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A NOTE TO A PAPER OF DUDLEY

by

L. LOVÁSZ and P. MAJOR

R.M. DUDLEY proved [1] some results about realization of probability measures on a metric space as distribution of random variables. Among others he showed, generalizing a result of Skorohod, that, given a separable metric space S and a sequence $\{P_n\}$ of probability measures weakly converging to a probability measure P_0 on S, there exists a probability space $(\Omega, \mathcal{A}, P)$ and S-valued random variables X_n such that X_n has distribution P_n $(n=0, 1, \ldots)$ and $X_n \to X_0$ with probability 1.

In the present note we prove the following

THEOREM. *Let S be a compact connected metric space; let the probability measures P_n (on the Borel sets of S) converge weakly to a probability measure P_0 such that every open set has positive P_0 measure. Then there is a probability space $(\Omega, \mathcal{A}, P)$ and random variables X_n, $(a=0, 1, \ldots)$ such that X_n has distribution P_n and $X_n \to X_0$ uniformly.*

It is a surprising fact, occuring already in DUDLEY's paper, that the proof of the theorem is based on a combinatorial lemma, the KÖNIG—EGERVÁRY theorem. However, here we use its "continuous" form:

Let G be a bipartite graph with vertices x_i and y_j $(i, j=1, 2, \ldots, n)$ such that the edges join x_i's with y_j's; associate a non-negative real number $p(u)$ with every vertex u of G. It is possible to associate a non-negative real $v(e)$ with every edge e of G in such a way that the sum of values $v(e)$ associated with edges incident with u be $p(u)$, if and only if (i) $\Sigma p(y_i) = \Sigma p(x_i)$ *and* (ii) *if* $I \subseteq \{1, \ldots, n\}$ *and J is the set of indices of points y_j connected to points x_i $(i \in I)$ then* $\sum_{j \in J} p(y_j) \geqq \sum_{i \in I} p(x_i)$.

PROOF of the Theorem. Let $k \geqq 1$. Divide S into disjoint sets $F_1^{(k)}, \ldots, F_{r_k}^{(k)}$ with diameters $< \frac{1}{k}$, with boundaries of P_0-probability 0 and having an inner point. Then

$$\lim_{n \to \infty} P_n(F_i^{(k)}) = P_0(F_i^{(k)}) > 0. \tag{1}$$

Let us choose an n_k such that

$$P_m(F_i^{(k)}) > (1 - \min_{j \leqq r_k} P_m(F_j^{(k)})) P_0(F_i^{(k)}) \tag{2}$$

for $m \geqq n_k$. We may assume $n_1 < n_2 < \cdots$.

For $n_k \leqq m < n_{k+1}$, consider the following bipartite graph G_m: let x_i, y_j, $(i, j=1, 2, \ldots, r_k)$ be its vertices, x_i and y_j connected by an edge iff $\overline{F_i^{(k)}} \cap \overline{F_j^{(k)}} \neq 0$, and let $p(x_i) = P_0(F_i^{(k)})$, $p(y_i) = P_m(F_i^{(k)})$. Then the conditions of the KÖNIG—EGER-

VÁRY theorem are fulfiled:

$$\sum_{i=1}^{r_k} p(x_i) = \sum_{i=1}^{r_k} p(y_i) = 1$$

and if $I \subset \{1, \dots, n\}$ then the points x_i $(i \in I)$ are connected to every point y_i $(i \in I)$ and, because of the connectedness of S, to at least one point y_j, $j \notin I$. Thus (2) implies (ii).

Let $a_{ij}^{(m)}$ denote the value associated with the edge connecting x_i to y_j, and $a_{ij}^{(m)}=0$ if no such edge exists. Consider the Cartesian product $S \times S \times \dots = \Omega'$ and the product σ-algebra $\mathscr{A}'$ on it. Denote by $X_n(\omega)$ the $(a+1)$-st coordinate of $\omega \in \Omega'$. We define a probability measure P on Ω' as follows: let $P(X_0 \in A) = P_0(A)$ and

$$P(X_m \in A | X_0 = x_0, \dots, X_{m-1} = x_{m-1}) = P(X_m \in A | X_0 = x_0) = = \sum_{j=1}^{r_k} \frac{a_{ij}^{(m)}}{P_0(F_i^{(k)})} P_m(A|F_j^{(k)})$$

for $n_k \leqq m < n_{k+1}$ and $x_0 \in F_i^{(k)}$. Then

$$P(X_m \in A) = \sum_i \sum_j a_{ij}^{(m)} \frac{P_m(A \cap F_j^{(k)})}{P_m(F_j^{(k)})} = \sum_j P_m(A \cap F_j^{(k)}) = P_m(A).$$

On the other hand, from the construction it follows that $P\left(d(X_0, X_m) > \frac{2}{k}\right) = 0$ for $m \geqq n_k$ and thus the set $\Omega = \bigcap_{\substack{k, m \\ n_k \leqq m}} \left\{d(X_0, X_m) \leqq \frac{2}{k}\right\}$ has probability 1. Thus, the functions X_n restricted to Ω give the required construction.

Remark: The condition that S be connected and P_0 be positive seems to be somewhat artificial. However, it is easy to see that they are necessary. For assume S is not connected, and let P_0 be a positive probability measure on it. Let $S = S_1 \cup S_2$, where $d(S_1, S_2) = \delta > 0$. Define the probability measure P_n by

$$P_n(A) = \left(1 + \frac{P_0(S_2)}{n}\right) P_0(A \cap S_1) + \left(1 - \frac{P_0(S_1)}{n}\right) P_0(A \cap S_2).$$

Then there are no random variables X_n with the properties required in the theorem; really, for each n there would be an ω such that $X_n(\omega) \in S_1$ but $X_0(\omega) \in S_2$ (because $P_0(S_1) < P_n(S_1)$), i.e. $d(X_n, X_0) \geqq \delta$.

On the other hand, imbed the preceding space S into a compact connected metric space S_1, and define all measures to be 0 on $S_1 - S$. Then, obviously, the preceding argument remains valid.

REFERENCES

[1] DUDLEY, R. M.: Distances of probability measures and random variables. *Ann. Math. Statist.* **39** (1968), 1563—1572.

Eötvös L. University, Budapest
Mathematical Institute of the Hungarian Academy of Sciences, Budapest

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