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A LAW OF THE ITERATED LOGARITHM FOR THE ROBBINS—MONRO METHOD

by

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Let a real number α and a function $H(x, y)$ be given which is a distribution function for fixed x and measurable for fixed y. Set

$$M(x) = \int_{-\infty}^{\infty} yH(dy, x).$$

Let $a_n \geqq 0$ be a sequence of real numbers for which $\Sigma a_n = \infty$, $\Sigma a_n^2 < \infty$. Let us define the random variables $x_1, x_2, \dots\ y_1, y_2, \dots$ with the following properties: x_1 is an arbitrary constant $x_{n+1} = x_n - a_n(y_n - \alpha)$, and

$$\mathsf{P}(y_n < y | x_1, y_1 \dots y_{n-1}, x_n) = \mathsf{P}(y_n < y | x_n) = H(y, x_n).$$

This construction of the x_i's and y_i's is the Robbins—Monro method [1]. J. R. BLUM proved in [2] that if $M(x)$ and the root θ of the equation $M(x) = \alpha$ satisfy the following conditions:

(I) $|M(x)| \leqq A|x| + B$ for all x and suitable A, B;

(II) $\inf\limits_{\varepsilon < x-\theta < \frac{1}{\varepsilon}} M(x) > \alpha$ and $\sup\limits_{-\varepsilon > x-\theta > -\frac{1}{\varepsilon}} M(x) < \alpha$ for every $\varepsilon > 0$;

(III) $\int_{-\infty}^{\infty} (y - M(x))^2 H(dy, x) \leqq K < \infty$ for every x

then $x_n \to \theta$ with probability 1.

We shall deal with the special case, when the errors are bounded, that is $H\big(M(x) - K, x\big) = 0$, $H\big(M(x) + K, x\big) = 1$ for every x, and $a_n = \frac{1}{n}$ are chosen. In this case, using mainly the idea of J. R. BLUM [2] we can strengthen his result in the following way.

THEOREM 1. *Let us suppose that $M'(\theta) > \frac{1}{2}$, $a_n = \frac{1}{n}$, $M(x)$ is bounded in a neighbourhood of θ and $x_n \to \theta$ with probability* 1. *Then there exists a number L (depending on $M'(\theta)$ and the bound K of the error) such that*

$$\mathsf{P}\left(\overline{\lim} \sqrt{\frac{n}{\log\log n}}\, |x_n - \theta| < L\right) = 1.$$

On the other hand, if we suppose that the errors are identically distributed, that is the distribution-function $H(y-M(x), x)$ does not depend on x then we can state the following

THEOREM 2. *If the errors are bounded having the same distribution and $M'(\theta)<\infty$, then there exists an $L'>0$ such that*

$$\mathsf{P}\left(\overline{\lim}\sqrt{\frac{n}{\log\log n}}|x_n-\theta|>L'\right)=1.$$

For the sake of simplicity we shall suppose that $\alpha=\theta=0$.

It is easy to see that for $m>n$

$$x_m = x_n-\sum_{i=n}^{m-1}\frac{1}{i}y_i = x_n-\sum_{i=n}^{m-1}\frac{1}{i}M(x_i)-\sum_{i=n}^{m-1}\frac{1}{i}(y_i-M(x_i)). \tag{1}$$

Let $F_i=B(x_1, y_1, \dots, x_i, y_i)$ be the smallest σ-algebra with respect to which the random variables $x_1, y_1, \dots, x_i, y_i$ are measurable. Then the system

$$\left(\frac{1}{i}(y_i-M(x_i)), F_i\right)$$

is a martingale difference series. If the errors are identically distributed then the random variables $\frac{1}{i}(y_i-M(x_i))$ are even independent. Using these facts we give a sharper bound than that of BLUM and these boundings will enable us to prove our statement. This way we need the following

LEMMA 1. *Let $(\xi_1, F_1)\dots(\xi_n, F_n)\dots$ be martingale difference series with $\mathsf{P}(|\xi_i|\leqq K)=1$. Then for almost all ω and for all $\varepsilon>0$ there exists an $n(\omega)=n(\omega, \varepsilon)$ such that for every $m>n\geqq n(\omega)$*

$$\left|\sum_{i=n}^{m}\frac{1}{i}\xi_i\right|\leqq 2\sqrt{2}K+\varepsilon)\sqrt{\frac{\log\log n}{n}}.$$

PROOF.

$$\mathsf{P}\left(\sup_{N\geqq m\geqq n}\sum_{i=n}^{m}\frac{\xi_i}{i}>u\sqrt{\frac{\log\log n}{n}}\right)=$$

$$=\mathsf{P}\left(\sup_{N\geqq m\geqq n}\exp f(n)\sum_{i=n}^{m}\frac{\xi_i}{i}>\exp u\cdot f(n)\sqrt{\frac{\log\log n}{n}}\right)$$

where $f(n)\geqq 0$ is arbitrary constant.

But
$$\left\{\exp\sum_{i=n}^{m}\frac{f(n)}{i}\xi_i, F_m\right\}\quad m=n, n+1, \dots N$$

is a semimartingale and by the Kolmogorov inequality for semimartingale (see [4] Chapter VII) we get that

$$\mathsf{P}\left(\sup_{N\geq m\geq n}\sum_{i=n}^{m}\frac{\xi_i}{i}\geq u\sqrt{\frac{\log\log n}{n}}\right)\leq$$

$$\leq\frac{\mathsf{E}\left\{\exp\left(f(n)\sum_{i=1}^{N}\frac{\xi_i}{i}\right)\right\}}{\exp\left(u\cdot f(n)\sqrt{\frac{\log\log n}{n}}\right)}.$$

For $|x|<\delta(\varepsilon)$ we have $e^x<1+x+\frac{x^2}{2-\varepsilon}$, and choosing $f(n)=v\cdot\sqrt{n\log\log n}$ we get that if n is big enough and $i\geq n$, then $\left|\frac{f(n)\xi_i}{i}\right|\leq\delta(\varepsilon)$.

Hence

$$\mathsf{E}\exp\sum_{i=n}^{N}\frac{f(n)\xi_i}{i}\leq\mathsf{E}\left[\left(\exp\sum_{i=n}^{N-1}\frac{f(n)\xi_i}{i}\right)\cdot\left(1+\frac{f(n)}{N}\xi_N+\frac{f^2(n)\xi_N^2}{(2-\varepsilon)N^2}\right)\right]\leq$$

$$\leq\left(1+\frac{f^2(n)K^2}{(2-\varepsilon)N^2}\right)\cdot\mathsf{E}\left(\exp\sum_{i=n}^{N-1}\frac{f(n)\xi_i}{i}\right)\leq\cdots$$

$$\leq\prod_{i=n}^{N}\left(1+\frac{f^2(n)K^2}{(2-\varepsilon)i^2}\right)\leq\exp\frac{f(n)^2K^2}{(2-\varepsilon)(n-1)}.$$

So for large n

$$\mathsf{P}\left(\sup_{N\geq m\geq n}\sum_{i=n}^{m}\frac{\xi_i}{i}\geq u\sqrt{\frac{\log\log n}{n}}\right)\leq\exp\left[\frac{f^2(n)K^2}{(2-\varepsilon)(n-1)}-u\cdot f(n)\sqrt{\frac{\log\log n}{n}}\right].$$

Tending with N to infinity, for large n

$$\mathsf{P}\left(\sup_{m\geq n}\sum_{i=n}^{m}\frac{\xi_i}{i}\geq u\sqrt{\frac{\log\log n}{n}}\right)\leq\exp\left\{\left[v^2\left(1+\frac{1}{n-1}\right)\frac{K^2}{2-\varepsilon}-uv\right]\log\log n\right\}\leq$$

$$\leq\frac{1}{(\log n)^{1+\varepsilon}}\quad\text{if we choose }u=\frac{2(1+\varepsilon)}{\sqrt{2-\varepsilon}}K,\ v=\frac{\sqrt{2-\varepsilon}}{K}.$$

Using the Borel—Cantelli lemma for arbitrary $\theta>1$

$$\sum_{n=1}^{\infty}\mathsf{P}\left(\sup_{m\geq\theta^n}\sum_{i=\theta^n}^{m}\frac{\xi_i}{i}\geq 2\frac{(1+\varepsilon)}{\sqrt{2-\varepsilon}}K\sqrt{\frac{\log\log\theta^n}{\theta^n}}\right)\leq C+\frac{1}{\log\theta}\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}<\infty$$

so for almost all ω there is an $n(\omega)$ such that for every

$$m\geq\theta^k\geq n(\omega)\sum_{i=\theta^k}^{m}\frac{\xi_i}{i}\leq(\sqrt{2}K+\varepsilon)\sqrt{\frac{\log\log\theta^k}{\theta^k}}.$$

In the same way

$$\sum_{i=\theta^k}^{m} \frac{\xi_i}{i} \geqq -(\sqrt{2}K+\varepsilon)\sqrt{\frac{\log\log\theta^k}{\theta^k}}.$$

Let $m>n>n(\omega)$ and $n(\omega)\leqq\theta^k<\theta^{k+1}$. Then

$$\left|\sum_{i=n}^{m} \frac{\xi_i}{i}\right| \leqq \left|\sum_{i=\theta^k}^{n} \frac{\xi_i}{i}\right| + \left|\sum_{i=\theta^k}^{m} \frac{\xi_i}{i}\right| \leqq$$

$$\leqq 2(\sqrt{2}K+\varepsilon)\sqrt{\frac{\log\log\theta^k}{\theta^k}} \leqq \frac{2(\sqrt{2}K+\varepsilon)}{\theta}\sqrt{\frac{\log\log n}{n}}.$$

Since ε and θ can be chosen as near to 0 and to 1, respectively as we want, our lemma is proved.

PROOF of Theorem 1. First we prove that $\mathsf{P}\left(\overline{\lim}\sqrt{\frac{n}{\log\log n}}|x_n|<\infty\right)=1$. We shall prove this relation for every ω for which $y_i(\omega)-M(x_i(\omega))$ satisfies the lemma and $x_n(\omega)\to 0$ as $n\to\infty$. Let us choose an $\varepsilon>0$ in such a way that

(2) $$|M(y)|>\beta|y| \quad \text{and} \quad |M(y)|<a \quad \text{if} \quad |y|<\varepsilon$$

where $\beta>\frac{1}{2}$, $a>0$ are constants. Let us choose a number c $0<c<1$ such that $2\beta c>1$. Then it is easy to see that

$$\lim_{\delta\to 0}\left[\frac{\beta\sqrt{1+\delta}\log(1+c\delta)}{\sqrt{1+\delta}-1}-\sqrt{1+\delta}\right]=2\beta c-1>0.$$

Let us choose a $\delta>0$ so, that

(3) $$\frac{\beta\sqrt{1+\delta}\log(1+c\delta)}{\sqrt{1+\delta}-1}-\sqrt{1+\delta}=u>0.$$

Let us choose an $n_0=n_0(\omega)$ so large, that

(4) $$|x_n(\omega)|<\varepsilon \quad \text{for} \quad n\geqq n_0$$

(5) $$\left|\sum_{i=n}^{m}\frac{1}{i}(y_i-M(x_i))\right|\leqq 3K\sqrt{\frac{\log\log n}{n}} \quad \text{for} \quad m\geqq n\geqq n_0$$

(6) $$\frac{K+a}{\sqrt{n_0}}\leqq 3K\frac{1+\delta}{(\sqrt{1+\delta}-1)n}.$$

Let us define

$$L(\omega)=\max\left\{3K\sqrt{1+\delta}\,\frac{1+\delta+u(\sqrt{1+\delta}-1)}{u(\sqrt{1+\delta}-1)},\ \max_{n_0\leqq n\leqq n_0(1+\delta)}\sqrt{\frac{n}{\log\log n}}|x_n|\right\}.$$

We state that

$$(7) \qquad |x_n| \leqq L(\omega)\sqrt{\frac{\log\log n}{n}} \quad \text{for every} \quad n \geqq n_0.$$

For $n_0 \leqq n \leqq n_0(1+\delta)$ (7) follows from the definition of $L(\omega)$. As a first step we prove that the validity of (7) for an $n \geqq n_0$ implies its validity for every $n(1+c\delta) \leqq \leqq \tilde{n} \leqq (1+\delta)n$ .Our statement is a straight consequence of this fact.

In order to prove this we introduce the following notation:

$$L^* = \frac{u(\sqrt{1+\delta}-1)}{\sqrt{1+\delta}[1+\delta+u(\sqrt{1+\delta}-1)]} L(\omega) \quad \text{and} \quad d = L^* \frac{1+\delta}{(\sqrt{1+\delta}-1)u}.$$

Obviously $L^* \geqq 3K$ and $d \leqq L(\omega)$. Let us consider two different cases:

First case: $x_j > d\sqrt{\frac{\log\log n}{n}}$ for every $n \leqq j < \tilde{n}$.

Since (7) is true for n by (1), (2) and (5) in this case we have

$$x_{\tilde{n}} = x_n + \sum_{j=n}^{\tilde{n}-1} \frac{1}{j}[y_j - M(x_j)] - \sum_{j=n}^{\tilde{n}-1} \frac{1}{j} M(x_j) \leqq$$

$$\leqq L(\omega)\sqrt{\frac{\log\log n}{n}} + L^*\sqrt{\frac{\log\log n}{n}} - \beta d \sum_{j=n}^{\tilde{n}-1} \frac{1}{j}\sqrt{\frac{\log\log n}{n}}.$$

Thus by the definition of d and by (3) and finally using the definition of L^* we get

$$x_{\tilde{n}} \leqq \sqrt{\frac{\log\log n}{n}}[L(\omega)+L^* - \beta d \log(1+\delta c)] =$$

$$= \sqrt{\frac{\log\log n}{n}}\left(L(\omega)+L^* - L^*\sqrt{1+\delta} - L^*\frac{1+\delta}{n}\right) = \frac{L(\omega)}{\sqrt{1+\delta}}\sqrt{\frac{\log\log n}{n}} \leqq$$

$$\leqq L(\omega)\sqrt{\frac{\log\log \tilde{n}}{\tilde{n}}}.$$

Second case: There is a j, $n \leqq j < \tilde{n}$ for which $x_j < \sqrt{\frac{\log\log n}{n}}\, d$. In this case either $x_{\tilde{n}-1} < 0$ and

$$x_{\tilde{n}} \leqq \frac{K+a}{\tilde{n}-1} \leqq d\frac{1}{\sqrt{n-1}} \leqq L(\omega)\sqrt{\frac{\log\log \tilde{n}}{\tilde{n}}}$$

because of (6) or there is a $n \leqq j^* < \tilde{n}$ for which $0 \leqq x_{j^*} \leqq d\sqrt{\frac{\log\log n}{n}}$ and $x_i \geqq 0$ for every $j^* \leqq i < \tilde{n}$ also because of (6). So $M(x_i) \geqq 0$ for these i-s. Therefore

$$x_{\tilde{n}} \leqq x_{j^*} + \left|\sum_{i=j^*}^{\tilde{n}-1} \frac{1}{i}(y_i - M(x_i))\right| \leqq d\sqrt{\frac{\log\log n}{n}} + L^*\sqrt{\frac{\log\log n}{n}} =$$

$$=\sqrt{\frac{\log\log n}{n}}\frac{L(\omega)}{\sqrt{1+\delta}}\leqq L(\omega)\sqrt{\frac{\log\log\tilde{n}}{\tilde{n}}}.$$

In the same way it can be proved that $x_n > -L(\omega)\sqrt{\frac{\log\log n}{n}}$. By a little modification the original statement can be proved too.

Actually we proved the following: There is a constant L

$$\left(\text{explicitely } L = 3K\sqrt{1+\delta}\left(1+\frac{1+\delta}{u(\sqrt{1+\delta}-1)}\right)\right),$$

such that if $|x_n| \leqq K_n\sqrt{\frac{\log\log n}{n}}$ for an $n \geqq n_0(\omega)$ and an appropriate number K_n then $x_{\tilde{n}} \leqq \max(K_n, L)\sqrt{\frac{\log\log n(1+\delta)}{n(1+\delta)}}$ for $n(1+c\delta) < \tilde{n} \leqq n(1+\delta)$. But choosing a $c < \varrho < 1$ constant we get that $x_{\tilde{n}} \leqq \sqrt{\frac{1+\delta\varrho}{1+\delta}}\max(L, K_n)\sqrt{\frac{\log\log\tilde{n}}{\tilde{n}}}$ for $n(1+c\delta) \leqq \tilde{n} \leqq n(1+\delta\varrho)$; If the relation

$$x_n \leqq L(\omega)\sqrt{\frac{\log\log n}{n}}$$

were true only for $L(\omega) > L$ if $n_0 \leqq n < n_0(1+\delta)$ then the unequality $|x_n| \leqq \leqq \sqrt{\frac{1+\delta d}{1+\delta}}L(\omega)\sqrt{\frac{\log\log n}{n}}$ would be true for $n_0(1+\varrho\delta) \leqq n \leqq n_0(1+\varrho\delta)^2$. Iterating this result we get our statement.

By an affine transformation we get the following corollary.

COROLLARY. Let $a_n = \frac{c}{n}$, $2M'(\theta) > c$ and the other conditions of the theorem be fulfilled. Then there is an L such that

$$\mathsf{P}\left(\overline{\lim}\sqrt{\frac{n}{\log\log n}}|x_n-\theta| < L\right) = 1.$$

Remark: If $M'(\theta) = \beta \leqq \frac{1}{2}$, $a_n = \frac{1}{n}$ the same argument gives that

$$\mathsf{P}(\lim n^{\beta-\varepsilon}(x_n-\theta) = 0) = 1.$$

The simplest case is when $M(x) = \beta\cdot x$ we look for the root of $M(x)=0$ and there is no error. In this case

$$x_n = (1-\beta)\left(1-\frac{\beta}{2}\right)\cdot\ \left(1-\frac{\beta}{n-1}\right)x_0 = x_0\exp\sum_{i=1}^{n-1}\log\left(1-\frac{\beta}{i}\right)\sim C\cdot n^{-\beta}$$

with a constant C that is

$$\mathsf{P}(\lim n^{\beta}(x_n-\theta) = c) = 1.$$

and this shows that the exponent can not be improved.

To prove theorem 2 we need the following lemma.

LEMMA 2. *Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be independent identically distributed random variables, so that $\mathsf{P}(|\xi_i| \leqq K) = 1$ for a constant K and $\mathsf{E}\xi_i = 0$.*
Then there is a $\theta > 1$ and $t > 0$ so that

$$\sum_{i=\theta^n+1}^{\theta^{n+1}} \frac{1}{i} \xi_i > t \sqrt{\frac{\log\log\theta^n}{\theta^n}}$$

for infinitely many n with probability 1.

PROOF. Let us denote $\mathsf{D}\xi_i$ by D. Clearly

$$\sum_{i=\theta^n+1}^{\theta^{n+1}} \frac{1}{i} \xi_i = \sum_{i=\theta^n+1}^{\theta^{n+1}} \frac{1}{\theta^n} \xi_i + \sum_{i=\theta^n+1}^{\theta^{n+1}} \left(\frac{1}{i} - \frac{1}{\theta^n}\right) \xi_i .$$

To estimate the first member of the sum we need the inequality $1 - \Phi(x) \geqq \geqq \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right) e^{\frac{-x^2}{2}}$ ($\Phi(x)$ is the normal distribution) and the theorem of large deviation for equal components (see [3] p. 517)

$$\mathsf{P}\left(\sum_{\theta^n+1}^{\theta^{n+1}} \frac{1}{\theta^n} \xi_i > C_1 D \sqrt{\frac{\log\log\theta^n}{\theta^n}}\right) = \mathsf{P}\left(\frac{\sum\limits_{i=1}^{\theta^n(\theta-1)} \xi_i}{D\sqrt{\theta^n(\theta-1)}} > C_1 \sqrt{\frac{\log\log\theta^n}{\theta^n}}\right) \geqq$$

$$\geqq 1 - \Phi\left[(C_1+\varepsilon)\sqrt{\frac{\log\log\theta^n}{\theta - 1}}\right] \geqq \exp\left[-(C_1+\varepsilon)^2 \frac{\log\log\theta^n}{2(\theta-1)}\right] = (n\log\theta)^{-\frac{(C_1+\varepsilon)^2}{2(\theta-1)}} .$$

So by the Borel—Cantelli lemma (the sums for different n's are independent) for arbitrary $\varepsilon > 0$

$$\sum_{i=\theta^n+1}^{\theta^{n+1}} \frac{1}{\theta^n} \xi_i > \sqrt{(2-\varepsilon)(\theta-1)}\, D \sqrt{\frac{\log\log\theta^n}{\theta^n}} \tag{8}$$

infinitely many times with probability 1.
Estimating the second member of the sum by the same method as in Lemma 1 we get with $f(n) = \gamma\sqrt{\theta^n \log\log\theta^n}$.

$$\mathsf{P}\left(\sum_{i=\theta^n+1}^{\theta^{n+1}} \left(\frac{1}{\theta^n} - \frac{1}{i}\right) \xi_i \geqq C_2 \sqrt{\frac{\log\log\theta^n}{\theta^n}}\right) =$$

$$= \mathsf{P}\left(\exp f(n) \sum_{\theta^n+1}^{\theta^{n+1}} \left(\frac{1}{\theta^n} - \frac{1}{i}\right) \xi_i \geqq \exp C_2 f(n) \sqrt{\frac{\log\log\theta^n}{\theta^n}}\right) \leqq$$

$$\leqq \prod_{\theta^n+1}^{\theta^{n+1}} \mathsf{E}\left(\exp f(n) \left(\frac{1}{\theta^n} - \frac{1}{i}\right) \xi_i\right) \mathsf{E}\left[\exp\left(-C_2 f(n) \sqrt{\frac{\log\log\theta^n}{\theta^n}}\right)\right] \leqq$$

$$\leqq \exp\left\{\frac{D^2}{2-\varepsilon} \gamma^2 \sum_{i=\theta^n+1}^{\theta^{n+1}} \left(\frac{1}{\theta^n} - \frac{1}{i}\right)^2 \log\log\theta^n - C_2\gamma \log\log\theta^n\right\}$$

for sufficiently large n's ($\varepsilon > 0$ is fixed).

But

$$\sum_{i=\theta^n+1}^{\theta^{n+1}} \left(\frac{1}{\theta^n}-\frac{1}{i}\right)^2 \leqq \frac{\sum_{i=1}^{\theta^{n+1}-\theta^n} i^2}{\theta^{4n}} \leqq (1+\varepsilon)\frac{(\theta^{n+1}-\theta^n)^3}{3\theta^{4n}} = \frac{(\theta-1)^3}{3\theta^n}(1+\varepsilon).$$

So by a little calculation, using the Borel—Cantelli lemma we get that

$$\sum_{i=\theta^n}^{\theta^{n+1}} \left(\frac{1}{\theta^n}-\frac{1}{i}\right)\xi_i \leqq \left[\frac{\sqrt{6}}{3} D(\theta-1)^{\frac{3}{2}}-\varepsilon\right]\sqrt{\frac{\log\log\theta^n}{\theta^n}} \quad \text{for} \tag{9}$$

sufficiently large n with probability 1.

Since we can choose θ and ε as near to 1, and to 0, respectively as we want, Lemma 2 follows from (8) and (9).

Using lemma 2 theorem 2 can be proved easily.

Proof of theorem 2. Apply lemma 2 for the sequence $y_i - M(x_i)$ and suppose that the statement of the theorem does not hold. Let $|M(x)| < \beta|x|$ if $|x| < \delta$ where β is a positive constant. Then with probability 1 for large k $|x_k(\omega)| \leqq \delta$ and $|x_k(\omega)| \leqq L' \sqrt{\frac{\log\log k}{k}}$ with a constant L' such that $L'(2+\beta\log\theta) < t$. But then

$$|x_{\theta^{n+1}+1} - x_{\theta^n+1}| \leqq 2L'\sqrt{\frac{\log\log\theta^n}{\theta^n}}$$

$$\left|\sum_{i=\theta^n+1}^{\theta^{n+1}} \frac{1}{i} M(x_i)\right| \leqq L'\sqrt{\frac{\log\log\theta^n}{\theta^n}} \sum_{i=\theta^n+1}^{\theta^{n+1}} \frac{\beta}{i} \leqq L'\beta\sqrt{\frac{\log\log\theta^n}{\theta^n}}\log\theta$$

$$\sum_{i=\theta^n+1}^{\theta^{n+1}} \frac{1}{i}[y_i - M(x_i)] = -x_{\theta^{n+1}+1} + x_{\theta^n+1} - \sum_{i=\theta^n+1}^{\theta^{n+1}} \frac{1}{i} M(x_i) \leqq$$

$$\leqq (2L'+\beta\log\theta)\sqrt{\frac{\log\log\theta^n}{\theta^n}}$$

for sufficiently large n contradicting to lemma 2.

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