simple. Indeed, since the function $g \in \mathcal{S}^c$ can be calculated as the inverse Fourier transform of its Fourier transform $\tilde{g} \in \mathcal{S}^c$, i.e. $g(x) = \frac{1}{(2\pi)^\nu} \int e^{-i(u,x)} \tilde{g}(u)\, du$, we can write

$$\begin{aligned}\int \overline{f(x)} g(x)\, dx &= \int \overline{f(x)} \left[\frac{1}{(2\pi)^\nu} \int e^{-i(u,x)} \tilde{g}(u)\, du \right] dx \\ &= \int \tilde{g}(u) \left[\frac{1}{(2\pi)^\nu} \int \overline{e^{i(u,x)} f(x)}\, dx \right] du = \frac{1}{(2\pi)^\nu} \int \overline{\tilde{f}(u)} \tilde{g}(u)\, du.\end{aligned}$$

Let us also remark that the Fourier transform $f \to \tilde{f}$ is a bicontinuous map from $\mathcal{S}^c$ to $\mathcal{S}^c$. (This means that this transformation is invertible, and both the Fourier transform and its inverse are continuous maps from $\mathcal{S}^c$ to $\mathcal{S}^c$.) (The restriction of the Fourier transform to the space $\mathcal{S}$ of real valued functions is a bicontinuous map from $\mathcal{S}$ to the subspace of $\mathcal{S}^c$ consisting of those functions $f \in \mathcal{S}^c$ for which $f(-x) = \overline{f(x)}$ for all $x \in R^\nu$.)

The above results make natural the following definition of the Fourier transform $\tilde{F}$ of a generalized function $F \in \mathcal{S}'$.

$$(\tilde{F}, \tilde{\varphi}) = (2\pi)^\nu (F, \varphi) \quad \text{for all } \varphi \in \mathcal{S}^c.$$

Indeed, if $F \in \mathcal{S}'$ then $\tilde{F}$ is also a continuous linear map on $\mathcal{S}^c$, i.e. it is also an element of $\mathcal{S}'$. Beside this, the above proved version of the Parseval formula implies that if we consider an integrable function f on R^ν both as a usual function and as a (regular) generalized function, its Fourier transform agrees in the two cases.

There are other classes of test functions and spaces of generalized functions studied in the literature. The most popular among them is the space $\mathcal{D}$ of infinitely many differentiable functions with compact support and its dual space $\mathcal{D}'$, the space of continuous linear transformations on the space $\mathcal{D}$. (These spaces are generally denoted by $\mathcal{D}$ and $\mathcal{D}'$ in the literature, although just the book [15] that we use as our main reference in this subject applies the notation $\mathcal{K}$ and $\mathcal{K}'$ for them.) We shall discuss this space only very briefly.

The space $\mathcal{D}$ consists of the infinitely many times differentiable functions with compact support. Thus it is a subspace of $\mathcal{S}$. A sequence $\varphi_n \in \mathcal{D}$, $n = 1, 2, \dots$, converges to a function φ, if there is a compact set $A \subset R^\nu$ which is the support of all these functions φ_n, and the functions φ_n together with all their derivatives converge uniformly to the function φ and to its corresponding derivatives. It is not difficult to see that also $\varphi \in \mathcal{D}$, and if the functions φ_n converge to φ in the space $\mathcal{D}$, then they also converge to φ in the space $\mathcal{S}$. Moreover, $\mathcal{D}$ is an everywhere dense subspace of $\mathcal{S}$. The space $\mathcal{D}'$ consists of the continuous linear functionals in $\mathcal{D}$.

The results describing the behaviour of $\mathcal{D}$ and $\mathcal{D}'$ are very similar to those describing the behaviour of $\mathcal{S}$ and $\mathcal{S}'$. There is one difference that deserves some attention. The

Fourier transforms of the functions in $\mathcal{D}$ may not belong to $\mathcal{D}$. The class of these Fourier transforms can be described by means of some results in complex analysis. A topological space $\mathcal{Z}$ can be defined on the set of Fourier transforms of the functions from the space $\mathcal{D}$. If we want to apply Fourier analysis in the space $\mathcal{D}$, then we also have to study this space $\mathcal{Z}$ and its dual space $\mathcal{Z}'$. I omit the details.

2. Wick polynomials.

In this section we consider the so-called Wick polynomials, a multi-dimensional generalization of Hermite polynomials. They are closely related to multiple Wiener–Itô integrals.

Let X_t, $t \in T$, be a set of jointly Gaussian random variables indexed by a parameter set T. Let $EX_t = 0$ for all $t \in T$. We define the real Hilbert space $\mathcal{H}_1$ and $\mathcal{H}$ in the following way: A square integrable random variable is in $\mathcal{H}$ if and only if it is measurable with respect to the σ-algebra $\mathcal{B} = \mathcal{B}(X_t,\ t \in T)$, and the scalar product in $\mathcal{H}$ is defined as $(\xi, \eta) = E\xi\eta$, $\xi,\ \eta \in \mathcal{H}$. The Hilbert space $\mathcal{H}_1 \subset \mathcal{H}$ is the subspace of $\mathcal{H}$ generated by the finite linear combinations $\sum c_j X_{t_j}$, $t_j \in T$. We consider only such sets of Gaussian random variables X_t for which $\mathcal{H}_1$ is separable. Otherwise X_t, $t \in T$, can be arbitrary, but the most interesting case for us is when $T = \mathcal{S}_\nu$ or $\mathbb{Z}_\nu$, and X_t, $t \in T$, is a stationary Gaussian field.

Let $Y_1, Y_2, \dots$ be an orthonormal basis in $\mathcal{H}_1$. The uncorrelated random variables $Y_1, Y_2, \dots$ are independent, since they are (jointly) Gaussian. Moreover, $\mathcal{B}(Y_1, Y_2, \dots) = \mathcal{B}(X_t,\ t \in T)$. Let $H_n(x)$ denote the n-th Hermite polynomial with leading coefficient 1, i.e. let $H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2})$. We recall the following results from analysis and measure theory.

Theorem 2A. *The Hermite polynomials $H_n(x)$, $n = 0, 1, 2, \dots$, form a complete orthogonal system in $L_2\left(R, \mathcal{B}, \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right)$. (Here $\mathcal{B}$ denotes the Borel σ-algebra on the real line.)*

Let $(X_j, \mathcal{X}_j, \mu_j)$, $j = 1, 2, \dots$, be countably many independent copies of a probability space $(X, \mathcal{X}, \mu)$. (We denote the points of X_j by x_j.) Let $(X^\infty, \mathcal{X}^\infty, \mu^\infty) = \prod_{j=1}^{\infty} (X_j, \mathcal{X}_j, \mu_j)$. With such a notation the following result holds.

Theorem 2B. *Let $\varphi_0, \varphi_1, \dots$, $\varphi_0(x) \equiv 1$, be a complete orthonormal system in the Hilbert space $L_2(X, \mathcal{X}, \mu)$. Then the functions $\prod_{j=1}^{\infty} \varphi_{k_j}(x_j)$, where only finitely many indices k_j differ from 0, form a complete orthonormal basis in $L_2(X^\infty, \mathcal{X}^\infty, \mu^\infty)$.*

Theorem 2C. *Let $Y_1, Y_2, \dots$ be random variables on a probability space $(\Omega, \mathcal{A}, P)$ taking values in a measurable space $(X, \mathcal{X})$. Let ξ be a real valued random variable measurable with respect to the σ-algebra $\mathcal{B}(Y_1, Y_2, \dots)$, and let $(X^\infty, \mathcal{X}^\infty)$ denote the infinite product $(X \times X \times \cdots, \mathcal{X} \times \mathcal{X} \times \cdots)$ of the space $(X, \mathcal{X})$ with itself. Then there exists a real valued, measurable function f on the space $(X^\infty, \mathcal{X}^\infty)$ such that $\xi = f(Y_1, Y_2, \dots)$.*

Remark. Let us have a stationary random field $X_n(\omega)$, $n \in \mathbb{Z}_\nu$. Theorem 2C enables us to extend the shift transformation T_m, defined as $T_m X_n(\omega) = X_{n+m}(\omega)$, n, $m \in \mathbb{Z}_\nu$, for all random variables $\xi(\omega)$, measurable with respect to the σ-algebra $\mathcal{B}(X_n(\omega),\, n \in \mathbb{Z}_\nu)$. Indeed, by Theorem 2C we can write $\xi(\omega) = f(X_n(\omega),\, n \in \mathbb{Z}_\nu)$, and define $T_m\xi(\omega) = f(X_{n+m}(\omega),\, n \in \mathbb{Z}_\nu)$. We still have to understand, that although the function f is not unique in the representation of the random variable $\xi(\omega)$, the above definition of $T_m\xi(\omega)$ is meaningful. To see this we have to observe that if $f_1(X_n(\omega),\, n \in \mathbb{Z}_\nu) = f_2(X_n(\omega),\, n \in \mathbb{Z}_\nu)$ for two functions f_1 and f_2 with probability 1, then also $f_1(X_{n+m}(\omega),\, n \in \mathbb{Z}_\nu) = f_2(X_{n+m}(\omega),\, n \in \mathbb{Z}_\nu)$ with probability 1 because of the stationarity of the random field $X_n(\omega)$, $n \in \mathbb{Z}_\nu$. Let us also observe that $\xi(\omega) \stackrel{\Delta}{=} T_m\xi(\omega)$ for all $m \in \mathbb{Z}_\nu$. Beside this, T_m is a linear operator on the linear space of random variables, measurable with respect to the σ-algebra $\mathcal{B}(X_n,\, n \in \mathbb{Z}_\nu)$. If we restrict it to the space of square integrable random variables, then T_m is a unitary operator, and the operators T_m, $m \in \mathbb{Z}_\nu$, constitute a unitary group.

Let a stationary generalized field $X = \{X(\varphi),\, \varphi \in \mathcal{S}\}$ be given. The shift $T_t\xi$ of a random variable ξ, measurable with respect to the σ-algebra $\mathcal{B}(X(\varphi),\, \varphi \in \mathcal{S})$ can be defined for all $t \in R^\nu$ similarly to the discrete case with the help of Theorem 2C and the following observation: If $\xi \in \mathcal{B}(X(\varphi),\, \varphi \in \mathcal{S})$ for a random variable ξ, then there exists such a countable subset $\{\varphi_1, \varphi_2, \dots\} \subset \mathcal{S}$ (depending on the random variable ξ) for which ξ is $\mathcal{B}(X(\varphi_1), X(\varphi_2), \dots)$ measurable. (We write $\xi(\omega) = f(X(\varphi_1)(\omega), X(\varphi_2)(\omega), \dots)$ with appropriate functions f, and $\varphi_1 \in \mathcal{S}$, $\varphi_2 \in \mathcal{S}$, $\dots$, and define the shift $T_t\xi$ as $T_t\xi(\omega) = f(X(T_t\varphi_1)(\omega), X(T_t\varphi_2)(\omega), \dots)$, where $T_t\varphi(x) = \varphi(x-t)$ for $\varphi \in \mathcal{S}$.) The transformations T_t, $t \in R^\nu$, are linear operators over the space of random variables measurable with respect to the σ-algebra $\mathcal{B}(X(\varphi),\, \varphi \in \mathcal{S})$ with similar properties as their discrete counterpart.

Theorems 2A, 2B and 2C have the following important consequence.

Theorem 2.1. *Let $Y_1, Y_2, \dots$ be an orthonormal basis in the Hilbert space $\mathcal{H}_1$ defined above with the help of a set of Gaussian random variables X_t, $t \in T$. Then the set of all possible finite products $H_{j_1}(Y_{l_1}) \cdots H_{j_k}(Y_{l_k})$ is a complete orthogonal system in the Hilbert space $\mathcal{H}$ defined above. (Here $H_j(\cdot)$ denotes the j-th Hermite polynomial.)*

The proof of Theorem 2.1. By Theorems 2A and 2B the set of all possible products $\prod_{j=1}^{\infty} H_{k_j}(x_j)$, where only finitely many indices k_j differ from 0, is a complete orthonormal system in $L_2\left(R^\infty, \mathcal{B}^\infty, \prod_{j=1}^{\infty} \frac{e^{-x_j^2/2}}{\sqrt{2\pi}}\, dx_j\right)$. Since $\mathcal{B}(X_t,\ t \in T) = \mathcal{B}(Y_1, Y_2, \dots)$, Theorem 2C implies that the mapping $f(x_1, x_2, \dots,) \to f(Y_1, Y_2, \dots)$ is a unitary transformation from $L_2\left(R^\infty, \mathcal{B}^\infty, \prod_{j=1}^{\infty} \frac{e^{-x_j^2/2}}{\sqrt{2\pi}}\, dx_j\right)$ to $\mathcal{H}$. (We call a transformation from a Hilbert space to another Hilbert space unitary if it is norm preserving and invertible.) Since the image of a complete orthogonal system under a unitary transformation is again a complete orthogonal system, Theorem 2.1 is proved.

Let $\mathcal{H}_{\le n} \subset \mathcal{H}$, $n = 1, 2, \dots$, (with the previously introduced Hilbert space $\mathcal{H}$) denote the Hilbert space which is the closure of the linear space consisting of the elements $P_n(X_{t_1}, \dots, X_{t_m})$, where P_n runs through all polynomials of degree less than or equal to n, and the integer m and indices $t_1, \dots, t_m \in T$ are arbitrary. Let $\mathcal{H}_0 = \mathcal{H}_{\le 0}$ consist of the constant functions, and let $\mathcal{H}_n = \mathcal{H}_{\le n} \ominus \mathcal{H}_{\le n-1}$, $n = 1, 2, \dots$, where $\ominus$ denotes orthogonal completion. It is clear that the Hilbert space $\mathcal{H}_1$ given in this definition agrees with the previously defined Hilbert space $\mathcal{H}_1$. If $\xi_1, \dots, \xi_m \in \mathcal{H}_1$, and $P_n(x_1, \dots, x_m)$ is a polynomial of degree n, then $P_n(\xi_1, \dots, \xi_m) \in \mathcal{H}_{\le n}$. Hence Theorem 2.1 implies that

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \cdots, \tag{2.1}$$

where + denotes direct sum. Now we introduce the following

Definition of Wick polynomials. *Given a polynomial $P(x_1, \dots, x_m)$ of degree n and a set of (jointly Gaussian) random variables $\xi_1, \dots, \xi_m \in \mathcal{H}_1$, the Wick polynomial $:P(\xi_1, \dots, \xi_m):$ is the orthogonal projection of the random variable $P(\xi_1, \dots, \xi_m)$ to the above defined subspace $\mathcal{H}_n$ of the Hilbert space $\mathcal{H}$.*

It is clear that Wick polynomials of different degree are orthogonal. Given some $\xi_1, \dots, \xi_m \in \mathcal{H}_1$ define the subspaces $\mathcal{H}_{\le n}(\xi_1, \dots, \xi_m) \subset \mathcal{H}_{\le n}$, $n = 1, 2, \dots$, as the set of all polynomials of the random variables $\xi_1, \dots, \xi_m$ with degree less than or equal to n. Let $\mathcal{H}_{\le 0}(\xi_1, \dots, \xi_m) = \mathcal{H}_0(\xi_1, \dots, \xi_m) = \mathcal{H}_0$, and $\mathcal{H}_n(\xi_1, \dots, \xi_m) = \mathcal{H}_{\le n}(\xi_1, \dots, \xi_m) \ominus \mathcal{H}_{\le n-1}(\xi_1, \dots, \xi_m)$. With the help of this notation we formulate the following

Proposition 2.2. *Let $P(x_1, \dots, x_m)$ be a polynomial of degree n. Then $:P(\xi_1, \dots, \xi_m):$ equals the orthogonal projection of $P(\xi_1, \dots, \xi_m)$ to $\mathcal{H}_n(\xi_1, \dots, \xi_m)$.*

The proof of Proposition 2.2. Let $:\bar{P}(\xi_1, \dots, \xi_m):$ denote the projection of $P(\xi_1, \dots, \xi_m)$ to $\mathcal{H}_n(\xi_1, \dots, \xi_m)$. Obviously

$$P(\xi_1, \dots, \xi_m) - :\bar{P}(\xi_1, \dots, \xi_m): \in \mathcal{H}_{\le n-1}(\xi_1, \dots, \xi_m) \subseteq \mathcal{H}_{\le n-1}.$$

Hence in order to prove Proposition 2.2 it is enough to show that for all $\eta \in \mathcal{H}_{\le n-1}$

$$E:\bar{P}(\xi_1, \dots, \xi_m): \eta = 0, \tag{2.2}$$

since this means that $:\bar{P}(\xi_1, \dots, \xi_m):$ is the orthogonal projection of $P(\xi_1, \dots, \xi_m) \in \mathcal{H}_{\le n}$ to $\mathcal{H}_{\le n-1}$.

Let $\varepsilon_1, \varepsilon_2, \dots$ be an orthonormal system in $\mathcal{H}_1$, also orthonormal to $\xi_1, \dots, \xi_m$, and such that $\xi_1, \dots, \xi_m, \varepsilon_1, \varepsilon_2, \dots$ form a basis in $\mathcal{H}_1$. If $\eta = \prod_{i=1}^{m} \xi_i^{l_i} \prod_{j=1}^{\infty} \varepsilon_j^{k_j}$ with such exponents l_i and k_j that $\sum l_i + \sum k_j \le n - 1$, then (2.2) holds for this random variable η because of the independence of the random variables ξ_i and ε_j. Since the linear combinations of such η are dense in $\mathcal{H}_{\le n-1}$, formula (2.2) and Proposition (2.2) are proved.

Corollary 2.3. *Let $\xi_1, \dots, \xi_m$ be an orthonormal system in $\mathcal{H}_1$, and let $P(x_1, \dots, x_m) = \sum c_{j_1,\dots,j_m} x^{j_1} \cdots x_m^{j_m}$ be a homogeneous polynomial, i.e. let $j_1 + \cdots j_m = n$ with some fixed number n for all sets $(j_1, \dots, j_m)$ appearing in this summation. Then*

$$:P(\xi_1, \dots, \xi_m): \; = \sum c_{j_1,\dots,j_m} H_{j_1}(\xi_1) \cdots H_{j_m}(\xi_m).$$

In particular,

$$:\xi^n: \; = H_n(\xi) \quad \text{if } \xi \in \mathcal{H}_1, \text{ and } E\xi^2 = 1.$$

Remark. Although we have defined the Wick polynomial (of degree n) for all polynomials $P(\xi_1, \dots, \xi_m)$ of degree n, we could have restricted our attention only to homogeneous polynomials of degree n, since the contribution of each terms $c(j_1, \dots j_m)\xi_1^{l_1} \cdots \xi_m^{l_m}$ of the polynomial $P(\xi_1, \dots, \xi_m)$ such that $l_1 + \cdots + l_m < n$ has a zero contribution in the definition of the Wick polynomial $:P(\xi_1, \dots, \xi_m):$.

Proof of Corollary 2.3. Let the degree of the polynomial P be n. Then

$$P(\xi_1, \dots, \xi_m) - \sum c_{j_1,\dots,j_m} H_{j_1}(\xi_1) \cdots H_{j_m}(\xi_m) \in \mathcal{H}_{\le n-1}(\xi_1, \dots, \xi_m), \tag{2.3}$$

since $P(\xi_1, \dots, x_m) - \sum c_{j_1,\dots,j_m} H_{j_1}(\xi_1) \cdots H_{j_m}(\xi_m)$ is a polynomial whose degree is less than n. Let $\eta = \xi_1^{l_1} \cdots \xi_m^{l_m}$, $\sum\limits_{i=1}^{m} l_i \le n-1$. Then

$$E\eta H_{j_1}(\xi_1) \cdots H_{j_m}(\xi_m) = \prod_{i=1}^{m} E\xi_i^{l_i} H_{j_i}(\xi_i) = 0,$$

since $l_i < j_i$ for at least one index i. Therefore

$$E\eta \sum c_{j_1,\dots,j_m} H_{j_1}(\xi_1) \cdots H_{j_m}(\xi_m) = 0. \tag{2.4}$$

Since every element of $\mathcal{H}_{\le n-1}(\xi_1, \dots, \xi_m)$ can be written as the sum of such elements η, relation (2.4) holds for all $\eta \in \mathcal{H}_{\le n-1}(\xi_1, \dots, \xi_m)$. Relations (2.3) and (2.4) imply Corollary 2.3.

The following statement is a simple consequence of the previous results.

Corollary 2.4. *Let $\xi_1, \xi_2, \dots$ be an orthonormal basis in $\mathcal{H}_1$. Then the random variables $H_{j_1}(\xi_1) \cdots H_{j_k}(\xi_k)$, $k = 1, 2, \dots$, $j_1 + \cdots + j_k = n$, form a complete orthogonal basis in $\mathcal{H}_n$.*

Proof of Corollary 2.4. It follows from Corollary 2.3 that

$$H_{j_1}(\xi_1) \cdots H_{j_k}(\xi_k) = :\xi_1^{j_1} \cdots \xi_k^{j_k}: \; \in \mathcal{H}_n \quad \text{for all } k = 1, 2, \dots$$

if $j_1 + \cdots + j_k = n$. These random variables are orthogonal, and all Wick polynomials $:P(\xi_1, \dots, \xi_m):$ of degree n of the random variables $\xi_1, \xi_2, \dots$ can be represented as the linear combination of such terms. Since these Wick polynomials are dense in $\mathcal{H}_n$, this implies Corollary 2.4.

The arguments of this section exploited heavily some properties of Gaussian random variables. Namely, that the linear combination of Gaussian random variables is again Gaussian, and in Gaussian case orthogonality implies independence. This means, in particular, that the rotation of a standard normal vector leaves its distribution invariant. We finish this section with an observation based on these facts. This may illuminate the content of formula (2.1) from another point of view. We shall not use the results of the subsequent considerations in the rest of this work.

Let U be a unitary transformation over $\mathcal{H}_1$. It can be extended to a unitary transformation $\mathcal{U}$ over $\mathcal{H}$ in a natural way. Fix an orthonormal basis $\xi_1, \xi_2, \dots$ in $\mathcal{H}_1$, and define $\mathcal{U}1 = 1$, $\mathcal{U}\xi_{j_1}^{l_1} \cdots \xi_{j_k}^{l_k} = (U\xi_{j_1})^{l_1} \cdots (U\xi_{j_k})^{l_k}$. This transformation can be extended to a linear transformation $\mathcal{U}$ over $\mathcal{H}$ in a unique way. The transformation $\mathcal{U}$ is norm preserving, since the joint distributions of $(\xi_{j_1}, \xi_{j_2}, \dots)$ and $(U\xi_{j_1}, U\xi_{j_2}, \dots)$ coincide. Moreover, it is unitary, since $U\xi_1, U\xi_2, \dots$ is an orthonormal basis in $\mathcal{H}_1$. It is not difficult to see that if $P(x_1, \dots, x_m)$ is an arbitrary polynomial, and $\eta_1, \eta_2 \dots, \eta_m \in \mathcal{H}_1$, then $\mathcal{U}P(\eta_1, \dots, \eta_m) = P(U\eta_1, \dots, U\eta_m)$. This relation means in particular that the transformation $\mathcal{U}$ does not depend on the choice of the basis in $\mathcal{H}_1$. If the transformations $\mathcal{U}_1$ and $\mathcal{U}_2$ correspond to two unitary transformations U_1 and U_2 on $\mathcal{H}_1$, then the transformation $\mathcal{U}_1\mathcal{U}_2$ corresponds to U_1U_2. The subspaces $\mathcal{H}_{\le n}$ and therefore the subspaces $\mathcal{H}_n$ remain invariant under the transformations $\mathcal{U}$.

The shift transformations of a stationary Gaussian field, and their extensions to $\mathcal{H}$ are the most interesting examples for such unitary transformations U and $\mathcal{U}$. In the terminology of group representations the above facts can be formulated in the following way: The mapping $U \to \mathcal{U}$ is a group representation of $U(\mathcal{H}_1)$ over $\mathcal{H}$, where $U(\mathcal{H}_1)$ denotes the group of unitary transformations over $\mathcal{H}_1$. Formula (2.1) gives a decomposition of $\mathcal{H}$ into orthogonal invariant subspaces of this representation.

3. Random spectral measures.

Some standard theorems of probability theory state that the correlation function of a stationary random field can be expressed as the Fourier transform of a so-called spectral measure. In this section we construct a random measure with the help of these results, and express the random field itself as the Fourier transform of this random measure in some sense. We restrict ourselves to the Gaussian case, although most of the results in this section are valid for arbitrary stationary random field with finite second moment if independence is replaced by orthogonality. In the next section we define the multiple Wiener–Itô integrals with respect to this random measure. In the definition of multiple stochastic integrals the Gaussian property will be heavily exploited. First we recall two results about the spectral representation of the covariance function.

Given a stationary Gaussian field X_n, $n \in \mathbb{Z}_\nu$, or $X(\varphi)$, $\varphi \in \mathcal{S}$, we shall assume throughout the paper that $EX_n = 0$, $EX_n^2 = 1$ in the discrete and $EX(\varphi) = 0$ in the generalized field case.

Theorem 3A. (Bochner) *Let X_n, $n \in \mathbb{Z}_\nu$, be a discrete (Gaussian) stationary field. There exists a unique probability measure G on $[-\pi, \pi)^\nu$ such that the correlation function $r(n) = EX_0X_n = EX_kX_{k+n}$, $n \in \mathbb{Z}_\nu$, $k \in \mathbb{Z}_\nu$, can be written in the form*

$$r(n) = \int e^{i(n,x)} G(\,dx), \tag{3.1}$$

where $(\cdot, \cdot)$ denotes scalar product. Further $G(A) = G(-A)$ for all $A \in [-\pi, \pi)^\nu$.

We can identify $[-\pi, \pi)^\nu$ with the torus $R^\nu / 2\pi\mathbb{Z}_\nu$. Thus e.g. $-(-\pi, \dots, -\pi) = (-\pi, \dots, -\pi)$.

Theorem 3B. (Bochner–Schwartz) *Let $X(\varphi)$, $\varphi \in \mathcal{S}$, be a generalized (Gaussian) stationary random field over $\mathcal{S} = \mathcal{S}_\nu$, which satisfies the condition $E(X(\varphi_n) - X(\varphi))^2 \to 0$ if $\varphi_n \to \varphi$ in the topology of the Schwartz space $\mathcal{S}$. There exists a unique σ-finite measure G on R^ν such that*

$$EX(\varphi)X(\psi) = \int \tilde{\varphi}(x)\bar{\tilde{\psi}}(x) G(\,dx) \qquad \text{for all } \varphi, \psi \in \mathcal{S}, \tag{3.2}$$

where $\tilde{}$ denotes Fourier transform and $\bar{}$ complex conjugate. The measure G has the properties $G(A) = G(-A)$ for all $A \in \mathcal{B}^\nu$, and

$$\int (1 + |x|)^{-r} G(\,dx) < \infty \quad \text{with an appropriate } r > 0. \tag{3.3}$$

Remark. The above formulated results are actually not the Bochner and Bochner–Schwartz theorem in their original form, they are their consequences. In an Adjustment to Section 3 I formulate the classical form of these theorems, and explain how the above formulated results follow from them.

The measure G appearing in Theorems 3A and 3B is called the spectral measure of the stationary field. A measure G with the same properties as the measure G in Theorem 3A or 3B will also be called a spectral measure. This terminology is justified, since there exists a stationary random field with spectral measure G for all such G.

Let us now consider a stationary Gaussian random field (discrete or generalized one) with spectral measure G. We shall denote the space $L_2([-\pi,\pi)^\nu, \mathcal{B}^\nu, G)$ or $L_2(R^\nu, \mathcal{B}^\nu, G)$ simply by L_G^2. Let $\mathcal{H}_1$ be the real Hilbert space defined by means of the stationary random field, as it was done in Section 2. Let $\mathcal{H}_1^c$ denote its complexification, i.e. the elements of $\mathcal{H}_1^c$ are of the form $X+iY,\ X,\ Y \in \mathcal{H}_1$, and the scalar product is defined as $(X_1+iY_1, X_2+iY_2) = EX_1X_2+EY_1Y_2+i(EY_1X_2-EX_1Y_2)$. We are going to construct a unitary transformation I from L_G^2 to $\mathcal{H}_1^c$. We shall define the random spectral measure via this transformation.

Let $\mathcal{S}^c$ denote the Schwartz space of rapidly decreasing, smooth, complex valued functions with the usual topology of the Schwartz space. (The elements of $\mathcal{S}^c$ are of the form $\varphi + i\psi,\ \ \varphi, \psi \in \mathcal{S}$.) We make the following observation. The finite linear combinations $\sum c_n e^{i(n,x)}$ are dense in L_G^2 in the discrete field, and the functions $\varphi \in \mathcal{S}^c$ are dense in L_G^2 in the generalized field case. In the discrete field case this follows from the Weierstrass approximation theorem, which states that all continuous functions on $[-\pi,\pi)^\nu$ can be approximated by trigonometrical polynomials. In the generalized field case let us first observe that the continuous functions with compact support are dense in L_G^2. We claim that also the functions of the space $\mathcal{D}$ are dense in L_G^2, where $\mathcal{D}$ denotes the class of (complex valued) infinitely many times differentiable functions with compact support. Indeed, if $\varphi \in \mathcal{D}$ is real valued, $\varphi(x) \geq 0$ for all $x \in R^\nu$, $\int \varphi(x)\,dx = 1$, we define $\varphi_t(x) = t^\nu \varphi\left(\frac{x}{t}\right)$, and f is a continuous function with compact support, then $f * \varphi_t \to f$ uniformly as $t \to \infty$. Here $*$ denotes convolution. On the other hand, $f * \varphi_t \in \mathcal{D}$ for all $t > 0$. Hence $\mathcal{D} \subset \mathcal{S}^c$ is dense in L_G^2.

Finally we recall the following result from the theory of distributions. The mapping $\varphi \to \tilde{\varphi}$ is an invertible, bicontinuous transformation from $\mathcal{S}^c$ into $\mathcal{S}^c$. In particular, the set of functions $\tilde{\varphi},\ \ \varphi \in \mathcal{S}$, is also dense in L_G^2.

Now we define the mapping

$$I\left(\sum c_n e^{i(n,x)}\right) = \sum c_n X_n \tag{3.4}$$

in the discrete case, where the sum is finite, and

$$I(\widetilde{\varphi + i\psi}) = X(\varphi) + iX(\psi), \quad \varphi, \psi \in \mathcal{S} \tag{3.4'}$$

in the generalized case.

Obviously,

$$\begin{aligned}\left\|\sum c_n e^{i(n,x)}\right\|_{L_G^2}^2 &= \sum\sum c_n \bar{c}_m \int e^{i(n-m),x} G(\,dx) \\ &= \sum\sum c_n \bar{c}_m EX_nX_m = E\left|\sum c_n X_n\right|^2,\end{aligned}$$

and

$$\|\widetilde{\varphi + i\psi}\|^2_{L^2_G} = \int [\tilde{\varphi}(x)\bar{\tilde{\varphi}}(x) - i\tilde{\varphi}(x)\bar{\tilde{\psi}}(x) + i\tilde{\psi}(x)\bar{\tilde{\varphi}}(x) + \tilde{\psi}(x)\bar{\tilde{\psi}}(x)]G(\,dx)$$
$$= EX(\varphi)^2 - iEX(\varphi)X(\psi) + iEX(\psi)X(\varphi) + EX(\psi)^2 = E\left(|X(\varphi) + iX(\psi)|\right)^2.$$

This means that the mapping I from a linear subspace of L^2_G to $\mathcal{H}^c_1$ is norm preserving. Beside this, the subspace where I was defined is dense in L^2_G, since the space of continuous functions is dense in L^2_G if G is a finite measure on the torus $R^\nu/2\pi\mathbb{Z}_\nu$, and the space of continuous functions with a compact support is dense in $L^2_G(R^\nu)$ if the measure G satisfies relation (3.3). Hence the mapping I can be uniquely extended to a norm preserving transformation from L^2_G to $\mathcal{H}^c_1$. Since the random variables X_n or $X(\varphi)$ are obtained as the image of some element from L^2_G under this transformation, I is a unitary transformation from L^2_G to $\mathcal{H}^c_1$. A unitary transformation preserves not only the norm, but also the scalar product. Hence $\int f(x)\bar{g}(x)G(\,dx) = EI(f)\overline{I(g)}$ for all f, $g \in L^2_G$.

Now we define the random spectral measure $Z_G(A)$ for all $A \in \mathcal{B}^\nu$ such that $G(A) < \infty$ by the formula

$$Z_G(A) = I(\chi_A),$$

where χ_A denotes the indicator function of the set A. It is clear that

(i) The random variables $Z_G(A)$ are complex valued, jointly Gaussian random variables. (The random variables $\operatorname{Re} Z_G(A)$ and $\operatorname{Im} G(A)$ with possibly different sets A are jointly Gaussian.)

(ii) $EZ_G(A) = 0$,

(iii) $EZ_G(A)\overline{Z_G(B)} = G(A \cap B)$,

(iv) $\sum_{j=1}^{n} Z_G(A_j) = Z_G\left(\bigcup_{j=1}^{n} A_j\right)$ if $A_1, \dots, A_n$ are disjoint sets.

Also the following relation holds.

(v) $Z_G(A) = \overline{Z_G(-A)}$.

This follows from the relation

(v′) $I(f) = \overline{I(f_-)}$ for all $f \in L^2_G$, where $f_-(x) = \overline{f(-x)}$.

Relation (v′) can be simply checked if f is a finite trigonometrical polynomial in the discrete field case, or if $f = \tilde{\varphi}$, $\varphi \in \mathcal{S}^c$, in the generalized field case. (In the case $f = \tilde{\varphi}$, $\varphi \in \mathcal{S}^c$, the following argument works. Put $f(x) = \tilde{\varphi}_1(x) + i\tilde{\varphi}_2(x)$ with $\varphi_1, \varphi_2 \in \mathcal{S}$. Then $I(f) = X(\varphi_1) + iX(\varphi_2)$, and $f_-(x) = \bar{\tilde{\varphi}}_1(-x) - i\bar{\tilde{\varphi}}_2(-x) = \tilde{\varphi}_1(x) + i\widetilde{(-\varphi_2)}(x)$, hence $I(f_-) = X(\varphi_1) + iX(-\varphi_2) = X(\varphi_1) - iX(\varphi_2) = \overline{I(f)}$.) Then a simple limiting procedure implies (v′) in the general case. Relation (iii) follows from the identity $EZ_G(A)\overline{Z_G(B)} = EI(\chi_A)\overline{I(\chi_B)} = \int \chi_A(x)\overline{\chi_B(x)}G(\,dx) = G(A \cap B)$. The remaining properties of $Z_G(\cdot)$ are simple consequences of the definition.

Remark. Property (iv) could have been omitted from the definition of random spectral measures, since it follows from property (iii). To show this it is enough to check that if

$A_1, \dots, A_n$ are disjoint sets, and property (iii) holds, then

$$E\left(\sum_{j=1}^{n} Z_G(A_j) - Z_G\left(\bigcup_{j=1}^{n} A_j\right)\right)\overline{\left(\sum_{j=1}^{n} Z_G(A_j) - Z_G\left(\bigcup_{j=1}^{n} A_j\right)\right)} = 0.$$

Now we introduce the following

Definition of random spectral measure. *Let G be a spectral measure. A set of random variables $Z_G(A)$, $G(A) < \infty$, satisfying (i)–(v) is called a (Gaussian) random spectral measure corresponding to the spectral measure G.*

Given a Gaussian random spectral measure Z_G corresponding to a spectral measure G we define the stochastic integral $\int f(x) Z_G(\,dx)$ for an appropriate class of functions f. Let us first consider simple functions of the form $f(x) = \sum c_i \chi_{A_i}(x)$, where the sum is finite, and $G(A_i) < \infty$ for all indices i. In this case we define

$$\int f(x) Z_G(\,dx) = \sum c_i Z_G(A_i).$$

Then we have

$$E\left|\int f(x) Z_G(\,dx)\right|^2 = \sum c_i \bar{c}_j G(A_i \cap A_j) = \int |f(x)|^2 G(\,dx). \tag{3.5}$$

Since the simple functions are dense in L^2_G, relation (3.5) enables us to define $\int f(x) Z_G(\,dx)$ for all $f \in L^2_G$ via L_2-continuity. It can be checked that the expressions

$$X_n = \int e^{i(n,x)} Z_G(\,dx), \quad n \in \mathbb{Z}_\nu, \tag{3.6}$$

and

$$X(\varphi) = \int \tilde{\varphi}(x) Z_G(\,dx), \quad \varphi \in \mathcal{S}, \tag{3.6'}$$

defined with the help of the above defined (random) integral and spectral measure Z_G are Gaussian stationary random discrete and generalized field with spectral measure G.

We also have

$$\int f(x) Z_G(\,dx) = I(f) \quad \text{for all } f \in L^2_G$$

if we consider the previously defined mapping $I(f)$ with the stationary random fields defined in (3.6) and (3.6′). Now we formulate the following

Theorem 3.1. *For a stationary Gaussian random field (a discrete or generalized one) with a spectral measure G there exists a unique Gaussian random spectral measure Z_G corresponding to the spectral measure G on the same probability space as the Gaussian*

random field such that relation (3.6) or (3.6′) *holds in the discrete or generalized field case respectively.*

Furthermore

$$\mathcal{B}(Z_G(A),\ G(A)<\infty)=\begin{cases}\mathcal{B}(X_n,\ n\in\mathbb{Z}_\nu) \text{ in the discrete field case,}\\ \mathcal{B}(X(\varphi),\ \varphi\in\mathcal{S}) \text{ in the generalized field case.}\end{cases} \tag{3.7}$$

We shall say that the random spectral measure Z_G satisfying Theorem 3.1 together with a Gaussian random field is adapted to this random field.

Proof of Theorem 3.1. Given a stationary Gaussian random field (discrete or stationary one) with a spectral measure G, we have constructed a random spectral measure Z_G corresponding to the spectral measure G. Moreover, the random integrals given in formulas (3.6) or (3.6′) define the original stationary random field. Since all random variables $Z_G(A)$ are measurable with respect to the original random field, relation (3.6) or (3.6′) implies (3.7).

To prove the uniqueness, it is enough to observe that because of the linearity and L_2 continuity of stochastic integrals relation (3.6) or (3.6′) implies that

$$Z_G(A)=\int \chi_A(x)Z_G(\,dx)=I(\chi_A)$$

for a Gaussian random spectral measure corresponding to the spectral measure G appearing in Theorem 3.1.

Finally we list some additional properties of Gaussian random spectral measures.

(vi) The random variables $\operatorname{Re} Z_G(A)$ are independent of the random variables $\operatorname{Im} Z_G(A)$.

(vii) Random variables of the form $Z_G(A\cup(-A))$ are real valued. If the sets $A_1\cup(-A_1)$, $\dots$, $A_n\cup(-A_n)$ are disjoint, then the random variables $Z_G(A_1)$, $\dots$, $Z_G(A_n)$ are independent.

(viii) If $A\cap(-A)=\emptyset$, then $\operatorname{Re} Z_G(-A)=\operatorname{Re} Z_G(A)$, $\operatorname{Im} Z_G(-A)=-\operatorname{Im} Z_G(A)$, and the (Gaussian) random variables $\operatorname{Re} Z_G(A)$ and $\operatorname{Im} Z_G(A)$ are independent with expectation zero and variance $G(A)/2$.

These properties easily follow from (i)–(v). Since $Z_G(\cdot)$ are complex valued Gaussian random variables, to prove the above formulated independence it is enough to show that the real and imaginary parts are uncorrelated. We show, as an example, the proof of (vi).

$$\begin{aligned}E\operatorname{Re} Z_G(A)\operatorname{Im} Z_G(B)&=\frac{1}{4i}E(Z_G(A)+\overline{Z_G(A)})(Z_G(B)-\overline{Z_G(B)})\\&=\frac{1}{4i}E(Z_G(A)+Z_G(-A))(\overline{Z_G(-B)}-\overline{Z_G(B)})\\&=\frac{1}{4i}G(A\cap(-B))-\frac{1}{4i}G(A\cap B)\\&\quad+\frac{1}{4i}G((-A)\cap(-B))-\frac{1}{4i}G((-A)\cap B)=0\end{aligned}$$

for all pairs of sets A and B such that $G(A) < \infty$, $G(B) < \infty$, since $G(D) = G(-D)$ for all $D \in \mathcal{B}^\nu$. The fact that $Z_G(A \cup (-A))$ is real valued random variable, and the relations $\operatorname{Re} Z_G(-A) = \operatorname{Re} Z_G(A)$, $\operatorname{Im} Z_G(-A) = -\operatorname{Im} Z_G(A)$ under the conditions of (viii) follow directly from (v). The remaining statements of (vii) and (viii) can be proved similarly to (vi) only the calculations are simpler in this case.

The properties of the random spectral measure Z_G listed above imply in particular that the spectral measure G determines the joint distribution of the corresponding random variables $Z_G(B)$, $B \in \mathcal{B}^\nu$.

Attachment to Section 3. *A more detailed discussion about the spectral representation of the covariance function of stationary random fields.*

The results formulated under the name of Bochner and Bochner–Schwartz theorem (I write this, because actually I presented not these theorems but an important consequence of them) have the following content. Given a finite, even measure G on the torus $R^\nu/2\pi\mathbb{Z}_\nu$ one can define a (Gaussian) discrete stationary field with correlation function satisfying (3.1) with this measure G. For an even measure G on R^ν satisfying (3.3) there exists a (Gaussian) generalized stationary field with correlation function defined in formula (3.2) with this measure G. The Bochner and Bochner–Schwartz theorems state that the correlation function of all (Gaussian) discrete stationary fields, respectively of all stationary generalized fields can be represented in such a way. Let us explain this in more detail.

First I formulate the following

Proposition 3C. *Let G be a finite measure on the torus $R^\nu/2\pi\mathbb{Z}_\nu$ such that $G(A) = G(-A)$ for all measurable sets A. Then there exists a Gaussian discrete stationary field X_n, $n \in \mathbb{Z}_\nu$, with expectation zero such that its correlation function $r(n) = EX_kX_{k+n}$, $n, k \in \mathbb{Z}_\nu$, is given by formula (3.1) with this measure G.*

Let G be a measure on R^ν satisfying (3.3) and such that $G(A) = G(-A)$ for all measurable sets A. Then there exists a Gaussian stationary generalized field $X(\varphi)$, $\varphi \in \mathcal{S}$, with expectation $EX(\varphi) = 0$ for all $\varphi \in \mathcal{S}$ such that its covariance function $EX(\varphi)X(\psi)$, $\varphi, \psi \in \mathcal{S}$, satisfies formula (3.2) with this measure G.

Moreover, the correlation function $r(n)$ or $EX(\varphi)X(\psi)$, $\varphi, \psi \in \mathcal{S}$, determines the measure G uniquely.

Proof of Proposition 3C. By Kolmogorov's theorem about the existence of random processes with consistent finite dimensional distributions it is enough to prove the following statement to show the existence of the Gaussian discrete stationary field with the demanded properties. For any points $n_1, \dots, n_p \in \mathbb{Z}_\nu$ there exists a Gaussian random vector $(X_{n_1}, \dots, X_{n_p})$ with expectation zero and covariance matrix $EX_{n_j}X_{n_k} = r(n_j - n_k)$. (Observe that the function $r(n)$ is real valued, $r(n) = r(-n)$, because of the evenness of the spectral measure G.) Hence it is enough to check that the corresponding matrix is positive definite, i.e. $\sum\limits_{j,k} c_j c_k r(n_j - n_k) \ge 0$ for all real vectors $(c_1, \dots, c_p)$. This relation holds, because $\sum\limits_{j,k} c_j c_k r(n_j - n_k) = \int |\sum\limits_j c_j e^{i(n_j, x)}|^2 \, G(\, dx) \ge 0$ by formula (3.1).

It can be proved similarly that in the generalized field case there exists a Gaussian random field with expectation zero whose covariance function satisfies formula (3.2). (Let us observe that the relation $G(A) = G(-A)$ implies that $EX(\varphi)X(\psi)$ is a real number for all φ, $\psi \in \mathcal{S}$, since $EX(\varphi)X(\psi) = \overline{EX(\varphi)X(\psi)}$ in this case. In the proof of this identity we exploit that $\tilde{\tilde{f}}(x) = \tilde{f}(-x)$ for a real valued function f.) We also have to show that a random field with such a distribution is a generalized field, i.e. it satisfies properties a) and b) given in the definition of generalized fields. It is not difficult to show that if $\varphi_n \to \varphi$ in the topology of the space $\mathcal{S}$, then $E[X(\varphi_n) - X(\varphi)]^2 = \int |\tilde{\varphi}_n(x) - \tilde{\varphi}(x)|^2 G(\,dx) \to 0$ as $n \to \infty$, hence property b) holds. (Here we exploit that the transformation $\varphi \to \tilde{\varphi}$ is bicontinuous in the space $\mathcal{S}$.) Property a) also holds, because, as it is not difficult to check with the help of formula (3.2),

$$E[a_1X(\varphi_1) + a_2X(\varphi_2) - X(\varphi(a_1\varphi_1 + a_2\varphi_2)]^2$$
$$= \int \left|a_1\tilde{\varphi}_1(x) + a_2\tilde{\varphi}_2(x) - \widetilde{(a_1\varphi_1 + a_2\varphi_2)}(x)\right|^2 G(\,dx) = 0.$$

It is clear that the Gaussian random field constructed in such a way is stationary.

Finally, as we have seen in our considerations in the main text, the correlation function determines the integral $\int f(x)\, G(\,dx)$ for all continuous functions f with a bounded support, hence it also determines the measure G.

The Bochner and Bochner–Schwartz theorems enable us to show that the correlation function of all stationary (Gaussian) fields (discrete or generalized one) can be presented in the above way with an appropriate spectral measure G. To see this let us formulate these results in their original form.

To formulate Bochner's theorem first we have to introduce the following notion.

Definition of positive definite functions. *Let $f(x)$ be a (complex valued) function on $\mathbb{Z}_\nu$ (or on R^ν). We say that $f(\cdot)$ is a positive definite function if for all parameters p, complex numbers $c_1, \dots, c_p$ and points $x_1, \dots, x_p$ in $\mathbb{Z}_\nu$ (or in R^ν) the inequality*

$$\sum_{j=1}^{p}\sum_{k=1}^{p} c_j\bar{c}_k f(x_j - x_k) \ge 0$$

holds.

A simple example for positive definite functions is the function $f(x) = e^{i(t,x)}$, where $t \in \mathbb{Z}_\nu$ in the discrete, and $t \in R^\nu$ in the continuous case. Bochner's theorem provides a complete description of positive definite functions.

Bochner's theorem. (Its original form) *A complex valued function $f(x)$ defined on $\mathbb{Z}_\nu$ is positive definite if and only if it can be written in the form $f(x) = \int e^{i(t,x)} G(\,dx)$ for all $x \in \mathbb{Z}_\nu$ with a finite measure G on the torus $R^\nu/2\pi\mathbb{Z}_\nu$. The measure G is uniquely determined.*

A complex valued function $f(x)$ defined on R^ν is continuous and positive definite if and only if it can be written in the form $f(x) = \int e^{i(t,x)} G(\,dx)$ for all $x \in R^\nu$ with a finite measure G on R^ν. The measure G is uniquely determined.

It is not difficult to see that the covariance function $r(n) = EX_k X_{k+n}$, $(EX_n = 0)$, $k, n \in \mathbb{Z}_\nu$, of a stationary (Gaussian) random field X_n is a positive definite function, since $\sum\limits_{j,k} c_j \bar{c}_k r(n_j - n_k) = E|\sum\limits_j c_j X_{n_j}|^2 > 0$ for any vector $(c_1, \dots, c_p)$. Hence Bochner's theorem can be applied for it. Beside this, the relation $r(n) = r(-n)$ together with the uniqueness of the measure G appearing in Bochner's theorem imply that the identity $G(A) = G(-A)$ holds for all measurable sets G. This implies the result formulated in the main text under the name Bochner's theorem.

The Bochner–Schwartz theorem formulates an analogous representation of positive definite generalized functions in $\mathcal{S}'$ as the Fourier transforms of positive generalized functions in $\mathcal{S}'$ together with an analogous result about generalized functions in the space $\mathcal{D}'$. To formulate it we have to introduce some definitions. First we have to clarify what a positive generalized function means. We introduce this notion both in the space $\mathcal{S}'$ and $\mathcal{D}'$, and then we characterize them in a Theorem.

Definition of positive generalized functions. *A linear functional $F \in \mathcal{S}'$ (or $F \in \mathcal{D}'$) is called a positive definite generalized function if for all such $\varphi \in \mathcal{S}$ (or $\varphi \in \mathcal{D}$) test functions for which $\varphi(x) \ge 0$ for all $x \in R^\nu$ $(F, \varphi) \ge 0$.*

Theorem about the representation of positive generalized functions. *All positive generalized functions $F \in \mathcal{S}'$ can be given in the form $(F, \varphi) = \int \varphi(x) \mu(\,dx)$, where μ is a polynomially increasing measure on R^ν, i.e. it satisfies the relation $\int (1 + |x|^2)^{-p} \mu(\,dx) < \infty$ with some $p > 0$. Similarly, all positive generalized functions in $\mathcal{D}'$ can be given in the form $(F, \varphi) = \int \varphi(x) \mu(\,dx)$ with such a measure μ on R^ν which is finite in all bounded regions. The generalized function F uniquely determines the measure μ in both cases.*

We also need the introduction of a technical notion and a result related to it. Let us remark that if $\varphi \in \mathcal{S}^c$ and $\psi \in \mathcal{S}^c$, then also their product $\varphi\psi \in \mathcal{S}^c$. The analogous result also holds in the space $\mathcal{D}$.

Definition of multiplicatively positive generalized functions. *A generalized function $\mathcal{F} \in \mathcal{S}'$ (or $F \in \mathcal{D}'$) is multiplicatively positive if $(F, \varphi\bar{\varphi}) = (F, |\varphi|^2) \ge 0$ for all $\varphi \in \mathcal{S}^c$ (or in $\varphi \in \mathcal{D}$).*

Theorem about the characterization of multiplicatively positive generalized functions. *A generalized function $F \in \mathcal{S}'$ (or $F \in \mathcal{D}'$) is multiplicatively positive if and only if it is positive.*

Now I introduce the definition of positive definite generalized functions.

Definition of positive definite generalized functions. A generalized function $F \in \mathcal{S}'$ (or $F \in \mathcal{D}'$) is positive definite if $(F, \varphi * \varphi^*) \ge 0$ for all $\varphi \in \mathcal{S}^c$ (of $\varphi \in \mathcal{D}$), where $\varphi^*(x) = \overline{\varphi(-x)}$, and $*$ denotes convolution, i.e. $\varphi * \varphi^*(x) = \int \varphi(t) \overline{\varphi(t - x)}\, dt$.

We refer to [15] for an explanation why this definition of positive definite generalized functions is natural. Let us remark that if $\varphi, \psi \in \mathcal{S}^c$, then $\varphi * \psi \in \mathcal{S}^c$, and the analogous result holds in $\mathcal{D}$. The original version of the Bochner–Schwartz theorem has the following form.

Bochner–Schwartz theorem. (Original form) *Let F be a positive definite generalized function in the space $\mathcal{S}'$ (or $\mathcal{D}'$). Then it is the Fourier transform of a polynomially increasing measure μ on R^ν, i.e. the identity $(F, \varphi) = \int \tilde{\varphi}(x)\, \mu(\, dx)$ holds for all $\varphi \in \mathcal{S}^c$ (or $\varphi \in \mathcal{D}$) with a measure μ that satisfies the relation $\int (1 + |x|^2)^{-p} \mu(\, dx) < \infty$ with an appropriate $p > 0$. The generalized function F uniquely determines the measure μ. On the other hand, if μ is a polynomially increasing measure on R^ν, then the formula $(F, \varphi) = \int \tilde{\varphi}(x) \mu(\, dx)$ with $\varphi \in \mathcal{S}^c$ (or $\varphi \in \mathcal{D}$) defines a positive definite generalized function F in the space $\mathcal{S}'$ (or $\mathcal{D}'$).*

Remark. It is a remarkable and surprising fact that the class of positive definite generalized functions are represented by the same class of measures μ in the spaces $\mathcal{S}'$ and $\mathcal{D}'$. (In the representation of positive generalized functions the class of measures μ considered in the case of $\mathcal{D}'$ is much larger, than in the case of $\mathcal{S}'$.) Let us remark that in the representation of the positive definite generalized functions in $\mathcal{D}'$ the function $\tilde{\varphi}$ we integrate is not in the class $\mathcal{D}$, but in the space $\mathcal{Z}$ consisting of the Fourier transforms of the functions in $\mathcal{D}$.

It is relatively simple to prove the representation of positive definite generalized functions given in the Bochner–Schwartz theorem for the class $\mathcal{S}'$. Some calculation shows that if F is a positive definite generalized function, then its Fourier transform is a multiplicatively positive generalized function. Indeed, since the Fourier transform of the convolution $\varphi * \psi(x)$ equals $\tilde{\varphi}(t)\tilde{\psi}(t)$, and the Fourier transform of $\varphi^*(x) = \overline{\varphi(-x)}$ equals $\overline{\tilde{\varphi}(t)}$, the Fourier transform of $\varphi * \varphi^*(x)$ equals $\tilde{\varphi}(t)\bar{\tilde{\varphi}}(t)$. Hence the positive definitiveness property of the generalized function F and the definition of the Fourier transform of generalized functions imply that $(\tilde{F}, \tilde{\varphi}\bar{\tilde{\varphi}}) = (2\pi)^\nu (F, \varphi * \varphi^*) \geq 0$ for all $\varphi \in \mathcal{S}^c$. Since every function of $\mathcal{S}^c$ is the Fourier transform $\tilde{\varphi}$ of some function $\varphi \in \mathcal{S}^c$ this implies that $\tilde{F}$ is a multiplicatively positive and as a consequence a positive generalized function in $\mathcal{S}'$. Such generalized functions have a good representation with the help of a polynomially increasing positive measure μ. Since $(F, \varphi) = (2\pi)^{-\nu}(\tilde{F}, \tilde{\varphi})$ it is not difficult to prove the Bochner–Schwartz theorem for the space $\mathcal{S}'$ with the help of this fact. The proof is much harder if the space $\mathcal{D}'$ is considered, but we do not need that result.

The Bochner–Schwartz theorem in itself is not sufficient to describe the correlation function of a generalized random fields. We still need another important result of Laurent Schwartz which gives useful information about the behaviour of (Hermitian) bilinear functionals in $\mathcal{S}^c$ and some additional information about the behaviour of translation invariant (Hermitian) bilinear functionals in this space. To formulate these results first we introduce the following definition.

Definition of Hermitian bilinear and translation invariant Hermitian bilinear functionals in the space $\mathcal{S}^c$. *A function $B(\varphi, \psi)$, $\varphi, \psi \in \mathcal{S}^c$, is a Hermitian bilinear functional in the space $\mathcal{S}^c$ if for all fixed $\psi \in \mathcal{S}^c$ $B(\varphi, \psi)$ is a continuous linear functional*

of the variable ψ in the topology of $\mathcal{S}^c$, and for all fixed $\varphi \in \mathcal{S}^c$ $\overline{B(\varphi,\psi)}$ is a continuous linear functional of the variable ψ in the topology of $\mathcal{S}^c$.

A Hermitian bilinear functional $B(\varphi,\psi)$ in $\mathcal{S}^c$ is translation invariant if it does not change by a simultaneous shift of its variables φ and ψ, i.e. if $B(\varphi(x),\psi(x)) = B(\varphi(x+h),\psi(x+h))$ for all $h \in R^\nu$.

Definition of positive definite Hermitian bilinear functionals. *We say that a Hermitian bilinear functional $B(\varphi,\psi)$ in $\mathcal{S}^c$ is positive definite if $B(\varphi,\varphi) \ge 0$ for all $\varphi \in \mathcal{S}^c$.*

The next result characterizes the Hermitian bilinear and translation invariant Hermitian bilinear functionals in $\mathcal{S}^c$.

Theorem 3D. *All Hermitian bilinear functionals $B(\varphi,\psi)$ in $\mathcal{S}^c$ can be given in the form $B(\varphi,\psi) = (F_1, \varphi(x)\overline{\psi(y)})$, $\varphi,\psi \in \mathcal{S}^c$, where F_1 is a continuous linear functional on $\mathcal{S}^c \times \mathcal{S}^c$, i.e. it is a generalized function in ${\mathcal{S}_{2\nu}}'$.*

A translation invariant Hermitian bilinear functional in $\mathcal{S}^c$ can be given in the form $\mathcal{B}(\varphi,\psi) = (F, \varphi * \psi^*)$, $\varphi,\psi \in \mathcal{S}^c$, where $F \in \mathcal{S},$, $\psi^*(x) = \overline{\psi}(-x)$, and $*$ denotes convolution.

The Hermitian bilinear form $B(\varphi,\psi)$ determines the generalized functions F_1, and if it is translation invariant then also the generalized function F uniquely. Beside this, for all functionals $F_1 \in \mathcal{S}^{2\nu\prime}$ and $F \in \mathcal{S}'$ the above formulas define a Hermitian bilinear functional and a translation invariant Hermitian bilinear functional in $\mathcal{S}^c$ respectively.

Let us consider a Gaussian generalized random field $X(\varphi)$, $\varphi \in \mathcal{S}$, with expectation zero together with its correlation function $B(\varphi,\psi) = EX\varphi)X(\psi)$, $\varphi,\psi \in \mathcal{S}$. More precisely, let us consider the complexification $X(\varphi_1 + i\varphi_2) = X(\varphi_1) + iX(\varphi_2)$ of this random field and its correlation function $B(\varphi,\psi) = EX(\varphi)\overline{X(\psi)}$, $\varphi,\psi \in \mathcal{S}^c$. This correlation function $B(\varphi,\psi)$ is a translation invariant Hermitian bilinear functional in $\mathcal{S}^c$, hence it can be written in the form $B(\varphi,\psi) = (F, \varphi * \psi^*)$ with an appropriate $F \in \mathcal{S}'$. Moreover, $B(\varphi,\varphi) \ge 0$ for all $\varphi \in \mathcal{S}^c$, and this means that the generalized function $F \in \mathcal{S}'$ corresponding to $B(\varphi,\psi)$ is positive definite. Hence the Bochner–Schwartz theorem can be applied for it, and it yields that

$$EX(\varphi)X(\psi) = \int \widetilde{\varphi * \psi^*}(x)\, G(\,dx) = \int \tilde{\varphi}(x)\bar{\tilde{\psi}}(x)\, G(\,dx) \quad \text{for all } \varphi,\, \psi \in \mathcal{S}^c$$

with a uniquely determined, polynomially increasing measure G on R^ν. To prove Theorem 3B we still have to show that G is an even measure. In the proof of this statement we exploit that for a real valued function $\varphi \in \mathcal{S}$ the random variable $X(\varphi)$ is also real valued. Hence if $\varphi,\psi \in \mathcal{S}$, then $EX(\varphi)X(\psi) = \overline{EX(\varphi)X(\psi)}$. Beside this $\tilde{\varphi}(-x) = \bar{\tilde{\varphi}}(x)$ and $\tilde{\psi}(-x) = \bar{\tilde{\psi}}(x)$ in this case. Hence

$$\begin{aligned}\int \tilde{\varphi}(x)\bar{\tilde{\psi}}(x)\, G(\,dx) &= \int \bar{\tilde{\varphi}}(x)\tilde{\psi}(x)\, G(\,dx) \\ &= \int \tilde{\varphi}(-x)\bar{\tilde{\psi}}(-x)\, G(\,dx) = \int \tilde{\varphi}(x)\bar{\tilde{\psi}}(x)\, G^-(\,dx)\end{aligned}$$

for all $\varphi, \psi \in \mathcal{S}$, where $G^-(A) = G(-A)$ for all $A \in \mathcal{B}^\nu$. This relation implies that the measures G and G^- agree. The proof of Theorem 3B is completed.

4. Multiple Wiener–Itô integrals.

In this section we define the so-called multiple Wiener–Itô integrals, and we prove their most important properties with the help of Itô's formula, whose proof is postponed to the next section. More precisely, we discuss in this section a modified version of the Wiener–Itô integrals with respect to a random spectral measure rather than with respect to a random measure with independent increments. This modification makes it necessary to slightly change the definition of the integral. This modified Wiener–Itô integral seems to be a more useful tool than the original one or the Wick polynomials, because it enables us to describe the action of shift transformations.

Let G be the spectral measure of a stationary Gaussian field (discrete or generalized one). We define the following *real* Hilbert spaces $\bar{\mathcal{H}}_G^n$ and $\mathcal{H}_G^n$, $n = 1, 2, \dots$. We have $f_n \in \bar{\mathcal{H}}_G^n$ if and only if $f_n = f_n(x_1, \dots, x_n)$, $x_j \in R^\nu$, $j = 1, 2, \dots, n$, is a complex valued function of n variables, and

(a) $f_n(-x_1, \dots, -x_n) = \overline{f_n(x_1, \dots, x_n)}$,
(b) $\|f_n\|^2 = \int |f_n(x_1, \dots, x_n)|^2 G(\,dx_1) \dots G(\,dx_n) < \infty$.

Relation (b) also defines the norm in $\bar{\mathcal{H}}_G^n$. The subspace $\mathcal{H}_G^n \subset \bar{\mathcal{H}}_G^n$ contains those functions $f_n \in \bar{\mathcal{H}}_G^n$ which are invariant under permutations of their arguments, i.e.

(c) $f_n(x_{\pi(1)}, \dots, x_{\pi(n)})) = f_n(x_1, \dots, x_n)$ for all $\pi \in \Pi_n$, where Π_n denotes the group of all permutations of the set $\{1, 2, \dots, n\}$.

The norm in $\mathcal{H}_G^n$ is defined in the same way as in $\bar{\mathcal{H}}_G^n$. Moreover, the scalar product is also similarly defined, namely if f, $g \in \bar{\mathcal{H}}_G^n$, then

$$\begin{aligned}(f, g) &= \int f(x_1, \dots, x_n)\overline{g(x_1, \dots, x_n)} G(\,dx_1) \dots G(\,dx_n) \\ &= \int f(x_1, \dots, x_n) g(-x_1, \dots, -x_n) G(\,dx_1) \dots G(\,dx_n).\end{aligned}$$

Because of the symmetry $G(A) = G(-A)$ of the spectral measure $(f, g) = \overline{(f, g)}$, i.e. the scalar product (f, g) is a real number for all f, $g \in \bar{\mathcal{H}}_G^n$. This means that $\bar{\mathcal{H}}_G^n$ is a real Hilbert space. We also define $\mathcal{H}_G^0 = \bar{\mathcal{H}}_G^0$ as the space of real constants with the norm $\|c\| = |c|$. We remark that $\bar{\mathcal{H}}_G^n$ is actually the n-fold direct product of $\mathcal{H}_G^1$, while $\mathcal{H}_G^n$ is the n-fold symmetrical direct product of $\mathcal{H}_G^1$. Condition (a) means heuristically that f_n is the Fourier transform of a real valued function.

Finally we define the so-called Fock space $\operatorname{Exp} \mathcal{H}_G$ whose elements are sequences of functions $f = (f_0, f_1, \dots)$, $f_n \in \mathcal{H}_G^n$ for all $n = 0, 1, 2, \dots$, such that

$$\|f\|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \|f_n\|^2 < \infty.$$

Given a function $f \in \bar{\mathcal{H}}_G^n$ we define $\operatorname{Sym} f$ as

$$\operatorname{Sym} f(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\pi \in \Pi_n} f(x_{\pi(1)}, \dots, x_{\pi(n)}).$$

Clearly, $\operatorname{Sym} f \in \mathcal{H}_G^n$, and

$$\|\operatorname{Sym} f\| \le \|f\|. \tag{4.1}$$

Let Z_G be a Gaussian random spectral measure corresponding to the spectral measure G on a probability space $(\Omega, \mathcal{A}, P)$. We shall define the n-fold Wiener–Itô integrals

$$I_G(f_n) = \frac{1}{n!} \int f_n(x_1, \dots, x_n) Z_G(\,dx_1) \dots Z_G(\,dx_n) \quad \text{for } f_n \in \bar{\mathcal{H}}_G^n$$

and

$$I_G(f) = \sum_{n=0}^{\infty} I_G(f_n) \quad \text{for } f = (f_0, f_1, \dots) \in \operatorname{Exp} \mathcal{H}_G.$$

We shall see that $I_G(f_n) = I_G(\operatorname{Sym} f_n)$ for all $f_n \in \bar{\mathcal{H}}_G^n$. Therefore, it would have been sufficient to define the Wiener–Itô integral only for functions in $\mathcal{H}_G^n$. Nevertheless, some arguments become simpler if we work in $\bar{\mathcal{H}}_G^n$. In the definition of Wiener–Itô integrals first we restrict ourselves to the case when the spectral measure is non-atomic, i.e. $G(\{x\}) = 0$ for all $x \in R^\nu$. This condition is satisfied in all interesting cases. However, we shall later show how one can get rid of this restriction.

First we define a subclass $\hat{\bar{\mathcal{H}}}_G^n \subset \bar{\mathcal{H}}_G^n$ of simple functions, and define the Wiener–Itô integrals for the functions of this subclass.

Let $\mathcal{D} = \{\Delta_j,\ j = \pm 1, \pm 2, \dots, \pm N\}$ be a finite collection of bounded, measurable sets in R^ν indexed by the integers $\pm 1, \dots, \pm N$. We say that $\mathcal{D}$ is a regular system if $\Delta_j = -\Delta_{-j}$, and $\Delta_j \cap \Delta_l = \emptyset$ if $j \ne l$ for all $j, l = \pm 1, \pm 2, \dots, \pm N$. A function $f \in \bar{\mathcal{H}}_G^n$ is adapted to this system $\mathcal{D}$ if $f(x_1, \dots, x_n)$ is constant on the sets $\Delta_{j_1} \times \Delta_{j_2} \times \dots \times \Delta_{j_n}$, $j_l = \pm 1, \dots, \pm N$, $l = 1, 2, \dots, n$, it vanishes outside these sets and also on the sets for which $j_l = \pm j_{l'}$ for some $l \ne l'$. A function $f \in \bar{\mathcal{H}}_G^n$ is in the class $\hat{\bar{\mathcal{H}}}_G^n$ of simple functions if it is adapted to some regular systems $\mathcal{D} = \{\Delta_j,\ j \pm 1, \dots, \pm N\}$, and its Wiener–Itô integral with respect to Z_G is defined as

$$\begin{aligned} &\int f(x_1, \dots, x_n) Z_G(\,dx_1) \dots Z_G(\,dx_n) \\ &\qquad = n! I_G(f) = \sum_{\substack{j_l = \pm 1, \dots, \pm N \\ l = 1, 2, \dots, n}} f(x_{j_1}, \dots, x_{j_n}) Z_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n}), \end{aligned} \tag{4.2}$$

where $x_j \in \Delta_j$, $j = \pm 1, \dots, \pm N$. We remark that although the regular system $\mathcal{D}$ to which f is adapted, is not uniquely determined (the elements of $\mathcal{D}$ can be divided to smaller sets), the integral defined in (4.2) is meaningful, i.e. it does not depend

on the choice of $\mathcal{D}$. This can be seen by observing that a refinement of a regular system $\mathcal{D}$ adapted to the function f yields the same value for the sum defining $n!I_G(f)$ in formula (4.2) as the original one. This follows from the additivity of the random spectral measure Z_G formulated in its property (iv), since this implies that each term $f(x_{j_1},\dots,x_{j_n})Z_G(\Delta_{j_1})\cdots Z_G(\Delta_{j_n})$ in the sum at the right-hand side of formula (4.2) corresponding to the original regular system equals the sum of all such terms $f(x_{j_1},\dots,x_{j_n})Z_G(\Delta'_{j'_1})\cdots Z_G(\Delta'_{j'_n})$ in the sum corresponding to the refined partition for which $\Delta'_{j'_1}\times\cdots\times\Delta'_{j'_n}\subset\Delta_{j_1}\times\cdots\times\Delta_{j_n}$.

By property (vii) of the random spectral measures all products $Z_G(\Delta_{j_1})\cdots Z_G(\Delta_{j_n})$ with non-zero coefficient in (4.2) are products of independent random variables. We had this property in mind when requiring the condition that the function f vanishes on a product $\Delta_{j_1}\times\cdots\times\Delta_{j_n}$ if $j_l=\pm j_{l'}$ for some $l\neq l'$. This condition is interpreted in the literature as discarding the hyperplanes $x_l=x_{l'}$ and $x_l=-x_{l'}$, $l,l'=1,2,\dots,n$, $l\neq l'$, from the domain of integration. Property (a) of the functions in $\bar{\mathcal{H}}_G^n$ and property (v) of the random spectral measures imply that $I_G(f)=\overline{I_G(f)}$, i.e. $I_G(f)$ is a real valued random variable for all $f\in\hat{\bar{\mathcal{H}}}_G^n$. The relation

$$EI_G(f)=0,\quad \text{for } f\in\hat{\bar{\mathcal{H}}}_G^n,\quad n=1,2,\dots \tag{4.3}$$

also holds. Let $\hat{\mathcal{H}}_G^n=\mathcal{H}_G^n\cap\hat{\bar{\mathcal{H}}}_G^n$. If $f\in\hat{\bar{\mathcal{H}}}_G^n$, then $\operatorname{Sym}f\in\hat{\mathcal{H}}_G^n$, and

$$I_G(f)=I_G(\operatorname{Sym}f). \tag{4.4}$$

Relation (4.4) follows immediately from the observation that $Z_G(\Delta_{j_1})\cdots Z_G(\Delta_{j_n})=Z_G(\Delta_{\pi(j_1)})\cdots Z_G(\Delta_{\pi(j_n)})$ for all $\pi\in\Pi_n$. We also claim that

$$EI_G(f)^2\le\frac{1}{n!}\|f\|^2\quad \textit{for } f\in\hat{\bar{\mathcal{H}}}_G^n, \tag{4.5}$$

and

$$EI_G(f)^2=\frac{1}{n!}\|f\|^2\quad \textit{for } f\in\hat{\mathcal{H}}_G^n. \tag{4.5'}$$

Because of (4.1) and (4.4) it is enough to check (4.5').

Let $\mathcal{D}$ be a regular system of sets in R^ν, $j_1,\dots,j_n$ and $k_1,\dots,k_n$ be indices such that $j_l\neq\pm j_{l'}$, $k_l\neq\pm k_{l'}$ if $l\neq l'$. Then

$$\begin{aligned}&EZ_G(\Delta_{j_1})\cdots Z_G(\Delta_{j_n})\overline{Z_G(\Delta_{k_1})\cdots Z_G(\Delta_{k_n})}\\&\qquad=\begin{cases}G(\Delta_{j_1})\cdots G(\Delta_{j_n})&\text{if } \{j_1,\dots,j_n\}=\{k_1,\dots,k_n\},\\0\quad\text{otherwise.}\end{cases}\end{aligned}$$

To see the last relation one has to observe that the product on the left-hand side can be written as a product of independent random variables because of property (vii) of the random spectral measures. If $\{j_1,\dots,j_n\}\neq\{k_1,\dots,k_n\}$, then there is an index l such that either $j_l\neq\pm k_{l'}$ for all $1\le l'\le n$, or there exists an index l', $1\le l'\le n$, such

that $j_l = -k_{l'}$. In the first case $Z_G(\Delta_{j_l})$ is independent of the remaining coordinates of the vector $(Z_G(\Delta_{j_1}), \dots, Z_G(\Delta_{j_n}), \overline{Z_G(\Delta_{k_1})}, \dots, \overline{Z_G(\Delta_{k_n})})$, and $EZ_G(\Delta_{j_l}) = 0$. Hence the expectation of the investigated product equals zero, as we claimed. If $j_l = -k_{l'}$ with some index l', then a different argument is needed, since $Z_G(\Delta_{j_l})$ and $Z_G(-\Delta_{j_l})$ are not independent. In this case we can state that since $j_p \neq \pm j_l$ if $p \neq l$, and $k_q \neq \pm j_l$ if $q \neq l'$, the vector $(Z_G(\Delta_{j_l}), Z_G(-\Delta_{j_l}))$ is independent of the remaining coordinates of the above random vector. On the other hand, the product $Z_G(\Delta_{j_l})\overline{Z_G(-\Delta_{j_l})}$ has zero expectation, since $EZ_G(\Delta_{j_l})\overline{Z_G(-\Delta_{j_l})} = G(\Delta_{j_l} \cap (-\Delta_{j_l})) = 0$ by property (iii) of the random spectral measures and the relation $\Delta_{j_l} \cap (-\Delta_{j_l}) = \emptyset$. Hence the expectation of the considered product equals zero also in this case. If $\{j_1, \dots, j_n\} = \{k_1, \dots, k_n\}$, then

$$EZ_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n})\overline{Z_G(\Delta_{k_1}) \cdots Z_G(\Delta_{k_n})} = \prod_{l=1}^{n} EZ_G(\Delta_{j_l})\overline{Z_G(\Delta_{j_l})} = \prod_{l=1}^{n} G(\Delta_{j_l}).$$

Therefore for a function $f \in \hat{\mathcal{H}}_G^n$

$$\begin{aligned} EI_G(f)^2 &= \left(\frac{1}{n}\right)^2 \sum \sum f(x_{j_1}, \dots, x_{j_n})\overline{f(x_{k_1}, \dots, x_{k_n})} \\ &\qquad EZ_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n})\overline{Z_G(\Delta_{k_1}) \cdots Z_G(\Delta_{k_n})} \\ &= \left(\frac{1}{n!}\right)^2 \sum |f(x_{j_1}, \dots, x_{j_n})|^2 G(\Delta_{j_1}) \cdots G(\Delta_{j_n}) \cdot n! \\ &= \frac{1}{n!} \int |f(x_1, \dots, x_n)|^2 G(\, dx_1) \cdots G(\, dx_n) = \frac{1}{n!} \|f\|^2. \end{aligned}$$

We claim that Wiener–Itô integrals of different order are uncorrelated. More explicitly, take two functions $f \in \hat{\hat{\mathcal{H}}}_G^n$ and $f' \in \hat{\hat{\mathcal{H}}}_G^{n'}$ such that $n \neq n'$. Then we have

$$EI_G(f)I_G(f') = 0 \quad \text{if} \quad f \in \hat{\hat{\mathcal{H}}}_G^n, \quad f' \in \hat{\hat{\mathcal{H}}}_G^{n'}, \text{ and } \quad n \neq n'. \tag{4.6}$$

To see this relation observe that a regular system $\mathcal{D}$ can be chosen is such a way that both f and f' are adapted to it. Then a similar, but simpler argument as the previous one shows that

$$EZ_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n})\overline{Z_G(\Delta_{k_1}) \cdots Z_G(\Delta_{k_{n'}})} = 0$$

for all sets of indices $\{j_1, \dots, j_n\}$ and $\{k_1, \dots, k_{n'}\}$ if $n \neq n'$, hence the sum expressing $EI_G(f)I_G(f')$ in this case equals zero.

We show that $\hat{\hat{\mathcal{H}}}_G^n$ is dense in $\bar{\mathcal{H}}_G^n$ (and $\hat{\mathcal{H}}_G^n$ is dense in $\mathcal{H}_G^n$). First we show that this property can be reduced to Statement A formulated below. In Statement A we reduce the statement about the good approximability of a general function $f \in \bar{\mathcal{H}}_G^n$ to the good approximability of the indicator function χ_A of a bounded set $A \in \mathcal{B}^{n\nu}$ such that

$A = -A$ by a function of the form $g = \chi_B \in \hat{\bar{\mathcal{H}}}_G^n$. (Observe that $\chi_A \in \bar{\mathcal{H}}_G^n$ for a bounded set $A \in \mathcal{B}^{n\nu}$ if and only if $A = -A$.) However, we have to formulate Statement A in a more complicated form, because only in such a way can we reduce the statement about the good approximability of a general, possibly complex valued function $f \in \bar{\mathcal{H}}_G^n$ by a function in $g \in \hat{\bar{\mathcal{H}}}_G^n$ to Statement A.

Statement A. Let $A \in \mathcal{B}^{n\nu}$ be a bounded, symmetric set, i.e. let $A = -A$. Then for any $\varepsilon > 0$ there is a function $g \in \hat{\bar{\mathcal{H}}}_G^n$ such that $g = \chi_B$ with some set $B \in \mathcal{B}^{n\nu}$, i.e. g is the indicator function of the set B, which satisfies the inequality $\|g - \chi_A\| < \varepsilon$ with the norm of the space $\bar{\mathcal{H}}_G^n$. (Here χ_A denotes the indicator function of the set A, and have $\chi_A \in \bar{\mathcal{H}}_G^n$.) Moreover, if the set A can written in the form $A = A_1 \cup (-A_1)$ with such a set A_1 for which the sets A_1 and $-A_1$ have a positive distance from each other, i.e. there is a number $\delta > 0$ such that $\rho(A_1, -A_1) = \inf_{x \in A_1,\, y \in -A_1} \rho(x, y) > \delta$, where ρ denotes the Euclidean distance in $R^{n\nu}$, then a good approximation of χ_A can be given with such a function $g = \chi_{B \cup (-B)} \in \hat{\bar{\mathcal{H}}}_G^n$ which has some additional good properties. Namely, there is a set $B \in \mathcal{B}^{n\nu}$ such that $B \subset A_1^{\delta/2} = \{x\colon\ \rho(x, A_1) \le \frac{\delta}{2}\}$, $G^n(A_1 \,\Delta\, B) < \frac{\varepsilon}{2}$, where $A \Delta B$ denotes the symmetric difference of the sets A and B, and G^n is the n-times direct product of the spectral measure G on the space $R^{n\nu}$, and $g = \chi_{B \cup (-B)} \in \hat{\bar{\mathcal{H}}}_G^n$. These properties of the set B imply that the function $g = \chi_{B \cup (-B)} \in \hat{\bar{\mathcal{H}}}_G^n$ satisfies the relation $\|g - \chi_A\| < \varepsilon$.

To justify this reduction to Statement A let us observe that if two functions $f_1 \in \bar{\mathcal{H}}_G^n$ and $f_2 \in \bar{\mathcal{H}}_G^n$ can be arbitrary well approximated by functions from $\hat{\bar{\mathcal{H}}}_G^n$ in the $\bar{\mathcal{H}}_G^n$ norm, then the same relation holds for any linear combination $c_1 f_1 + c_2 f_2$ with real coefficients c_1 and c_2. (If the functions f_i are approximated by some functions $g_i \in \hat{\bar{\mathcal{H}}}_G^n$, $i = 1, 2$, then we may assume, by applying some refinement of the partitions if it is necessary, that the regular partitions appearing in the definition of the approximating functions are the same.) Hence the proof about the arbitrary good approximability of a function $f \in \bar{\mathcal{H}}_G^n$ by functions $g \in \hat{\bar{\mathcal{H}}}_G^n$ can be reduced to the proof about the arbitrary good approximability of its real part $\operatorname{Re} f \in \bar{\mathcal{H}}_G^n$ and its imaginary part $\operatorname{Im} f \in \bar{\mathcal{H}}_G^n$. Moreover, since the real part and imaginary part of the function f can be arbitrary well approximated by such real or imaginary valued functions from the space $\bar{\mathcal{H}}_G^n$ which take only finitely many values, the desired approximation result can be reduced to the case when f is the indicator function of a set $A \in \mathcal{B}^{n\nu}$ such that $A = -A$ (if f is real valued), or it takes three values, the value i on a set $A_1 \in \mathcal{B}^{n\nu}$, the value $-i$ on the set $-A_1$, and it equals zero on $R^{n\nu} \setminus (A_1 \cup (-A_1))$ (if f is purely imaginary valued). Beside this, the inequalities $G^n(A) < \infty$ and $G^n(A_1) < \infty$ hold. We can even assume that A and A_1 are bounded sets, because $G^n(A) = \lim_{K \to \infty} G^n(A \cap [-K, K]^{n\nu})$, and the same argument applies for A_1.

Hence Statement A immediately implies the desired approximation result in the first case when f is the indicator function of a set A. In the second case, when such a function f is considered that takes the values $\pm i$ and zero, observe that the sets $A_1 = \{x\colon\ f(x) = i\}$ and $-A_1 = \{x\colon\ f(x) = -i\}$ are disjoint. Moreover, we may assume

that they have positive distance from each other, because there are such compact sets $A_N \subset A$, $N = 1, 2, \dots$, for which $\lim_{N\to\infty} G^n(A \setminus A_N) = 0$, and two disjoint compact sets have positive distance. Then the approximation result also holds in the second case if we take the approximation of the pair $(A_1, -A_1)$ by the pair $(B, -B)$ appearing in Statement A, and define $g(x) = i$ if $x \in B$, $g(x) = -i$ if $x \in -B$ and $g(x) = 0$ otherwise.

In the next step we reduce the proof of Statement A to the proof of a result formulated under the name Statement B. We show that to prove Property A it is enough to prove the good approximability of some very special indicator functions $\chi_B \in \bar{\mathcal{H}}_G^n$ by a function $g \in \hat{\bar{\mathcal{H}}}_G^n$. We have to handle such sets $B \in \mathcal{B}^{n\nu}$ where the proof is simpler.

Statement B. Let $B = D_1 \times \cdots \times D_n$ be the direct product of bounded sets $D_j \in \mathcal{B}^\nu$ such that $D_j \cap (-D_j) = \emptyset$ for all $1 \le j \le n$. Then for all $\varepsilon > 0$ there is a set $F \subset B \cup (-B)$, $F \in \mathcal{B}^{n\nu}$ such that $\chi_F \in \hat{\bar{\mathcal{H}}}_G^n$, and $\|\chi_{B\cup(-B)} - \chi_F\| \le \varepsilon$, with the norm of the space $\bar{\mathcal{H}}_G^n$.

To deduce Statement A from statement B let us first remark that we may reduce our attention to such sets A in Statement A for which all coordinates of the points of the set A are separated from the origin. More explicitly, we may assume the existence of a number $\eta > 0$ with the property $A \cap K(\eta) = \emptyset$, where $K(\eta) = \bigcup_{j=1}^n K_j(\eta)$ with $K_j(\eta) = \{(x_1, \dots, x_n)\colon\ x_l \in R^\nu,\ l = 1, \dots, n,\ \rho(x_j, 0) \le \eta\}$. To see our right to make such a reduction observe that the relation $G(\{0\}) = 0$ implies that $\lim_{\eta\to 0} G^n(K(\eta)) = 0$, hence $\lim_{\eta\to 0} G^n(A \setminus K(\eta)) = G^n(A)$. At this point we exploited a weakened form of the non-atomic property of the spectral measure G, namely the relation $G(\{0\}) = 0$.

To prove Statement A with the help of Statement B it is enough to show that for all numbers $\varepsilon > 0$ and bounded sets $A \in \mathcal{B}^{n\nu}$ such that $A = -A$ there is a finite sequence of bounded sets $B_j \in \mathcal{B}^{n\nu}$, $j = \pm 1, \dots, \pm N$, such that the sets B_j are disjoint, $B_{-j} = -B_j$, $j = \pm 1, \dots, \pm N$, each set B_j can be written in the form $B_j = D_1^{(j)} \times \cdots \times D_n^{(j)}$ with $D_k^{(j)} \in \mathcal{B}^\nu$, and $D_k^{(-j)} \cap (-D_k^{(j)}) = \emptyset$ for all $1 \le j \le N$ and $1 \le k \le n$, and finally the set $B = \bigcup_{j=1}^N (B_j \cup B_{-j})$ satisfies the relation $G^n(A \Delta B) \le \varepsilon$. Indeed, since we can choose $\varepsilon > 0$ arbitrary small, the application of Statement B for all pairs $(B_j, -B_j)$ supplies an arbitrary good approximation of the function χ_A by a function of the form $\chi_{\bar{B}} \in \hat{\bar{\mathcal{H}}}_G^n$ in the norm of the space $\bar{\mathcal{H}}_G^n$.

If the set A can be written in the form $A = A_1 \cup (-A_1)$ such that $\rho(A_1, -A_1) > \delta$, then we can show the existence of a good approximation of the set A with the extra properties formulated in Statement A in the following way. We may assume that all sets B_j in the above sequence have a non-empty intersection with the set A. Otherwise the pair (B_j, B_{-j}) could have been omitted from this sequence. We may also assume, by applying a refinement of the sets B_j if it is necessary that all sets B_j have a diameter less that $\frac{\delta}{4}$. Then for a pair (B_j, B_{-j}) one of these sets has a non-empty intersection with A_1 and an empty intersection with $-A_1$, while the other set has a non-empty intersection with $-A_1$ and an empty intersection with A_1. Take the indexes of these sets so that

$B_j \cap A_1 \neq \emptyset$. Then it is not difficult to see that the application of Statement B for the pairs (B_j, B_{-j}) with such an indexation supplies this part of Statement A.

To find a sequence B_j with the above properties for a set A satisfying the conditions of Statement A observe that there is a sequence of finitely many bounded sets B_j of the form $B_j = D_1^{(j)} \times \cdots \times D_n^{(j)}$, $D_l^{(j)} \in \mathcal{B}^\nu$, whose union $B = \bigcup B_j$ satisfies the relation $G^{(n)}(A \,\Delta\, B) < \frac{\varepsilon}{2}$. Because of the symmetry property $A = -A$ of the set A we may assume that these sets B_j have such an indexation with both positive and negative integers for which $B_j = -B_{-j}$. We may also demand that $B_j \cap A \neq \emptyset$ for all sets B_j. Beside this, we may assume, by dividing the sets $D_l^{(j)}$ appearing in the definition of the sets B_j into smaller sets if this is needed that their diameter $\rho(D_l^{(j)}) = \sup\limits_{x \in D_l^{(j)}} \rho(x, 0) < \frac{\eta}{2}$. This implies because of the relation $A \cap K(\eta) = \emptyset$ that $D_l^{(j)} \cap (-D_l^{(j)}) = \emptyset$ for all $1 \le l \le n$. Because of these properties of the sets B_j it can be seen that with the help of their appropriate further splitting the set B can be represented as the union of disjoint sets B_j indexed by some numbers $j = \pm 1, \dots, \pm N$ such that $B_j = -B_{-j}$ for all $1 \le j \le N$, and the pairs (B_j, B_{-j}) have the additional property $D_l^{(j)} \cap (-D_l^{(j)}) = \emptyset$ for all $1 \le l \le n$, and this is what we had to show. For the sake of completeness we present a partition of the set B with the properties we need.

Let us first take the following partition of R^ν for all $1 \le l \le n$. For a fixed number l this partition consists of all sets $\bar{D}_r^{(l)}$ of the form $\bigcap\limits_{j>0} F_{l,j}^{r(j)}$, where $r(j) = 1, 2$ or 3, and $F_{l,j}^{(1)} = D_l^{(j)}$, $F_{l,j}^{(2)} = -D_l^{(j)}$, $F_{l,j}^{(3)} = R^\nu \setminus (D_l^{(j)} \cup (-D_l^{(j)}))$. Then B can be represented as the union of those sets of the form $\bar{D}_{r_1}^{(1)} \times \cdots \times \bar{D}_{r_n}^{(n)}$ which are contained in B.

To prove Statement B first we show that for all $\bar{\varepsilon} > 0$ there is a regular system $\mathcal{D} = \{\Delta_l,\, l = \pm 1, \dots, \pm N\}$ such that all sets D_j and $-D_j$, $1 \le j \le n$, can be expressed as the union of some elements Δ_l of $\mathcal{D}$, and $G(\Delta_l) \le \bar{\varepsilon}$ for all $\Delta_l \in \mathcal{D}$.

First we show that there is a regular system $\bar{\mathcal{D}} = \{\Delta_l',\, l = \pm 1, \dots, \pm N'\}$ such that all sets D_j and $-D_j$ can be expressed as the union of some sets Δ_l' of $\bar{\mathcal{D}}$. But we say nothing about the measure $G(\Delta_l')$ of the elements of this regular system. To get such a regular system we define the sets $\Delta'(\varepsilon_s,\, 1 \le |s| \le n) = D_1^{\varepsilon_1} \cap (-D_1)^{\varepsilon_{-1}} \cap \cdots \cap D_n^{\varepsilon_n} \cap (-D_n)^{\varepsilon_{-n}}$ for all vectors $(\varepsilon_s,\, 1 \le |s| \le n)$ such that $\varepsilon_s = \pm 1$ for all $1 \le |s| \le n$, and the vector $(\varepsilon_s,\, 1 \le |s| \le n)$ contains at least one coordinate $+1$, and $D^1 = D$, $D^{-1} = R^\nu \setminus D$ for all sets $D \in \mathcal{B}^\nu$. Then taking an appropriate reindexation of the sets $\Delta'(\varepsilon_s,\, 1 \le |s| \le n)$ we get a regular system $\bar{\mathcal{D}}$ with the desired properties. (In this construction the sets $\Delta'(\varepsilon_s,\, 1 \le |s| \le n)$ are disjoint, and during their reindexation we drop those of them which equal the empty set.) To see that $\bar{\mathcal{D}}$ with a good indexation is a regular system observe that for a set $\Delta_l = \Delta'(\varepsilon_s,\, 1 \le |s| \le n) \in \bar{\mathcal{D}}$ we have $-\Delta_l = \Delta'(\varepsilon_{-s},\, 1 \le |s| \le n) \in \bar{\mathcal{D}}$, and $\Delta_l \cap (-\Delta_l) \subset D_j \cap (-D_j) = \emptyset$ with some index $1 \le j \le n$. (We had to exclude the possibility $\Delta_l = -\Delta_l$.)

Next we show that by appropriately refining the above regular system $\bar{\mathcal{D}}$ we can get such a regular system $\mathcal{D} = \{\Delta_l,\, l = \pm 1, \dots, \pm N\}$ which satisfies also the property $G(\Delta_l) \le \bar{\varepsilon}$ for all $\Delta_l \in \mathcal{D}$. To show this let us observe that there is a finite partition

$\{E_1, \dots, E_l\}$ of $\bigcup_{j=1}^{n} (D_j \cup (-D_j))$ such that $G(E_j) \le \bar{\varepsilon}$ for all $1 \le j \le l$. Indeed, the closure of $D = \bigcup_{j=1}^{n} (D_j \cup (-D_j))$ can be covered by open sets $H_i \subset R^\nu$ such that $G(H_i) \le \bar{\varepsilon}$ for all sets H_i because of the non-atomic property of the measure G, and by the Heyne–Borel theorem this covering can be chosen finite. With the help of these sets H_i we can get a partition $\{E_1, \dots, E_l\}$ of $\bigcup_{j=1}^{n} (D_j \cup (-D_j))$ with the desired properties.

Then we can make the following construction with the help of the above sets E_j. Take a pair of elements $(\Delta'_l, \Delta'_{-l}) = (\Delta'_l, -\Delta'_l)$, of $\bar{\mathcal{D}}$, and split up the set Δ'_l with the help of the sets E_j to the union of finitely many disjoint sets of the form $\Delta_{l,j} = \Delta'_l \cap E_j$. Then $G(\Delta_{l,j}) < \bar{\varepsilon}$ for all sets $\Delta_{l,j}$, and we can write the set Δ'_{-l} as the union of the disjoint sets $-\Delta_{l,j}$. By applying this procedure for all pairs $(\Delta'_l, \Delta'_{-l})$ and by reindexing the sets $\Delta_{l,j}$ obtained by this procedure in an appropriate way we get a regular system $\mathcal{D}$ with the desired properties.

Let us write $B \cup (-B)$ as the union of products of sets of the form $\Delta_{l_1} \times \cdots \times \Delta_{l_n}$ with sets $\Delta_{l_j} \in \mathcal{D}$, $1 \le j \le n$, and let us discard those products for which $l_j = \pm l_{j'}$ for some pair (j, j'), $j \ne j'$. We define the set F about which we claim that it satisfies Property B as the union of the remaining sets $\Delta_{l_1} \times \cdots \times \Delta_{l_n}$. Then $\chi_F \in \hat{\bar{\mathcal{H}}}_G^n$. Hence to prove that Statement B holds with this set F if $\bar{\varepsilon} > 0$ is chosen sufficiently small it is enough to show that the sum of the terms $G(\Delta_{l_1}) \cdots G(\Delta_{l_n})$ for which $l_j = \pm l_{j'}$ with some $j \ne j'$ is less than $n^2 \bar{\varepsilon} M^{n-1}$, where $M = \max G(D_j \cup (-D_j)) = 2 \max G(D_j)$. To see this observe that for a fixed pair (j, j'), $j \ne j'$, the sum of all products $G(\Delta_{l_1}) \cdots G(\Delta_{l_n})$ such that $l_j = l_{j'}$ can be bounded by $\bar{\varepsilon} M^{n-1}$, and the same estimate holds if summation is taken for products with the property $l_j = -l_{j'}$. Indeed, each term of this sum can be bounded by $\bar{\varepsilon} G^{n-1} \left(\prod_{1 \le p \le n,\, p \ne j} \Delta_{l_p} \right)$, and the events whose G^{n-1} measure is considered in the investigated sum are disjoint. Beside this their union is in the product set $\prod_{1 \le p \le n,\, p \ne j} (D_p \cup D_{-p})$, whose measure is bounded by M^{n-1}.

As the transformation $I_G(f)$ is a contraction from $\hat{\bar{\mathcal{H}}}_G^n$ into $L_2(G, \mathcal{A}, P)$, it can uniquely be extended to the closure of $\hat{\bar{\mathcal{H}}}_G^n$, i.e. to $\bar{\mathcal{H}}_G^n$. We define the n-fold Wiener–Itô integral in the general case via this extension. The expression $I_G(f)$ is a real valued random variable for all $f \in \bar{\mathcal{H}}_G^n$, and relations (4.3), (4.5), (4.5′) remain valid for $f, f' \in \bar{\mathcal{H}}_G^n$ or $f \in \mathcal{H}_G^n$ instead of $f, f' \in \hat{\bar{\mathcal{H}}}_G^n$ of $f \in \hat{\mathcal{H}}_G^n$. Relations (4.5′) and (4.6) imply that the transformation I_G: $\operatorname{Exp} \mathcal{H}_G \to L^2(\Omega, \mathcal{A}, P)$ is an isometry. We shall show that also the following result holds.

Theorem 4.1. *Let a stationary Gaussian random field be given (discrete or generalized one), and let Z_G denote the random spectral measure adapted to it. If we integrate with respect to this Z_G, then the transformation I_G: $\operatorname{Exp} \mathcal{H}_G \to \mathcal{H}$ is unitary. The transformation $(n!)^{1/2} I_G$: $\mathcal{H}_G^n \to \mathcal{H}_n$ is also unitary.*

In the proof of Theorem 4.1 we need an identity whose proof is postponed to the next section.

Theorem 4.2. (Itô's formula) *Let $\varphi_1, \dots, \varphi_m$, $\varphi_j \in \mathcal{H}_G^1$, $1 \le j \le m$, be an orthonormal system in L_G^2. Let some positive integers $j_1, \dots, j_m$ be given, and let $j_1 + \dots + j_m = N$. Define for all $i = 1, \dots, N$ the function g_i as $g_i = \varphi_s$ for $j_1 + \dots + j_{s-1} < i \le j_1 + \dots + j_s$, $1 \le s \le m$. (In particular, $g_i = \varphi_1$ for $0 < i \le j_1$.) Then*

$$\begin{aligned} H_{j_1}\left(\int \varphi_1(x) Z_G(\,dx)\right) \cdots H_{j_m}\left(\int \varphi_m(x) Z_G(\,dx)\right) \\ = \int g_1(x_1) \cdots g_N(x_N)\, Z_G(\,dx_1) \cdots Z_G(\,dx_N) \\ = \int \mathrm{Sym}\,[g_1(x_1) \cdots g_N(x_N)]\, Z_G(\,dx_1) \cdots Z_G(\,dx_N) \end{aligned}$$

($H_j(x)$ denotes again the j-th Hermite polynomial with leading coefficient 1.)

Proof of Theorem 4.1. The one-fold integral $I_G(f)$, $f \in \mathcal{H}_G^1$, agrees with the stochastic integral $I(f)$ defined in Section 3. Hence $I_G(e^{i(n,x)}) = X(n)$ in the discrete field case, and $I_G(\tilde{\varphi}) = X(\varphi)$, $\varphi \in \mathcal{S}$, in the generalized field case. Hence I_G: $\mathcal{H}_G^1 \to \mathcal{H}_1$ is a unitary transformation. Let $\varphi_1, \varphi_2, \dots$ be a complete orthonormal basis in $\mathcal{H}_G^1$. Then $\xi_j = \int \varphi_j(x)\, Z_G(\,dx)$, $j = 1, 2, \dots$, is a complete orthonormal basis in $\mathcal{H}_G^1$. Itô's formula implies that for all sets of positive integers $(j_1, \dots, j_m)$ the random variable $H_{j_1}(\xi_1) \cdots H_{j_m}(\xi_m)$ can be written as a $j_1 + \dots + j_m$-fold Wiener–Itô integral. Therefore Theorem 2.1 implies that the image of $\mathrm{Exp}\,\mathcal{H}_G$ is the whole space $\mathcal{H}$, and I_G: $\mathrm{Exp}\,\mathcal{H}_G$ is unitary.

The image of $\mathcal{H}_G^n$ contains $\mathcal{H}_n$, because of Corollary 2.4 and Itó's formula. Since these images are orthogonal for different n, formula (2.1) implies that the image of $\mathcal{H}_G^n$ coincides with $\mathcal{H}_n$. Hence $(n!)^{1/2} I_G$: $\mathcal{H}_G^n \to \mathcal{H}_n$ is a unitary transformation.

The next result describes the action of shift transformations in $\mathcal{H}$. We know by Theorem 4.1 that all $\eta \in \mathcal{H}$ can be written in the form

$$\eta = f_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int f_n(x_1, \dots, x_n) Z_G(\,dx_1) \dots Z_G(\,dx_n) \tag{4.7}$$

with $f = (f_0, f_1, \dots) \in \mathrm{Exp}\,\mathcal{H}_G$ in a unique way, where Z_G is the random measure adapted to the stationary Gaussian field.

Theorem 4.3. *Let $\eta \in \mathcal{H}$ have the form (4.7). Then*

$$T_t \eta = f_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int e^{i(t, x_1 + \dots + x_n)} f_n(x_1, \dots, x_n) Z_G(\,dx_1) \dots Z_G(\,dx_n)$$

for all $t \in R^\nu$ in the generalized field and for all $t \in \mathbb{Z}_\nu$ in the discrete field case.

Proof of Theorem 4.3. Because of formulas (3.6) and (3.6′) and the definition of the shift operator T_t we have

$$T_t\left(\int e^{i(n,x)}Z_G(\,dx)\right) = T_tX_n = X_{n+t} = \int e^{i(t,x)}e^{i(n,x)}Z_G(\,dx), \quad t \in \mathbb{Z}_\nu,$$

and because of the identity $\widetilde{T_t\varphi}(x) = \int e^{(i(u,x)}\varphi(u-t)\,du = e^{i(t,x)}\tilde{\varphi}(x)$ for $\varphi \in \mathcal{S}$

$$T_t\left(\int \tilde{\varphi}(x)\,Z_G(\,dx)\right) = T_tX(\varphi) = X(T_t\varphi) = \int e^{i(t,x)}\tilde{\varphi}(x)Z_G(\,dx), \quad \varphi \in \mathcal{S}, \quad t \in R^\nu,$$

in the discrete and generalized field cases respectively. Hence

$$T_t\left(\int f(x)Z_G(\,dx)\right) = \int e^{i(t,x)}f(x)Z_G(\,dx) \quad \text{if } f \in \mathcal{H}_G^1$$

for all $t \in \mathbb{Z}_\nu$ in the discrete field and for all $t \in R^\nu$ in the generalized field case. This means that Theorem 4.3 holds in the special case when η is a one-fold Wiener–Itô integral. Let $f_1(x), \dots, f_m(x)$ be an orthogonal system in $\mathcal{H}_G^1$. The set of functions $e^{i(t,x)}f_1(x), \dots, e^{i(t,x)}f_m(x)$ is also an orthogonal system in $\mathcal{H}_G^1$. ($t \in \mathbb{Z}_\nu$ in the discrete and $t \in R^\nu$ in the generalized field case.) Hence Itô's formula implies that Theorem 4.3 also holds for random variables of the form

$$\eta = H_{j_1}\left(\int f_1(x)Z_G(\,dx)\right)\cdots H_{j_m}\left(\int f_m(x)Z_G(\,dx)\right)$$

and for their finite linear combinations. Since these linear combinations are dense in $\mathcal{H}$, Theorem 4.3 holds true.

The next result is a formula for the change of variables in Wiener–Itô integrals.

Theorem 4.4. *Let G and G' be two non-atomic spectral measures such that G is absolutely continuous with respect to G', and let $g(x)$ be a complex valued function such that*

$$g(x) = \overline{g(-x)},$$
$$|g^2(x)| = \frac{dG(x)}{dG'(x)}.$$

For every $f = (f_0, f_1, \dots) \in \operatorname{Exp}\mathcal{H}_G$, we define

$$f_n'(x_1, \dots, x_n) = f_n(x_1, \dots, x_n)g(x_1)\cdots g(x_n), \quad n = 1, 2, \dots, \quad f_0' = f_0.$$

Then $f' = (f_0', f_1', \dots) \in \operatorname{Exp}\mathcal{H}_{G'}^n$, and

$$f_0 + \sum_{n=1}^{\infty}\int \frac{1}{n!}f_n(x_1, \dots, x_n)Z_G(\,dx_1)\dots Z_G(\,dx_n)$$
$$\stackrel{\Delta}{=} f_0' + \sum_{n=1}^{\infty}\frac{1}{n!}\int f_n'(x_1, \dots, x_n)Z_{G'}(\,dx_1)\dots Z_{G'}(\,dx_n),$$

where Z_G and $Z_{G'}$ are Gaussian random spectral measures corresponding to G and G'.

Proof of Theorem 4.4. We have $\|f'_n\|_{G'} = \|f_n\|_G$, hence $f' \in \operatorname{Exp} \mathcal{H}_{G'}$. Let $\varphi_1, \varphi_2, \dots$ be a complete orthonormal system in $\mathcal{H}^1_G$. Then $\varphi'_1, \varphi'_2, \dots$, $\varphi'_j(x) = \varphi_j(x)g(x)$ for all $j = 1, 2, \dots$ is a complete orthonormal system in $\mathcal{H}^1_{G'}$. All functions $f_n \in \mathcal{H}^n_G$ can be written in the form $f(x_1, \dots, x_n) = \sum c_{j_1,\dots,j_n} \operatorname{Sym}(\varphi_{j_1}(x_1) \cdots \varphi_{j_n}(x_n))$. Then $f'(x_1, \dots, x_n) = \sum c_{j_1,\dots,j_n} \operatorname{Sym}(\varphi'_{j_1}(x_1) \cdots \varphi'_{j_n}(x_n))$. Rewriting all terms

$$\int \operatorname{Sym}(\varphi_{j_1}(x_1) \cdots \varphi_{j_n}(x_n)) Z_G(\,dx_1) \dots Z_G(,dx_n)$$

and

$$\int \operatorname{Sym}(\varphi'_{j_1}(x_1) \cdots \varphi'_{j_n}(x_n)) Z_{G'}(\,dx_1) \dots Z_{G'}(,dx_n)$$

by means of Itô's formula we get that f and f' depend on a sequence of independent standard normal random variables in the same way. Theorem 4.4 is proved.

For the sake of completeness I present in the next Lemma 4.5 another type of change of variable result. I formulate it only in that simple case in which we need it in some later considerations.

Lemma 4.5. *Define for all $t > 0$ the (multiplication) transformation $T_t x = tx$ either from R^ν to R^ν or from the torus $[-\pi, \pi)^\nu$ to the torus $[-\pi t, \pi t)^\nu$. Given a spectral measure G on R^ν or on $[-\pi, \pi)^\nu$ define the spectral measure G_t on R^ν or on $[-\pi t, \pi t)^\nu$ by the formula $G_t(A) = G(\frac{A}{t})$ for all measurable sets A, and similarly define the function $f_{k,t}(x_1, \dots, x_k) = f_k(tx_1, \dots, tx_k)$ for all measurable functions f_k of k variables, $k = 1, 2, \dots$, with $x_j \in R^\nu$ or $x_j \in [-\pi, \pi)^\nu$ for all $1 \le j \le k$, and put $f_{0,t} = f_0$. If $f = (f_0, f_1, \dots) \in \operatorname{Exp} \mathcal{H}_G$, then $f_t = (f_{0,t}, f_{1,t}, \dots) \in \operatorname{Exp} \mathcal{H}_{G_t}$, and*

$$\begin{aligned} f_0 + \sum_{n=1}^{\infty} \int \frac{1}{n!} f_n(x_1, \dots, x_n) Z_G(\,dx_1) \dots Z_G(\,dx_n) \\ \stackrel{\Delta}{=} f_{0,t} + \sum_{n=1}^{\infty} \frac{1}{n!} \int f_{n,t}(x_1, \dots, x_n) Z_{G_t}(\,dx_1) \dots Z_{G_t}(\,dx_n), \end{aligned}$$

where Z_G and Z_{G_t} are Gaussian random spectral measures corresponding to G and G'.

Proof of Lemma 4.5. It is easy to see that $f_t = (f_{0,t}, f_{1,t}, \dots) \in \operatorname{Exp} \mathcal{H}_{G_t}$. Moreover, we may define the random spectral measure Z_{G_t} in the identity we want to prove by the formula $Z_{G_t}(A) = Z_G(\frac{A}{t})$. But with such a choice of Z_{G_t} we can write even $=$ instead of $\stackrel{\Delta}{=}$ in this formula.

The next result shows a relation between Wick polynomials and Wiener–Itô integrals.

Theorem 4.6. *Let a stationary Gaussian field be given, and let Z_G denote the random spectral measure adapted to it. Let $P(x_1, \dots, x_m) = \sum c_{j_1,\dots,j_n} x_{j_1} \cdots x_{j_n}$ be a homogeneous polynomial of degree n, and let $h_1, \dots, h_m \in \mathcal{H}^1_G$. (Here $j_1, \dots, j_n$ are n indices*

such that $1 \le j_l \le m$ *for all* $1 \le l \le n$. *It is possible that* $j_l = j_{l'}$ *also if* $l \ne l'$.) *Define the random variables* $\xi_j = \int h_j(x) Z_G(\,dx)$, $j = 1, 2, \dots, m$, *and the function* $\tilde{P}(u_1, \dots, u_n) = \sum c_{j_1,\dots,j_n} h_{j_1}(u_1) \cdots h_{j_n}(u_n)$. *Then*

$$:P(\xi_1, \dots, \xi_m): \; = \int \tilde{P}(u_1, \dots, u_n) Z_G(\,du_1) \dots Z_G(\,du_n).$$

Remark. If P is a polynomial of degree n, then it can be written as $P = P_1 + P_2$, where P_1 is a homogeneous polynomial of degree n, and P_2 is a polynomial of degree less than n. Obviously,

$$:P(\xi_1, \dots, \xi_m): \; =: P_1(\xi_1, \dots, \xi_m):$$

Proof of Theorem 4.6. It is enough to show that

$$:\xi_{j_1} \cdots \xi_{j_n}: \; = \int h_{j_1}(u_1) \cdots h_{j_n}(u_n) Z_G(\,du_1) \dots Z_G(\,du_n).$$

If $h_1, \dots, h_m \in \mathcal{H}_G^1$ are orthonormal, (all functions h_l have norm 1, and if $l \ne l'$, then h_l and $h_{l'}$ are either orthogonal or $h_l = h_{l'}$), then this relation follows from a comparison of Corollary 2.3 with Itô's formula. In the general case an orthonormal system $\bar{h}_1, \dots, \bar{h}_m$ can be found such that

$$h_j = \sum_{k=1}^{m} c_{j,k} \bar{h}_k, \quad j = 1, \dots, m$$

with some real constants $c_{j,k}$. Set $\eta_k = \int \bar{h}_j Z_G(\,dx)$. Then

$$\begin{aligned} :\xi_{j_1} \cdots \xi_{j_n}: \; &=: \left(\sum_{k=1}^{m} c_{j_1,k} \eta_k \right) \cdots \left(\sum_{k=1}^{m} c_{j_n,k} \eta_k \right): \; = \sum_{k_1,\dots,k_n} c_{j_1,k_1} \cdots c_{j_n,k_n} :\eta_{k_1} \cdots \eta_{k_n}: \\ &= \sum_{k_1,\dots,k_n} c_{j_1,k_1} \cdots c_{j_n,k_n} \int \bar{h}_{k_1}(u_1) \cdots \bar{h}_{k_n}(u_n) Z_G(\,du_1) \dots Z_G(du_n) \\ &= \int h_{j_1}(u_1) \cdots h_{j_n}(u_n) Z_G(\,du_1) \dots Z_G(du_n) \end{aligned}$$

as we claimed.

We finish this section by showing how the Wiener–Itô integral can be defined if the spectral measure G may have atoms. We do this although such a construction seems to have a limited importance as in most applications the restriction that we apply the Wiener–Itô integral only in the case of a non-atomic spectral measure G causes no serious problem. If we try to give this definition by modifying the original one, then we have to split up the atoms. The simplest way we found for this splitting up, was the use of randomization.

Let G be a spectral measure on R^ν, and let Z_G be a corresponding Gaussian spectral random measure on a probability space $(\Omega, \mathcal{A}, P)$. Let us define a new spectral measure $\hat{G} = G \times \lambda_{[-\frac{1}{2},\frac{1}{2}]}$ on $R^{\nu+1}$, where $\lambda_{[-\frac{1}{2},\frac{1}{2}]}$ denotes the uniform distribution on the interval $[-\frac{1}{2}, \frac{1}{2}]$. If the probability space $(\Omega, \mathcal{A}, P)$ is sufficiently rich, a random spectral measure $Z_{\hat{G}}$ corresponding to $\hat{G}$ can be defined on it in such a way that $Z_{\hat{G}}(A \times [-\frac{1}{2}, \frac{1}{2}]) = Z_G(A)$ for all $A \in \mathcal{B}^\nu$. For $f \in \bar{\mathcal{H}}_G^n$ we define the function $\hat{f} \in \bar{\mathcal{H}}_{\hat{G}}^n$ by the formula $\hat{f}(y_1, \dots, y_n) = f(x_1, \dots, x_n)$ if y_j is the juxtaposition (x_j, u_j), $x_j \in R^\nu$, $u_j \in R^1$, $j = 1, 2, \dots, n$. Finally we define the Wiener–Itô integral in the general case by the formula

$$\int f(x_1, \dots, x_n) Z_G(\,dx_1) \dots Z_G(\,dx_n) = \int \hat{f}(y_1, \dots, y_n) Z_{\hat{G}}(\,dy_1) \dots Z_{\hat{G}}(dy_n).$$

(What we actually have done was to introduce a virtual new coordinate u. With the help of this new coordinate we could reduce the general case to the special case when G is non-atomic.) If G is a non-atomic spectral measure, then the new definition of Wiener–Itô integrals coincides with the original one. It is easy to check this fact for one-fold integrals, and then Itô's formula proves it for multiple integrals. It can be seen with the help of Itô's formula again, that all results of this section remain valid for the new definition of Wiener–Itô integrals. In particular, we formulate the following result.

Given a stationary Gaussian field let Z_G be the random spectral measure adapted to it. All $f \in \mathcal{H}_G^n$ can be written in the form

$$f(x_1, \dots, x_n) = \sum c_{j_1,\dots,j_n} \varphi_{j_1}(x_1) \cdots \varphi_{j_n}(x_n) \tag{4.8}$$

with some functions $\varphi_j \in \mathcal{H}_G^1$, $j = 1, 2, \dots$. Define $\xi_j = \int \varphi_j(x) Z_G(\,dx)$. If f has the form (4.8), then

$$\int f(x_1, \dots, x_n) Z_G(\,dx_1) \dots Z_G(\,dx_n) = \sum c_{j_1,\dots,j_n} {:}\, \xi_{j_1} \cdots \xi_{j_n} {:}\,.$$

The last identity would provide another possibility for defining Wiener–Itô integrals.

5. The proof of Itô's formula. The diagram formula and some of its consequences.

We shall prove Itô's formula with the help of the following

Proposition 5.1. *Let $f \in \bar{\mathcal{H}}_G^n$ and $h \in \bar{\mathcal{H}}_G^1$. Let us define the functions*

$$f \underset{k}{\times} h(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) = \int f(x_1, \dots, x_n)\overline{h(x_k)}G(\,dx_k), \quad k = 1, \dots, n,$$

and

$$fh(x_1, \dots, x_{n+1}) = f(x_1, \dots, x_n)h(x_{n+1}).$$

Then $f \underset{k}{\times} h$, $k = 1, \dots, n$, and fh are in $\bar{\mathcal{H}}_G^{n-1}$ and $\bar{\mathcal{H}}_G^{n+1}$ respectively, and their norm satisfies the inequality $\|f \underset{k}{\times} h\| \le \|f\| \cdot \|h\|$ and $\|fh\| \le \|f\| \cdot \|h\|$. The relation

$$n! I_G(f) I_G(h) = (n+1)! I_G(fh) + \sum_{k=1}^{n} (n-1)! I_G\left(f \underset{k}{\times} h\right)$$

holds true.

Remark. There is a small inaccuracy in the formulation of Lemma 5.1. We considered the Wiener–Itô integral of the function $f \underset{k}{\times} h$ with arguments $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$, while we defined this integral for functions with arguments $x_1, \dots, x_{n-1}$. We can correct this inaccuracy by reindexing the variables of $f \underset{k}{\times} h$ and to work with the function $(f \underset{k}{\times} h)'(x_1, \dots, x_{n-1}) = f \underset{k}{\times} h(x_{\alpha_k(1)}, \dots, x_{\alpha_k(k-1)}, x_{\alpha_k(k+1)}, \dots, x_{\alpha_k(n)})$ instead of $f \underset{k}{\times} h$, where $\alpha_k(j) = j$ for $1 \le j \le k-1$ and $\alpha_k(j) = j-1$ for $k+1 \le j \le n$.

We also need the following recursion formula for Hermite polynomials.

Lemma 5.2.

$$H_n(x) = xH_{n-1}(x) - (n-1)H_{n-2}(x) \quad \textit{for} \quad n = 1, 2, \dots,$$

with the notation $H_{-1}(x) \equiv 0$.

Proof of Lemma 5.2.

$$\begin{aligned} H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}) &= -e^{x^2/2} \frac{d}{dx}\left(H_{n-1}(x)e^{-x^2/2}\right) \\ &= xH_{n-1}(x) - \frac{d}{dx}H_{n-1}(x). \end{aligned}$$

Since $\frac{d}{dx}H_{n-1}(x)$ is a polynomial of order $n-2$ with leading coefficient $n-1$ we can write

$$\frac{d}{dx}H_{n-1}(x) = (n-1)H_{n-2}(x) + \sum_{j=0}^{n-3} c_j H_j(x).$$

To complete the proof of Lemma 5.2 it remains to show that in the last expansion all coefficients c_j are zero. This follows from the orthogonality of the Hermite polynomials and the calculation

$$\begin{aligned}\int e^{-x^2/2} H_j(x) \frac{d}{dx} H_{n-1}(x)\, dx &= -\int H_{n-1}(x) \frac{d}{dx}(e^{-x^2/2} H_j(x))\, dx \\ &= \int e^{-x^2/2} H_{n-1}(x) P_{j+1}(x)\, dx = 0\end{aligned}$$

with the polynomial $P_{j+1}(x) = xH_j(x) - \frac{d}{dx} H_j(x)$ of order $j+1$ for $j \le n-3$.

Proof of Theorem 4.2 via Proposition 5.1. We prove Theorem 4.2 by induction. Theorem 4.2 holds for $N = 1$. Assume that it holds for $N-1$. Let us define the functions

$$\begin{aligned} f(x_1, \dots, x_{N-1}) &= g_1(x_1) \cdots g_{N-1}(x_{N-1}) \\ h(x) &= g_N(x). \end{aligned}$$

Then

$$\begin{aligned} J &= \int g_1(x_1) \cdots g_N(x_N) Z_G(\, dx_1) \dots Z_G(\, dx_N) \\ &= N!\, I_G(fh) = (N-1)!\, I_G(f) I_G(h) - \sum_{k=1}^{N-1} (N-2)!\, I_G\left(f \underset{k}{\times} h\right) \end{aligned}$$

by Proposition 5.1. The induction hypothesis implies that

$$\begin{aligned} J =& H_{j_1}\left(\int \varphi_1(x) Z_G(\, dx)\right) \cdots H_{j_{m-1}}\left(\int \varphi_{m-1}(x) Z_G(\, dx)\right) \\ & \qquad H_{j_m - 1}\left(\int \varphi_m(x) Z_G(\, dx)\right) \int \varphi_m(x) Z_G(\, dx) \\ & - (j_m - 1) H_{j_1}\left(\int \varphi_1(x) Z_G(\, dx)\right) \cdots H_{j_{m-1}}\left(\int \varphi_{m-1}(x) Z_G(\, dx)\right) \\ & \qquad H_{j_m - 2}\left(\int \varphi_m(x) Z_G(\, dx)\right), \end{aligned}$$

where $H_{j_m-2}(x) = H_{-1}(x) \equiv 0$ if $j_m = 1$. This relation holds, since

$$\begin{aligned} f \underset{k}{\times} h(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{N-1}) &= \int g_1(x_1) \cdots g_{N-1}(x_{N-1}) \overline{\varphi_m(x_k)} G(\, dx_k) \\ &= \begin{cases} 0 \quad \text{if} \quad k \le N - j_m \\ g_1(x_1) \cdots g_{k-1}(x_{k-1}) g_{k+1}(x_{k+1}) \cdots g_{N-1}(x_{N-1}) \quad \text{if } N - j_m < k \le N-1. \end{cases} \end{aligned}$$

Hence Lemma 5.2 implies that

$$\begin{aligned} J = \prod_{s=1}^{m-1} H_{j_s}\left(\int \varphi_s(x) Z_G(\, dx)\right) \Big[& H_{j_m - 1}\left(\int \varphi_m(x) Z_G(\, dx)\right) \int \varphi_m(x) Z_G(\, dx) \\ & - (j_m - 1) H_{j_m - 2}\left(\int \varphi_m(x) Z_G(\, dx)\right)\Big] = \prod_{s=1}^{m} H_{j_s}\left(\int \varphi_s(x) Z_G(\, dx)\right), \end{aligned}$$

as claimed.

Let us fix some functions $h_1 \in \bar{\mathcal{H}}_G^{n_1}$, ... , $h_m \in \bar{\mathcal{H}}_G^{n_m}$. In the next result, in the so-called diagram formula, we express the product $n_1!I_G(h_1)\cdots n_m!I_G(h_m)$ as the sum of Wiener–Itô integrals. This result contains Proposition 5.1 as a special case. There is no unique terminology for this result in the literature. We shall follow the notation of Dobrushin in [7].

We shall use the term diagram of order $(n_1,\dots,n_m)$ for an undirected graph of $n_1+\cdots+n_m$ vertices such that its vertices are indexed by the pairs of integers (j,l), $l=1,\dots,m,\ j=1,\dots,n_l$, with the properties that no more than one edge enters into each vertex, and edges can connect only pairs of vertices (j_1,l_1) and (j_2,l_2) for which $l_1\neq l_2$. Let $\Gamma=\Gamma(n_1,\dots,n_m)$ denote the set of all diagrams. Given a diagram $\gamma\in\Gamma$ $|\gamma|$ denotes the number of edges in γ. Let there be given a set of functions $h_1\in\bar{\mathcal{H}}_G^{n_1},\dots,$ $h_m\in\bar{\mathcal{H}}_G^{n_m}$. Sometimes we denote the variables of the function h_l by $x_{(j,l)}$ instead of x_j, i.e. we write $h_l(x_{(1,l)},\dots,x_{(n_l,l)})$ instead of $h_l(x_1,\dots,x_{n_l})$. Put $N=n_1+\cdots+n_m$. We introduce the function of N variables corresponding to the vertices of the diagram by the formula

$$h(x_{(j,l)},\ l=1,\dots,m,\ j=1,\dots,n_l)=\prod_{l=1}^m h_l(x_{(j,l)},\ j=1,\dots,n_l).$$

Fixing a diagram $\gamma\in\Gamma$ we enumerate the variables $x_{(j,l)}$ in such a way that the vertices into which no edges enter will have the numbers $1,2,\dots,N-2|\gamma|$ and the vertices connected by an edge will have the numbers p and $p+|\gamma|$, where $p=N-2|\gamma|+1,\dots,N-|\gamma|$. Let

$$\begin{aligned}h_\gamma(x_1,\dots,x_{N-2|\gamma|})=\int\cdots\int h(x_1,\dots,x_{N-|\gamma|},-x_{N-2|\gamma|+1},\dots,-x_{N-|\gamma|})\\ G(\,dx_{N-2|\gamma|+1})\dots G(\,dx_{N-|\gamma|}).\end{aligned}\tag{5.1}$$

The function h_γ depends only on the variables $x_1,\dots,x_{N-2|\gamma|}$, i.e. it is independent of how the vertices connected by edges are indexed. Indeed, it follows from the evenness of the spectral measure that by interchanging the indices s and $s+\gamma$ of two vertices connected by an edge does not change the value of the integral h_γ. Let us now consider $I_G(h_\gamma)$. The function h_γ may depend on the numbering of those vertices of γ from which no edge starts, but $\operatorname{Sym}h_\gamma$ and therefore $I_G(h_\gamma)$ does not depend on it. Now we formulate the following

Theorem 5.3. (Diagram formula) *For all functions $h_1\in\bar{\mathcal{H}}_G^{n_1}$, ... , $h_m\in\bar{\mathcal{H}}_G^{n_m}$, $n_1,\dots,n_m=1,2,\dots$, the following relations hold:*

A) $h_\gamma\in\bar{\mathcal{H}}_G^{n-2|\gamma|}$, *and* $\|h_\gamma\|\le\prod_{j=1}^m\|h_j\|$ *for all* $\gamma\in\Gamma$.

B) $n_1!I_G(h_1)\cdots n_m!I_G(h_m)=\sum_{\gamma\in\Gamma}(N-2|\gamma|)!I_G(h_\gamma)$.

Remark. In the special case $m = 2$, $n_1 = n$, $n_2 = 1$ Theorem 5.3 coincides with Proposition 5.1. To see this it is enough to observe that $h(-x) = \overline{h(x)}$ for all $h \in \bar{\mathcal{H}}_G^1$.

Proof of Theorem 5.3. It suffices to prove Theorem 5.3 in the special case $m = 2$. Then the case $m > 2$ follows by induction.

We shall use the notation $n_1 = n$, $n_2 = m$, and we write $x_1, \dots, x_{n+m}$ instead of $x_{(1,1)}, \dots, x_{(n,1)}, x_{(1,2)} \dots, x_{(m,2)}$. It is clear that the function h_γ satisfies property (a) of the classes $\bar{\mathcal{H}}_G^j$ defined in Section 4. We show that Part A of Theorem 5.3 is a consequence of the Schwartz inequality. To prove this estimate on the norm of h_γ it is enough to restrict ourselves to such diagrams γ in which the vertices $(n, 1)$ and $(m, 2)$, $(n-1, 1)$ and $(m-1, 2)$, $\dots$, $(n-k, 1)$ and $(m-k, 2)$ are connected by edges with some $0 \le k \le \min(n, m)$. In this case we can write

$$
\begin{aligned}
&|h_\gamma(x_1, \dots, x_{n-k-1}, x_{n+1}, \dots, x_{n+m-k-1})|^2 \\
&\quad = \Big| \int h_1(x_1, \dots, x_n) h_2(x_{n+1}, \dots, x_{n+m-k-1}, -x_{n-k}, \dots, -x_n) \\
&\qquad\qquad G(\,dx_{n-k}) \dots G(\,dx_n) \Big|^2 \\
&\quad \le \int |h_1(x_1, \dots, x_n)|^2 G(\,dx_{n-k}) \dots G(\,dx_n) \\
&\qquad\qquad \int |h_2(x_{n+1}, \dots, x_{n+m})|^2 G(\,dx_{n+m-k}) \dots G(\,dx_{n+m})
\end{aligned}
$$

by the Schwartz inequality and the symmetry $G(-A) = G(A)$ of the spectral measure G. Integrating this inequality with respect to the free variables we get part A) of Theorem 5.3.

In the proof of part B) first we restrict ourselves to the case when $h_1 \in \hat{\bar{\mathcal{H}}}_G^n$ and $h_2 \in \hat{\bar{\mathcal{H}}}_G^m$. Assume that they are adapted to a regular system $\mathcal{D} = \{\Delta_j,\ j = \pm 1, \dots, \pm N\}$ of subsets of R^n with finite measure G. We may even assume that all $\Delta_j \in \mathcal{D}$ satisfy the inequality $G(\Delta_j) < \varepsilon$ with some $\varepsilon > 0$ to be chosen later, because otherwise we could split up the sets Δ_j into smaller ones. Let us fix a point $u_j \in \Delta_j$ in all sets $\Delta_j \in \mathcal{D}$. Put $K_i = \sup_x |h_i(x)|$, $i = 1, 2$, and let A be a cube containing all Δ_j.

We can write

$$
\begin{aligned}
I = n! I_G(h_1) m! I_G(h_2) = {\sum}' & h_1(u_{j_1}, \dots, u_{j_n}) h_2(u_{k_1}, \dots, u_{k_m}) \\
& Z_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n}) Z_G(\Delta_{k_1}) \cdots Z_G(\Delta_{k_m})
\end{aligned}
$$

with the numbers $u_{j_p} \in \Delta_{j_p}$ and $u_{k_r} \in \Delta_{k_r}$ we have fixed, where the summation in $\sum'$ goes through all pairs $((j_1, \dots, j_n), (k_1, \dots, k_m))$, j_p, $k_r \in \{\pm 1, \dots, \pm N\}$, $p = 1, \dots, n$, $r = 1, \dots, m$, such that $j_p \ne \pm j_{\bar{p}}$ and $k_r \ne \pm k_{\bar{r}}$ if $p \ne \bar{p}$ or $r \ne \bar{r}$.

Write

$$
\begin{aligned}
I = \sum_{\gamma \in \Gamma} {\sum}^{\gamma} & h_1(u_{j_1}, \dots, u_{j_n}) h_2(u_{k_1}, \dots, u_{k_m}) \\
& Z_G(\Delta_{j_1}) \cdots Z_G(\Delta_{j_n}) Z_G(\Delta_{k_1}) \cdots Z_G(\Delta_{k_m}),
\end{aligned}
$$

where $\sum^{\gamma}$ contains those terms of $\sum'$ for which $j_p = k_r$ or $j_p = -k_r$ if the vertices $(1,p)$ and $(2,r)$ are connected in γ, and $j_p \neq \pm k_r$ if $(1,p)$ and $(2,r)$ are not connected. Let us define the sets

$$A_1 = A_1(\gamma) = \{p\colon\ p \in \{1,\dots,n\},\ \text{and no edge starts from } (p,1) \text{ in } \gamma\},$$
$$A_2 = A_2(\gamma) = \{r\colon\ r \in \{1,\dots,m\},\ \text{and no edge starts from } (r,2) \text{ in } \gamma\}$$

and

$$B = B(\gamma) = \{(p,r)\colon\ p \in \{1,\dots,n\},\ r \in \{1,\dots,m\},\ (p,1) \text{ and } (r,2) \text{ are connected in } \gamma\}$$

together with the map $\alpha\colon\ \{1,\dots,n\} \setminus A_1 \to \{1,\dots,m\} \setminus A_2$ defined as

$$\alpha(p) = r \quad \text{if } (p,r) \in B \quad \text{for all } \ p \in \{1,\dots,n\} \setminus A_1. \tag{5.2}$$

Let Σ^{γ} denote the value of the inner sum $\sum^{\gamma}$ for some $\gamma \in \Gamma$ in the last summation formula, and write it in the form

$$\Sigma^{\gamma} = \Sigma_1^{\gamma} + \Sigma_2^{\gamma}$$

with

$$\Sigma_1^{\gamma} = \sum\nolimits^{\gamma} h_1(u_{j_1},\dots,u_{j_n}) h_2(u_{k_1},\dots,u_{k_m}) \prod_{p\in A_1} Z_G(\Delta_{j_p}) \prod_{r\in A_2} Z_G(\Delta_{k_r}) \cdot \prod_{(p,r)\in B} E\left(Z_G(\Delta_{j_p}) Z_G(\Delta_{k_r})\right)$$

and

$$\Sigma_2^{\gamma} = \sum\nolimits^{\gamma} h_1(u_{j_1},\dots,u_{j_n}) h_2(u_{k_1},\dots,u_{k_m}) \prod_{p\in A_1} Z_G(\Delta_{j_p}) \prod_{r\in A_2} Z_G(\Delta_{k_r}) \cdot \left[\prod_{(p,r)\in B} Z_G(\Delta_{j_p}) Z_G(\Delta_{k_r}) - E\left(\prod_{(p,r)\in B} Z_G(\Delta_{j_p}) Z_G(\Delta_{k_r}) \right) \right].$$

The random variables Σ_1^{γ} and Σ_2^{γ} are real valued. To see this observe that if the sum defining these expressions contains a term with arguments Δ_{j_p}, and Δ_{k_r}, then it also contains the term with arguments $-\Delta_{j_p}$ and $-\Delta_{k_r}$. This fact together with property (v) of the random spectral measure Z_G and the analogous property of the functions h_1 and h_2 imply that $\Sigma_1^{\gamma} = \overline{\Sigma_1^{\gamma}}$ and $\Sigma_2^{\gamma} = \overline{\Sigma_2^{\gamma}}$. Hence these random variables are real valued. As a consequence, we can bound $(n+m-2|\gamma|)! I_G(h_\gamma) - \Sigma_1^{\gamma}$ and Σ_2^{γ} by means of their second moment.

We are going to show that Σ_1^γ is a good approximation of $(n+m-2|\gamma|)!\, I_G(h_\gamma)$, and Σ_2^γ is negligibly small. This implies that $(n+m-2|\gamma|)!I_G(h_\gamma)$ well approximates Σ^γ.

To estimate $(n+m-2|\gamma|)!I_G(h_\gamma)-\Sigma_1^\gamma$ we rewrite Σ_1^γ as a Wiener–Itô integral with a kernel function adapted to the regular system $\mathcal{D}$ which is close to h_γ. To find this kernel function we rewrite the sum defining Σ_1^γ by first fixing the variables u_{j_p}, $p \in A_1$, and u_{k_r}, $r \in A_2$, and summing up by the remaining variables, and after this summing by the variables fixed at the first step. We get that

$$\Sigma_1^\gamma = \sum_{\substack{j_p\colon 1\le|j_p|\le N \text{ for all } p\in A_1\\ k_r\colon 1\le|k_r|\le N \text{ for all } r\in A_2}} h_{\gamma,1}(j_p,\ p\in A_1,\ k_r,\ r\in A_2) \prod_{p\in A_1} Z_G(\Delta_{j_p}) \prod_{r\in A_2} Z_G(\Delta_{k_r}) \tag{5.3}$$

with a function $h_{\gamma,1}$ depending on the arguments j_p, $p \in A_1$, and k_r, $r \in A_2$, with values $j_p, k_r \in \{\pm1,\dots,\pm N\}$ defined with the help another function $h_{\gamma,2}$ described below. It also depends on the arguments j_p, $p \in A_1$, and k_r, $r \in A_2$, with values $j_p, k_r \in \{\pm1,\dots,\pm N\}$. Formula (5.3) holds with

$$h_{\gamma,1}(j_p,\ p\in A_1,\ k_r,\ r\in A_2) = 0 \tag{5.4a}$$

if the numbers of the set $\{\pm j_p\colon\ p\in A_1\}\cup\{\pm k_r\colon\ r\in A_2\}$ are not all different, and

$$h_{\gamma,1}(j_p,\ p\in A_1,\ k_r,\ r\in A_2) = h_{\gamma,2}(j_p,\ p\in A_1,\ k_r,\ r\in A_2) \tag{5.4b}$$

if all numbers $\pm j_p$, $p \in A_1$, and $\pm k_r$, $r \in A_2$ are different with the function $h_{\gamma,2}(j_p,\ p\in A_1,\ k_r,\ r\in A_2)$ defined for all sequences j_p, $p \in A_1$ and k_r, $r \in A_2$, with $j_p, k_r \in \{\pm1,\dots,\pm N\}$ (i.e. also in the case when some of the arguments j_p, $p \in A_1$, or k_r, $r \in A_2$, agree) by the formula

$$\begin{aligned} h_{\gamma,2}(j_p,\ p\in A_1,\ k_r,\ r\in A_2) = {}& \sum\nolimits^{\gamma,1} h_1(u_{j_1},\dots,u_{j_n})h_2(u_{k_1},\dots,u_{k_m}) \\ & \cdot \prod_{(p,r)\in B} E\left(Z_G(\Delta_{j_p})Z_G(\Delta_{k_r})\right). \end{aligned} \tag{5.5}$$

The summation $\sum^{\gamma,1}$ in formula (5.5) which depends on the arguments j_p, $p \in A_1$, and k_r, $r \in A_2$, is defined in the following way. We sum up for such sequences j_p, k_r with indices $p \in \{1,\dots,n\}\setminus A_1$ and $r \in \{1,\dots,m\}\setminus A_2$ which satisfy the following conditions. Put $C = \{\pm j_p,\ p\in A_1\}\cup\{\pm k_r,\ r\in A_2\}$. We demand that all numbers j_p and k_r with indices $p \in \{1,\dots,n\}\setminus A_1$ and $r \in \{1,\dots,m\}\setminus A_2$ are such that $j_p, k_r \in \{\pm1,\dots,\pm N\}\setminus C$. Let us write all numbers $r \in \{1,\dots,m\}\setminus A_2$ in the form $r = \alpha(p)$, $p \in \{1,\dots,n\}\setminus A_1$, with the map α defined in (5.2). We also demand that only such $k_r = k_{\alpha(p)}$ appear in the summation for which $k_{\alpha(p)} = \pm j_p$ for all $p \in \{1,\dots,n\}\setminus A_1$. Beside this, all numbers $\pm j_p$, $p \in \{1,\dots,n\}\setminus A_1$, must be different. The summation in $\sum^{\gamma,1}$ is taken for all such sequences j_p, $p \in \{1,\dots,n\}\setminus A_1$ and k_r, $r \in \{1,\dots,m\}\setminus A_2$ which satisfy the above conditions.

Formula (5.5) can be rewritten in a simpler form. To do this let us first observe that the condition $k_{\alpha(p)} = \pm j_p$ can be replaced by the condition $k_{\alpha(p)} = -j_p$ in it, and we can

write $G(\Delta_{j_p})$ instead of the term $EZ_G(\Delta_{j_p})Z_G(\Delta_{k_r})$ (with $(p,r)\in B$) in the product at the end of (5.5). This follows from the fact that $EZ_G(\Delta_{j_p})Z_G(\Delta_{k_r}) = EZ_G(\Delta_{j_p})^2 = 0$ if $k_r = j_p$ and $EZ_G(\Delta_{j_p})Z_G(\Delta_{k_r}) = EZ_G(\Delta_{j_p}Z_G(-\Delta_{j_p}) = G(\Delta_{j_p})$ if $k_r = -j_p$. Beside this, the expression in (5.5) does not change if we take summation for all sequences j_p, $p\in\{1,\dots,n\}\setminus A$, with $j_p\in\{\pm1,\dots,\pm N\}$, because in such a way we only attach such terms to the sum which equal zero. This follows from the fact that both functions h_1 and h_2 are adapted to the regular system $\mathcal{D}$, hence $h_1(u_{j_1},\dots,u_{j_n})h_2(u_{k_1},\dots,u_{k_m}) = 0$ if for an index $p\in\{1,\dots,n\}\setminus A_1$ $j_p = \pm j_{p'}$ with $p\neq p'$ or $j_p = -k_r$ with $(p,r)\in B$, and beside this there exists some $r'\in A_2$ such that $j_p = \pm k_{r'}$.

The above relations enable us to rewrite (5.5) in the following way. Let us define that map α^{-1} on the set $\{1,\dots,m\}\setminus A_2$ which is the inverse of the map α defined in (5.2), i.e. $\alpha^{-1}(r) = p$ if $(p,r)\in B$. With this notation we can write

$$\begin{aligned} &h_{\gamma,2}(j_p,\ p\in A_1,\ k_r,\ r\in A_2)\\ &\quad= \sum_{\substack{j_p,\ p\in\{1,\dots,n\}\setminus A_1,\\ 1\le|j_p|\le N \text{ for all indices } p}} h_1(u_{j_1},\dots,u_{j_n})h_2(u_{k_r},\ r\in A_2, -u_{j_{\alpha^{-1}(r)}},\ r\in\{1,\dots,m\}\setminus A_2)\\ &\qquad\qquad \prod_{p\in\{1,\dots,n\}\setminus A_1} G(\Delta_{j_p}). \end{aligned} \tag{5.6}$$

Formula (5.6) can be rewritten as

$$\begin{aligned} &h_{\gamma,2}(j_p,\ p\in A_1,\ k_r,\ r\in A_2)\\ &\quad= \int h_1(u_{j_p},\ p\in A_1,\ x_p,\ p\in\{1,\dots,n\}\setminus A_1)\\ &\qquad h_2(u_{k_r},\ r\in A_2,\ -x_{\alpha^{-1}(r)},\ r\in\{1,\dots,m\}\setminus A_2) \prod_{p\in\{1,\dots,n\}\setminus A_1} G(\,dx_p). \end{aligned} \tag{5.7}$$

We define with the help of $h_{\gamma,1}$ and $h_{\gamma,2}$ two new functions on $R^{(n+m-2|\gamma|)\nu}$ with arguments $x_1,\dots,x_{n+m-2|\gamma|}$. The first one will be the kernel function of the Wiener–Itô integral expressing Σ_1^γ and the second one will be equal to the function h_γ defined in (5.1). We define these functions in two steps. In the first step we reindex the arguments of the functions $h_{1,\gamma}$ and h_2,γ to get functions depending on sequences $j_1,\dots,j_{n+m-2|\gamma|}$. For this goal we list the elements of the sets A_1 and A_2 as $A_1 = \{p_1,\dots,p_{n-|\gamma|}\}$ with $1\le p_1<p_2<\dots<p_{n-|\gamma|}\le n$ and $A_2 = \{r_1,\dots,r_{m-|\gamma|}\}$ with $1\le r_1<r_2<\dots<p_{m-|\gamma|}\le m$ and define the maps $\beta_1\colon A_1\to\{1,\dots,n-|\gamma|\}$ and $\beta_2\colon A_2\to\{n-|\gamma|+1,\dots,n+m-2|\gamma|\}$ by the formulas $\beta_1(p_l) = l$ if $1\le l\le n-\gamma$, $1\le l\le n-|\gamma|$, and $\beta_2(r_l) = l+n-|\gamma|$, $1\le l\le m-|\gamma|$, if $n-|\gamma|+1\le l\le n+m-2|\gamma|$. We define with the help of the maps β_1 and β_2 the functions

$$h_{\gamma,3}(j_1,\dots,j_{n+m-2|\gamma|}) = h_{\gamma,1}(j_{\beta_1(r_1)},\dots,j_{\beta_1(n-|\gamma|)}),k_{\beta_2(1)},\dots,k_{\beta_2(m-|\gamma|)})$$

and

$$h_{\gamma,4}(j_1,\dots,j_{n+m-2|\gamma|}) = h_{\gamma,2}(j_{\beta_1(r_1)},\dots,j_{\beta_1(n-|\gamma|)}),k_{\beta_2(1)},\dots,k_{\beta_2(m-|\gamma|)}),$$

where the arguments of the functions $h_{\gamma,3}$ and $h_{\gamma,4}$ are sequences $j_1, \dots, j_{n+m-2|\gamma|}$ with $j_s \in \{\pm 1, \dots, \pm N\}$ for all $1 \le s \le n+m-2|\gamma|$.

With the help of the above functions we define the following functions $h_{\gamma,5}$ and $h_{\gamma,6}$ on $R^{(n+m-2|\gamma|)\nu}$.

$$h_{\gamma,5}(x_1, \dots, x_{n+m-2|\gamma|}) = \begin{cases} h_{\gamma,3}(j_1, \dots, j_{n+m-2|\gamma|}) & \text{if } x_l \in \Delta_{j_l}, \\ \qquad \text{for all } 1 \le l \le n+m-2|\gamma| \\ 0 \quad \text{otherwise,} \end{cases}$$

and

$$h_{\gamma,6}(x_1, \dots, x_{n+m-2|\gamma|}) = \begin{cases} h_{\gamma,4}(j_1, \dots, j_{n+m-2|\gamma|}) & \text{if } x_l \in \Delta_{j_l}, \\ \qquad \text{for all } 1 \le l \le n+m-2|\gamma| \\ 0 \quad \text{otherwise.} \end{cases}$$

It follows from relation (5.4a) and the definition of the function $h_{\gamma,5}$ (with the help of the definition of the functions $h_{\gamma,1}$ and $h_{\gamma,3}$) that $h_{\gamma,5} \in \hat{\hat{\mathcal{H}}}_G^n$, and it is adapted to the regular system $\mathcal{D}$. Then relations (5.3) and the definition of $h_{\gamma,5}$ also imply that $\Sigma_1^\gamma = (n+m-2|\gamma|)! I(h_{\gamma,5})$.

On the other hand, I claim that the function h_γ defined in (5.1) satisfies the identity $h_\gamma = h_{\gamma,6}$. This statement must be formulated in a more precise form, because the definition of the function h_γ is not unique, we have some freedom in choosing the indices of its variables. I shall define such a version of h_γ which provides an appropriate enumeration of its variables, and the identity $h_\gamma = h_{\gamma,6}$ holds for this version.

In the definition of h_γ we shall work with the function $\hat{h}_2(x_{n+1}, \dots, x_{n+m}) = h_2(x_{\delta(1)}, \dots, x_{\delta(m)})$, with $\delta(l) = l-n$, $n+1 \le l \le n+m$, i.e. we work with a function with arguments x_{l+n} instead of arguments x_l, $1 \le l \le m$. We also replace the set A_2 by its shift $\bar{A}_2$ defined as $\bar{A}_2 = A_2 + n = \{\bar{r}_1, \dots, \bar{r}_{m-|\gamma|}\}$ with $\bar{r}_j = r_j + n$, $1 \le j \le m-|\gamma|$, where $1 \le r_1 < r_2 < \dots < r_{m-|\gamma|} \le m$ are the elements of the set A_2. We define the following two functions $\varepsilon(\cdot)$ and $\eta(\cdot)$ on the set $\{n+1, \dots, n+m\}$. We put $\varepsilon(l) = 1$ if $l \in \bar{A}_2$, and $\varepsilon(l) = -1$ if $l \in \{n+1, \dots, n+m\} \setminus A_2$, and we also introduce the function $\eta(\cdot)$ defined as $\eta(l) = l$ if $l \in \bar{A}_2$ and $\eta(l) = p$ with that number p, $1 \le p \le n$, for which $(p, l-n) \in B$ if $l \in \{n+1, \dots, n+m\} \setminus \bar{A}_2$. With the help of these functions we define the function

$$\hat{h}_\gamma(x_l, l \in A_1 \cup \bar{A}_2) = \int h_1(x_1, \dots, x_n) \\ \hat{h}_2(\varepsilon(n+1)x_{\eta(n+1)}, \dots, \varepsilon(n+m)x_{\eta(n+m)}) \prod_{l \in \{1,\dots,n\} \setminus A_1} G(\,dx_l),$$

and define the function h_γ by introducing the 'right' enumeration of the variables of the function $\hat{h}_\gamma$. For this goal we define (similarly to the maps β_1 and β_2 defined before) the map $\bar{\beta}$: $A_1 \cup \bar{A}_2 \to \{1, \dots, n+m-2|\gamma|\}$ as $\bar{\beta}(p_l) = l$ for $1 \le l \le n-|\gamma|$, and $\bar{\beta}(\bar{r}_l) = l+n-|\gamma|$ for $1 \le l \le m-|\gamma|$, where $A_1 = \{p_1, \dots, p_{n-|\gamma|}$ with $1 \le p_1 < \dots <$

$p_{n|\gamma|} \le n$, and $\bar{A}_2 = A_2 + n = \{\bar{r}_1, \dots, \bar{r}_{m-|\gamma|}\}$ with $n+1 \le \bar{r}_1 < \dots < \bar{r}_{m-|\gamma|} \le n+m$. Then we define

$$h_\gamma(x_1, \dots, x_{n+m-2|\gamma|}) = \hat{h}_\gamma(x_{\bar{\beta}(l)},\ l \in A_1 \cup \bar{A}_2).$$

To see that $h_{\gamma,6} = h_\gamma$ with the above defined function h_γ let us first observe that

$$h_{\gamma,6}(x_1, \dots, x_{n+m-2|\gamma|}) = h_{\gamma,2}(j_{\beta_1(r_1)}, \dots, j_{\beta_1(n-|\gamma|)}), k_{\beta_2(1)}, \dots, k_{\beta_2(m-|\gamma|)}),$$

if $x_l \in \Delta_{j_l}$ for all $1 \le n+m-2|\gamma|$. On the other hand, we get, since both functions h_1 and h_2 are adapted to $\mathcal{D}$, by applying the definition of the functions h_γ and $\hat{h}_\gamma$ and relation (5.7) together with a comparison of the function h_2 with $\hat{h}_2$ and of the pair of maps β_1 and β_2 with the map $\bar{\beta}$ that

$$\begin{aligned} h_\gamma(x_1, \dots, x_{n+m-2|\gamma|}) = h_\gamma(u_{j_1}, \dots, u_{j_{n+m-2|\gamma|}}) = \hat{h}_\gamma(u_{\bar{\beta}(l)},\ l \in A_1 \cup \bar{A}_2) \\ h_{\gamma,2}(j_{\beta_1(r_1)}, \dots, j_{\beta_1(n-|\gamma|)}), k_{\beta_2(1)}, \dots, k_{\beta_2(m-|\gamma|)}) \end{aligned}$$

if $x_l \in \Delta_{j_l}$ for all $1 \le n+m-2|\gamma|$.

By these identities $h_{\gamma,6}(x_1, \dots, x_{n+m-2|\gamma|})$ and $h_\gamma(x_1, \dots, x_{n+m-2|\gamma|})$ agree in such points $(x_1, \dots, x_{n+m-2|\gamma|})$ for which $x_l \in \Delta_{j_l}$ with some $j_l \in \{\pm 1, \dots, \pm N\}$ for all $1 \le l \le n+m-2|\gamma|$. Since both functions $h_{\gamma,6}$ equal zero in other points, this implies that $h_{\gamma,6} = h_\gamma$, as claimed.

Observe that the function h_γ disappears also in such points $(x_1, \dots, x_{n+m-2|\gamma|})$ for which $x_l \in \Delta_{j_l}$ for all $1 \le l \le n+m-2|\gamma|$ with such indices j_l for which some of the numbers in the set $\{\pm j_1, \dots, \pm j_{n-|\gamma|}\}$ or in the set $\{\pm j_{n-|\gamma|+1}, \dots, \pm j_{n+m-2|\gamma|}\}$ agree. This fact together with the identity $h_\gamma = h_{\gamma,6}$ and the relation between the functions $h_{\gamma,5}$ and $h_{\gamma,6}$ yield the identity

$$h_\gamma(x_1, \dots, x_{n+m-2|\gamma|}) = h_{\gamma,5}(x_1, \dots, x_{n+m-2|\gamma|}) + h_{\gamma,7}(x_1, \dots, x_{n+m-2|\gamma|})$$

with

$$\begin{aligned} &h_{\gamma,7}(x_1, \dots, x_{n+m-2|\gamma|}) \\ &\quad = \begin{cases} h_\gamma(x_1, \dots, x_{n+m-2|\gamma|}) \quad \text{if there exist indices } j_l,\ 1 \le |j_l| \le N, \\ \qquad 1 \le l \le n+m-2|\gamma| \text{ such that } x_l \in \Delta_{j_l},\ 1 \le l \le n+m-2|\gamma|, \\ \qquad \text{all numbers } \pm j_1, \dots, \pm j_{n-2|\gamma|} \text{ are different}, \\ \qquad \text{all numbers } \pm j_{n-|\gamma|+1}, \dots, \pm j_{n+m-2|\gamma|} \text{ are different}, \\ \qquad \text{and } \{\pm j_1, \dots, \pm j_{n-|\gamma|}\} \cap \{\pm j_{n-|\gamma|+1}, \dots, \pm j_{n+m-2|\gamma|}\} \ne \emptyset \\ 0 \quad \text{otherwise.} \end{cases} \end{aligned}$$

Since $\Sigma_1^\gamma = (n+m-2|\gamma|)! I_G(h_{\gamma,5})$, we have

$$(n+m-2|\gamma|)! I_G(h_\gamma) - \Sigma_1^\gamma = (n+m-2|\gamma|)! I_G(h_{\gamma,7}),$$

and

$$E(\Sigma_1^\gamma - (n+m-2|\gamma|)!I_G(h_\gamma))^2 \le (n+m-2|\gamma|)!\|(h_{\gamma,7})\|^2$$

with the norm $\|\cdot\|$ in $\bar{\mathcal{H}}_G^{n+m-2|\gamma|}$.

On the other hand,

$$\sup|h_{\gamma,7}(x_1,\dots,x_{n+m-2|\gamma|})| \le \sup|h_\gamma(x_1,\dots,x_{n+m-2|\gamma|})| \le K_1K_2L^{|\gamma|},$$

with $K_1 = \sup|h_1|$, $K_2 = \sup|h_2|$, and $L = G(A)$, where A is a fixed cube containing all Δ_j. Hence

$$\begin{aligned} E(\Sigma_1^\gamma - (n+m-2|\gamma|)!I_G(h_\gamma))^2 &\le C_1\|(h_{\gamma,7})\|^2 \le C_2{\sum}'' G(\Delta_{j_1})\cdots G(\Delta_{j_{n+m-2|\gamma|}}) \\ &\le C\sup_j G(\Delta_j) \le C\varepsilon, \end{aligned} \tag{5.8}$$

where the summation $\sum''$ goes for such sequences $j_1,\dots,j_{n+m-2|\gamma|}$, $1 \le |j_l| \le N$ for all $1 \le l \le n+m-2|\gamma|$, for which all numbers $\pm j_1,\dots,\pm j_{n-|\gamma|}$ are different, the same relation holds for the elements of the sequence $\pm j_{n-|\gamma|+1},\dots,\pm j_{n+m-2|\gamma|}$, and

$$\{\pm j_1,\dots,\pm j_{n-|\gamma|}\} \cap \{\pm j_{n-|\gamma|+1},\dots,\pm j_{n+m-2|\gamma|}\} \ne \emptyset.$$

The constants C_1, C_2 and C may depend on the functions h_1, h_2 and spectral measure G, but they do not depend on the regular system $\mathcal{D}$, hence in particular on the parameter ε. In the verification of (5.8) we can exploit that each term in the sum $\sum''$ is a product which contains a factor $G(\Delta_j)^2 \le \varepsilon G(\Delta_j)$. Here an argument can be applied which is similar to the closing step in the proof of *Statement B* in the proof of the fact that $\hat{\bar{\mathcal{H}}}_G^n$ is dense in the space $\bar{\mathcal{H}}_G^n$.

Now we turn to the estimation of $E(\Sigma_2^\gamma)^2$. It can be expressed as a linear combination of terms of the form

$$\begin{aligned} &\Sigma_3^\gamma(j_p,k_r,j_{\bar p},k_{\bar r},\ p,\bar p \in \{1,\dots,n\},\ r,\bar r \in \{1,\dots,m\}) \\ &\quad = E\Bigg(\Bigg(\prod_{p\in A_1} Z_G(\Delta_{j_p}) \prod_{r\in A_2} Z_G(\Delta_{k_r}) \prod_{\bar p\in A_1} Z_G(\Delta_{j_{\bar p}}) \prod_{\bar r\in A_2} Z_G(\Delta_{k_{\bar r}})\Bigg) \\ &\qquad \Bigg[\prod_{(p,r)\in B} Z_G(\Delta_{j_p})Z_G(\Delta_{k_r}) - E\prod_{(p,r)\in B} Z_G(\Delta_{j_p})Z_G(\Delta_{k_r})\Bigg] \\ &\qquad \Bigg[\prod_{(\bar p,\bar r)\in B} Z_G(\Delta_{j_{\bar p}})Z_G(\Delta_{k_{\bar r}}) - E\prod_{(\bar p,\bar r)\in B} Z_G(\Delta_{j_{\bar p}})Z_G(\Delta_{k_{\bar r}})\Bigg]\Bigg), \end{aligned} \tag{5.9}$$

where Σ_3^γ depends on such sequences of numbers j_p, k_r, $j_{\bar p}$, $k_{\bar r}$ with indices $1 \le p,\bar p \le n$ and $1 \le r,\bar r \le m$ for which $j_p,k_r,j_{\bar p},k_{\bar r} \in \{\pm1,\dots,\pm N\}$ for all indices $p,r,\bar p$ and $\bar r$,

$j_p = k_r$ or $j_p = -k_r$ if $(p, r) \in B$, otherwise all numbers $\pm j_p$, $\pm k_r$ are different, and the same relations hold for the indices $j_{\bar p}$ and $k_{\bar r}$ if p is replaced by $\bar p$ and r is replaced by $\bar r$. Moreover the absolute value of all coefficients in this linear combination is bounded by $\sup |h_1(x)|^2 \sup |h_2(x)|^2$.

We want to show that for most sets of arguments $(j_p,\, k_r,\, j_{\bar p},\, k_{\bar r})$ Σ_3^γ equals zero, and it is also small in the remaining cases.

Let us fix a sequence of arguments $j_p,\, k_r,\, j_{\bar p},\, k_{\bar r}$ of Σ_3^γ, and let us estimate its value with these arguments. Define the sets

$$\mathcal{A} = \{j_p\colon\, p \in A_1\} \cup \{k_r\colon\, r \in A_2\} \quad \text{and} \quad \bar{\mathcal{A}} = \{j_{\bar p}\colon\, \bar p \in A_1\} \cup \{k_{\bar r}\colon\, \bar r \in A_2\}.$$

We claim that Σ_3^γ equals zero if $\bar{\mathcal{A}} \neq -\mathcal{A}$. In this case there exists an index $l \in \mathcal{A}$ such that $-l \notin \bar{\mathcal{A}}$. Let us carry out the multiplication in (5.9). Because of the independence properties of random spectral measures each product in this expression can be written as the product of independent factors, and the independent factor containing the term $Z_G(\Delta_l)$ has zero expectation. To see this observe that the set Δ_l appears exactly once among the arguments of the terms $Z_G(\Delta_{j_p})$ and $Z_G(\Delta_{k_r})$, and none of these terms contains the argument $-\Delta_l = \Delta_{-l}$. Although $-l \notin \bar{\mathcal{A}}$, it may happen that $l \in \bar{\mathcal{A}}$. In this case the product under investigation contains the independent factor $Z_G(\Delta_l)^2$ with $EZ_G(\Delta_l)^2 = 0$. If $l \notin \bar{\mathcal{A}}$, then there are two possibilities. Either this product contains an independent factor of the form $Z_G(\Delta_l)$ with $EZ_G(\Delta_l) = 0$, or there is a pair $(\bar p, \bar r) \in B$ such that $(j_{\bar p}, k_{\bar r}) = (\pm l, \pm l)$, and an independent factor of the form $Z_G(\Delta_l)Z_G(\pm\Delta_{-l})Z_G(\pm\Delta_l)$ with the property $EZ_G(\Delta_l)Z_G(\pm\Delta_{-l})Z_G(\pm\Delta_l) = 0$ appears.

Let

$$\mathcal{F} = \bigcup_{(p,r)\in B} \{j_p, k_r\} \quad \text{and} \quad \bar{\mathcal{F}} = \bigcup_{(\bar p,\bar r)\in B} \{(j_{\bar p}, k_{\bar r}\}.$$

A factorization argument shows again that the expression in (5.9) equals zero if the sets $\mathcal{F} \cup (-\mathcal{F})$ and $\bar{\mathcal{F}} \cup (-\bar{\mathcal{F}})$ are disjoint. We can restrict ourselves to the case $\mathcal{A} = -\bar{\mathcal{A}}$, and in this case $\pm\mathcal{A}$ is disjoint both of $\mathcal{F} \cup (-\mathcal{F})$ and $\bar{\mathcal{F}} \cup (-\bar{\mathcal{F}})$, and the product under investigation contains the independent factor $\prod_{(p,r)\in B} Z_G(\Delta_{j_p})Z_G(\Delta_{k_r}) - E\prod_{(p,r)\in B} Z_G(\Delta_{j_p})Z_G(\Delta_{k_r})$ with expectation zero.

Moreover, if $\mathcal{F} \cup (-\mathcal{F})$ and $\bar{\mathcal{F}} \cup (-\bar{\mathcal{F}})$ are not disjoint, (and $\mathcal{A} = -\bar{\mathcal{A}}$), then the absolute value of the expression in (5.9) can be estimated from above by

$$C\,\varepsilon \prod G(\Delta_{j_p})G(\Delta_{k_r})G(\Delta_{j_{\bar p}})G(\Delta_{k_{\bar r}}) \tag{5.10}$$

with a universal constant $C < \infty$ depending only on the parameters n and m, where the indices $j_p,\, k_r,\, j_{\bar p},\, k_{\bar r}$ are the same as in (5.9) with the following difference: All indices appear in (5.10) with multiplicity 1, and if both indices l and $-l$ are present in (5.9), then one of them is omitted form (5.10). The multiplying term ε appears in (5.10), since by carrying out the multiplications in (5.9) and factorizing each term,

we get that all non-zero terms have a factor $EZ_G(\Delta)^2 Z_G(-\Delta)^2 = E(\operatorname{Re} Z_G(\Delta)^2 + \operatorname{Im} Z_G(\Delta)^2)^2 = E \operatorname{Re} Z_G(\Delta)^4 + E \operatorname{Im} Z_G(\Delta)^4 + 2E \operatorname{Re} Z_G(\Delta)^2 E \operatorname{Im} Z_G(\Delta)^2 = 8G(\Delta)^2$ or $\big(E|Z_G(\Delta)|^2\big)^2 = G(\Delta)^2$, and $G(\Delta) < \varepsilon$ for $\Delta \in \mathcal{D}$. (We did not mention the possibility of an independent factor of the form $EZ_G(\Delta)^4$ or $EZ_G(\Delta)^3 Z_G(-\Delta)$ with $\Delta \in \mathcal{D}$, because as some calculation shows, $EZ_G(\Delta)^4 = 0$ and $EZ_G(\Delta)^3 Z_G(-\Delta) = 0$.)

Let us express $E(\Sigma_2^\gamma)^2$ as the linear combination of the quantities Σ_3^γ, and let us bound each term Σ_3^γ in the above way. This supplies an upper bound for $E(\Sigma_2^\gamma)^2$ by means of a sum of terms of the form (5.10). Moreover, each of these terms appears only with a multiplicity less than $C(n,m)$ with an appropriate constant $C(n,m)$. Hence we can write

$$E(\Sigma_2^\gamma)^2 \le K_1^2 K_2^2 C(n,m) C\varepsilon \sum_{r=1}^{n+m} {\sum_{j_1,\dots,j_r}}''' G(\Delta_{j_1}) \cdots G(\Delta_{j_r}),$$

where the indices $j_1, \dots, j_r \in \{\pm 1, \dots, \pm N\}$ in the sum $\sum'''$ are all different, and $K_j = \sup |h_j(x)|$, $j = 1, 2$. Hence

$$E(\Sigma_2^\gamma)^2 \le C_1 \varepsilon \sum_{r=1}^{n+m} G(A)^r \le C_2 \varepsilon$$

with some appropriate constants C_1 and C_2. Because of the inequality (5.8), the identity $n! I_G(h_1) m! I_G(h_2) = \sum\limits_{\gamma \in \Gamma} (\Sigma_1^\gamma + \Sigma_\gamma^2)$ and the last relation one has

$$\begin{aligned} E &\left(n! I_G(h_1) m! I_G(h_2) - \sum_{\gamma \in \Gamma} (n+m-2|\gamma|)! I_G(h_\gamma) \right)^2 \\ &= E \left(\sum_{\gamma \in \Gamma} (\Sigma_1^\gamma + \Sigma_2^\gamma - (n+m-2|\gamma|)!\, I_G(h_\gamma)) \right)^2 \\ &\le C_3 \left(\sum_{\gamma \in \Gamma} E((m+n-2|\gamma|)!\, I_G(h_\gamma) - \Sigma_1^\gamma)^2 + E(\Sigma_2^\gamma)^2 \right) \le C_4 \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrary small, part B is proved in the special case $h_1 \in \hat{\bar{\mathcal{H}}}_G^n$, $h_2 \in \hat{\bar{\mathcal{H}}}_G^m$.

If $h_1 \in \bar{\mathcal{H}}_G^n$ and $h_2 \in \bar{\mathcal{H}}_G^m$, then let us choose a sequence of functions $h_{1,r} \in \hat{\bar{\mathcal{H}}}_G^n$ and $h_{2,r} \in \hat{\bar{\mathcal{H}}}_G^m$ such that $h_{1,r} \to h_1$ and $h_{2,r} \to h_2$ in the norm of the spaces $\bar{\mathcal{H}}_G^n$ and $\bar{\mathcal{H}}_G^m$ respectively. Define the functions $\hat{h}_\gamma(r)$ and $h_\gamma(r)$ in the same way as h_γ, but substitute the pair of functions (h_1, h_2) by $(h_{1,r}, h_2)$ and $(h_{1,r}, h_{2,r})$ in their definition. We shall show by the help of Part A) that

$$E|I_G(h_1) I_G(h_2) - I_G(h_{1,r}) I_G(h_{2,r})| \to 0,$$

and

$$E|I_G(h_\gamma) - I_G(h_\gamma(r))| \to 0 \quad \text{for all } \gamma \in \Gamma$$

as $r \to \infty$. Then a simple limiting procedure shows that Theorem 5.3 holds for all $h_1 \in \bar{\mathcal{H}}_G^n$ and $h_2 \in \bar{\mathcal{H}}_G^m$.

We have

$$\begin{aligned} &E|I_G(h_1)I_G(h_2) - I_G(h_{1,r})I_G(h_{2,r})| \\ &\quad \le E|(I_G(h_1 - h_{1,r}))I_G(h_2)| + E|I_G(h_{1,r})I_G(h_2 - h_{2,r})| \\ &\quad \le \frac{1}{n!\,m!}\left(\|h_1 - h_{1,r}\|^{1/2}\|h_2\|^{1/2} + \|h_2 - h_{2,r}\|^{1/2}\|h_{1,r}\|\right) \to 0, \end{aligned}$$

and by part A) of Theorem 5.3

$$\begin{aligned} E|I_G(h_\gamma) - I_G(h_\gamma(r))| &\le E|I_G(h_\gamma) - I_G(\hat{h}_\gamma(r))| + E|I_G(h_\gamma(r)) - I_G(\hat{h}_\gamma(r))| \\ &\le \|h_\gamma - \hat{h}_\gamma(r)\|^{1/2} + \|h_\gamma(r) - \hat{h}_\gamma(r)\|^{1/2} \\ &\le \|h_1 - \hat{h}_{1,r}\|^{1/2}\|h_2\|^{1/2} + \|h_2 - \hat{h}_{2,r}\|^{1/2}\|h_{1,r}\|^{1/2} \to 0. \end{aligned}$$

Theorem 5.3 is proved.

We formulate some consequences of Theorem 5.3. Let $\bar{\Gamma} \subset \Gamma$ denote the set of complete diagrams, i.e. let a diagram $\gamma \in \bar{\Gamma}$ if an edge enters in each vertex of γ. We have $EI(h_\gamma) = 0$ for all $\gamma \in \Gamma \setminus \bar{\Gamma}$, since (4.3) holds for all $f \in \bar{\mathcal{H}}_G^n$, $n \ge 1$. If $\gamma \in \bar{\Gamma}$, then $I(h_\gamma) \in \bar{\mathcal{H}}_G^0$. Let h_γ denote the value of $I(h_\gamma)$ in this case. Now we have the following

Corollary 5.4. *For all* $h_1 \in \bar{\mathcal{H}}_G^{n_1}, \dots, h_n \in \bar{\mathcal{H}}_G^{n_m}$

$$En_1!I_G(h_1)\cdots n_m!I_G(h_m) = \sum_{\gamma \in \bar{\Gamma}} h_\gamma.$$

(The sum on the right-hand side equals zero if $\bar{\Gamma}$ *is empty.)*

As a consequence of Corollary 5.4 we can calculate the expectation of products of Wick polynomials of Gaussian random variables.

Let $X_{k,j}$, $EX_{k,j} = 0$, $1 \le k \le p$, $1 \le j \le n_k$, be a sequence of Gaussian random variables. We want to calculate the expected value of the Wick polynomials $:X_{k,1}\cdots X_{k,n_k}:$, $1 \le k \le p$, if we know all covariances $EX_{k,j}X_{\bar{k},\bar{j}} = a((k,j),(\bar{k},\bar{j}))$, $1 \le k, \bar{k}, \le p$, $1 \le j \le n_k$, $1 \le \bar{j} \le \bar{n}_k$. For this goal let us consider the class of closed diagrams $\bar{\Gamma}(k_1, \dots, k_p)$, and define the following quantity $\gamma(A)$ depending on the closed diagrams γ and the set A of all covariances $EX_{k,j}X_{\bar{k},\bar{j}} = a((k,j),(\bar{k},\bar{j}))$.

$$\gamma(A) = \prod_{((k,j),(\bar{k},\bar{j})) \text{ is an edge of } \gamma} a((k,j),(\bar{k},\bar{j})), \quad \gamma \in \Gamma.$$

With the above notation we can formulate the following result.

Corollary 5.5. *Let $X_{k,j}$, $EX_{k,j} = 0$, $1 \le k \le p$, $1 \le j \le n_k$, be a sequence of Gaussian random variables. Let $a((k,j),(\bar k,\bar j)) = EX_{k,j}X_{\bar k,\bar j}$, $1 \le k, \bar k, \le p$, $1 \le j \le n_k$, $1 \le \bar j \le \bar n_k$ denote the covariances of these random variables. Then the expected value of the Wick polynomials $:X_{k,1}\cdots X_{k,n_k}:$, $1 \le k \le p$, can be expressed as*

$$E\left(\prod_{k=1}^{p} :X_{k,1}\cdots X_{k,n_k}:\right) = \sum_{\gamma\in\bar\Gamma(k_1,\dots,k_p)} \gamma(A)$$

with the above defined quantities $\gamma(A)$. In the case when $\bar\Gamma(k_1,\dots,k_p)$ is empty, e.g. if $k_1+\cdots+k_p$ is an odd number, the above expectation equals zero.

Remark. In the special case when $X_{k,1} = \cdots = X_{k,n_k} = X_k$, and $EX_k^2 = 1$ for all indices $1 \le k \le p$ Corollary 5.5 provides a formula for the expectation of the product of Hermite polynomials of standard normal random variables. In this case we have $a((k,j),(\bar k,\bar j)) = \bar a(k,\bar k)$ with a function $\bar a(\cdot,\cdot)$ not depending on the arguments j and $\bar j$, and the left-hand side of the identity in Corollary 5.5 equals $EH_{n_1}(X_1)\cdots H_{n_p}(X_p)$ with standard normal random variables $X_1,\dots,X_n$ with correlations $EX_kX_{\bar k} = \bar a(k,\bar k)$.

Proof of Corollary 5.5. We can represent the random variables $X_{k,j}$ in the form $X_{k,j} = \sum_p c_{k,j,p}\xi_p$ with some appropriate coefficients $c_{k,j,p}$, where $\xi_1,\xi_2,\dots$ is a sequence of independent standard normal random variables. Let $Z(\,dx)$ denote a random spectral measure corresponding to the one-dimensional spectral measure with density function $g(x) = \frac{1}{2\pi}$ for $|x| < \pi$, and $g(x) = 0$ for $|x| \ge \pi$. The random integrals $\int e^{ipx}Z(\,dx)$, $p = 0, \pm1, \pm2, \dots$, are independent standard normal random variables. Define $h_{k,j}(x) = \sum_p c_{k,j,p}e^{ipx}$, $k = 1,\dots,p$, $1 \le j \le n_k$. The random variables $X_{k,j}$ can be identified with the random integrals $\int h_{k,j}(x)Z(\,dx)$, $k = 1,\dots,p$, $1 \le j \le n_k$, since their joint distributions coincide. Put $\hat h_k(x_1,\dots,x_{n_k}) = \prod_{j=1}^{n_k} h_{k,j}(x_j)$. It follows from Theorem 4.6 that

$$:X_{k,1}\cdots X_{k,n_k}: \;= \int \hat h_k(x_1,\dots,x_{n_k})Z(\,dx_1)\dots Z(\,dx_{n_k}) = n_k!I(\hat h_k(x_1,\dots,x_{n_k}))$$

for all $1 \le k \le p$. Hence an application of Corollary 5.4 yields Corollary 5.5. One only has to observe that $\int_{-\pi}^{\pi} h_{k,j}(x)\overline{h_{\bar k,\bar j}(x)}\,dx = a((k,j),(\bar k,\bar j))$ for all k, $k = 1,\dots,p$ and $1 \le j \le n_k$.

Theorem 5.3 states in particular that the product of Wiener–Itô integrals with respect to a random spectral measure of a stationary Gaussian fields belongs to the Hilbert space $\mathcal{H}$ defined by this field, since it can be written as a sum of Wiener–Itô integrals. This means a trivial measurability condition, and also that the product has a finite second moment, which is not so trivial. Theorem 5.3 actually gives the following non-trivial inequality.

Let $h_1 \in \mathcal{H}_G^{n_1}, \dots, h_m \in \mathcal{H}_G^{n_m}$. Let $|\bar{\Gamma}(n_1, n_1, \dots, n_m, n_m)|$ denote the number of complete diagrams in $\bar{\Gamma}(n_1, n_1, \dots, n_m, n_m)$, and put

$$C(n_1, \dots, n_m) = \frac{|\bar{\Gamma}(n_1, n_1, \dots, n_m, n_m)|}{n_1! \cdots n_m!}.$$

In the special case $n_1 = \cdots = n_m = n$ let $\bar{C}(n, m) = C(n_1, \dots, n_m)$. Then

Corollary 5.6.

$$E\left[(n_1! I_G(h_1))^2 \cdots (n_m! I_G(h_m))^2\right] \le C(n_1, \dots, n_m) E(n_1! I_G(h_1))^2 \cdots (n_m! E(I_G(h_m))^2.$$

In particular,

$$E\left[(n! I_G(h))^{2m}\right] \le \bar{C}(n, m)(E(n! I_G(h))^2)^m \quad \text{if } h \in \mathcal{H}_G^n.$$

Corollary 5.6 follows immediately from Corollary 5.4 by applying it first for the sequence $h_1, h_1, \dots, h_m, h_m$ and then for the pair h_j, h_j which yields that $E(n_j! I_G(h_j))^2 = n_j! \|h_j\|^2$, $1 \le j \le m$. One only has to observe that $|h_\gamma| \le \|h_1\|^2 \cdots \|h_m\|^2$ for all complete diagrams by Part A) of Theorem 5.3.

The inequality in Corollary 5.6 is sharp. If G is a finite measure and $h_1 \in H_G^{n_1}, \dots, h_m \in H_G^{n_m}$ are constant functions, then equality can be written in Corollary 5.6. We remark that in this case $I_G(h_1), \dots, I_G(h_m)$ are constant times the n_1-th, $\dots$, n_m-th Hermite polynomials of the same standard normal random variable. Let us emphasize that the constant $C(n_1, \dots, n_m)$ depends only on the parameters $n_1, \dots, n_m$ and not on the form of the functions $h_1, \dots, h_m$. The function $C(n_1, \dots, n_m)$ is monotone in its arguments. The following argument shows that

$$C(n_1 + 1, n_2, \dots, n_m) \ge C(n_1, \dots, n_m)$$

Let us say that two complete diagrams in $\bar{\Gamma}(n_1, n_1, \dots, n_m, n_m)$ or in $\bar{\Gamma}(n_1+1, n_1+1, \dots, n_m, n_m)$ are equivalent if they can be transformed into each other by permuting the vertices $(1,1), \dots, (1, n_1)$ in $\bar{\Gamma}(n_1, n_1, \dots, n_m, n_m)$ or the vertices $(1,1), \dots, (1, n_1+1)$ in $\bar{\Gamma}(n_1+1, n_1+1, \dots, n_m, n_m)$. The equivalence classes have $n_1!$ elements in the first case and $(n_1+1)!$ elements in the second one. Moreover, the number of equivalence classes is less in the first case than in the second one. (They would agree if we counted only those equivalence classes in the second case which contain a diagram where $(1, n_1+1)$ and $(2, n_1, 1)$ are connected by an edge. Hence

$$\frac{1}{n_1!}|\bar{\Gamma}(n_1, n_1, \dots, n_m, n_m)| \le \frac{1}{(n_1+1)!}|\bar{\Gamma}(n_1+1, n_1+1, \dots, n_m, n_m)|$$

as we claimed.

The next result, formulated in a more elementary way, may better illuminate the content of Corollary 5.6.

Corollary 5.7. *Let $\xi_1, \dots, \xi_k$ be a normal random vector, and $P(x_1, \dots, x_k)$ a polynomial of degree n. Then*

$$E\left[P(\xi_1, \dots, \xi_k)^{2m}\right] \le \bar{C}(n,m)(n+1)^m \left(EP(\xi_1, \dots, \xi_k)^2\right)^m.$$

The multiplying constant $\bar{C}(n,m)(n+1)^m$ is not sharp in this case. Observe that it does not depend on k.

Proof of Corollary 5.7. We can write $\xi_j = \int f_j(x) Z(\,dx)$ with some $f_j \in \mathcal{H}^1$, $j = 1, 2, \dots, k$, where $Z(\,dx)$ is the same as in the proof of Corollary 5.5. There exist some $h_j \in \mathcal{H}^j$, $j = 0, 1, \dots, n$, such that

$$P(\xi_1, \dots, \xi_k) = \sum_{j=0}^{n} j! I(h_j).$$

Then

$$\begin{aligned}
EP(\xi_1, \dots, \xi_k)^{2m} &= E\left[\left(\sum_{j=0}^{n} j! I(h_j)\right)^{2m}\right] \le (n+1)^m E\left[\sum_{j=0}^{n} (j! I(h_j))^2\right]^m \\
&\le (n+1)^m \sum_{p_1+\dots+p_n=m} C(p_1, \dots, p_n) (EI(h_0)^2)^{p_0} \cdots (En! I(h_n)^2)^{p_n} \frac{m!}{p_1! \cdots p_n!} \\
&\le (n+1)^m \bar{C}(n,m) \sum_{p_1+\dots+p_n=m} (EI(h_0)^2)^{p_0} \cdots (EI(n! h_n)^2)^{p_n} \frac{m!}{p_1! \cdots p_n!} \\
&= (n+1)^m \bar{C}(n,m) \left[\sum E(j! I(h_j))^2\right]^m = (n+1)^m \bar{C}(n,m) \left(EP(\xi_1, \dots, \xi_k)^2\right)^m.
\end{aligned}$$

6. Subordinated random fields. Construction of self-similar fields.

Let X_n, $n \in \mathbb{Z}_\nu$, be a discrete stationary Gaussian random field, and let the random field ξ_n, $n \in \mathbb{Z}_\nu$, be subordinated to it. Let Z_G denote the random spectral measure adapted to the random field X_n. By Theorem 4.1 the random variable ξ_0 can be represented as

$$\xi_0 = f_0 + \sum_{k=1}^{\infty} \frac{1}{k!} \int f_k(x_1, \dots, x_k) Z_G(\, dx_1) \dots Z_G(\, dx_k)$$

with an appropriate $f = (f_0, f_1, \dots) \in \operatorname{Exp} \mathcal{H}_G$ in a unique way. This formula together with Theorem 4.3 yields the following

Theorem 6.1. *A random field ξ_n, $n \in \mathbb{Z}_\nu$, subordinated to the stationary Gaussian random field X_n, $n \in \mathbb{Z}_\nu$, can be written in the form*

$$\xi_n = f_0 + \sum_{k=1}^{\infty} \frac{1}{k!} \int e^{i((n, x_1 + \dots + x_k)} f_k(x_1, \dots, x_k) Z_G(\, dx_1) \dots Z_G(\, dx_k), \quad n \in Z_\nu, \tag{6.1}$$

with some $f = (f_0, f_1, \dots) \in \operatorname{Exp} \mathcal{H}_G$, where G is the spectral measure of the field X_n, and Z_G is the random spectral measure adapted to it. This representation is unique. It is also clear that formula (6.1) defines a subordinated field for all $f \in \operatorname{Exp} \mathcal{H}_G$.

If the spectral measure G has the property $G(\{x\colon\ x_p = u\}) = 0$ for all $u \in R^1$ and $1 \le p \le k$, where $x = (x_1, \dots, x_k)$ (this is a strengthened form of the non-atomic property), then the functions

$$\bar{f}_k(x_1, \dots, x_k) = f_k(x_1, \dots, x_k) \tilde{\chi}_0^{-1}(x_1 + \dots + x_k), \quad k = 1, 2, \dots,$$

are meaningful, as functions in the measure space $(R^{k\nu}, \mathcal{B}^{k\nu}, G^k)$, where $\tilde{\chi}_n(x) = e^{i(n,x)} \prod_{p=0}^{\nu} \frac{e^{ix^{(p)}} - 1}{ix^{(p)}}$, $n \in \mathbb{Z}_\nu$, denotes the Fourier transform of the uniform distribution on the ν-dimensional unit cube $\prod_{p=1}^{\nu} [n^{(p)}, n^{(p)} + 1]$. Then the random variable ξ_n in formula (6.1) can be rewritten in the form

$$\xi_n = f_0 + \sum_{k=1}^{\infty} \frac{1}{k!} \int \tilde{\chi}_n(x_1 + \dots + x_k) \bar{f}_k(x_1, \dots, x_k) Z_G(\, dx_1) \dots Z_G(\, dx_k).$$

Hence the following Theorem 6.1′ can be considered as the continuous time version of Theorem 6.1.

Theorem 6.1′**.** *Let the generalized random field $\xi(\varphi)$, $\varphi \in \mathcal{S}$, be subordinated to the stationary Gaussian generalized random field $X(\varphi)$, $\varphi \in \mathcal{S}$. Let G denote the spectral measure of the field $X(\varphi)$, and let Z_G be the random spectral measure adapted to it. Then $\xi(\varphi)$ can be written in the form*

$$\xi(\varphi) = f_0 \cdot \tilde{\varphi}(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \int \tilde{\varphi}(x_1 + \dots + x_k) f_k(x_1, \dots, x_k) Z_G(\, dx_1) \dots Z_G(\, dx_k), \tag{6.1′}$$

where the functions f_k are invariant under all permutations of their variables,

$$f_k(-x_1,\dots,-x_k)=\overline{f_k(x_1,\dots,x_k)},\quad k=1,2,\dots,$$

and

$$\sum_{k=1}^{\infty}\frac{1}{k!}\int(1+|x_1+\dots+x_k|^2)^{-p}|f_k(x_1+\dots+x_k)|^2G(\,dx_1)\dots G(\,dx_k)<\infty \tag{6.2}$$

with an appropriate number $p>0$. This representation is unique.

Contrariwise, all random fields $\xi(\varphi)$, $\varphi\in\mathcal{S}$, defined by formulas (6.1′) *and (6.2) are subordinated to the stationary, Gaussian random field $X(\varphi)$, $\varphi\in\mathcal{S}$.*

Proof of Theorem 6.1′. It is clear that a random field $\xi(\varphi)$, $\varphi\in\mathcal{S}$, defined by (6.1′) and (6.2) is subordinated to $X(\varphi)$. One has to check that the definition of $\xi(\varphi)$ in formula (6.1′) is meaningful for all $\varphi\in\mathcal{S}$, because of (6.2), $\xi(T_t\varphi)=T_t\xi(\varphi)$ for all shifts T_t, $t\in R^\nu$, by Theorem 4.3, and also the following continuity property holds. For all $\varepsilon>0$ there is a small neighbourhood H of the origin in the space $\mathcal{S}$ such that if $\varphi=\varphi_1-\varphi_2\in H$ for some $\varphi_1,\varphi_2\in\mathcal{S}$ then $E[\xi(\varphi_1)-\xi(\varphi_2)]^2=E\xi\varphi)^2<\varepsilon^2$.

Since the Fourier transform $\varphi(\cdot)\to\tilde\varphi(\cdot)$ is a bicontinuous map in $\mathcal{S}$, to prove the above continuity property it is enough to check that $E\xi(\varphi)^2<\varepsilon^2$ if $\tilde\varphi\in H$ for an appropriate small neighbourhood H of the origin in $\mathcal{S}$. But this relation holds with the choice $H=\{\varphi\colon(1+|x|^2)^p|\varphi(x)|\le\frac{\varepsilon^2}{K}$ for all $x\in R^\nu\}$ with a sufficiently large $K>0$ because of condition (6.2).

To prove that all subordinated fields have the above representation observe that the relation

$$\xi(\varphi)=\Psi_{\varphi,0}+\sum_{k=1}^{\infty}\frac{1}{k!}\int\Psi_{\varphi,k}(x_1,\dots,x_k)Z_G(\,dx_1)\dots Z_G(\,dx_k) \tag{6.3}$$

holds for all $\varphi\in\mathcal{S}$ with some $(\Psi_{\varphi,0},\Psi_{\varphi,1},\dots)\in\operatorname{Exp}\mathcal{H}_G$ depending on the function φ. We are going to show that these functions $\Psi_{\varphi,k}$ can be given in the form

$$\Psi_{\varphi,k}(x_1,\dots,x_k)=f_k(x_1,\dots,x_k)\cdot\tilde\varphi(x_1+\dots+x_k),\quad k=1,2,\dots,$$

with some functions $f_k\in\mathcal{B}^{k\nu}$, and

$$\Psi_{\varphi,0}=f_0\cdot\tilde\varphi(0)$$

for all $\varphi\in\mathcal{S}$ with a sequence of functions $f_0,f_1,\dots$ not depending on φ.

To show this let us choose a $\varphi_0\in\mathcal{S}$ such that $\tilde\varphi_0(x)>0$ for all $x\in R^\nu$. (We can make for instance the choice $\varphi_0(x)=e^{-(x,x)}$.) We claim that the finite linear combinations $\sum a_p\varphi_0(x-t_p)=\sum a_pT_{t_p}\varphi_0(x)$ are dense in $\mathcal{S}$. To prove this it is enough to show that the functions ψ whose Fourier transforms $\tilde\psi$ have a compact support can well be approximated by such linear combinations, because these functions ψ are dense

in $\mathcal{S}$. (The statement that these functions ψ are dense in $\mathcal{S}$ is equivalent to the statement that their Fourier transform $\tilde{\psi}$ are dense in the space $\tilde{\mathcal{S}} \subset \mathcal{S}^c$ consisting of the Fourier transforms of the (real valued) functions in the space $\mathcal{S}$.) We have $\frac{\tilde{\psi}}{\tilde{\varphi}_0} \in \mathcal{S}^c$ for such functions ψ, where $\mathcal{S}^c$ denotes the Schwartz-space of complex valued, at infinity strongly decreasing, smooth functions again, because $\tilde{\varphi}_0(x) \neq 0$, and $\tilde{\psi}$ has a compact support. There exists a function $\chi \in \mathcal{S}$ such that $\tilde{\chi} = \frac{\tilde{\psi}}{\tilde{\varphi}_0}$. (Here we exploit that the space of Fourier transforms of the functions from $\mathcal{S}$ agrees with the space of those functions $f \in \mathcal{S}^c$ for which $f(-x) = \overline{f(x)}$.) Therefore $\psi(x) = \chi * \varphi_0(x) = \int \chi(t)\varphi_0(x-t)\,dt$, where $*$ denotes convolution. It can be seen by exploiting this relation together with the rapid decrease of χ and φ_0 together of its derivatives at infinity, and approximating the integral defining the convolution by an appropriate final sum that for all integers $r > 0$, $s > 0$ and real numbers $\varepsilon > 0$ there exists a finite linear combination $\hat{\psi}(x) = \hat{\psi}_{r,s,\varepsilon}(x) = \sum_p a_p \varphi_0(x - t_p)$ such that $(1 + |x|^s)|\psi(x) - \hat{\psi}(x)| < \varepsilon$ for all $x \in R^\nu$, and the same estimate holds for all derivatives of $\psi(x) - \hat{\psi}(x)$ of order less than r. Beside this, also the relation $\hat{\psi}(-x) = \overline{\hat{\psi}(x)}$ holds (similarly to the relation $\psi(-x) = \overline{\psi(x)}$).

I only briefly explain why such an approximation exists. Some calculation enables us to reduce this statement to the case when $\psi = \chi * \varphi_0$ with a function $\chi \in \mathcal{D}$, which has compact support. To give the desired approximation choose a small number $\delta > 0$, introduce the cube $\Delta = \Delta(\delta) = [-\delta, \delta)^\nu \subset R^\nu$ and define the vectors $k(\delta) = (2k_1\delta, \dots, 2k_\nu\delta) \in R^\nu$ for all $k = (k_1, \dots, k_\nu) \in \mathbb{Z}_\nu$. Given a fixed vector $x \in R^\nu$ let us define the vector $u(x) \in R^\nu$ for all $u \in R^\nu$ as $u(x) = x + k(\delta)$ with that vector $k \in \mathbb{Z}_\nu$ for which $x + k(\delta) - u \in \Delta$, and put $\varphi_{0,x}(u) = \varphi_0(u(x))$. It can be seen that $\hat{\psi}(x) = \chi * \varphi_{0,x}(x)$ is a finite linear combination of numbers of the form $\varphi_0(x - t_k)$ (with $t_k = k(\delta)$) with coefficients not depending on x. Moreover, if $\delta > 0$ is chosen sufficiently small (depending on r, s and ε), then $\hat{\psi}(x) = \hat{\psi}_{r,s,\varepsilon}(x)$ has all properties we demanded.

The above argument implies that there is a sequence of functions $\hat{\psi}_{r,s,\varepsilon}$ which converges to the function ψ in the topology of the space $\mathcal{S}$. As a consequence, the finite linear combinations $\sum a_p \varphi_0(x - t_p)$ are dense in $\mathcal{S}$.

Define

$$f_k(x_1, \dots, x_k) = \frac{\Psi_{\varphi_0,k}(x_1, \dots, x_k)}{\tilde{\varphi}_0(x_1 + \dots + x_k)}, \quad k = 1, 2, \dots, \quad \text{and } f_0 = \frac{\Psi_{\varphi_0,0}}{\tilde{\varphi}_0(0)}.$$

If $\varphi(x) = \sum a_p \varphi_0(x - t_p) = \sum a_p T_{t_p} \varphi_0(x)$, and the sum defining φ is finite, then by Theorem 4.3

$$\begin{aligned}\xi(\varphi) &= \Big(\sum a_p\Big) f_0 \cdot \tilde{\varphi}_0(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \int \sum_p a_p e^{i(t_p, x_1 + \dots + x_k)} \tilde{\varphi}_0(x_1 + \dots + x_k) \\ &\qquad \cdot f_k(x_1, \dots, x_k) Z_G(\,dx_1) \dots Z_G(\,dx_k), \\ &= f_0 \cdot \tilde{\varphi}(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \int \tilde{\varphi}(x_1 + \dots + x_k) f_k(x_1, \dots, x_k) Z_G(\,dx_1) \dots Z_G(\,dx_k).\end{aligned}$$

Relation (6.3) holds for all $\varphi \in \mathcal{S}$, and there exists a sequence $\varphi_j(x) = \sum_p a_p^{(j)} \varphi_0(x - t_p^{(j)}) \in \mathcal{S}$ satisfying (6.1′) such that $\varphi_j \to \varphi$ in the topology of $\mathcal{S}$. This implies that $\lim E[\xi(\varphi_j) - \xi(\varphi)]^2 \to 0$, and in particular $EI_G(\Psi_{\varphi,k} - \hat{\varphi}_{j,k} f_k)^2 \to 0$ with $\hat{\varphi}_{j,k}(x_1, \dots, x_k) = \tilde{\varphi}_j(x_1 + \cdots + x_k)$ as $j \to \infty$ for all $k = 1, 2, \dots$. (To carry out some further argument we restricted the domain of integration to a bounded set A.) Hence

$$\int_A |\Psi_{\varphi,k}(x_1, \dots, x_k) - \tilde{\varphi}_j(x_1 + \cdots + x_k) f_k(x_1, \dots, x_k)|^2 G(\,dx_1) \dots G(\,dx_k) \to 0$$

as $j \to \infty$ for all k and for all bounded sets $A \in R^{k\nu}$. On the other hand,

$$\int_A |\tilde{\varphi}(x_1 + \cdots + x_k) - \tilde{\varphi}_j(x_1 + \cdots + x_k)|^2 |f_k(x_1, \cdots, x_k)|^2 G(\,dx_1) \dots G(\,dx_k) \to 0,$$

since $\tilde{\varphi}_j(x) - \tilde{\varphi}(x) \to 0$ in the supremum norm if $\tilde{\varphi}_j \to \tilde{\varphi}$ in the topology of $\mathcal{S}$, and the property $\tilde{\varphi}_0(x) > 0$ together with the continuity of $\tilde{\varphi}_0$ and the inequality $EI_G(\hat{\varphi}_{0,k} f_k)^2 < \infty$ imply that $\int_A |f_k(x_1, \dots, x_k)|^2 G(\,dx_1) \dots G(dx_k) < \infty$ on all bounded sets A. The last two relations yield that

$$\Psi_{\varphi,k}(x_1, \dots, x_k) = \tilde{\varphi}(x_1 + \cdots + x_k) f_k(x_1, \dots, x_k), \quad k = 1, 2, \dots.$$

Similarly,

$$\psi_{\varphi,0} = \tilde{\varphi}(0) f_0.$$

These relations imply (6.1′).

To complete the proof of Theorem 6.1′ we show that (6.2) follows from the continuity of the transformation F: $\varphi \to \xi(\varphi)$ from the space $\mathcal{S}$ into the space $L^2(\Omega, \mathcal{A}, P)$.

We recall that the transformation $\varphi \to \tilde{\varphi}$ is bicontinuous in $\mathcal{S}^c$. Hence the transformation $\tilde{\varphi} \to \xi(\varphi)$ is a continuous map from the space of the Fourier transforms of the functions in the space $\mathcal{S}$ to $L^2(\Omega, \mathcal{A}, P)$. This continuity implies that there exist some integers $p > 0$, $r > 0$ and real number $\delta > 0$ such that if

$$(1 + |x^2|)^p \left| \frac{\partial^{s_1 + \cdots + s_\nu}}{\partial x^{(1)^{s_1}} \dots \partial x^{(\nu)^{s_\nu}}} \tilde{\varphi}(x) \right| < \delta \quad \text{for all } s_1 + \cdots + s_\nu \le r, \tag{6.4}$$

then $E\xi(\varphi)^2 \le 1$.

Let us choose a $\psi \in \mathcal{S}$ such that ψ has a compact support, $\psi(x) = \psi(-x)$, $\psi(x) \ge 0$ for all $x \in R^\nu$, and $\psi(x) = 1$ if $|x| \le 1$. (There exist such functions.) Define the functions $\tilde{\varphi}_m(x) = C(1 + |x|^2)^{-p} \psi(\frac{x}{m})$. Then $\varphi_m \in \mathcal{S}$, since its Fourier transform $\tilde{\varphi}_m$ is an even function, and it is in the space $\mathcal{S}$ being an infinite many times differentiable function with compact support. Moreover, φ_m satisfies (6.4) for all $m = 1, 2, \dots$ if the number $C > 0$ in its definition is chosen sufficiently small. This number C can be chosen independently of m. (To see this observe that $(1 + |x^2|)^{-p}$ together with all of its

derivatives of order not bigger than r can be bounded by $\frac{C(p,r)}{(1+|x|^2)^p}$ with an appropriate constant $C(p,r)$.) Hence

$$E\xi(\varphi_m)^2 = \sum \frac{1}{k!} \int |\tilde{\varphi}_m(x_1 + \cdots + x_k)|^2 |f_k(x_1, \cdots, x_k)|^2 G(dx_1) \dots G(dx_k) \le 1$$

for all $m = 1, 2, \dots$.

As $\tilde{\varphi}_m(x) \to C(|1 + |x|^2)^{-p}$ as $m \to \infty$, and $\tilde{\varphi}_k(x) \ge 0$, an $m \to \infty$ limiting procedure in the last relation together with Fatou's lemma imply that

$$C \sum \frac{1}{k!} \int (1 + |x_1 + \cdots + x_k)|^2)^{-p} |f_k(x_1, \cdots, x_k)|^2 G(dx_1) \dots G(dx_k) \le 1.$$

Theorem 6.1′ is proved.

We shall call the representations given in Theorems 6.1 and 6.1′ the canonical representation of a subordinated field. From now on we restrict ourselves to the case $E\xi_n = 0$ or $E\xi(\varphi) = 0$ respectively, i.e. to the case when $f_0 = 0$ in the canonical representation. If

$$\xi(\varphi) = \sum_{k=1}^{\infty} \frac{1}{k!} \int \tilde{\varphi}(x_1 + \cdots + x_k) f_k(x_1, \dots, x_k) Z_G(\, dx_1) \dots Z_G(\, dx_k),$$

then

$$\xi(\varphi_t^A) = \sum_{k=1}^{\infty} \frac{1}{k!} \frac{t^\nu}{A(t)} \int \tilde{\varphi}(t(x_1 + \cdots + x_k)) f_k(x_1, \dots, x_k) Z_G(\, dx_1) \dots Z_G(\, dx_k)$$

with the function φ_t^A defined in (1.3). Define the spectral measures G_t by the formula $G_t(A) = G(tA)$. Then we have by Lemma 4.5

$$\xi(\varphi_t^A) \stackrel{\Delta}{=} \sum_{k=1}^{\infty} \frac{1}{k!} \frac{t^\nu}{A(t)} \int \tilde{\varphi}(x_1 + \cdots + x_k) f_k\left(\frac{x_1}{t}, \dots, \frac{x_k}{t}\right) Z_{G_t}(\, dx_1) \dots Z_{G_t}(\, dx_k).$$

If $G(tB) = t^{2\kappa} G(B)$ with some $\kappa > 0$ for all $t > 0$ and $B \in \mathcal{B}^\nu$, $f_k(\lambda x_1, \dots, \lambda x_k) = \lambda^{\nu - \kappa k - \alpha} f_k(x_1, \dots, x_k)$, and $A(t)$ is chosen as $A(t) = t^\alpha$, then Theorem 4.4 (with the choice $G'(B) = G(tB) = t^{2\kappa} G(B)$) implies that $\xi(\varphi_t^A) \stackrel{\Delta}{=} \xi(\varphi)$. Hence we obtain the following

Theorem 6.2. *Let the generalized random field $\xi(\varphi)$ be given by the formula*

$$\xi(\varphi) = \sum_{k=1}^{\infty} \frac{1}{k!} \int \tilde{\varphi}(x_1 + \cdots + x_k) f_k(x_1, \dots, x_k) Z_G(\, dx_1) \dots Z_G(\, dx_k). \tag{6.5}$$

If $f_k(\lambda x_1, \dots, \lambda x_k) = \lambda^{\nu-\kappa k-\alpha} f_k(x_1, \dots, x_k)$ *for all* k, $(x_1, \dots, x_k) \in R^{k\nu}$ *and* $\lambda > 0$, $G(\lambda A) = \lambda^{2\kappa} G(A)$ *for all* $\lambda > 0$ *and* $A \in \mathcal{B}^\nu$, *then* ξ *is a self-similar random field with parameter* α.

The discrete time version of this result can be proved in the same way. It states the following

Theorem 6.2′. *If the discrete random field* ξ_n, $n \in \mathbb{Z}_\nu$, *has the form*

$$\xi_n = \sum_{k=1}^{\infty} \frac{1}{k!} \int \tilde{\chi}_n(x_1 + \dots + x_k) f_k(x_1, \dots, x_k) Z_G(\,dx_1) \dots Z_G(\,dx_k), \quad n \in \mathbb{Z}_\nu, \quad (6.5',)$$

and $f_k(\lambda x_1, \dots, \lambda x_k) = \lambda^{\nu-\kappa k-\alpha} f_k(x_1, \dots, x_k)$ *for all* k, $G(\lambda A) = \lambda^{2\kappa} G(A)$, *then* ξ_n *is a self-similar random field with parameter* α.

Theorems 6.2 and 6.2′ enable us to construct self-similar random fields. Nevertheless, we have to check whether formulas (6.5) and (6.5′) are meaningful. The hard part of this problem is to check whether

$$\sum \frac{1}{k!} \int |\tilde{\chi}_n(x_1 + \dots + x_k)|^2 |f_k(x_1, \dots, x_k)|^2 G(\,dx_1) \dots G(\,dx_k) < \infty,$$

or whether

$$\sum \frac{1}{k!} \int |\tilde{\varphi}(x_1 + \dots + x_k)|^2 |f_k(x_1, \dots, x_k)|^2 G(\,dx_1) \dots G(\,dx_k) < \infty \quad \text{for all } \varphi \in \mathcal{S}.$$

To investigate when these expressions are finite is a rather hard problem in the general case. The next result enables us to prove the finiteness of these expressions in some interesting cases.

Let us define the measure G

$$G(A) = \int_A |x|^{2\kappa-\nu} a\left(\frac{x}{|x|}\right) dx, \quad A \in \mathcal{B}^\nu, \tag{6.6}$$

where $a(\cdot)$ is a non-negative, measurable and even function on the ν-dimensional unit sphere $S_{\nu-1}$, and $\kappa > 0$. (The condition $\kappa > 0$ is imposed to guarantee the relation $G(A) < \infty$ for all bounded sets $A \in \mathcal{B}^\nu$.) We prove the following

Proposition 6.3. *Let the measure* G *be the same as in formula (6.6).*

a) If the function $a(\cdot)$ *is bounded on the unit sphere* $S_{\nu-1}$, *and* $\frac{\nu}{k} > 2\kappa > 0$, *then*

$$\begin{aligned} D(\varphi) &= \int |\tilde{\varphi}(x_1 + \dots + x_k)|^2 G(\,dx_1) \dots G(\,dx_k) \\ &\le C \int (1 + |x_1 + \dots + x_k)|^2)^{-p} G(\,dx_1) \dots G(\,dx_k) < \infty \end{aligned}$$

if $\varphi \in \mathcal{S}$ for all $p > \frac{\nu}{2}$ with some $C = C(\varphi, p) < \infty$, and

$$D(n) = \int |\tilde{\chi}_n(x_1 + \cdots + x_k)|^2 G(\,dx_1) \dots G(\,dx_k) < \infty \quad \textit{for all} \quad n \in \mathbb{Z}_\nu.$$

b) If there is a constant $C > 0$ such that $a(x) > D$ in a neighbourhood of a point $x_0 \in S_{\nu-1}$, and either $2\kappa \le 0$ of $2\kappa \ge \frac{\nu}{k}$, then the integrals $D(n)$ and some $D(\varphi)$, $\varphi \in \mathcal{S}$, are divergent.

Proof of Proposition 6.3. Proof of Part a)
We may assume that $a(x) = 1$ for all $x \in S_{\nu-1}$. Define

$$J_{\kappa,k}(x) = \int_{x_1+\cdots+x_k=x} |x_1|^{2\kappa-\nu} \cdots |x_k|^{2\kappa-\nu}\, dx_1 \dots\, dx_k, \quad x \in R^\nu$$

for $k \ge 2$, where $dx_1 \dots\, dx_k$ denotes the Lebesgue measure on the hyperplane $x_1 + \cdots + x_k = x$, and let $J_{\kappa,1}(x) = |x|^{2\kappa-\nu}$. We have

$$J_{\kappa,k}(\lambda x) = |\lambda|^{k(2\kappa-\nu)+(k-1)\nu} J_{\kappa,k}(x), = |\lambda|^{2k\kappa-\nu} J_{\kappa,k}(x), \quad x \in R^\nu \ \ \lambda > 0,$$

because of the homogeneity of the integral. Beside this

$$\begin{aligned} D(n) &= \int_{R^\nu} |\tilde{\chi}_n(x)|^2 J_{\kappa,k}(x)\, dx, \\ \int (1 + |x_1 + \cdots + x_k|^2)^{-p} G(\,dx_1) \dots G(\,dx_k) &= \int (1 + |x|^2)^{-p} J_{\kappa,k}(x)\, dx. \end{aligned} \tag{6.7}$$

We prove by induction on k that

$$J_{\kappa,k}(x) \le C(\kappa, k)|x|^{2\kappa k-\nu} \tag{6.8}$$

with an appropriate constant $C(\kappa, k) < \infty$ if $\frac{\nu}{k} > 2\kappa > 0$.

We have

$$J_{\kappa,k}(x) = \int J_{\kappa,k-1}(y)|x - y|^{2\kappa-\nu}\, dy.$$

Hence

$$\begin{aligned} J_{\kappa,k} &\le C(\kappa, k-1) \int |y|^{(2\kappa(k-1)-\nu}|x - y|^{2\kappa-\nu}\, dy \\ &= C(\kappa, k-1)|x|^{2\kappa k-\nu} \int |y|^{(2\kappa(k-1)-\nu} \left| \frac{x}{|x|} - y \right|^{2\kappa-\nu} dy = C(\kappa, k)|x|^{2\kappa k-\nu}, \end{aligned}$$

since $\int |y|^{(2\kappa(k-1)-\nu} \left| \frac{x}{|x|} - y \right|^{2\kappa-\nu} dy < \infty$.

The last integral is finite, since its integrand behaves at zero asymptotically as $C|y|^{2\kappa(k-1)-\nu}$, at the point $e = \frac{x}{|x|} \in S_{\nu-1}$ as $C_2|y-e|^{2\kappa-\nu}$ and at infinity as $C_3|y|^{2\kappa k-2\nu}$. Relations (6.7) and (6.8) imply that

$$\begin{aligned} D(n) &\le C' \int |\tilde{\chi}_0(x)|^2 |x|^{2\kappa k-\nu}\, dx \le C'' \int |x|^{2\kappa k-\nu} \prod_{l=1}^{\nu} \frac{1}{1+|x^{(l)}|^2}\, dx \\ &\le C''' \int_{|x^{(1)}|=\max\limits_{1\le l\le\nu} |x^{(l}|} |x^{(1)}|^{2\kappa k-\nu} \prod_{l=1}^{\nu} \frac{1}{1+|x^{(l)}|^2}\, dx \\ &= \sum_{p=0}^{\infty} C''' \int_{|x^{(1)}|=\max\limits_{1\le l\le\nu} |x^{(l}|,\ 2^p\le|x^{(1)}|<2^{p+1}} + C''' \int_{|x^{(1)}|=\max\limits_{1\le l\le\nu} |x^{(l}|,|x^{(1)}|<1} . \end{aligned}$$

The second term in the last sum can be simply bounded by a constant, since $B = \left\{x\colon\ |x^{(1)}| = \max\limits_{1\le l\le\nu} |x^{(l}|,\ |x^{(1)}| < 1\right\} \subset \{x\colon\ |x| \le \sqrt{\nu}\}$, and $|x^{(1)}|^{2\kappa k-\nu} \prod\limits_{l=1}^{\nu} \frac{1}{1+|x^{(l)}|^2} \le$ const. $|x|^{2\kappa k-\nu}$ on the set B. Hence

$$D(n) \le C_1 \sum_{p=0}^{\infty} 2^{p(2\kappa k-\nu)} \left[\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx\right]^{\nu} + C_2 < \infty.$$

We have $|\varphi(x)| \le C(1+|x^2|)^{-p}$ with some $C > 0$ and $D > 0$ if $\varphi \in \mathcal{S}$. The proof of the estimate $D(\varphi) < \infty$ for $\varphi \in \mathcal{S}$ is similar but simpler.

Proof of part b). Define, similarly to the function $J_{\kappa,k}$,

$$J_{\kappa,k,a}(x) = \int_{x_1+\dots+x_k=x} |x_1|^{2\kappa-\nu} a\left(\frac{x_1}{|x_1|}\right) \cdots |x_k|^{2\kappa-\nu} a\left(\frac{x_k}{|x_k|}\right) dx_1 \dots\, dx_k, \quad x \in R^{\nu},$$

where $dx_1 \dots\, dx_k$ denotes the Lebesgue measure on the hyperplane $x_1 + \dots + x_k = x$. Since

$$J_{\kappa,k,a}(x) \ge \int_{y\colon\ |y|<(\frac{1}{2}+\alpha)|x|,\ |y-x|<(\frac{1}{2}+\alpha)|x|} J_{\kappa,k-1,a}(y) a\left(\frac{x-y}{|x-y|}\right) |x-y|^{2\kappa-\nu}\, dy$$

with an arbitrary $\alpha > 0$ an argument similar to the one in part a) shows that

$$J_{\kappa,k,a}(x) \begin{cases} \ge \bar{C}(\kappa,k)|x|^{2\kappa k-\nu} & \text{if } \dfrac{\nu}{k} > 2\kappa > 0, \\ = \infty \qquad \text{if } \kappa \le 0 \text{ or } 2\kappa \ge \dfrac{\nu}{k} \end{cases}$$

if $\frac{x}{|x|}$ is close to such a point $x_0 \in S_{\nu-1}$ in whose small neighbourhood the function $a(\cdot)$ is separated from zero. Since $|\tilde{\chi}_n(x)|^2 > 0$ for almost all $x \in R^{\nu}$,

$$D(n) = \int |\tilde{\chi}_n(x)|^2 J_{\kappa,k,a}(x)\, dx = \infty$$

under the conditions of part b). Similarly $D(\varphi) = \infty$ if $|\tilde{\varphi}(x)|^2 > 0$ for almost all $x \in R^\nu$. We remark that the conditions in part b) can be weakened. It would have been enough to assume that $a(x) > 0$ on a set of positive Lebesgue measure in $S_{\nu-1}$.

Theorem 6.2 and 6.2′ together with Proposition 6.3 have the following

Corollary 6.4. *The formulae*

$$\xi_n = \sum_{k=1}^{M} \int \tilde{\chi}_n(x_1 + \cdots + x_k) \prod_{l=1}^{k} \left(|x_l|^{-\kappa+(\nu-\alpha)/k} \cdot b_k\left(\frac{x_l}{|x_l|}\right)\right) Z_G(\,dx_1) \dots Z_G(\,dx_k), \quad n \in \mathbb{Z}_\nu,$$

and

$$\xi(\varphi) = \sum_{k=1}^{M} \int \tilde{\varphi}(x_1 + \cdots + x_k) \prod_{l=1}^{k} \left(|x_l|^{-\kappa+(\nu-\alpha)/k} \cdot b_k\left(\frac{x_l}{|x_l|}\right)\right) Z_G(\,dx_1) \dots Z_G(\,dx_k), \quad \varphi \in \mathcal{S},$$

define self-similar random fields with self-similarity parameter α if G is defined by formula (6.6), the parameter α satisfies the inequality $\frac{\nu}{2} < \alpha < \nu$, and the functions $a(\cdot)$ (in the definition of the measure $G(\cdot)$ in (6.6)), $b_1(\cdot)$, ... $b_k(\cdot)$ are bounded even functions on $S_{\nu-1}$.

The following observation may be useful when we want to prove Corollary 6.4. We can replace ξ_n by another random field with the same distribution. Thus we can write, by exploiting Theorem 4.4,

$$\xi_n = \sum_{k=1}^{M} \tilde{\chi}_n(x_1 + \cdots + x_k) Z_{G'}(\,dx_1) \dots Z_{G'}(\,dx_k), \quad n \in \mathbb{Z}_\nu,$$

with random spectral measure $Z_{G'}$ corresponding to the spectral measure $G'(\,dx) = b(\frac{x}{|x|})^2 |x|^{-2\kappa+2(\nu-\alpha)/k} G(\,dx) = a(\frac{x}{|x|}) b(\frac{x}{|x|})^2 |x|^{-\nu+2(\nu-\alpha)/k}\,dx$. In the case of generalized random fields a similar argument can be applied.

Remark 6.5. The estimate on $J_{\kappa,k}$ and the end of the of part a) in Proposition (6.3) show that the self-similar random fields

$$\xi(\varphi) = \sum_{k=1}^{M} \int \tilde{\varphi}(x_1 + \cdots + x_k) |x_1 + \cdots + x_k|^p \, u\left(\frac{x_1 + \cdots + x_k}{|x_1 + \cdots + x_k|}\right) \prod_{l=1}^{k} \left(|x_l|^{-\kappa+(\nu-\alpha)/k} \cdot b_k\left(\frac{x_l}{|x_l|}\right)\right) Z_G(\,dx_1) \dots Z_G(\,dx_k), \quad \varphi \in \mathcal{S},$$

and

$$\xi_n = \sum_{k=1}^{M} \int \tilde{\chi}_n(x_1 + \cdots + x_k)|x_1 + \cdots + x_k|^p \, u\left(\frac{x_1 + \cdots + x_k}{|x_1 + \cdots + x_k|}\right) \prod_{l=1}^{k} \left(|x_l|^{-\kappa+(\nu-\alpha)/k} \cdot b_k\left(\frac{x_l}{|x_l|}\right)\right) Z_G(\,dx_1) \dots Z_G(\,dx_k), \quad n \in \mathbb{Z}_\nu,$$

are well defined if G is defined by formula (6.6), $a(\cdot)$, $b(\cdot)$ and $u(\cdot)$ are bounded even functions on $S_{\nu-1}$, $\frac{\nu}{2} < \alpha < \nu$, and $\alpha - p < \nu$ in the generalized and $\frac{\nu-1}{2} < \alpha - p < \nu$ is the discrete random field case. The self-similarity parameter of these random fields is $\alpha - p$. We remark that in the case $p > 0$ this class of self-similar fields also contains self-similar fields with self-similarity parameter less than $\frac{\nu}{2}$.

In proving the statement of Remark 6.5 we have to check the integrability conditions needed for the existence of the Wiener–Itô integrals $\xi(\varphi)$ and ξ_n. To check them it is worth remarking that in the proof of part a) of Proposition 6.2 we proved the estimate $J_{\bar{\kappa},k}(x) \le C(\bar{\kappa}, k)|x|^{2\bar{\kappa}k-\nu}$. We want to apply this inequality in the present case with the choice $\bar{\kappa} = \frac{\nu-\alpha}{k}$. Then arguing similarly to the proof of part a) of Proposition 6.2 we get to the problem whether the relations $\int |\tilde{\chi}_n(x)|^2|x|^{2p+2(\nu-\alpha)-\nu}\,dx < \infty$ and $\int |\tilde{\varphi}(x)|^2|x|^{2p+2(\nu-\alpha)-\nu}\,dx < \infty$ if $\varphi \in \mathcal{S}$ hold under the conditions of Remark 6.5. They can be proved by the argument applied at the end of the proof of part a) of Proposition 6.2.

The following question arises in a natural way. When do different formulas satisfying the conditions of Theorem 6.2 or Theorem 6.2′ define self-similar random fields with different distributions? In particular: Are the self-similar random fields constructed via multiple Wiener–Itô integrals necessarily non-Gaussian? We cannot give a completely satisfactory answer for the above question, but our former results yield some useful information. Let us substitute the spectral measure G by G' such that $\frac{G(\,dx)}{G'(\,dx)} = |g^2(x)|^2$, $g(-x) = \overline{g(x)}$ and the functions $|x_l|^{-\kappa+(\nu-\alpha)/k} b(\frac{x_l}{|x_l|})$ by $(\frac{x_l}{|x_l|}) g(x_l)|x_l|^{-\kappa+(\nu-\alpha)/k}$ in Corollary 6.4. By Theorem 4.4 the new field has the same distribution as the original one. On the other hand, Corollary 5.4 helps us to decide whether two random variables have different moments, and therefore different distributions. Let us consider e.g. a moment of odd order of the random variables ξ_n or $\xi(\varphi)$ defined in Corollary 6.4. It is clear that all $h_\gamma \ge 0$. Moreover, if $b_k(x)$ does not vanish for some even number k, then there exists a $h_\gamma > 0$ in the sum expressing an odd moment of ξ_n or $\xi(\varphi)$. Hence the odd moments of ξ_n or $\xi(\varphi)$ are positive in this case. This means in particular that the self-similar random fields defined in Corollary 6.4 are non-Gaussian if b_k is non-vanishing for some even k. The next result shows that the tail behaviour of multiple Wiener–Itô integrals of different order is different.

Theorem 6.6. *Let G be a spectral measure and Z_G a random spectral measure corresponding to G. For all $h \in \mathcal{H}_G^m$ there exist some constants $K_1 > K_2 > 0$ and $x_0 > 0$ depending on the function h such that*

$$e^{-K_1 x^{2/m}} \le P(|I_G(h)| > x) \le e^{-K_2 x^{2/m}}$$

for all $x > x_0$.

Remark. As the proof of Theorem 6.6 shows the constant K_2 in the upper bound of the above estimate can be chosen as $K_m = C_m(EI_G(h)^2)^{-1/m}$ with a constant C_m depending only on the order m of the Wiener–Itô integral of $I_G(h)$. This means that for a fixed number m the constant K_2 in the above estimate can be chosen as a constant depending only on the variance of the random variable $I_G(h)$. On the other hand, no simple characterization of the constant $K_1 > 0$ appearing in the lower bound of this estimate is known.

Proof of Theorem 6.6. a) Proof of the upper estimate.

We have

$$P(|I_G(h)| > x) \le x^{2N} E(I_G(h)|^{2N}).$$

By Corollary 5.6

$$E(I_G(h)|^{2N}) \le \bar{C}(m,N)[E(I_G(h)^2)]^N \le \bar{C}(m,N)C_1^N,$$

and by a simple combinatorial argument we obtain that

$$\bar{C}(m,N) \le \frac{(2Nm-1)(2Nm-3)\cdots 1}{(m!\,)^N},$$

since the numerator on the right-hand side of this inequality equals the number of complete diagrams $|\bar{\Gamma}(\underbrace{m,\dots,m}_{2N \text{ times}})|$ if vertices from the same row can also be connected. Multiplying the inequalities

$$(2nM-2j-1)(2Nm-2j-1-2N)\cdots(2Nm-2j-1-2N(m-1)) \le (2N)^m m!,$$

$j = 1,\dots,N$, we obtain that

$$\bar{C}(m,N) \le (2N)^{mN}.$$

(This inequality could be sharpened, but it is sufficient for our purpose.) Choose a sufficiently small number $\alpha > 0$, and define $N = [\alpha x^{2/m}]$, where $[\cdot]$ denotes integer part. With this choice we have

$$P(|I_G(h)| > x) \le (x^{-2}(2\alpha)^m x^2)^N C_1^N = [C_1(2\alpha)^m]^N \le e^{-K_2 x^{2/m}},$$

if α is chosen in such a way that $C_1(2\alpha)^m \le \frac{1}{e}$, $K_2 = \frac{\alpha}{2}$, and $x > x_0$ with an appropriate $x_0 > 0$.

b) Proof of the lower estimate.

First we reduce this inequality to the following statement. Let $Q(x_1,\dots,x_k)$ be a homogeneous polynomial of order m (the number k is arbitrary), and $\xi = (\xi_1,\dots,\xi_k)$ a k-dimensional standard normal variable. Then

$$P(Q(\xi_1,\dots,\xi_k) > x) \ge e^{-Kx^{2/m}} \tag{6.9}$$

if $x > x_0$, where the constants $K > 0$ and $x_0 > 0$ may depend on the polynomial Q.

By the results of Section 4, $I_G(h)$ can be written in the form

$$I_G(h) = \sum_{j_1+\cdots+j_l=m} C_{j_1,\dots,j_l}^{k_1,\dots,k_l} H_{j_1}(\xi_{k_1}) \cdots H_{j_k}(\xi_{k_l}), \tag{6.10}$$

where $\xi_1, \xi_2, \dots$ are independent standard normal random variables, $C_{j_1,\dots,j_l}^{k_1,\dots,k_l}$ are appropriate coefficients, and the right-hand side of (6.10) is convergent in L_2 sense. Let us fix a sufficiently large integer k, and let us consider the conditional distribution of the right-hand side of (6.10) under the condition $\xi_{k+1} = x_{k+1}, \xi_{k+2} = x_{k+2}, \dots$, where the numbers $x_{k+1}, x_{k+2}, \dots$ are arbitrary. This conditional distribution coincides with the distribution of the random variable $Q(\xi_1, \dots, \xi_k, x_{k+1}, x_{k+2}, \dots)$ with probability 1, where the polynomial Q is obtained by substituting $\xi_{k+1} = x_{k+1}$, $\xi_{k+2} = x_{k=2}, \dots$ into the right-hand side of (6.10). It is clear that all these polynomials $Q(\xi_1, \dots, \xi_k, x_{k+1}, x_{k+2}, \dots)$ are of order m if k is sufficiently large. It is sufficient to prove that

$$P(|Q(\xi_1, \dots, \xi_k, x_{k+1}, x_{k+2}, \dots)| > x) \ge e^{-Kx^{2/m}}$$

for $x > x_0$, where the constants $K > 0$ and $x_0 > 0$ may depend on the polynomial Q. Write

$$Q(\xi_1, \dots, \xi_k, x_{k+1}, x_{k+2}, \dots) = Q_1(\xi_1, \dots, \xi_k) + Q_2(\xi_1, \dots, \xi_k)$$

where Q_1 is a homogeneous polynomial of order m, and Q_2 is a polynomial of order less than m. The polynomial Q_2 can be rewritten as the sum of finitely many Wiener–Itô integrals with multiplicity less than m. Hence the already proved part of Theorem 6.6 implies that

$$P(Q_2(\xi_1, \dots, \xi_k) > x) \le e^{-\bar{q}Kx^{2/(m-1)}}.$$

(We may assume that $m \ge 2$). Then an application of relation (6.9) to Q_1 implies the remaining part of Theorem 6.6, thus it suffices to prove (6.9).

If $Q(\cdot)$ is a polynomial of k variables, then there exist some $\alpha > 0$ and $\beta > 0$ such that

$$\lambda\left(\left|Q\left(\frac{x_1}{|x|}, \dots, \frac{x_k}{|x|}\right)\right| > \alpha\right) > \beta,$$

where $|x|^2 = \sum_{j=1}^{k} x_j^2$, and λ denotes the Lebesgue measure on the k-dimensional unit sphere S_{k-1}. Exploiting that $|\xi|$ and $\frac{\xi}{|\xi|}$ are independent, $\frac{\xi}{|\xi|}$ is uniformly distributed on the unit sphere S_{k-1}, and $P(|\xi| > x) \ge ce^{-x^2}$ for a k-dimensional standard normal random variable, we obtain that

$$P(|Q(\xi_1, \dots, \xi_k)| > x) \ge P\left(|\xi|^m > \frac{x}{\alpha}\right)\beta > e^{-Kx^{2/m}},$$

if the constants K and x are sufficiently large. Theorem 6.6 is proved.

Theorem 6.6 implies in particular that Wiener–Itô integrals of different multiplicity have different distributions. A bounded random variable measurable with respect to the σ-algebra generated by a stationary Gaussian field can be expressed as a sum of multiple Wiener–Itô integrals. Another consequence of Theorem 6.6 is the fact that the number of terms in this sum must be infinite.

In Theorems 6.2 and 6.2′ we have defined a large class of self-similar fields. The question arises whether this class contains self-similar fields such that the distributions of their random variables tend to one (or zero) at infinity (at minus infinity) much faster than the normal distribution functions do. This question has been unsolved by now. By Theorem 6.6 such fields, if any, must be expressed as a sum of infinitely many Wiener–Itô integrals. The above question is of much greater importance than it may seem at first instant. Some considerations suggest that in some important models of statistical physics self-similar fields with very fast decreasing tail distributions appear as limit, when the so-called renormalization group transformations are applied for the probability measure describing the state of the model at critical temperature. (The renormalization group transformations are the transformations over the distribution of stationary fields induced by formula (1.1) or (1.3), when $A_N = N^\alpha$, $A(t) = t^\alpha$ with some α.) No rigorous proof about the existence of such self-similar fields is known yet. Thus the real problem behind the above question is whether the self-similar fields interesting for statistical physics can be constructed via multiple Wiener–Itô integrals.

7. On the original Wiener–Itô integral.

In this section the definition of the original Wiener–Itô integral introduced by Itô in [18] is explained. As the arguments are very similar to those of Sections 4 and 5 (only the notations become simpler) most proofs will be omitted.

Let a measure space $(M, \mathcal{M}, \mu)$ with a σ-finite measure μ be given. Let μ satisfy the following continuity property: For all $\varepsilon > 0$ and $A \in \mathcal{M}$, $\mu(A) < \infty$, there exist some disjoint sets $B_j \in \mathcal{M}$, $j = 1, \dots, N$, with some integer N such that $\mu(B_j) < \varepsilon$ for all $1 \le j \le N$, and $A = \bigcup_{j=1}^{N} B_j$. We introduce the following definition.

Definition of (Gaussian) random orthogonal measures. *A system of random variables $Z_\mu(A)$, $A \in \mathcal{M}$, $\mu(A) < \infty$, is called a Gaussian random orthogonal measure corresponding to the measure μ if*

(i) $Z_\mu(A_1), \dots, Z_\mu(A_k)$ *are independent Gaussian random variables if the sets $A_j \in \mathcal{M}$, $\mu(A_j) < \infty$, $j = 1, \dots, k$, are disjoint.*

(ii) $EZ_\mu(A) = 0$, $EZ_\mu(A)^2 = \mu(A)$.

(iii) $Z_\mu\left(\bigcup_{j=1}^{k} A_j\right) = \sum_{j=1}^{k} Z_\mu(A_k)$ *with probability 1 if $A_1, \dots, A_k$ are disjoint sets.*

Remark. There is the following equivalent version for the definition of random orthogonal measures: The system of random variables system of random variables $Z_\mu(A)$,

$A \in \mathcal{M}$, $\mu(A) < \infty$, is a Gaussian random orthogonal measure corresponding to the measure μ if

(i′) $Z_\mu(A_1), \dots, Z_\mu(A_k)$ are (jointly) Gaussian random variables for all sets $A_j \in \mathcal{M}$, $\mu(A_j) < \infty$, $j = 1, \dots, k$.

(ii′) $EZ_\mu(A) = 0$, and $EZ_\mu(A)Z_\mu(B) = \mu(A \cap B)$ if A, $B \in \mathcal{M}$, $\mu(A) < \infty$, $\mu(B) < \infty$.

It is not difficult to see that properties (i), (ii) and (iii) imply relations (i′) and (ii′). On the other hand, it is clear that (i′) and (ii′) imply (i) and (ii). To see that they also imply relation (iii) observe that under these conditions

$$E\left[Z_\mu\left(\bigcup_{j=1}^k A_j\right) - \sum_{j=1}^k Z_\mu(A_k)\right]^2 = 0$$

if $A_1, \dots, A_k$ are disjoint sets.

The second characterization of random orthogonal measures may help to show that for any measure space $(M, \mathcal{M}, \mu)$ with a σ-finite measure μ there exists a Gaussian random orthogonal measure corresponding to the measure μ. The main point in checking this statement is the proof that for any sets $A_1, \dots, A_k \in \mathcal{M}$, $\mu(A_j) < \infty$, $1 \le j \le k$, there exists a Gaussian random vector $(Z_\mu(A_1), \dots, Z_\mu(A_k))$, $EZ_\mu(A_j) = 0$, with correlation $EZ_\mu(A_i)Z_\mu(A_j) = \mu(A_i \cap A_j)$ for all $1 \le i, j \le k$. To prove this we have to show that the corresponding covariance matrix is really positive definite, i.e. $\sum_{i,j} c_i \bar{c}_j \mu(A_i \cap A_j) \ge 0$ for an arbitrary vector $(c_1, \dots, c_k)$. But this follows from the observation $\sum_{i,j} c_i \bar{c}_j \chi_{A_i \cap A_j}(x) = \sum_{i,j} c_i \bar{c}_j \chi_{A_i}(x)\overline{\chi_{A_j}(x)} = \left|\sum_i c_i \chi_{A_i}(x)\right|^2 \ge 0$ for all $x \in M$, if we integrate this inequality with respect to the measure μ in the space M.

We define the real Hilbert spaces $\bar{\mathcal{K}}_\mu^n$, $n = 1, 2, \dots$. The space $\bar{\mathcal{K}}_\mu^n$ consists of the real-valued measurable functions over $(\underbrace{M \times \cdots \times M}_{n \text{ times}}, \underbrace{\mathcal{M} \times \cdots \times \mathcal{M}}_{n \text{ times}})$ such that

$$\|f\|^2 = \int |f(x_1, \dots, x_n)|^2 \mu(\,dx_1) \dots \mu(\,dx_n) < \infty,$$

and the last formula defines the norm in $\bar{\mathcal{K}}_\mu^n$. Let $\mathcal{K}_\mu^n$ denote the subspace of $\bar{\mathcal{K}}_\mu^n$ consisting of the functions $f \in \bar{\mathcal{K}}_\mu^n$ such that

$$f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)}) \quad \text{for all } \pi \in \Pi_n.$$

Let the spaces $\bar{\mathcal{K}}_\mu^0$ and $\mathcal{K}_\mu^0$ consist of the real constants with the norm $\|c\| = |c|$. Finally we define the Fock space $\operatorname{Exp} \mathcal{K}_\mu$ which consists of the sequences $f = (f_0, f_1, \dots)$, $f_n \in \mathcal{K}_\mu^n$, $n = 0, 1, 2, \dots$, such that

$$\|f\|^2 = \sum_{n=0}^\infty \frac{1}{n!} \|f_n\|^2 < \infty.$$

Given a random orthogonal measure Z_μ corresponding to μ, let us introduce the σ-algebra $\mathcal{F} = \sigma(Z_\mu(A)\colon\ A \in \mathcal{M},\ \mu(A) < \infty)$. Let $\mathcal{K}$ denote the real Hilbert space of square integrable random variables measurable with respect to the σ-algebra $\mathcal{F}$. Let $\mathcal{K}_{\le n}$ denote the subspace that is the closure of the linear space containing the polynomials of the random variables $Z_\mu(A)$ of order less than or equal to n. Let $\mathcal{K}_n$ be the orthogonal completion of $\mathcal{K}_{\le n-1}$ to $\mathcal{K}_{\le n}$. (The norm is defined as $\|\xi\|^2 = E\xi^2$ in these Hilbert spaces.)

The multiple Wiener–Itô integrals with respect to the random orthogonal measure Z_μ, to be defined below, give a unitary transformation from $\operatorname{Exp}\mathcal{K}_\mu$ to $\mathcal{K}$. We shall denote these integrals by $\int'$ to distinguish them from the Wiener–Itô integrals defined in Section 4.

First we define the class of elementary functions $\hat{\bar{\mathcal{K}}}_\mu^n \subset \bar{\mathcal{K}}_\mu^n$. A function $f \in \bar{\mathcal{K}}_\mu^n$ is in $\hat{\bar{\mathcal{K}}}_\mu^n$ if there exists a finite system of disjoint sets $\Delta_1, \dots, \Delta_N$, with $\Delta_j \in \mathcal{M}$, $\mu(\Delta_j) < \infty$, $j = 1, \dots, N$, such that $f(x_1, \dots, x_n)$ is constant on the sets $\Delta_{j_1} \times \dots \times \Delta_{j_n}$ if the indices $j_1, \dots, j_n$ are disjoint, and $f(x_1, \dots, x_n)$ equals zero outside these sets. We define

$$\int' f(x_1, \dots, x_n) Z_\mu(\,dx_1) \dots Z_\mu(\,dx_n) = \sum f(x_{j_1}, \dots, x_{j_n}) Z_\mu(\Delta_{j_1}) \cdots Z_\mu(\Delta_{j_n})$$

for $f \in \hat{\bar{\mathcal{K}}}_\mu^n$, where $x_k \in \Delta_k$, $k = 1, \dots, N$.

Let $\hat{\mathcal{K}}_\mu^n = \hat{\bar{\mathcal{K}}}_\mu^n \cap \mathcal{K}_\mu^n$. The random variables

$$I'_\mu(f) = \frac{1}{n!} \int' f(x_1, \dots, x_n) Z_\mu(\,dx_1) \dots Z_\mu(\,dx_n), \quad f \in \hat{\bar{\mathcal{K}}}_\mu^n,$$

have zero expectation, integrals of different order are orthogonal,

$$I'_\mu(f) = I'_\mu(\operatorname{Sym} f), \quad \text{and } \operatorname{Sym} f \in \hat{\mathcal{K}}_\mu^n \text{ if } f \in \hat{\bar{\mathcal{K}}}_\mu^n,$$

$$EI'_\mu(f)^2 \le \frac{1}{n!}\|f\|^2 \quad \text{if } f \in \hat{\bar{\mathcal{K}}}_\mu^n, \tag{7.1}$$

and (7.1) holds with equality if $f \in \hat{\mathcal{K}}_\mu^n$.

It can be seen that $\hat{\bar{\mathcal{K}}}_\mu^n$ is dense in $\bar{\mathcal{K}}_\mu^n$, hence relation (7.1) enables us to extend the definition of the n-fold Wiener–Itô integrals over $\bar{\mathcal{K}}_\mu^n$. All the above mentioned relations remain valid if $f \in \hat{\bar{\mathcal{K}}}_\mu^n$ is substituted by $f \in \bar{\mathcal{K}}_\mu^n$, and $f \in \hat{\mathcal{K}}_\mu^n$ is substituted by $f \in \mathcal{K}_\mu^n$. We formulate Itô's formula for these integrals. It can be proved similarly to Theorem 4.2.

Theorem 7.1. (Itô's formula) *Let $\varphi_1, \dots, \varphi_m$, $\varphi_j \in \mathcal{K}_\mu^1$ for all $1 \le j \le m$, be an orthonormal system in L_μ^2. Let some positive integers $j_1, \dots, j_m$ be given, put $j_1 + \dots + j_m = N$, and define for all $i = 1, \dots, N$*

$$g_i = \varphi_1 \text{ for } 1 \le i \le j_1, \quad \text{and} \quad g_i = \varphi_s \quad \text{for } j_1 + \dots + j_{s-1} < i \le j_1 + \dots + j_s.$$

Then

$$H_{j_1}\left(\int' \varphi_1(x)Z_\mu(\,dx)\right)\cdots H_{j_m}\left(\int' \varphi_m(x)Z_\mu(\,dx)\right)$$
$$=\int' g_1(x_1)\cdots g_N(x_N)\,Z_\mu(\,dx_1)\dots Z_\mu(\,dx_N)$$
$$=\int' \mathrm{Sym}\,[g_1(x_1)\cdots g_N(x_N)]\,Z_\mu(\,dx_1)\dots Z_\mu(\,dx_N).$$

(Let me remark that the diagram formula (Theorem 5.3) also remains valid for this integral if we replace $-x_j$ is by x_j and $G(\,dx_j)$ by $\mu(\,dx_j)$, $N-2|\gamma|+1\le j\le N-|\gamma|$, in the definition of h_γ in formula (5.1).)

It can be seen with the help of Theorem 7.1 that the transformation $I'_\mu\colon \mathrm{Exp}\,\mathcal{K}_\mu\to\mathcal{K}$, where $I'_\mu(f)=\sum_{n=0}^{\infty} I'_\mu(f_n)$, $f=(f_0,f_1,\dots)\in\mathrm{Exp}\,\mathcal{K}_\mu$ is a unitary transformation, and so are the transformations $(n!)^{1/2}I'_\mu$ from $\mathcal{K}_\mu^n$ to $\mathcal{K}_n$.

Let us consider the special case $(M,\mathcal{M},\mu)=(R^\nu,\mathcal{B}^\nu,\lambda)$, where λ denotes the Lebesgue measure in R^ν. A random orthogonal measure corresponding to λ is called the white noise. A random *spectral measure* corresponding to λ, when the Lebesgue measure is considered as the spectral measure of a generalized field, is also called a white noise. The next result, that can be considered as a random Plancherel formula, establishes a connection between the two types of Wiener–Itô integrals with respect to white noise.

Proposition 7.2. *Let $f=(f_0,f_1,\dots,)\in\mathrm{Exp}\,\mathcal{K}_\lambda$ be an element of the Fock space corresponding to the Lebesgue measure in the Euclidean space $(R^\nu,\mathcal{B}^\nu)$. Then $f'=(f'_0,f'_1,\dots,)\in\mathrm{Exp}\,\mathcal{H}_\lambda$ with the functions $f'_0=f_0$ and $f'_n=(2\pi)^{-n\nu/2}\tilde f_n$, $n=1,2,\dots$, (where $\tilde f_n(u_1,\dots,u_n)=\int_{R^{n\nu}} e^{i(x,u)}f_n(x_1,\dots,x_n)\,dx_1\dots dx_n$ with $x=(x_1,\dots,x_n)$ and $u=(u_1,\dots,u_n)$), and*

$$\sum_{n=0}^{\infty}\frac{1}{n!}\int' f_n(x_1,\dots,x_n)\,Z_\lambda(\,dx_1)\dots Z_\lambda(\,dx_n)$$
$$\triangleq\sum_{n=0}^{\infty}\frac{1}{n!}\int f'_n(u_1,\dots,u_n)\,Z_\lambda(\,du_1)\dots Z_\lambda(\,du_n),$$

where $Z_\lambda(\,dx)$ is a white noise as a random orthogonal measure and $Z_\lambda(\,du)$ is a white noise as a random spectral measure.

Proof of Proposition 7.2. We have

$$(2\pi)^{-n\nu/2}\|\tilde f_n\|_{L_\lambda^2}=\|f_n\|_{L_\lambda^2},$$

hence $f' \in \operatorname{Exp} \mathcal{H}_\lambda$.

Let $\varphi_1, \varphi_2, \dots$ be a complete orthonormal system in L^2_λ. Then $\varphi'_1, \varphi'_2, \dots$ is also a complete orthonormal system in L^2_λ, and if

$$f_n(x_1, \dots, x_n) = \sum c_{j_1,\dots,j_n} \varphi_{j_1}(x_1) \cdots \varphi_{j_n}(x_n),$$

then

$$f'_n(u_1, \dots, u_n) = \sum c_{j_1,\dots,j_n} \varphi'_{j_1}(u_1) \cdots \varphi'_{j_n}(u_n).$$

Hence an application of Itô's formula for both types of integrals, (i.e. Theorems 4.2 and 7.1) imply Proposition 7.2.

Finally we restrict ourselves to the case $\nu = 1$. We formulate a result which reflects a connection between multiple Wiener–Itô integrals and classical Itô integrals. Let $W(t)$, $a \le t \le b$, be a Wiener process, and let us define the random orthogonal measure $Z(\,dx)$ as

$$Z(A) = \int \chi_A(x) W(\,dx), \quad A \subset [a, b), \quad A \in \mathcal{B}^1.$$

Then we have the following

Proposition 7.3. *Let $f \in \mathcal{K}^n_{\lambda[a,b)}$, where $\lambda[a, b)$ denotes the Lebesgue measure on the interval $[a, b)$. Then*

$$\begin{aligned} &\int{}' f(x_1, \dots, x_n)\, Z(\,dx_1) \dots Z(\,dx_n) \qquad (7.2) \\ &\quad = n! \int_a^b \left(\int_a^{t_n} \left(\cdots \left(\int_a^{t_3} \left(\int_a^{t_2} f(t_1, \dots, t_n) W(\,dt_1) \right) W(\,dt_2) \right) \dots \right) W(\,dt_n) \right). \end{aligned}$$

Proof of Proposition 7.3. Given a function $f \in \hat{\mathcal{K}}^n_{\lambda[a,b)}$, let the function $\hat{f}$ be defined as

$$\hat{f}(x_1, \dots, x_n) = \begin{cases} f(x_1, \dots, x_n) & \text{if } \ x_1 < x_2 < \cdots < x_n \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to check Proposition 7.3 for such special functions $f \in \hat{\mathcal{K}}^n_{\lambda[a,b)}$ for which the function $\hat{f}$is the indicator function of a rectangle of the form $\prod_{j=1}^n [a_j, b_j)$ with constants $a \le a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \le b$. Here we exploit the relation $I'(f) = n! I'(\hat{f})$. Beside this, we have to calculate the value of the right-hand side of formula (7.2) for such elementary functions $f \in \hat{\mathcal{K}}^n_{\lambda[a,b)}$. A simple inductive argument shows that it equals $\prod_{j=1}^n [W(b_j) - W(a_j)]$ if $a \le a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \le b$, and it equals zero otherwise. Then a simple limiting procedure with the help of the

approximation of general functions in $\mathcal{K}^n_{\lambda[a,b)}$ by the linear combinations of such functions proves Proposition 7.3 in the general case.

As a consequence of Proposition 7.3 in the case $\nu = 1$ multiple Wiener–Itô integrals can be substituted by Itô integrals in the investigation of most problems. In the case $\nu = 2$ there is no simple definition of Itô integrals. On the other hand, no problem arises in generalizing the definition of multiple Wiener–Itô integrals to the case $\nu \geq 2$.

8. Non-central limit theorems.

In this section we investigate the problem formulated in Section 1, and we show how the technique of Wiener–Itô integrals can be applied for the investigation of such a problem. We restrict ourselves to the case of discrete fields, although the case of generalized fields can be discussed in almost the same way. The proof of some details will be omitted. They can be found in [9]. First we recall the following

Definition 8A. (Definition of slowly varying functions.) *A function $L(t)$, $t \in [t_0, \infty)$, $t_0 > 0$, is said to be a slowly varying function (at infinity) if*

$$\lim_{t\to\infty} \frac{L(st)}{L(t)} = 1 \quad \textit{for all} \quad s > 0.$$

We shall apply the following description of slowly varying functions.

Theorem 8A. (Karamata's theorem.) *If a slowly varying function $L(t)$ is bounded on every finite interval, then it can be represented in the form*

$$L(t) = a(t) \exp\left\{\int_{t_0}^{t} \frac{\varepsilon(s)}{s}\, ds\right\},$$

where $a(t) \to a_0 \neq 0$, and $\varepsilon(t) \to 0$ as $t \to \infty$, and the functions $a(\cdot)$ and $\varepsilon(\cdot)$ are bounded in every finite interval.

Let X_n, $n \in \mathbb{Z}_\nu$, be a stationary Gaussian field with expectation zero and a correlation function

$$r(n) = EX_0X_n = |n|^{-\alpha} a\left(\frac{n}{|n|}\right) L(|n|), \quad n \in \mathbb{Z}_\nu, \tag{8.1}$$

where $0 < \alpha < \nu$, $L(t)$ is a slowly varying function, bounded in all finite intervals, and $a(t)$ is a continuous function on the unit sphere $\mathcal{S}_{\nu-1}$, satisfying the symmetry property $a(x) = a(-x)$ for all $x \in \mathcal{S}_{\nu-1}$. Let G denote the spectral measure of the field X_n, and let us define the measures G_N, $N = 1, 2, \dots$, by the formula

$$G_N(A) = \frac{N^\alpha}{L(N)} G\left(\frac{A}{N}\right), \quad A \in \mathcal{B}^\nu, \quad N = 1, 2, \dots. \tag{8.2}$$

Now we recall the definition of vague convergence of not necessarily finite measures on a Euclidean space.

Definition of vague convergence of measures. *Let G_n, $n = 1, 2, \dots$, be a sequence of locally finite measures over R^ν, i.e. let $G_n(A) < \infty$ for all measurable bounded sets A. We say that the sequence G_n vaguely converges to a locally finite measure G_0 (in notation $G_n \xrightarrow{v} G_0$) if*

$$\lim_{n\to\infty} \int f(x)\, G_n(\,dx) = \int f(x)\, G_0(\,dx)$$

for all continuous functions f with a bounded support.

We formulate the following

Lemma 8.1. *Let G be the spectral measure of a stationary field with a correlation function $r(n)$ of the form (8.1). Then the sequence of measures G_N defined in (8.2) tends vaguely to a locally finite measure G_0. The measure G_0 has the homogeneity property*

$$G_0(A) = t^{-\alpha} G_0(tA) \quad \text{for all } A \in \mathcal{B}^\nu \quad \text{and } t > 0, \tag{8.3}$$

and it satisfies the identity

$$\begin{aligned} &2^\nu \int e^{i(t,x)} \prod_{j=1}^{\nu} \frac{1-\cos x^{(j)}}{(x^{(j)})^2}\, G_0(\,dx) \\ &\quad = \int_{[-1,1]^\nu} (1-|x^{(1)}|)\cdots(1-|x^{(\nu)}|) \frac{a\left(\frac{x+t}{|x+t|}\right)}{|x+t|^\alpha}\, dx, \quad \text{for all } t \in R^\nu. \end{aligned} \tag{8.4}$$

We postpone the proof of Lemma 8.1 for a while.

Formulae (8.3) and (8.4) imply that the function $a(t)$ and the number α in the definition (8.1) of a correlation function $r(n)$ uniquely determine the measure G_0. Indeed, by formula (8.4) they determine the (finite) measure $\prod_{j=1}^{\nu} \frac{1-\cos x^{(j)}}{(x^{(j)})^2} G_0(\,dx)$, since they determine its Fourier transform. Hence they also determine the measure G_0. (Formula (8.3) shows that this is a locally finite measure). Let us also remark that since $G_N(A) = G_N(-A)$ for all $N = 1, 2, \dots$ and $A \in \mathcal{B}^\nu$, the relation $G_0(A) = G_0(-A)$, $A \in \mathcal{B}^\nu$ also holds. These properties of the measure G_0 imply that it can be considered as the spectral measure of a generalized random field. Now we formulate

Theorem 8.2. *Let X_n, $n \in \mathbb{Z}_\nu$, be a stationary Gaussian field with a correlation function $r(n)$ satisfying relation (8.1). Let us define the stationary random field $\xi_j = H_k(X_j)$, $j \in \mathbb{Z}_\nu$, with some positive integer k, where $H_k(x)$ denotes the k-th Hermite polynomial with leading coefficient 1, and assume that the parameter α appearing in (8.1) satisfies the relation $0 < \alpha < \frac{\nu}{k}$. If the random fields Z_n^N, $N = 1, 2, \dots$, $n \in \mathbb{Z}_\nu$,*

are defined by formula (1.1) with $A_N = N^{\nu - k\alpha/2} L(N)^{k/2}$ *and the above defined* $\xi_j = H_k(X_j)$*, then their multi-dimensional distributions tend to those of the random field* Z_n^**,*

$$Z_n^* = \int \tilde{\chi}_n(x_1 + \dots + x_k)\, Z_{G_0}(\, dx_1) \dots Z_{G_0}(\, dx_k), \quad n \in \mathbb{Z}_\nu.$$

Here Z_{G_0} *is a random spectral measure corresponding to the spectral measure* G_0 *which appeared in Lemma 8.1. The function* $\tilde{\chi}_n(\cdot)$*,* $n = (n^{(1)}, \dots, n^{(\nu)})$*, is (similarly to Section 6) the Fourier transform of the uniform distribution on the* ν*-dimensional unit cube* $\prod_{p=1}^{\nu} [n^{(p)}, n^{(p)} + 1]$*.*

Remark. The condition that the correlation function $r(n)$ of the random field X_n, $n \in \mathbb{Z}_\nu$, satisfies formula (8.1) can be weakened. Theorem 8.2 and Lemma 8.1 remain valid if (8.1) is replaced by the slightly weaker condition

$$\lim_{T \to \infty} \sup_{n:\, n \in \mathbb{Z}_\nu,\, |n| \ge T} \frac{r(n)}{|n|^{-\alpha} a\left(\frac{n}{|n|}\right) L(|n|)} = 1,$$

where $0 < \alpha < \nu$, $L(t)$ is a slowly varying function, bounded in all finite intervals, and $a(t)$ is a continuous function on the unit sphere $\mathcal{S}_{\nu-1}$, satisfying the symmetry property $a(x) = a(-x)$ for all $x \in \mathcal{S}_{\nu-1}$.

First we explain why the choice of the normalizing constant A_N in Theorem 8.2 was natural, then we explain the ideas of the proof, finally we work out the details.

Corollary 5.5 implies in particular that $EH_k(\xi)H_k(\eta) = k!(E\xi\eta)^k$ for a Gaussian random vector (ξ, η) with $E\xi = E\eta = 0$ and $E\xi^2 = E\eta^2 = 1$. Hence

$$E(Z_n^N)^2 = \frac{k!}{A_N^2} \sum_{j \in B_0^N,\, l \in B_0^N} r(j-l)^k \sim \frac{k!}{A_N^2} \sum_{j,\, l \in B_0^N} |j-l|^{-k\alpha} a^k \left(\frac{j-l}{|j-l|}\right) L(|j-l|)^k,$$

with the set B_0^N introduced after formula (1.1). Some calculation with the help of the above formula shows that with our choice of A_N the expectation $E(Z_n^N)^2$ is separated both from zero and infinity, therefore this is the natural norming factor. In this calculation we have to exploit the condition $k\alpha < \nu$, which implies that in the sum expressing $E(Z_n^N)^2$ those terms are dominant for which $j - l$ is relatively large, more explicitly it is of order N. There are const. $N^{2\nu}$ such terms.

The field ξ_n is subordinated to the Gaussian field X_n. It is natural to rewrite it in canonical form, and to express Z_n^N via multiple Wiener–Itô integrals. Itô's formula yields the relation

$$\xi_n = H_k\left(\int e^{i(n,x)} Z_G(\, dx)\right) = \int e^{i(n, x_1 + \dots + x_k)} Z_G(\, dx_1) \dots Z_G(\, dx_k),$$

where Z_G is the random spectral measure adapted to the random field X_n. Then

$$Z_n^N = \frac{1}{A_N} \sum_{j \in B_n^N} \int e^{i(j, x_1 + \dots + x_k)} Z_G(\, dx_1) \dots Z_G(\, dx_k)$$

$$= \frac{1}{A_N} \int e^{i(Nn, x_1 + \dots + x_k)} \prod_{j=1}^{\nu} \frac{e^{iN(x_1^{(j)} + \dots + x_k^{(j)})} - 1}{e^{i(x_1^{(j)} + \dots + x_k^{(j)})} - 1} Z_G(\, dx_1) \dots Z_G(dx_k).$$

Let us make the substitution $y_j = Nx_j$, $j = 1, \dots, k$, in the last formula, and let us rewrite it in a form resembling formula (6.5′). To this end, let us introduce the measures G_N defined in (8.2). By Lemma 4.5 we can write

$$Z_n^N \stackrel{\Delta}{=} \int f_N(y_1, \dots, y_k) \tilde{\chi}_n(y_1 + \dots + y_k)\, Z_{G_N}(\, dy_1) \dots Z_{G_N}(dy_k)$$

with

$$f_N(y_1, \dots, y_k) = \prod_{j=1}^{\nu} \frac{i(y_1^{(j)} + \dots + y_k^{(j)})}{\left(\exp\left\{i \frac{1}{N}(y_1^{(j)} + \dots + y_k^{(j)})\right\} - 1\right) N}. \tag{8.5}$$

(It follows from Lemma 8B formulated below and the Fubini theorem that the set, where the denominator of the function f_N disappears, i.e. the set where $y_1^{(j)} + \dots + y_k^{(j)} = 2lN\pi$ with some integer $l \neq 0$ and $1 \leq j \leq \nu$ has 0 $G_N \times \dots \times G_N$ measure. This means that the functions f_N are well defined.) The functions f_N tend to 1 uniformly in all bounded regions, and the measures G_N tend vaguely to G_0 as $N \to \infty$ by Lemma 8.1. These relations suggest the following limiting procedure. The limit of Z_n^N can be obtained by substituting f_N with 1 and G_N with G_0 in the Wiener–Itô integral expressing Z_n^N. We want to justify this formal limiting procedure. For this we have to show that the Wiener–Itô integral expressing Z_n^N is essentially concentrated in a large bounded region independent of N. The L_2 isomorphism of Wiener–Itô integrals can help us in showing that. The next lemma is a useful tool for the justification of the above limiting procedure.

Before formulating this lemma we make a small digression. It was explained that Wiener–Itô integrals can be defined also with respect to random stationary fields Z_G adapted to a stationary Gaussian random field whose spectral measure G may have atoms, and we can work with them similarly as in the case of non-atomic spectral measures. Here a lemma will be proved which shows that in the proof of Theorem 8.2 we do not need this observation, because if the correlation function of the random field satisfies (8.1), then its spectral measure is non-atomic.

Lemma 8B. *Let the correlation function of a stationary field X_n, $n \in \mathbb{Z}_\nu$, satisfy the relation $r(n) \leq A|n|^{-\alpha}$ with some $A > 0$ and $\alpha > 0$ for all $n \in \mathbb{Z}_\nu$, $n \neq 0$. Then its spectral measure G is non-atomic. Moreover, all hyperplanes $\sum_{j=1}^{\nu} c_j x^{(j)} = d$ defined with some constants c_j and d have zero G measure.*

Proof of Lemma 8B. Lemma 8B clearly holds if $\alpha > \nu$, because in this case the spectral measure G has even a density function $g(x) = \sum_{n \in \mathbb{Z}_\nu} e^{-i(n,x)} r(n)$. On the other hand, the p-fold convolution of the spectral measure G with itself (on the torus $R^\nu / 2\pi\mathbb{Z}_\nu$) has Fourier transform, $r(n)^p$, hence in the case $p > \frac{\nu}{\alpha}$ it is non-atomic. Hence it is enough to show that if the convolution $G * G$ is a non-atomic measure, then so is the measure G. But this is obvious, because if there were a point $x \in R^\nu / 2\pi\mathbb{Z}_\nu$ such that $G(\{x\}) > 0$, then $G * G(\{x + x\}) > 0$ would hold, and this is a contradiction. (Here addition is taken on the torus.) The proof of the zero G measure of all hyperplanes is similar.

Now we formulate the following result.

Lemma 8.3. *Let G_N, $N = 1, 2, \dots$, be a sequence of spectral measures on R^ν tending vaguely to a spectral measure G_0. Let a sequence of measurable functions $K_N = K_N(x_1, \dots, x_k)$, $N = 0, 1, 2, \dots$, be given such that $K_N \in \bar{\mathcal{H}}^k_{G_N}$ for $N = 1, 2, \dots$. Assume further that the following properties hold: For all $\varepsilon > 0$ there exist some constants $A = A(\varepsilon) > 0$ and $N_0 = N_0(\varepsilon) > 0$ and finitely many rectangles $P_1, \dots, P_M$ with some cardinality $M = M(\varepsilon)$ on $R^{k\nu}$ which satisfy the following conditions a) and b). (We call a set $P \in \mathcal{B}^{k\nu}$ a rectangle if it can be written in the form $P = L_1 \times \cdots \times L_k$ with some bounded open sets $L_s \in \mathcal{B}^\nu$, $1 \le s \le k$, with boundaries ∂L_s of zero G_0 measure, i.e. $G_0(\partial L_s) = 0$ for all $1 \le s \le k$.)*

a) The function K_0 is continuous on the set $B = [-A, A]^{k\nu} \setminus \bigcup_{j=1}^{M} P_j$, and $K_N \to K_0$ uniformly on the set B as $N \to \infty$. Beside this the hyperplanes $x_p = \pm A$ have zero G_0 measure for all $1 \le p \le \nu$.

b) $\int_{R^{k\nu} \setminus B} |K_N(x_1, \dots, x_k)|^2 G_N(\,dx_1) \dots G_N(dx_k) < \frac{\varepsilon^3}{k!}$ if $N = 0$ or $N \ge N_0$, and $K_0(-x_1, \dots, -x_k) = \overline{K_0(x_1, \dots, x_k)}$ for all $(x_1, \dots, x_k) \in R^{k\nu}$.

Then $K_0 \in \bar{\mathcal{H}}^k_{G_0}$, and

$$\int K_N(x_1, \dots, x_k)\, Z_{G_N}(\,dx_1) \dots Z_{G_N}(\,dx_k) \xrightarrow{\mathcal{D}} \int K_0(x_1, \dots, x_k)\, Z_{G_0}(\,dx_1) \dots Z_{G_0}(\,dx_k)$$

as $N \to \infty$, where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.

Remark. In the proof of Theorem 8.2 or of its generalization Theorem 8.2′ formulated later a simpler version of Lemma 8.3 with a simpler proof would suffice. We could work with such a version where the rectangles P_j do not appear. We formulated this somewhat more complicated result, because it can be applied in the proof of more general theorems, where the limit is given by such a Wiener–Itô integral whose kernel function may have discontinuities.

Proof of Lemma 8.3. Conditions a) and b) obviously imply that

$$\int |K_0(x_1, \dots, x_k)|^2\, G_0(\,dx_1) \dots G_0(\,dx_k) < \infty,$$

hence $K_0 \in \bar{\mathcal{H}}^k_{G_0}$. By using the same argument as in the definition of Wiener–Itô integrals with atomic spectral measure we can reduce the lemma to the case when the spectral measures G_N, $N = 0, 1, 2, \dots$, are non-atomic.

Let us fix an $\varepsilon > 0$, and let $A > 0$, $N_0 > 0$ and the rectangles $P_1, \dots, P_M$ satisfy conditions a) and b) with this ε. Then

$$\begin{aligned} E\left[\int [1 - \chi_B(x_1, \dots, x_k)] K_N(x_1, \dots, x_k)\, Z_{G_N}(\,dx_1) \dots Z_{G_N}(\,dx_k)\right]^2 \\ \le k! \int_{R^{k\nu} \setminus B} |K_N(x_1, \dots, x_k)|^2 G_N(\,dx_1) \dots G_N(\,dx_k) < \varepsilon^3 \end{aligned} \tag{8.6}$$

for $N = 0$ or $N > N_0$, where χ_B denotes the indicator function of the set B introduced in the formulation of condition a).

Since $B \subset [-A, A]^{k\nu}$, and $G_N \xrightarrow{v} G_0$, hence $G_N \times \dots \times G_N(B) < C(A)$ with an appropriate constant $C(A) < \infty$ for all $N = 0, 1, \dots$. Because of this estimate and the uniform convergence $K_N \to K_0$ on the set B we have

$$\begin{aligned} E\left[\int (K_N(x_1, \dots, x_k) - K_0(x_1, \dots, x_k)) \chi_B(x_1, \dots, x_k)\, Z_{G_N}(\,dx_1) \dots Z_{G_N}(\,dx_k)\right]^2 \\ \le k! \int_B |K_N(x_1, \dots, x_k) - K_0(x_1, \dots, x_k)|^2\, G_N(\,dx_1) \dots G_N(\,dx_k) < \varepsilon^3 \end{aligned} \tag{8.7}$$

for $N > N_1$ with some $N_1 = N_1(A, \varepsilon)$.

With the help of formulas (8.6) and (8.7) we reduce the proof of Lemma 8.3 to that of the relation

$$\begin{aligned} \int K_0(x_1, \dots, x_k) \chi_B(x_1, \dots, x_k)\, Z_{G_N}(\,dx_1) \dots Z_{G_N}(\,dx_k) \\ \xrightarrow{\mathcal{D}} \int K_0(x_1, \dots, x_k) \chi_B(x_1, \dots, x_k)\, Z_{G_0}(\,dx_1) \dots Z_{G_0}(\,dx_k). \end{aligned} \tag{8.8}$$

We do this with the help of a classical result of probability theory about the basic properties of the so-called Prokhorov metric defined in the following way. Given a complete separable metric space $(X, \mathcal{A})$ with some metric ρ let $\mathcal{S}$ denote the space of probability measures on it. The Prokhorov metric ρ_P is the metric in the space $\mathcal{S}$ defined by the formula $\rho_P(\mu, \nu) = \inf\{\varepsilon\colon\ \mu(A) \le \nu(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{A}\}$ for two probability measures $\mu, \nu \in \mathcal{S}$, where $A^\varepsilon = \{x\colon\ \rho(x, A) < \varepsilon\}$. It is known that ρ_P is a metric on $\mathcal{S}$ (in particular $\rho_P(\mu, \nu) = \rho_P(\nu, \mu)$) which metricizes the weak convergence of probability measures in the metric space $(X, \mathcal{A})$. (see R.M. Dudley Distances of probability measures and random variables. Ann. Math. Statist. 39, 1563–1572 (1968)).

I formulated the above result for probability measures in a general metric space, but I shall work on the real line. Given a random variable ξ let $\mu(\xi)$ denote its distribution.Put $\xi_N = k! I_{G_N}(K_N(x_1, \dots, x_k))$, $N = 0, 1, 2, \dots$. With such a notation we can

formulate the statement of Lemma 8.3 in the following way. For all $\varepsilon > 0$ there exists some index $N_0' = N_0'(\varepsilon)$ such that $\rho_P(\mu(\xi_N), \mu(\xi_0)) \le 4\varepsilon$ for all $N \ge N_0'$.

To prove this statement let us first show that for three random variables ξ, $\bar{\xi}$ and η such that $P(|\eta| \ge \varepsilon) \le \varepsilon$ the inequality $\rho_P(\mu(\xi+\eta), \mu(\bar{\xi})) \le \rho_P(\mu(\xi), \mu(\bar{\xi})) + \varepsilon$ holds. Indeed, since $\{\omega\colon\ \xi(\omega) + \eta(\omega) \in A\} \subset \{\omega\colon\ \xi(\omega) \in A^\varepsilon\} \cup \{\omega\colon\ |\eta(\omega)| \ge \varepsilon\}$, we have $P(\xi + \eta \in A) \le P(\xi \in A^\varepsilon) + \varepsilon$ for any set $A \in \mathcal{B}_1$ if $P(|\eta| \ge \varepsilon) \le \varepsilon$. Beside this, $P(\xi \in A^\varepsilon) \le P(\bar{\xi} \in A^{\varepsilon+\delta}) + \delta$ for all $\delta > \rho_P(\mu(\xi), \mu(\bar{\xi}))$. Hence $P(\xi + \eta \in A) \le P(\bar{\xi} \in A^{\varepsilon+\delta}) + \varepsilon + \delta$ for all $A \in \mathcal{B}_1$ and $\delta > \rho_P(\mu(\xi), \mu(\bar{\xi}))$, i.e. $\rho_P(\mu(\xi+\eta), \mu(\bar{\xi})) \le \varepsilon + \delta$, and this implies the inequality $\rho_P(\mu(\xi+\eta), \mu(\bar{\xi})) \le \rho_P(\mu(\xi), \mu(\bar{\xi})) + \varepsilon$.

Put

$$\xi_N^{(1)} = k! I_{G_N}(K_0(x_1, \dots, x_k)\chi_B(x_1, \dots, x_k)),$$
$$\xi_N^{(2)} = k! I_{G_N}(K_N(x_1, \dots, x_k) - K_0(x_1, \dots, x_k))\chi_B(x_1, \dots, x_k)),$$
$$\xi_N^{(3)} = k! I_{G_N}(1 - \chi_B(x_1, \dots, x_k))K_N(x_1, \dots, x_k))$$

for all $N = 0, 1, 2, \dots$. With this notation it follows from relation (8.8) and the fact that the Prokhorov metric metricizes the weak convergence that $\rho_P(\mu(\xi_N^{(1)}), \mu(\xi_0^{(1)})) \le \varepsilon$ if $N \ge N_1'(\varepsilon)$ with some threshold index $N_1'(\varepsilon)$. Formulas (8.6) and (8.7) together with the Chebishev inequality imply that $P(|\xi_N^{(2)}| \ge \varepsilon) \le \varepsilon$ and $P(|\xi_N^{(3)}| \ge \varepsilon) \le \varepsilon$ if $N \ge N_2'(\varepsilon)$ or $N = 0$ with some threshold index $N_2'(\varepsilon)$. Beside this, we have $\xi_0 = \xi_0^{(1)} + \xi_0^{(3)}$ and $\xi_N = \xi_N^{(1)} + \xi_N^{(2)} + \xi_N^{(3)}$ for $N = 1, 2, \dots$. The above mentioned properties of the random variables we considered together with the result of the previous paragraph imply that

$$\begin{aligned}
\rho_P(\mu(\xi_N), \mu(\xi_0)) &= \rho_P(\mu(\xi_N^{(1)} + \xi_N^{(2)} + \xi_N^{(3)}), \mu(\xi_0^{(1)} + \xi_0^{(3)})) \\
&\le \rho_P(\mu(\xi_N^{(1)} + \xi_N^{(2)} + \xi_N^{(3)}), \mu(\xi_0^{(3)})) + \varepsilon \\
&\le \rho_P(\mu(\xi_N^{(2)} + \xi_N^{(3)}), \mu(\xi_0^{(3)})) + 2\varepsilon \\
&\le \rho_P(\mu(\xi_N^{(3)}), \mu(\xi_0^{(3)})) + 3\varepsilon \le 4\varepsilon
\end{aligned}$$

if $N \ge N_0'(\varepsilon) = \max(N_1'(\varepsilon), N_2'(\varepsilon))$, and this is what we wanted to prove.

To prove (8.8) we will show that $K_0(x_1, \dots, x_k)\chi_B(x_1, \dots, x_k)$ can be well approximated by functions from $\hat{\hat{\mathcal{H}}}_{G_0}^k$ in the following sense. For all $\varepsilon > 0$ there exists such an (elementary) function $f_\varepsilon \in \hat{\hat{\mathcal{H}}}_{G_0}^k$ for which the $L^2_{G_0^k}$ norm of the difference $f_\varepsilon(x_1, \dots, x_k) - K_0(x_1, \dots, x_k)\chi_B(x_1, \dots, x_k)$ is less than $\frac{\varepsilon^3}{k!}$, and also the $L^2_{G_N^k}$ norm of this difference is smaller than $\frac{\varepsilon^3}{k!}$ if $N \ge N_2$ with some threshold $N_2 = N_2(\varepsilon)$. Moreover, the function f_ε has the following additional property. The function $f_\varepsilon \in \hat{\hat{\mathcal{H}}}_{G_0}^k$ is adapted to such a regular system $\mathcal{D} = \{\Delta_j,\ j = \pm 1, \dots, \pm M\}$ for which the boundaries of the sets Δ_j satisfy the relation $G_0(\partial\Delta_j) = 0$ for all $j = \pm 1, \dots, \pm M$.

First I claim that such a function $f_\varepsilon \in \hat{\hat{\mathcal{H}}}_{G_0}^k$ satisfies the relation

$$\int f_\varepsilon(x_1, \dots, x_k)\, Z_{G_N}(\,dx_1) \dots Z_{G_N}(\,dx_k) \xrightarrow{\mathcal{D}} \int f_\varepsilon(x_1, \dots, x_k)\, Z_{G_0}(\,dx_1) \dots Z_{G_0}(\,dx_k) \tag{8.9}$$

as $N \to \infty$. To prove (8.9) observe that for the regular system $\mathcal{D} = \{\Delta_j,\ j = \pm 1, \dots, \pm M\}$ to which the function $f_\varepsilon \in \hat{\mathcal{H}}^k_{G_0}$ is adapted has the following property: The (Gaussian) random vectors $(Z_{G_N}(\Delta_j),\ j = \pm 1, \dots, \pm M)$ converge in distribution to the (Gaussian) random vector $(Z_{G_0}(\Delta_j),\ j = \pm 1, \dots, \pm M)$ as $N \to \infty$. (We needed at this point the property $G_0(\partial\Delta_j) = 0$. It follows from the vague convergence $G_N \xrightarrow{v} G_0$, (similarly to the case of weak convergence) that $\lim_{N\to\infty} G_N(\Delta_j) = G_0(\Delta_j)$ if $G_0(\partial\Delta_j) = 0$, which also implies the weak convergence of the above mentioned random vectors, but the condition $G_0(\partial\Delta_j) = 0$ cannot be dropped here.) Beside this, the Wiener–Itô integrals in formula (8.9) are polynomials (not depending on the parameter N) of these random vectors. Hence relation (8.9) holds.

The approximation result formulated after formula (8.8) implies the existence of such a function $V_\varepsilon \in \bar{\mathcal{H}}^k_G$ for all $\varepsilon > 0$ for which

$$k!I_G(K_0(x_1, \dots, x_k)\chi_B(x_1, \dots, x_k)) = k!I_G(f_\varepsilon(x_1, \dots, x_k)) + k!I_G(V_\varepsilon(x_1, \dots, x_k))$$

with the above considered function f_ε, and

$$E\left(\int V_\varepsilon(x_1, \dots, x_k) Z_{G_N}(\,dx_1) \dots Z_{G_N}(\,dx_k)\right)^2$$
$$\le k! \int (K_0(x_1, \dots, x_k)\chi_B(x_1, \dots, x_k) - f_\varepsilon(x_1, \dots, x_k))^2 G_N(\,dx_1) \dots G_N(\,dx_k) \le \varepsilon^3$$

if $N \ge N_2$ with some $N_2 = N_2(\varepsilon)$ or $N = 0$. The last inequality together with formula (8.9) imply (8.8) in the same way as inequalities (8.6), (8.7) and (8.8) imply Lemma 8.3.

We still have to prove that $K_0(x_1, \dots, x_k)\chi_B(x_1, \dots, x_k)$ can be well approximated by an appropriate elementary function f_ε. This can be reduced with help of the relation $G_N \xrightarrow{v} G_0$ to the following simpler statement where only the limit measure G_0 is considered. For all $\varepsilon > 0$ there is a function $f_\varepsilon \in \hat{\mathcal{H}}^k_{G_0}$ for which $\int |K_0(x_1, \dots, x_k)\chi_B(x_1, \dots, x_k) - f_\varepsilon(x_1, \dots, x_k)|^2 G_0(\,dx_1) \dots G_0(\,dx_k) < \frac{\varepsilon^3}{k!}$, and it is adapted to such a regular system $\mathcal{D} = \{\Delta_j,\ j = \pm 1, \dots, \pm \bar{N}\}$ whose elements Δ_j have boundaries of zero G_0 measure.

Indeed, as $|K_0(x_1, \dots, x_k)\chi_B(x_1, \dots, x_k) - f_\varepsilon(x_1, \dots, x_k)|^2$ is a bounded function with a compact support which is continuous in almost all points with respect to the measure $\underbrace{G_0 \times \dots \times G_0}_{k}$ the relation $G_N \xrightarrow{v} G_0$ together with the above statement about G_0 also implies the inequality

$$\int |K_0(x_1, \dots, x_k)\chi_B(x_1, \dots, x_k) - f_\varepsilon(x_1, \dots, x_k)|^2 G_N(\,dx_1) \dots G_N(\,dx_k) < \frac{\varepsilon^3}{k!}$$

if $N \ge N_2(\varepsilon)$.

The existence of the approximating function $f_\varepsilon \in \hat{\bar{\mathcal{H}}}^k_{G_0}$ with the desired properties can be shown similarly to proof of the result that $\hat{\bar{\mathcal{H}}}^n_G$ is a dense subset of $\bar{\mathcal{H}}^n_G$. We can reduce the statement we want to verify first to a slightly modified version of Statement A and then to a slightly modified version of Statement B in the proof of the result about the good approximability of a function in $\bar{\mathcal{H}}^n_G$ by a function in $\hat{\bar{\mathcal{H}}}^n_G$. In the modified version of Statement A we are dealing with such sets A and A_1 whose boundaries have zero G_0^k measures, and we demand the same property about the set $B \in \mathcal{B}^{k\nu}$ appearing in their approximation. Similarly, in the modified version of Statement B we are dealing with such sets D_j whose boundaries have zero G_0 measure, and we demand that the set $F \in \mathcal{B}^{k\nu}$ providing a good approximation also must have boundaries of zero G_0^k measure. This can be proved similarly to the original statements. We leave to the reader to work out the details.

Remark. We have formulated this statement in the case when G_N is a spectral measure on R^ν. But it remains valid if G_N is a spectral measure on the torus of size $2C_N\pi$ with $C_N \to \infty$ if $N \to \infty$ if we identify this torus with the set $[-C_N\pi, C_N\pi)^\nu \subset R^\nu$ in a natural way.

Now we turn to the proof of Theorem 8.2.

The proof of Theorem 8.2. We want to prove that for all positive integers p, real numbers $c_1, \dots, c_p$ and $n_l \in \mathbb{Z}_\nu$, $l = 1, \dots, p$,

$$\sum_{l=1}^p c_l Z^N_{n_l} \xrightarrow{\mathcal{D}} \sum_{l=1}^p c_l Z^*_{n_l},$$

since this relation also implies the convergence of the multi-dimensional distributions. Applying the same calculation as before we get with the help of Lemma 4.5 that

$$\sum_{l=1}^p c_l Z^N_{n_l} = \frac{1}{A_N} \sum_{l=1}^p c_l \int \sum_{j \in B^N_{n_l}} e^{i(j, x_1 + \dots + x_k)}\, Z_G(\,dx_1) \dots Z_G(\,dx_k),$$

and

$$\sum_{l=1}^p c_l Z^N_{n_l} \stackrel{\Delta}{=} \int K_N(x_1, \dots, x_k)\, Z_{G_N}(\,dx_1) \dots Z_{G_N}(\,dx_k)$$

with

$$\begin{aligned} K_N(x_1, \dots, x_k) &= \frac{1}{N^\nu} \sum_{l=1}^p c_l \sum_{j \in B^N_{n_l}} \exp\left\{ i \left(\frac{j}{N}, x_1 + \dots + x_k \right) \right\} \\ &= f_N(x_1, \dots, x_k) \sum_{l=1}^p c_l \tilde{\chi}_{n_l}(x_1 + \dots + x_k). \end{aligned} \tag{8.10}$$

with the function f_N defined in (8.5) and the measure G_N defined in (8.2).

Let us define the function

$$K_0(x_1,\dots,x_k)=\sum_{l=1}^{p} c_l\tilde{\chi}_{n_l}(x_1+\dots+x_k)$$

and the measures μ_N on $R^{k\nu}$ by the formula

$$\mu_N(A)=\int_A |K_N(x_1,\dots,x_k)|^2 G_N(\,dx_1)\dots G_N(\,dx_k),\quad A\in\mathcal{B}^{k\nu} \text{ and } N=0,1,\dots, \tag{8.11}$$

where G_0 is the vague limit of the measures G_N.

To prove Theorem 8.2 it is enough to show that Lemma 8.3 can be applied with these spectral measures G_N and functions K_N. (We choose no exceptional rectangles P_j in this application of Lemma 8.3.) Since $G_N \xrightarrow{v} G_0$, and $K_N \to K_0$ uniformly in all bounded regions in $R^{k\nu}$, it is enough to show, beside the proof of Lemma 8.1, that the measures μ_N, $N=1,2,\dots$, tend weakly to the (necessary finite) measure μ_0, (in notation $\mu_N \xrightarrow{w} \mu_0$), i.e. $\int f(x)\mu_N(\,dx)\to\int f(x)\mu_0(\,dx)$ for all continuous and bounded functions f on $R^{k\nu}$. Then this convergence implies condition b) in Lemma 8.3. Moreover, it is enough to show the slightly weaker statement by which there exists some finite measure $\bar{\mu}_0$ such that $\mu_N \xrightarrow{w} \bar{\mu}_0$, since then $\bar{\mu}_0$ must coincide with μ_0 because of the relations $G_N \xrightarrow{v} G_0$ and $K_N\to K_0$ uniformly in all bounded regions of $R^{k\nu}$, and K_0 is a continuous function.

There is a well-known theorem in probability theory about the equivalence between weak convergence of finite measures and the convergence of their Fourier transforms. It would be natural to apply this theorem for proving $\mu_N \xrightarrow{w} \bar{\mu}_0$. On the other hand, we have the additional information that the measures μ_N, $N=1,2,\dots$, are concentrated in the cubes $[-N\pi,N\pi)^{k\nu}$, since the spectral measure G is concentrated in $[-\pi,\pi)^{\nu}$. It is more fruitful to apply a version of the above mentioned theorem, where we can exploit our additional information. We formulate the following

Lemma 8.4. *Let $\mu_1,\mu_2,\dots$ be a sequence of finite measures on R^l such that $\mu_N(R^l\setminus[-C_N\pi,C_N\pi)^l)=0$ for all $N=1,2,\dots$, with some sequence $C_N\to\infty$ as $N\to\infty$. Define the modified Fourier transform*

$$\varphi_N(t)=\int_{R^l}\exp\left\{i\left(\frac{[tC_N]}{C_N},x\right)\right\}\mu_N(\,dx),\quad t\in R^l,$$

where $[tC_N]$ is the integer part of the vector $tC_N\in R^l$. (For an $x\in R^l$ its integer part $[x]$ is the vector $n\in\mathbb{Z}_l$ for which $x^{(p)}-1<n^{(p)}\le x^{(p)}$ if $x^{(p)}\ge 0$, and $x^{(p)}-1\le n^{(p)}<x^{(p)}+1$ if $x^{(p)}<0$ for all $p=1,2,\dots,l$.) If for all $t\in R^l$ the sequence $\varphi_N(t)$ tends to a function $\varphi(t)$ continuous at the origin, then the measures μ_N weakly tend to a finite measure μ_0, and $\varphi(t)$ is the Fourier transform of μ_0.

I make some comments on the conditions of Lemma 8.4. Let us observe that if the measures μ_N or a part of them are shifted with a vector $2\pi C_N u$ with some $u\in\mathbb{Z}_l$, then

their modified Fourier transforms $\varphi_N(t)$ do not change because of the periodicity of the trigonometrical functions $e^{i(j/C_N,x)}$, $j \in \mathbb{Z}_l$. On the other hand, these new measures which are not concentrated in $[-C_N\pi, C_N\pi)^l$, have no limit. Lemma 8.4 states that if the measures μ_N are concentrated in the cubes $[-C_N\pi, C_N\pi)^l$, then the convergence of their modified Fourier transforms defined in Lemma 8.4, which is a weaker condition, than the convergence of their Fourier transforms, also implies their convergence to a limit measure.

Proof of Lemma 8.4. The proof is a natural modification of the proof about the equivalence of weak convergence of measures and the convergence of their Fourier transforms. First we show that for all $\varepsilon > 0$ there exits some $K = K(\varepsilon)$ such that

$$\mu_N(x\colon\, x \in R^l,\ |x^{(1)}| > K) < \varepsilon \quad \text{for all} \quad N \ge 1. \tag{8.12}$$

As $\varphi(t)$ is continuous at the origin there is some $\delta > 0$ such that

$$|\varphi(0,\dots,0) - \varphi(t,0,\dots,0)| < \frac{\varepsilon}{2} \quad \text{if} \quad |t| < \delta. \tag{8.13}$$

We have

$$0 \le \operatorname{Re}\left[\varphi_N(0,\dots,0) - \varphi_N(t,0,\dots,0)\right] \le 2\varphi_N(0,\dots,0) \tag{8.14}$$

for all $N = 1, 2, \dots$. The sequence in the middle term of (8.14) tends to $\operatorname{Re}\left[\varphi(0,\dots,0) - \varphi(t,0,\dots,0)\right]$ as $N \to \infty$. The right-hand side of (8.14) is a bounded function in the variable N, since it is convergent. Hence the dominated convergence theorem can be applied. We get because of the condition $C_N \to \infty$ and relation (8.13) that

$$\begin{aligned}\lim_{N\to\infty} \int_0^{[\delta C_N]/C_N} \frac{1}{\delta} \operatorname{Re}\left[\varphi_N(0,\dots,0) - \varphi_N(t,0,\dots,0)\right] dt \\ = \int_0^{\delta} \frac{1}{\delta} \operatorname{Re}\left[\varphi(0,\dots,0) - \varphi(t,0,\dots,0)\right] dt < \frac{\varepsilon}{2}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\varepsilon}{2} &> \lim_{N\to\infty} \int_0^{[\delta C_N]/C_N} \frac{1}{\delta} \operatorname{Re}\left[\varphi_N(0,\dots,0) - \varphi_N(t,0,\dots,0)\right] dt \\ &= \lim_{N\to\infty} \int \left(\frac{1}{\delta} \int_0^{[\delta C_N]/C_N} \operatorname{Re}\left[1 - e^{i[tC_N]x^{(1)}/C_N}\right] dt \right) \mu_N(\,dx) \\ &= \lim_{N\to\infty} \int \frac{1}{\delta C_N} \sum_{j=0}^{[\delta C_N]-1} \operatorname{Re}\left[1 - e^{ijx^{(1)}/C_N}\right] \mu_N(\,dx) \\ &\ge \limsup_{N\to\infty} \int_{\{|x^{(1)}|>K\}} \frac{1}{\delta C_N} \sum_{j=0}^{[\delta C_N]-1} \operatorname{Re}\left[1 - e^{ijx^{(1)}/C_N}\right] \mu_N(\,dx) \\ &= \limsup_{N\to\infty} \int_{\{|x^{(1)}|>K\}} \left(1 - \frac{1}{\delta C_N} \operatorname{Re} \frac{1 - e^{i[\delta C_N]x^{(1)}/C_N}}{1 - e^{ix^{(1)}/C_N}}\right) \mu_N(\,dx)\end{aligned}$$

with arbitrary $K > 0$. (In the last but one step of this calculation we have exploited that $\frac{1}{\delta C_N} \sum_{j=0}^{[\delta C_N]-1} \mathrm{Re}\,[1 - e^{ijx^{(1)}/C_N}] \ge 0$ for all $x^{(1)} \in R^1$.)

Since the measure μ_N is concentrated in $\{x\colon x \in R^l,\ |x^{(1)}| \le C_N\pi\}$, and

$$\mathrm{Re}\,\frac{1 - e^{i[\delta C_N]x^{(1)}/C_N}}{1 - e^{ix^{(1)}/C_N}} = \frac{\mathrm{Re}\,\left(ie^{-ix^{(1)}/2C_N}\left(1 - e^{i[\delta C_N]x^{(1)}/C_N}\right)\right)}{i(e^{-ix^{(1)}/2CN} - e^{ix^{(1)}/2C_N})}$$
$$\le \frac{1}{\left|\sin\left(\frac{x^{(1)}}{2C_N}\right)\right|} \le \frac{C_N\pi}{|x^{(1)}|}$$

if $|x^{(1)}| \le C_N\pi$, (here we exploit that $|\sin u| \ge \frac{2}{\pi}|u|$ if $|u| \le \frac{\pi}{2}$), hence we have with the choice $K = \frac{2\pi}{\delta}$

$$\frac{\varepsilon}{2} > \limsup_{N\to\infty} \int_{\{|x^{(1)}|>K\}} \left(1 - \left|\frac{\pi}{\delta x^{(1)}}\right|\right) \mu_N(\,dx) \ge \limsup_{N\to\infty} \frac{1}{2}\mu_N(|x^{(1)}| > K).$$

As the measures μ_N are finite the inequality $\mu_N(|x^{(1)}| > K) < \varepsilon$ holds for each index N with a constant $K = K(N)$ that may depend on N. Hence the above inequality implies that formula (8.12) holds for all $N \ge 1$ with a possibly larger index K that does not depend on N.

Applying the same argument to the other coordinates we find that for all $\varepsilon > 0$ there exists some $C(\varepsilon) < \infty$ such that

$$\mu_N\left(R^l \setminus [-C(\varepsilon), C(\varepsilon)]^l\right) < \varepsilon \quad \text{for all } N = 1, 2, \dots.$$

Consider the usual Fourier transforms

$$\tilde\varphi_N(t) = \int_{R^l} e^{i(t,x)}\mu_N(\,dx), \quad t \in R^l.$$

Then

$$|\varphi_N(t) - \tilde\varphi_N(t)| \le 2\varepsilon + \int_{[-C(\varepsilon),C(\varepsilon)]} \left|e^{i(t,x)} - e^{i([tC_N]/C_N,x)}\right| \mu_N(\,dx) \le 2\varepsilon + \frac{lC(\varepsilon)}{C_N}\mu_N(R^l)$$

for all $\varepsilon > 0$. Hence $\tilde\varphi_N(t) - \varphi_N(t) \to 0$ as $N \to \infty$, and $\tilde\varphi_N(t) \to \varphi(t)$. (Observe that $\mu_N(R^l) = \varphi_N(0) \to \varphi(0) < \infty$ as $N \to \infty$, hence the measures $\mu_N(R^l)$ are uniformly bounded, and $C_N \to \infty$ by the conditions of Lemma 8.4.) Then Lemma 8.4 follows from standard theorems on Fourier transforms.

We return to the proof of Theorem 8.2. We apply Lemma 8.4 with $C_N = N$ and $l = k\nu$ for the measures μ_N defined in (8.11). Because of the middle term in (8.10) we can write

$$\varphi_N(t_1, \dots, t_k) = \sum_{r=1}^{p}\sum_{s=1}^{p} c_r c_s \psi_N(t_1 + n_r - n_s, \dots, t_k + n_r - n_s)$$

with

$$\psi_N(t_1,\dots,t_r)=\frac{1}{N^{2\nu}}\int \exp\left\{i\frac{1}{N}((j_1,x_1)+\dots+(j_k,x_k))\right\}$$
$$\sum_{p\in B_0^N}\sum_{q\in B_0^N}\exp\left\{i\left(\frac{p-q}{N},x_1+\dots+x_k\right)\right\}G_N(\,dx_1)\dots G_N(\,dx_k)$$
$$=\frac{1}{N^{2\nu-k\alpha}L(N)^k}\sum_{p\in B_0^N}\sum_{q\in B_0^N}r(p-q+j_1)\cdots r(p-q+j_k), \tag{8.15}$$

where $j_p=[t_pN]$, $t_p\in R^\nu$, $p=1,\dots,k$.

The asymptotical behaviour of $\psi_N(t_1,\dots,t_k)$ for $N\to\infty$ can be investigated by the help of the last relation and formula (8.1). Rewriting the last double sum in the form of a single sum by fixing first the variable $l=p-q\in[-N,N]^\nu\cap\mathbb{Z}_\nu$, and then summing up for l one gets

$$\psi_N(t_1,\dots,t_k)=\int_{[-1,1]^\nu}f_N(t_1,\dots,t_k,x)\,dx$$

with

$$f_N(t_1,\dots,t_k,x)=\left(1-\frac{[|x^{(1)}N|]}{N}\right)\cdots\left(1-\frac{[|x^{(\nu)}N|]}{N}\right)\frac{r([xN]+j_1)}{N^{-\alpha}L(N)}\cdots\frac{r([xN]+j_k)}{N^{-\alpha}L(N)}.$$

(In the above calculation we exploited that in the last sum of formula (8.15) the number of pairs (p,q) for which $p-q=l=(l_1,\dots,l_\nu)$ equals $(N-|l_1|)\cdots(N-|l_\nu|)$.)

It can be seen with the help of formula (8.1) that

$$f_N(t_1,\dots,t_k,x)\to f_0(t_1,\dots,t_k,x) \tag{8.16}$$

with

$$f_0(t_1,\dots,t_k,x)=(1-|x^{(1)}|)\dots(1-|x^{(\nu)}|)\frac{a\left(\frac{x+t_1}{|x+t_1|}\right)}{|x+t_1|^\alpha}\cdots\frac{a\left(\frac{x+t_k}{|x+t_k|}\right)}{|x+t_k|^\alpha}$$

uniformly on the set $x\in[1,1]^\nu\setminus\bigcup_{p=1}^k\{x\colon\,|x+t_p|>\varepsilon\}$ for all $\varepsilon>0$.

We claim that

$$\psi_N(t_1,\dots,t_k)\to\psi_0(t_1,\dots,t_k)=\int_{[-1,1]^\nu}f_0(t_1,\dots,t_k,x)\,dx,$$

and ψ_0 is a continuous function.

This relation implies that $\mu_N\xrightarrow{w}\mu_0$. To prove it, it is enough to show beside formula (8.16) that

$$\left|\int_{|x+t_p|<\varepsilon}f_0(t_1,\dots,t_k,x)\,dx\right|<C(\varepsilon),\quad p=1,\dots,k, \tag{8.17}$$

and

$$\int_{|x+t_p|<\varepsilon} |f_N(t_1,\dots,t_k,x)|\,dx < C(\varepsilon), \quad p=1,\dots,k, \text{ and } N=1,2,\dots \tag{8.17'}$$

with a constant $C(\varepsilon)$ such that $C(\varepsilon)\to 0$ as $\varepsilon\to 0$.

By Hölder's inequality

$$\left|\int_{|x+t_p|<\varepsilon} f_0(t_1,\dots,t_k,x)\,dx\right| \le C \prod_{1\le l\le k,\, l\ne p} \left[\int_{x\in[-1,1]^\nu} |x+t_l|^{-k\alpha}\,dx\right]^{1/k} \left[\int_{|x+t_p|\le\varepsilon} |x+t_p|^{-k\alpha}\,dx\right]^{1/k} \le C' e^{\nu/k-\alpha}$$

with some appropriate $C>0$ and $C'>0$, since $\nu-k\alpha>0$, and $a(\cdot)$ is a bounded function. Similarly,

$$\int_{|x+t_p|<\varepsilon} |f_N(t_1,\dots,t_k,x)|\,dx \le \prod_{1\le l\le k,\, l\ne p} \left[\int_{x\in[-1,1]^\nu} \frac{|r([xN]+j_l)|^k}{N^{-k\alpha}L(N)^k}\,dx\right]^{1/k} \left[\int_{|x+t_p|\le\varepsilon} \frac{|r([xN]+j_p)|^k}{N^{-k\alpha}L(N)^k}\,dx\right]^{1/k},$$

It is not difficult to see, by using Karamata's theorem, that if $L(\cdot)$ is a slowly varying function which is bounded in all finite intervals, then for all $\eta>0$ there is a threshold index N_0 and a number $C=C(N_0,\eta)$ such that

$$L(tN)\le Ct^{-\eta}L(N) \quad \text{for all } t<1 \text{ and } N\ge N_0.$$

Hence formula (8.1) implies that

$$|r([xN]+j_l)| = |r([xN]+[t_lN]) \le CN^{-\alpha}L(N)(1+|x+t_l|^{-\alpha-\eta}), \tag{8.18}$$

and

$$\int_{|x+t_p|<\varepsilon} \frac{|r([xN]+j_p)|^k}{N^{-k\alpha}L(N)^k}\,dx \le B\int_{|x+t_p|<\varepsilon} (1+|x+t_p|^{-k(\alpha+\eta)})\,dx \le B'\varepsilon^{\nu-k(\alpha+\eta)}$$

$$\int_{x\in[-1,1]^\nu} \frac{|r([xN]+j_l)|^k}{N^{-k\alpha}L(N)^k}\,dx \le B''.$$

for a sufficiently small constant $\eta>0$ with some constants $B,B',B''<\infty$ depending on η and t_p, $1\le p\le k$. (Let us remark that (8.18) holds also for $|[xN]+j_l|\le K_1$ with some $K_1>0$ independent of N, i.e. when the argument of $r(\cdot)$ is relatively small,

because $|r(n)| \le 1$ for all $n \in \mathbb{Z}_\nu$.) Therefore we get, by choosing an $\eta > 0$ such that $k(\alpha+\eta) < \nu$, the inequality

$$\int_{|x+t_p|<\varepsilon} |f_N(t_1,\dots,t_k,x)|\,dx \le C\varepsilon^{\nu/k-(\alpha+\eta)}$$

with some $C < \infty$. The right-hand side of this inequality tends to zero as $\varepsilon \to 0$. Hence we proved beside (8.16) formulae (8.17) and (8.17′), therefore also the relation $\mu_N \xrightarrow{w} \mu_0$. To complete the proof of Theorem 8.2 it remains to prove Lemma 8.1.

Proof of Lemma 8.1. Introduce the notation

$$K_N(x) = \prod_{j=1}^{\nu} \frac{e^{ix^{(j)}}-1}{N(e^{ix^{(j)}/N}-1)}, \quad N = 1,2,\dots,$$

and

$$K_0(x) = \prod_{j=1}^{\nu} \frac{e^{ix^{(j)}}-1}{ix^{(j)}}.$$

Let us consider the measures μ_N defined in formula (8.11) in the special case $k=1$, $p=1$, $c_1=1$. Then

$$\mu_N(A) = \int_A |K_N(x)|^2\, G_N(\,dx).$$

We have already seen in the proof of Theorem 8.2 that $\mu_N \xrightarrow{w} \mu_0$ with some finite measure μ_0, and the Fourier transform of μ_0 is

$$\varphi_0(t) = \int_{[-1,1]^\nu} (1-|x^{(1)}|)\cdots(1-|x^{(\nu)}|)\frac{a\left(\frac{x+t}{|x+t|}\right)}{|x+t|^\alpha}\,dx.$$

First we show that for all $T \ge 1$ there is a finite measure G_0^T concentrated on $[-T\pi, T\pi]^\nu$ such that

$$\lim_{N\to\infty} \int f(x)\, G_N(\,dx) = \int f(x)\, G_0^T(\,dx) \tag{8.19}$$

for all continuous functions f which vanish outside the cube $[-T\pi, T\pi]^\nu$.

Let a continuous function f vanish outside the cube $[-T\pi, T\pi]^\nu$ with some $T \ge 1$. Let $M = [\frac{N}{2T}]$. Then

$$\begin{aligned}
\int f(x) G_N(\,dx) &= \frac{N^\alpha}{L(N)} \cdot \frac{L(M)}{M^\alpha} \int f\left(\frac{N}{M}x\right) G_M(\,dx) \\
&= \frac{N^\alpha L(M)}{M^\alpha L(N)} \int f\left(\frac{N}{M}x\right) |K_M(x)|^{-2} \mu_M(\,dx) \\
&\to (2T)^\alpha \int f(2Tx)|K_0(x)|^{-2}\mu_0(\,dx) \\
&= \int f(x) \frac{(2T)^\alpha}{|K_0(\frac{x}{2T})|^2} \mu_0\left(\frac{dx}{2T}\right) \quad \text{as } N \to \infty,
\end{aligned}$$

because $f(\frac{N}{M}x)|K_M(x)|^{-2}$ vanishes outside the cube $[-\pi,\pi]^\nu$, $f(\frac{N}{M}x)|K_M(x)|^{-2} \to f(2Tx)|K_0(x)^{-2}$ uniformly, (the function $K_0(\cdot)^{-2}$ is continuous in the cube $[-\pi,\pi]^\nu$,) and $\mu_M \xrightarrow{w} \mu_0$ as $N \to \infty$. Hence relation (8.19) holds. The measures G_0^T appearing in (8.19) are consistent for different parameters T, i.e. G_0^T is the restriction of the measure $G_0^{T'}$ to the cube $[-T\pi,T\pi]^\nu$ if $T' > T$. It can be seen with the help of these facts that there is a locally finite measure G_0 on R^ν such that G_0^T is its restriction to the cube $[-T\pi,T\pi]^\nu$, and $G_N \xrightarrow{v} G_0$.

As $G_N \xrightarrow{v} G_0$, and $|K_N(x)|^2 \to |K_0(x)|^2$ uniformly in all bounded regions, hence $\mu_N \xrightarrow{v} \bar{\mu}_0$, where $\bar{\mu}_0(A) = \int_A |K_0(x)|^2 G_0(\,dx)$, $A \in \mathcal{B}^\nu$. Since $\mu_N \xrightarrow{w} \mu_0$ the measures μ_0 and $\bar{\mu}_0$ must coincide, i.e.

$$\mu_0(A) = \int_A |K_0(x)|^2 \, G_0(\,dx), \quad A \in \mathcal{B}^\nu.$$

Relation (8.4) expresses the fact that φ_0 is the Fourier transform of μ_0.

Let us extend the definition of the measures G_N given in (8.2) to all non-negative real numbers u. It is easy to see that the relation $G_u \xrightarrow{v} G_0$ as $u \to \infty$ remains valid. Hence we get for all fixed $s > 0$ and continuous functions f with compact support that

$$\begin{aligned}\int f(x)G_0(\,dx) &= \lim_{u\to\infty} \int f(x)\, G_u(\,dx) = \lim_{u\to\infty} \frac{s^\alpha L(\frac{u}{s})}{L(u)} \int f(sx) G_{\frac{u}{s}}(\,dx) \\ &= s^\alpha \int f(sx) G_0(\,dx) = \int f(x) s^\alpha G_0\left(\frac{dx}{s}\right).\end{aligned}$$

This identity implies the homogeneity property (8.3) of G_0. Lemma 8.3 is proved.

The next result is a generalization of Theorem 8.2.

Theorem 8.2′. *Let X_n, $n \in \mathbb{Z}_\nu$, be a stationary Gaussian field with a correlation function $r(n)$ defined in (8.1). Let $H(x)$ be a real function with the properties $EH(X_n) = 0$ and $EH(X_n)^2 < \infty$. Let us consider the Fourier expansion*

$$H(x) = \sum_{j=1}^{\infty} c_j H_j(x), \quad \sum c_j^2 j! < \infty, \tag{8.20}$$

of the function $H(\cdot)$ by the Hermite polynomials H_j (with leading coefficients 1). Let k be the smallest index in this expansion such that $c_k \neq 0$. If $0 < k\alpha < \nu$ in (8.1), and the field Z_n^N is defined by the field $\xi_n = H(X_n)$, $n \in \mathbb{Z}_\nu$, and formula (1.1), then the multi-dimensional distributions of the fields Z_n^N with $A_N = N^{\nu - k\alpha/2} L(N)^{k/2}$ tend to those of the fields $c_k Z_n^*$, $n \in \mathbb{Z}_\nu$, where the field Z_n^* is the same as in Theorem 8.2.

Proof of Theorem 8.2′. Define $H'(x) = \sum\limits_{j=k+1}^{\infty} c_j H_j(x)$ and $Y_n^N = \frac{1}{A_N} \sum\limits_{l \in B_n^N} H'(X_l)$. Because of Theorem 8.2 in order to prove Theorem 8.2′ it is enough to show that

$$E(Y_n^N)^2 \to 0 \quad \text{as } N \to \infty.$$

It follows from Corollary 5.5 that

$$E(Y_n^N)^2 = \frac{1}{A_N^2} \sum_{j=k+1}^{\infty} j! \sum_{s,t \in B_n^N} [r(s-t)]^j.$$

Hence a simple calculation with the help of formula (8.1) yields

$$E(Y_n^N)^2 = \frac{1}{A_N^2} \left[O(N^{2\nu-(k+1)\alpha} L(N)^{k+1}) + O(N^{\nu}) \right] \to 0.$$

Theorem 8.2′ is proved.

Let us consider a slightly more general version of the problem investigated in Theorem 8.2′. Take a stationary Gaussian random field X_n, $EX_n = 0$, $EX_n^2 = 1$, $n \in \mathbb{Z}_\nu$ with a correlation function satisfying relation (8.1), and the field $\xi_n = H(X_n)$, $n \in \mathbb{Z}_\nu$, subordinated to it with a general function $H(x)$ such that $EH(X_n) = 0$ and $EH(X_n)^2 < \infty$. We are interested in the large-scale limit of such random fields. Take the Hermite expansion (8.20) of the function $H(x)$, and let k be the smallest such index for which $c_k \neq 0$ in the expansion (8.20). In Theorem 8.2′ we solved this problem if $0 < k\alpha < \nu$. We are interested in the question what happens in the case when $k\alpha > \nu$. Let me remark that in the case $k\alpha \geq \nu$ the field Z_n^*, $n \in \mathbb{Z}_\nu$, which appeared in the limit in Theorem 8.2′ does not exist. The Wiener-Itô integral defining Z_n^* is meaningless, because the integral which should be finite to guarantee the existence of the Wiener–Itô integral is divergent in this case. Next I formulate a general result which contains the answer to the above question as a special case.

Theorem 8.5. *Let us consider a stationary Gaussian random field X_n, $EX_n = 0$, $EX_n^2 = 1$, $n \in \mathbb{Z}_n$, with correlation function $r(n) = EX_m X_{m+n}$, $m, n \in \mathbb{Z}_\nu$. Take a function $H(x)$ on the real line such that $EH(X_n) = 0$ and $EH(X_n)^2 < \infty$. Take the Hermite expansion (8.20) of the function $H(x)$, and let k be smallest index in this expansion such that $c_k \neq 0$. If*

$$\sum_{n \in \mathbb{Z}_\nu} |r(n)|^k < \infty, \tag{8.21}$$

then the limit

$$\lim_{N \to \infty} EZ_n^N(H_l)^2 = \lim_{N \to \infty} N^{-\nu} \sum_{i \in B_n^N} \sum_{j \in B_n^N} r^l(i-j) = \sigma_l^2 l!$$

exists for all indices $l \geq k$, where $Z_n^N(H_l)$ is defined in (1.1) with $A_N = N^{\nu/2}$, and $\xi_n = H_l(X_n)$ with the l-th Hermite polynomial $H_l(x)$ with leading coefficient 1. Moreover, also the inequality

$$\sigma^2 = \sum_{l=k}^{\infty} c_l^2 l! \sigma_l^2 < \infty$$

holds.

The finite dimensional distributions of the random field $Z_n^N(H)$ defined in (1.1) with $A_N = N^{\nu/2}$ and $\xi_n = H(X_n)$ tend to the finite dimensional distributions of a random field σZ_n^* with the number σ defined in the previous relation, where Z_n^*, $n \in \mathbb{Z}_\nu$, are independent, standard normal random variables.

Theorem 8.5 can be applied if the conditions of Theorem 8.2′ hold with the only modification that the condition $k\alpha < \nu$ is replaced by the relation $k\alpha > \nu$. In this case the relation (8.21) holds, and the large-scale limit of the random field Z_n^N, $n \in \mathbb{Z}_\nu$ with normalization $A_N = N^{\nu/2}$ is a random field consisting of independent standard normal random variables multiplied with the number σ. There is a slight generalization of Theorem 8.5 which also covers the case $k\alpha = \nu$. In this result we assume instead of the condition (8.21) that $\sum_{n \in \bar{B}_N} r(n)^k = L(N)$ with a slowly varying function $L(\cdot)$, where $\bar{B}_N = \{(n_1, \dots, n_\nu) \in \mathbb{Z}_\nu\colon\ -N \le n_j \le N,\ 1 \le j \le \nu\}$, and some additional condition is imposed which states that an appropriately defined finite number $\sigma^2 = \lim_{N \to \infty} \sigma_N^2$, which plays the role of the variance of the random variables in the limiting field, exists. There is a similar large scale limit in this case as in Theorem 8.5, the only difference is that the norming constant in this case is $A_N = N^{\nu/2} L(N)^{1/2}$. This result has the consequence that if the conditions of Theorem 8.2′ hold with the only differnce that $k\alpha = \nu$ instead of $k\alpha < \nu$, then the large scale limit exists with norming constants $A_N = N^{\nu/2} L(N)$ with an appropriate slowly varying function $L(\cdot)$, and it consists of independent Gaussian random variables with expectation zero.

The proof of Theorem 8.5 and its generalization that we did not formulate here explicitly appeared in paper [3]. I omit its proof, I only make some short explanation about it.

In the proof we show that all moments of the random variables Z_n^N converge to the corresponding moments of the random variables Z_n^* as $N \to \infty$. The moments of the random variables Z_n^N can be calculated by means of the diagram formula if we either rewrite them in the form of a Wiener–Itô integral or apply a version of the diagram formula which gives the moments of Wick polynomials instead of Wiener–Itô integrals. In both cases the moments can be expressed explicitly by means of the correlation function of the underlying Gaussian random field. The most important step of the proof is to show that we can select a special subclass of (closed) diagrams, called regular diagrams in [3] which yield the main contribution to the moment $E(Z_n^N)^M$, and their contribution can be simply calculated. The contribution of all remaining diagrams is $o(1)$, hence it is negligible. For the sake of simplicity let us restrict our attention to the case $H(x) = H_k(x)$ when defining the regular diagrams. If M is an even number, then take a partion $\{k_1, k_2\}$, $\{k_3, k_4\}$, … , $\{k_{M-1}, k_M\}$ of the set $\{1, \dots, M\}$ to subsets consisting of exactly two elements, to define the regular diagrams. They are those (closed) diagrams which contain only edges connecting vertices from the k_{2j-1}-th and k_{2j}-th row of the diagram with some $1 \le j \le \frac{M}{2}$, where $\{k_{2j-1}, k_{2j}\}$ is an element of the above partition. If M is an odd number, then there is no regular diagram.

In Theorems 8.2 and 8.2′ we investigated some very special subordinated fields.

The next result shows that the same limiting field as the one in Theorem 8.2 appears in a much more general situation.

Let us define the field

$$\xi_n = \sum_{j=k}^{\infty} \frac{1}{j!} \int e^{i(n,x_1+\cdots+x_j)} \alpha_j(x_1,\dots,x_j)\, Z_G(\,dx_1)\dots Z_G(\,dx_j), \quad n \in \mathbb{Z}_\nu, \tag{8.22}$$

where Z_G is the random spectral measure adapted to a Gaussian field X_n, $n \in \mathbb{Z}_\nu$, with correlation function satisfying (8.1) with $0 < \alpha < \frac{\nu}{k}$.

Theorem 8.6. *Let the fields Z_n^N be defined by formulae (8.22) and (8.1) with $A_N = N^{\nu - k\alpha/2}$. The multi-dimensional distributions of the fields Z_n^N tend to those of the field $\alpha_k(0,\dots,0)Z_n^*$ where the field Z_n^* is the same as in Theorem 8.2 if the following conditions are fulfilled:*

(i) *$\alpha_k(x_1,\dots,x_k)$ is a bounded function, continuous at the origin, and such that $\alpha_k(0,\dots,0) \neq 0$.*

(ii)

$$\sum_{j=k=1}^{\infty} \frac{1}{j!} \frac{N^{-(j-k)\alpha}}{L(N)^{j-k}} \int_{R^{j\nu}} \left|\alpha_j\left(\frac{x_1}{N},\dots,\frac{x_j}{N}\right)\right|^2 \frac{1}{N^{2\nu}} \left| \sum_{j \in B_0^N} e^{i(l/N,x_1+\cdots+x_j)} \right|^2 G_N(\,dx_1)\dots G_N(\,dx_j) \to 0,$$

where G_N is defined in (8.2).

Proof of Theorem 8.6. The proof is very similar to those of Theorem 8.2 and 8.2′. The same argument as in the proof of Theorem 8.2′ shows that because of condition (ii) ξ_n can be substituted in the present proof by the following expression:

$$\xi_n' = \frac{1}{k!} \int e^{i(n,x_1+\cdots+x_k)} \alpha_k(x_1,\dots,x_k) Z_G(\,dx_1)\dots Z_G(\,dx_k), \quad n \in \mathbb{Z}_\nu.$$

Then a natural modification in the proof of Theorem 8.2 implies Theorem 8.6. The main point in this modification is that we have to substitute the measures μ_N defined in formula (8.11) by the following measure $\bar{\mu}_N$:

$$\bar{\mu}_N(A) = \int_A |K_N(x_1,\dots,x_k)|^2 \left|\alpha_k\left(\frac{x_1}{N},\dots,\frac{x_k}{N}\right)\right|^2 G_N(\,dx_1)\dots G_N(\,dx_k), \quad A \in \mathcal{B}^{k\nu},$$

and to observe that because of condition (i) the limit relation $\mu_N \xrightarrow{w} \mu_0$ implies that $\bar{\mu}_N \xrightarrow{w} |\alpha_k(0,\dots,0)|^2 \mu_0$.

The main problem in applying Theorem 8.6 is to check conditions (i) and (ii). We remark without proof that any field $\xi_n = H(X_{s_1+n},\dots,X_{s_p+n})$, $s_1,\dots,s_p \in \mathbb{Z}_\nu$ and $n \in \mathbb{Z}_\nu$, for which $E\xi_n^2 < \infty$ satisfies condition (ii). This is proved in Remark 6.2

of [9]. If the conditions (i) or (ii) are violated, then a limit of different type may appear. Finally we quote such a result without proof. (See [23] for a proof.) Here we restrict ourselves to the case $\nu = 1$. The limiting field appearing in this result belongs to the class of self-similar fields constructed in Remark 6.5.

Let a_n, $n = \dots, -1, 0, 1, \dots$, be a sequence of real numbers such that

$$\begin{aligned} a_n &= C(1)n^{-\beta-1} + o(n^{-\beta-1}) \quad \text{if } n \geq 0 \\ a_n &= C(2)|n|^{-\beta-1} + o(|n|^{-\beta-1}) \quad \text{if } n < 0 \end{aligned} \qquad -1 < \beta < 1. \tag{8.22}$$

Let X_n, $n = \dots, -1, 0, 1, \dots$, be a stationary Gaussian sequence with correlation function $r(n) = EX_0X_n = |n|^{-\alpha}L(|n|)$, $0 < \alpha < 1$, where $L(\cdot)$ is a slowly varying function. Define the field ξ_n, $n = \dots, -1, 0, 1, \dots$, as

$$\xi_n = \sum_{m=-\infty}^{\infty} a_m H_k(X_{m+n}). \tag{8.22}$$

Theorem 8.7. *Let a sequence ξ_n, $n = \dots, -1, 0, 1, \dots$, be defined by (8.23) and (8.24). Let $0 < k\alpha < 1$, $0 < 1 - \beta - \frac{k}{2}\alpha < 1$, and let one of the following conditions be satisfied.*

(a) $0 < \beta < 1$, *and* $\sum_{n=-\infty}^{\infty} a_n = 0$.

(b) $0 > \beta > -1$.

(c) $\beta = 0$, $C(1) = -C(2)$, *and* $\sum_{n=0}^{\infty} |a_n + a_{-n}| < \infty$.

Let us define the sequences Z_n^N by formula (1.1) with $A_N = N^{1-\beta-k\alpha/2}L(N)^{k/2}$ and the above defined field ξ_n. The multi-dimensional distributions of the sequences Z_n^N tend to those of the sequences $D^{-k}Z_n^*(\alpha, \beta, a, b, c)$, where

$$\begin{aligned} Z_n^*(\alpha, \beta, k, b, c) = \int \tilde{\chi}_n(x_1 + \dots + x_k) \\ \big[b|x_1 + \dots + x_k|^\beta + ic|x_1 + \dots + x_k|^\beta \operatorname{sign}(x_1 + \dots + x_k)\big] \\ |x_1|^{(\alpha-1)/2} \dots |x_k|^{(\alpha-1)/2} W(dx_1) \dots W(dx_k), \end{aligned}$$

$W(\cdot)$ denotes the white noise field, i.e. a random spectral measure corresponding to the Lebesgue measure, and the constants D, b and c are defined as $D = 2\Gamma(\alpha)\cos(\frac{\alpha}{2}\pi)$, and

$b = 2[C(1)+C(2)]\Gamma(-\beta)\sin(\frac{\beta+1}{2}\pi)$, $c = 2[C(1)-C(2)]\Gamma(-\beta)\cos(\frac{\beta+1}{2}\pi)$ in case (a) and (b), and

$b = \sum_{n=-\infty}^{\infty} a_n$, $c = C(1)$ in case (c).

9. History of the problems. Comments.

Section 1.

In statistical physics the problem formulated in this section appeared at the investigation of some physical models at critical temperature. A discussion of this problem and further references can be found in the fourth chapter of the forthcoming book of Ya. G. Sinai [33]. The first example of a limit theorem for partial sums of random variables which is considerably different form the independent case was given by M. Rosenblatt in [28]. Further results in this direction were proved by R. L. Dobrushin, H. Kesten and F. Spitzer, P. Major, M. Rosenblatt and M. S. Taqqu [7], [8], [9], [23], [29], [30], [34], [37]. In most of these papers only the one-dimensional case is considered, and it is formulated in a different but equivalent way. The joint distribution of the random variables $A_N^{-1} \sum_{j=1}^{Nt]} \xi_j$, $0 < t < \infty$, is considered.

Similar problems also appeared in the theory of infinite particle systems. The large-scale limit of the so-called voter model and of infinite particle branching Brownian motions were investigated in papers [2], [6], [17], [24]. It was proved that in these models the limit is, with a non-typical normalization, a Gaussian self-similar field. The investigation of the large-scale limit would be very natural for many other infinite particle systems, but in most cases this problem is hopelessly difficult.

The notion of subordinated fields in the present context first appeared at Dobrushin [7]. It is natural to expect that there exists a large class of self-similar fields which cannot be obtained as subordinated fields. Nevertheless the present techniques are not powerful enough for finding them.

The approach to the problem is different in statistical physics. In statistical physics one looks for self-similar fields which satisfy some conditions formulated in accordance to physical considerations. One tries to describe these fields with the help of a power series which is the Radon–Nykodim derivative of the field with respect to a Gaussian field. The deepest result in this direction is a recent paper of P. M. Bleher and M. D. Missarov [1] who can define the required formal power series. This result enables one to calculate several critical indices interesting for physicists, but the task of proving that this formal expression defines an existing field seems to be very hard. It is also an open problem whether the class of self-similar fields constructed via multiple Wiener–Itô integrals contains the non-Gaussian self-similar fields interesting for statistical physics. Some experts are very skeptical in this respect. The Gaussian self-similar fields are investigated in [7] and [32]. A more thorough investigation is under preparation in [11].

The notion of generalized fields was introduced by I. M. Gelfand. A detailed discussion can be found in the book [15], where the properties of Schwartz spaces we need can also be found.

In the definition of generalized fields the class of test functions $\mathcal{S}$ can be substituted by other linear topological spaces consisting of real valued functions. The most frequently considered space, beside the space $\mathcal{S}$, is the space $\mathcal{D}$ of infinitely many times differentiable functions with compact support. In paper [7] Dobrushin also considered

the space $\mathcal{S}^r \subset \mathcal{S}$, which consists of the functions $\varphi \in \mathcal{S}$ satisfying the additional relation $\int x^{(1)^{j_1}} \cdots x^{(\nu)^{j_\nu}} \varphi(x)\, dx = 0$ provided that $j_1 + \cdots + j_\nu < r$. He considered this class of test functions because there are much more continuous linear functionals over $\mathcal{S}^r$ than over $\mathcal{S}$, and this property of $\mathcal{S}^r$ can be exploited in certain investigations. Generally no problem arises in the proofs if the space of test functions $\mathcal{S}$ is substituted by $\mathcal{S}^r$ or $\mathcal{D}$ in the definition of generalized fields.

Two generalized fields $X(\varphi)$ and $\bar{X}(\varphi)$ can be identified if $X(\varphi) \stackrel{\Delta}{=} \bar{X}(\varphi)$ for all $\varphi \in \mathcal{S}$. Let me remark that this relation also implies that the multi-dimensional distributions of the random vectors $(X(\varphi_1), \dots, X(\varphi_n))$ and $(\bar{X}(\varphi_1), \dots, \bar{X}(\varphi_n))$ coincide for all $\varphi_1, \dots, \varphi_n \in \mathcal{S}$. As $\mathcal{S}$ is a linear space, this relation can be deduced from property a) of generalized fields by exploiting that two distribution functions on R^n agree if and only if their characteristic functions agree.

Let $\mathcal{S}'$ denote the space of continuous linear functionals over $\mathcal{S}$, and let $\mathcal{A}_{\mathcal{S}'}$ be the σ-algebra over $\mathcal{S}'$ generated by the sets $A(\varphi, a) = \{F\colon \ F \in \mathcal{S}';, F(\varphi) < a\}$, where $\varphi \in \mathcal{S}$ and $a \in R^1$ are arbitrary. Given a probability space $(\mathcal{S}', \mathcal{A}_{\mathcal{S}'}, P)$, a generalized field $\bar{X} = \bar{X}(\varphi)$ can be defined on it by the formula $\bar{X}(\varphi)(F) = F(\varphi), \ \ \varphi \in \mathcal{S}$, and $F \in \mathcal{S}'$. The following deep result is due to Minlos (see e.g. [15]).

Theorem. (Minlos) *Let $(X(\varphi),\, \varphi \in \mathcal{S})$ be a generalized random field. There exists a probability measure P on the measurable space $(\mathcal{S}, \mathcal{A}_{\mathcal{S}'})$ such the generalized field $\bar{X} = (\bar{X}(\varphi),\, \varphi \in \mathcal{S})$ defined on the probability space $(\bar{S}, \mathcal{A}_{\mathcal{S}'}, P)$ by the formula $\bar{X}(\varphi)(F) = F(\varphi)$, $\varphi \in \mathcal{S}$, $F \in \mathcal{S}'$, satisfies the relation $X(\varphi) \stackrel{\Delta}{=} \bar{X}(\varphi)$ for all $\varphi \in \mathcal{S}$.*

The generalized field $\bar{X}$ has some nice properties. Namely property a) in the definition of generalized fields holds for all $F \in \mathcal{S}'$. Moreover $\bar{X}$ satisfies the following strengthened version of property b):

b′) $\lim \bar{X}(\varphi_n) = \bar{X}(\varphi)$ in every point $F \in \mathcal{S}'$ if $\varphi_n \to \varphi$ in the topology of $\mathcal{S}$.

Because of this nice behaviour of the field $\bar{X}(\varphi)$ most authors define generalized fields as the versions $\bar{X}$ defined in Minlos' theorem. Since we have never needed the extra properties of the field $\bar{X}$ we have deliberately avoided the application of Minlos' theorem in the definition of generalized random fields. Minlos' theorem heavily depends on some topological properties of $\mathcal{S}$, namely that $\mathcal{S}$ is a so-called nuclear space. Minlos' theorem also holds if the space of test functions is substituted by $\mathcal{D}$ or $\mathcal{S}^r$ in the definition of generalized fields.

Let us finally remark that Lamperti [21] gave an interesting characterization of self-similar random fields. Let $X(t)$, $t \in R^1$, be a continuous time stationary random process, and define the random process $Y(t) = \frac{X(\log t)}{t^\alpha}$, $t > 0$, with some $\alpha > 0$. Then, as it is not difficult to see, the random processes $Y(t)$, $t > 0$, and $\frac{Y(ut)}{u^\alpha}$, $t > 0$, have the same finite dimensional distributions for all $u > 0$. This can be interpreted so that $Y(t)$ is a self-similar process with parameter $\alpha > 0$ on the half-line $t > 0$. Contrariwise, if the finite dimensional distributions of the processes $Y(t)$ and $\frac{Y(ut)}{u^\alpha}$, $t > 0$, agree for all $u > 0$, then the process $X(t) = \frac{X(e^t)}{e^{\alpha t}}$, $t \in R^1$, is stationary. These relations show some

connection between stationary and self-similar processes. But they have a rather limited importance in the investigations of this work, because here we are really interested in such random fields which are simultaneously stationary and self-similar.

Section 2.

Wick polynomials are widely used in the literature of statistical physics. A detailed discussion about Wick polynomials can be found in [12]. Theorems 2A and 2B are well-known, and they can be found in the standard literature. Theorem 2C can be found e.g. in Dynkin's book [13] (Lemma 1.5). Theorem 2.1 is due to Segal [31]. It is closely related to a result of Cameron and Martin [4]. The remarks at the end of the section about the content of formula 2.1 are related to [25].

Section 3.

Random spectral measures were independently introduced by Cramer and Kolmogorov [5], [20]. They could have been introduced by means of Stone's theorem about the spectral representation of one-parameter groups of unitary operators. Bochner's theorem can be found in any standard book on functional analysis, the proof of the Bochner–Schwartz theorem can be found in [15]. Let me remark that the same result holds true if the space of test functions $\mathcal{S}$ is substituted by $\mathcal{D}$.

Section 4.

The stochastic integral defined in this section is a version of that introduced by Itô in [18]. This modified integral first appeared in Totoki's lecture note [38] in a special form. Its definition is a little bit more difficult than the definition of the original stochastic integral introduced by Itô, but it has the advantage that the effect of the shift transformation can be better studied with its help. Most results of this section can be found in Dobrushin's paper [7]. The definition of Wiener–Itô integrals in the case when the spectral measure may have atoms is new. In the new version of this lecture note I worked out many arguments in a more detailed form than in the old text. In particular, I have given a much more detailed explanation of the statement that all kernel functions of Wiener–Itô integrals can be well approximated by elementary functions.

Section 5.

Proposition 5.1 is proved for the original Wiener–Itô integrals by Itô in [18]. Lemma 5.2 contains a well-known formula about Hermite polynomials. The main result of this section, Theorem 5.3, appeared in Dobrushin's work [7]. The proof given there is not complete. Several non-trivial details are omitted. I felt even necessary to present a more detailed proof in this note when I wrote down its new version. Theorem 5.3 is closely related to Feynman's diagram formula. The result of Corollary 5.5 was already known at the beginning of this century. It was proved with the help of some formal manipulations. This formal calculation was justified by Taqqu in [35] with the help of some deep inequalities. In the new version of this note I formulated a more general

result than in the older one. Here I gave a formula about the moment of products of Wick polynomials and not only of Hermite polynomials.

I could not find results similar to Propositions 5.6 and 5.7 in the literature of probability theory. On the other hand, such results are well-known in statistical physics, and they play an important role in constructive field theory. A sharpened form of these results is Nelson's deep hypercontractive inequality [27], which I formulate below.

Let X_t, $t \in T$, and $Y_{t'}$, $t' \in T'$ be two sets of jointly Gaussian random variables on some probability spaces $(\Omega, \mathcal{A}, P)$ and $(\Omega, \mathcal{A}', P')$. Let $\mathcal{H}_1$ and $\mathcal{H}_1'$ be the Hilbert spaces generated by the finite linear combinations $\sum c_j X_{t_j}$ and $\sum c_j Y_{t_j'}$. Let us define the σ-algebras $\mathcal{B} = \sigma(X_t,\, t \in T)$ and $\mathcal{B}' = \sigma(Y_{t'},\, t' \in T')$ and the Banach spaces $L_p(X) = L_p(\Omega, \mathcal{B}, P)$, $L_p(Y) = L_p(\Omega', \mathcal{B}', P')$, $1 \le p \le \infty$. Let A be linear transformation from $\mathcal{H}_1$ to $\mathcal{H}_1'$ with norm not exceeding 1. We define an operator $\Gamma(A)\colon L_p(X) \to L_{p'}(Y)$ for all $1 \le p, p' \le \infty$ in the following way. If η is a homogeneous polynomial of the variables X_t,

$$\eta = \sum C_{j_1,\dots,j_s}^{t_1,\dots,t_s} X_{t_1}^{j_1} \cdots X_{t_s}^{j_s}, \quad t_1, \dots, t_s \in T,$$

then

$$\Gamma(A)\colon \eta \colon = \sum C_{j_1,\dots,j_s}^{t_1,\dots,t_s} \colon (AX_{t_1})^{j_1} \cdots (AX_{t_s})^{j_s} \colon .$$

It can be proved that this definition is meaningful, i.e. $\Gamma(A)\colon \eta \colon$ does not depend on the representation of η, and $\Gamma(A)$ can be extended to a bounded operator from $L_1(X)$ to $L_1(Y)$ in a unique way. This means in particular that $\Gamma(A)\xi$ is defined for all $\xi \in L_p(X)$, $p \ge 1$. Nelson's hypercontractive inequality says the following. Let A be a contraction from $\mathcal{H}_1$ to $\mathcal{H}_1'$. Then $\Gamma(A)$ is a contraction from $L_q(X)$ to $L_p(Y)$ for $1 \le q \le p$ provided that

$$\|A\| \le \left(\frac{q-1}{p-1}\right)^{1/2}. \tag{+}$$

If (+) does not hold, then $\Gamma(A)$ is not a bounded operator from $L_q(X)$ to $L_p(Y)$.

A further generalization of this result can be found in [16].

The following discussion may help to understand the relation between Nelson's hypercontractive inequality and Corollary 5.6. Let us apply Nelson's inequality in the special case when $(X_t,\, t \in T) = (Y_{t'},\, t' \in T')$ is a stationary Gaussian field with spectral measure G, $q = 2$, $p = 2m$ with some positive integer m, $A = c \cdot \mathrm{Id}$, where Id denotes the identity operator, and $c = (2m-1)^{-1/2}$. Let $\mathcal{H}^c$ and $\mathcal{H}_n^c$ be the complexification of the real Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_n$ defined in Section 2. Then $L_2(X) = \mathcal{H}^c = \mathcal{H}_0^c + \mathcal{H}_1^c + \cdots$ by Theorem 2.1 and formula 2.1. The operator $\Gamma(c \cdot \mathrm{Id}\,)$ equals $c^n \cdot \mathrm{Id}$ on the subspace $\mathcal{H}_n^2$. If $h_n \in \mathcal{H}_G^n$, then $I_G(h_n) \in \mathcal{H}_n$, hence the application of Nelson's inequality for the operator $A = c \cdot \mathrm{Id}$ shows that

$$\left(EI_G(h_n)^{2m}\right)^{1/2m} = c^{-n} \left(E(\Gamma(c \cdot \mathrm{Id}) I_G(h_n))^{2m}\right)^{1/2m} \le c^{-n} \left(EI_G(h_n)^2\right)^{1/2}$$

i.e.

$$EI_G(h_n)^{2m} \le c^{-2nm} \left(EI_G(h_n)^2\right)^m = (2m-1)^{mn} \left(EI_G(h_n)^2\right)^m .$$

This inequality is very similar to the second inequality in Corollary 5.6, only the multiplying constants are different. Moreover, for large m these multiplying constants are near to each other. I remark that the following weakened form of Nelson's inequality could be deduced relatively easily from Corollary 5.6. Let A: $\mathcal{H}_1 \to \mathcal{H}_1'$ be a contraction $\|A\| = c < 1$. Then there exists a $\bar{p} = \bar{p}(c) > 2$ such that $\Gamma(A)$ is a bounded operator from $L_2(X)$ to $L_p(Y)$ for $p < \bar{p}$. This weakened form of Nelson's inequality is sufficient in many applications.

Section 6.

Theorems 6.1, 6.2 and Corollary 6.4 were proved by Dobrushin in [7]. Taqqu proved similar results in [36], but he gave a different representation. Theorem 6.6 was proved by H. P. Mc.Kean in [26]. The proof of the lower bound uses some ideas from [14]. Remark 6.5 is from [23]. As Proposition 6.3 also indicates, some non-trivial problems about the convergence of certain integrals must be solved when constructing self-similar fields. Such convergence problems are common in statistical physics. To tackle such problems the so-called power counting method (see e.g. [22]) was worked out. This method could also be applied in this section. Part b) of Proposition 6.3 implies that the self-similarity parameter α cannot be chosen in a larger domain in Corollary 6.4. One can ask about the behaviour of the random variables ξ_j and $\xi(\varphi)$ defined in Corollary 6.4 if the self-similarity parameter α tends to the critical value $\frac{\nu}{2}$. The variance of the random variables ξ_j and $\xi(\varphi)$ tends to infinity in this case, and the fields ξ_j, $j \in \mathbb{Z}_\nu$, and $\xi(\varphi)$, $\varphi \in \mathcal{S}$, tend, after an appropriate renormalization, to a field of independent normal random variables in the discrete, and to a white noise in the continuous case. The proof of these results with a more detailed discussion will appear in [10].

In a recent paper [19] Kesten and Spitzer have proved a limit theorem, where the limit field is a self-similar field which seems not to belong to the class of self-similar fields constructed in Section 6. (We cannot however, exclude the possibility that there exists some self-similar field in the class defined in Theorem 6.2 with the same distribution as this field, although it is given by a completely different form.) This self-similar field constructed by Kesten and Spitzer is the only rigorously constructed self-similar field known for us that does not belong to the fields constructed in Theorem 6.2. I describe this field, and then I make some comments.

Let $B_1(t)$ and $B_2(t)$, $-\infty < t < \infty$, be two independent Wiener processes. (We say that $B(t)$ is a Wiener process on the real line if $B(t)$, $t \ge 0$, and $B(-t)$, $t \ge 0$, are two independent Wiener processes.) Let $K(x, t_1, t_2)$, $x \in R^1$, $t_1 < t_2$, denote the local time of the process B_1 at the point x in the interval $[t_1, t_2]$. The one-dimensional field

$$Z_n = \int K(x, n, n+1)\, B_2(\, dx), \quad n = \dots, -1, 0, 1, \dots,$$

where the integral in the last formula is an Itô integral, is a stationary self-similar field with self-similarity parameter $\frac{3}{4}$.

To see the self-similarity property one has to observe that

$$K(\lambda^{1/2}x, \lambda t_1, \lambda t_2) \stackrel{\Delta}{=} \lambda^{1/2} K(x, t_1, t_2) \quad \text{for all } x \in R^1, \;\; t_1 < t_2, \text{ and } \lambda > 0$$

because of the relation $B_1(\lambda u) \stackrel{\Delta}{=} \lambda^{1/2} B_1(u)$. Hence

$$\sum_{j=0}^{n-1} Z_j \stackrel{\Delta}{=} n^{1/2} \int K(n^{-1/2}x, 0, 1) B_2(\,dx) \stackrel{\Delta}{=} n^{3/4} \int K(x, 0, 1)\, B_2(\,dx) = n^{3/4} Z_0.$$

The invariance of the multi-dimensional distributions of the field Z_n under the transformation (1.1) can be seen similarly.

To see the stationarity of the field Z_n we need the following two observations.

a) $K(x, s, t) \stackrel{\Delta}{=} K(x + \eta(s), 0, t - s)$ with $\eta(s) = -B_1(-s)$. (The form of η is not important for us. What we need is that the pair (η, K) is independent of B_2.)

b) If $\alpha(x)$, $-\infty < x < \infty$, is a process independent of B_2, then

$$\int \alpha(x + u) B_2(\,dx) \stackrel{\Delta}{=} \int \alpha(x) B_2(\,dx) \quad \text{for all } u \in R^1.$$

It is enough to show, because of Property a) that

$$\int K(x + \eta(s), 0, t - s)\, B_2(\,dx) \stackrel{\Delta}{=} \int K(x, 0, t - s)\, B_2(\,dx).$$

This relation follows from property b), because the conditional distributions of the left and right-hand sides agree under the condition $\eta(s) = u$, $u \in R^1$.

The generalized field version of the above field Z_n is the field

$$Z(\varphi) = -\int \left[K(x, 0, t) \frac{d\varphi}{dt}\, dt \right] B_2(\,dx), \quad \varphi \in \mathcal{S}.$$

To explain the analogy between the field Z_n and $Z(\varphi)$ we remark that the kernel of the integral defining Z_n can be written, at least formally, as

$$K(x, n, n + 1) = \int \chi_{[n,n+1)}(u) \frac{d}{du} K(x, n, u)\, du,$$

although K is a non-differentiable function. Substituting the function $\chi_{[n,n+1)}$ by $\varphi \in \mathcal{S}$, and integrating by parts (or precisely, considering $\frac{d}{du}K$ as the derivative of a distribution) we get the above definition of $Z(\varphi)$.

Using the same idea as before, a more general class of self-similar fields can be constructed. The integrand $K(x, n, n + 1)$ can be substituted by the local time of any self-similar field with stationary increments which is independent of B_2. Naturally, it must be clarified first that this local time really exists. One could enlarge this class also by integrating with respect to a self-similar field with stationary increments, independent of B_1. The integral with respect to a field independent of the field $K(x, s, t)$ can be defined without any difficulty.

There seems to be no natural way to represent the above random fields as random fields subordinated to a Gaussian random field. On the other hand, the local times $K(x, s, t)$ are measurable with respect to B_1, they have finite second moments, therefore they can be expressed by means of multiple Wiener–Itô integrals with respect to a white noise field. Then the process Z_n itself can also be represented via multiple Wiener–Itô integrals. It would be interesting to know whether the above defined self-similar fields, and probably a larger class of self-similar fields, can be constructed in a simple natural way via multiple Wiener–Itô integrals with the help of a randomization.

Section 7.

The definition of Wiener–Itô integrals together with the proof of Theorem 7.1 and Proposition 7.3 are given by Itô in [18]. Theorem 7.2 is proved in Taqqu's paper [37]. He needed this result to show that the self-similar fields defined in [9] by means of Wiener–Itô integrals coincide with the self-similar fields defined in [37] by means of modified Wiener–Itô integrals.

Section 8.

The results of this section, with the exception of Theorem 8.7 are proved in [9]. Theorem 8.7 is proved in [23]. This paper was strongly motivated by [29]. Lemma 8.3 is formulated in a slightly more general form than Lemma 3 in [9]. The present formulation is more complicated, but it is more useful in some applications. Let me explain this in more detail. The difference between the original and the present formulation of this lemma is that here we allow that the integrand K_0 in the limiting stochastic integral is discontinuous on a small subset of $R^{k\nu}$, and the functions K_N may not converge on this set. This freedom can be exploited in some applications. Indeed, let us consider e.g. the self-similar fields constructed in Remark 6.5. In case $p < 0$ the integrand in the formula expressing these fields is not continuous on the hyperplane $x_1 + \cdots + x_n = 0$. Hence, if we want to prove limit theorems where these fields appear as the limit, and this happens e.g. in Theorem 8.7 then we can apply Lemma 8.3, but not its original version Lemma 3 in [9].

The example for non-central limit theorems given by Rosenblatt in [28] and its generalization by Taqqu in [34] are special cases of Theorem 8.2. In these papers only the special case $H_k(x) = x^2 - 1$ is considered. Later Taqqu [37] proved a result similar to Theorem 8.2′, but he needed more restrictive conditions. The observation that Theorem 8.2′ can be deduced from Theorem 8.2 is from Taqqu [34].

The method of [28] and [34] does not apply for the proof of Theorem 8.2 in the case of $H_k(x)$, $k \geq 3$. In these papers it is proved that the moments of the random variables Z_n^N converge to the corresponding moments of Z_n^*. (Actually a different but equivalent statement is established in these papers.) This convergence of the moments implies the convergence $Z_n^N \xrightarrow{\mathcal{D}} Z_n^*$ if and only if the distribution of Z_n^* is uniquely determined by its moments.

Theorem 6.6 implies that the n-th moment of a k-fold Wiener–Itô integral equals to $e^{(kn \log n)/2 + O(n)}$. Hence some results about the so-called moment problem show that

the distribution of a k-fold Wiener–Itô integral is determined by its moments only for $k = 1$ and $k = 2$. Therefore the method of moments does not work in the proof of Theorem 8.2 for $H_k(x)$, $k \geq 3$.

Throughout Section 8 I have assumed that the correlation function of the underlying Gaussian field to which our fields are subordinated satisfies formula (8.1). This assumption seems natural, since it implies that the spectral measure of the Gaussian field satisfies Lemma 8.1, and such a condition is needed when Z_{G_N} is substituted by Z_{G_0} in the limit. It can be asked whether in Theorem 8.2 formula 8.1 can be substituted by the weaker assumption that the spectral measure of the Gaussian field satisfies Lemma 8.1. This question was investigated in Section 4 of [9]. The investigation of the moments shows that the answer is negative. The reason for it is that the validity of Lemma 8.1, unlike that of Theorem 8.2, does not depend on whether the spectral measure G has large singularities outside the origin or not. The discussion in [9] also shows that the Gaussian case, that is the case when $H_k(x) = H_1(x) = x$ in Theorem 8.2, is considerably different from the non-Gaussian case. A forthcoming paper of M. Rosenblatt [30] gives a better insight into the above question.

The limiting fields appearing in Theorem 8.2 and 8.6 belong to a special subclass of the self-similar fields defined in Theorem 6.2. These results indicate that the self-similar fields defined in formula (6.5) have a much greater range of attraction if the homogeneous function f_n in (6.5) is the constant function. The reason for the particular behaviour of these fields is that the constant function is analytic, while a general homogeneous function typically has a singularity at the origin. A more detailed discussion about this problem can be found in [23].

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Subject index.

Notations.