Z. Wahrscheinlichkeitstheorie verw. Gebiete
50, 27–52 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Non-Central Limit Theorems for Non-Linear Functionals of Gaussian Fields*

R.L. Dobrushin[1] and P. Major[2]

[1] Institute for Problems of Information Transmission, Moscow
[2] Mathematical Institute of the Hungarian Academy of Sciences, Budapest 1053, Hungary

Dedicated to Professor Leopold Schmetterer on his sixtieth Birthday

Summary. Let a stationary Gaussian sequence X_n, $n=\dots-1,0,1,\dots$ and a real function $H(x)$ be given. We define the sequences $Y_n^N = \frac{1}{A_N}\cdot \sum_{j=(n-1)N}^{nN-1} H(X_j)$, $n=\dots-1,0,1\dots$; $N=1,2,\dots$ where A_N are appropriate norming constants. We are interested in the limit behaviour as $N\to\infty$. The case when the correlation function $r(n)=EX_0X_n$ tends slowly to 0 is investigated. In this situation the norming constants A_N tend to infinity more rapidly than the usual norming sequence $A_N=\sqrt{N}$. Also the limit may be a non-Gaussian process. The results are generalized to the case when the parameter-set is multi-dimensional.

1. Introduction

Let a stationary Gaussian sequence X_n, $n=\dots-1, 0, 1, \dots$ $EX_n=0$, $EX_n^2=1$ be given. We assume that the correlation function $r(n)=EX_0X_n$ satisfies the relation

$$r(n)=n^{-\alpha}L(n), \qquad 0<\alpha<1, \tag{1.1}$$

where $L(t)$, $t\in(0,\infty)$ is a slowly varying function; i.e.

$$\lim_{s\to\infty}\frac{L(st)}{L(s)}=1 \qquad \text{for every } t\in(0,\infty), \tag{1.2}$$

and $L(t)$ is integrable on every finite interval. (See e.g. [5] Appendix 1.) We consider a real function $H(x)$ such that $H(x)$ does not vanish on a set of positive measure,

* This paper contains results closely connected to those of the paper by Taqqu, Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 53–83 (1979). The investigations were done independently and at about the same time. Different methods were used

0044-3719/79/0050/0027/$05.20

$$\int_{-\infty}^{\infty} H(x) \exp\left(-\frac{x^2}{2}\right) dx = 0, \tag{1.3}$$

and

$$\int_{-\infty}^{\infty} [H(x)]^2 \exp\left(-\frac{x^2}{2}\right) dx < \infty. \tag{1.4}$$

Throughout this paper $H_j(x)$ denotes the j-th Hermite polynomial with highest coefficient 1. Because of (1.3) and (1.4) we may expand $H(x)$ as

$$H(x) = \sum_{j=1}^{\infty} c_j H_j(x) \tag{1.5}$$

with

$$\sum_{j=1}^{\infty} c_j^2 j! < \infty. \tag{1.6}$$

We consider the sequence $H(X_n)$, $n = \ldots -1, 0, 1, \ldots$ and take the so-called renorm group transformation (see e.g. [1, 2]), i.e. we define the sequences

$$Y_n^N = \frac{1}{A_N} \sum_{j=N(n-1)}^{Nn-1} H(X_j), \qquad \begin{array}{l} n = \ldots -1, 0, 1, \ldots \\ N = 1, 2, \ldots \end{array} \tag{1.7}$$

where A_N is an appropriate positive norming constant. We consider the case $N \to \infty$, and we are interested in the limit process Y_n^* if it exists.

In our situation the mixing conditions guaranteeing the central limit theorem with the usual norming factor $\sqrt{N}$ for sums of weakly dependent random variables (see e.g. [5]) do not hold, and actually both the norming factors and the limit distribution may differ from the usual ones. (Let us remark that by the central limit theorem we mean a slightly stronger statement than it is usually done in the literature. We demand that the sequence defined in (1.7) tend to a sequence of independent normal random variables.)

It was Rosenblatt [6] who first observed these new possibilities (see also [5] 19.5). He proved that in case of $H(x) = x^2 - 1$ the limit distribution may be non-Gaussian. The problem was later investigated by Taqqu [8]. He proved that the case of a general $H(x)$ can be reduced to the case $H(x) = H_j(x)$, and gave a complete solution for the problem in case $j = 1, 2$.

In paper [2] it was proven that any such limit process has to be self-similar. In the present paper we show that in the case $c_1 = c_2 = \ldots = c_{k-1} = 0$, $c_k \neq 0$, $\alpha < \frac{1}{k}$ the limit process exists and belongs to a class of self-similar processes which was constructed in [1] by means of multiple Wiener-Itô integrals. (It was called Itô integral in [1].) Our method based on the properties of the Wiener-Itô integrals is different from that of the papers [6] and [8].

Now we formulate

Theorem 1. *Let* (1.1) *hold with* $\alpha<\frac{1}{k}$, *where k is the smallest index in the expansion* (1.5) *for which* $c_k \neq 0$. *Then, if* $N\to\infty$ *and we choose*

$$A_N = N^{1-\frac{k\alpha}{2}} L(N)^{\frac{k}{2}}, \tag{1.8}$$

the finite dimensional distributions of the sequence $Y_n^N, n=\dots-1,0,1,\dots$ *defined in* (1.7) *tend to that of the sequence* Y_n^* *given by the formula*

$$Y_n^* = D^{-\frac{k}{2}} c_k \int e^{in(x_1+\dots+x_k)} \frac{e^{i(x_1+\dots+x_k)}-1}{i(x_1+\dots+x_k)} |x_1|^{\frac{\alpha-1}{2}} \dots |x_k|^{\frac{\alpha-1}{2}} dW(x_1)\dots dW(x_k) \tag{1.9}$$

where

$$D=\int_{-\infty}^{\infty} \exp(ix)|x|^{\alpha-1}dx = 2\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right). \tag{1.10}$$

Formula (1.9) denotes multiple Wiener-Itô integral with respect to the random spectral measure W of the white-noise process.

The notion of the Wiener-Itô integral with respect to the random spectral measure of a stationary process (or of a stationary random field) is a slight modification of the usual Wiener-Itô (or = Wiener) integral with respect to a Gaussian orthogonal measure. This modification is needed because of the evenness of the spectral measure. The definition and the basic properties of this integral needed in the present paper can be found for example in [1].

We make some comments on the condition $\alpha<\frac{1}{k}$. We remark that if ξ and η are jointly Gaussian random variables $E\xi=E\eta=0$, $E\xi^2=E\eta^2=1$, $E\xi\eta=r$, then

$$E H_k(\xi) H_j(\eta) = \delta_{j,k} r^k k!. \tag{1.11}$$

(see e.g. [7], Theorem 1.3).

It is easy to see, applying (1.11), that in case $\alpha<\frac{1}{k}$ the variance

$$D\left(\sum_{j=1}^{N} H(X_j)\right) \asymp N^{2-k\alpha} L(N)^k. \tag{1.12}$$

(Here and in the following relation $\gamma_n \asymp \delta_n$ means that $c^{-1}\delta_n<\gamma_n<c\,\delta_n$ for some $0<c<\infty$.)
This explains the choice of A_N in (1.8).

On the other hand if $\alpha>\frac{1}{k}$, then $D\left(\sum_{j=1}^{N} H(X_j)\right)\asymp N$. This indicates that the dependence between distant $H(X_j)-s$ is sufficiently weak, therefore it is natural to expect that the central limit theorem holds with the usual normalization. We shall prove this fact in a subsequent paper.

The case $\alpha=\frac{1}{k}$ deserves special attention. It may happen in this case (e.g. if $L(n)\asymp 1$) that $N^{-1} D\left(\sum_{j=1}^{N} H(X_j)\right)\to\infty$, i.e. the behaviour of the variance is not similar to the weakly dependent case. On the other hand, if $\alpha=\frac{1}{k}$ formula (1.9) is meaningless, since

$$\int\left|\frac{e^{i(x_1+\ldots+x_k)}-1}{i(x_1+\ldots+x_k)}\right|^2 |x_1|^{\frac{1}{k}-1}\ldots|x_k|^{\frac{1}{k}-1}\,dx_1\ldots dx_k=\infty. \tag{1.13}$$

We give a short proof of the last relation. Let us define the sets D_n in the k-dimensional Euclidian space

$$D_n=\left\{(x_1,\ldots x_k)\left|-n-\frac{1}{3}<x_1<-n;\ \frac{n}{2k}<x_j<\frac{n}{k},\ j=2,\ldots k-1;\right.\right.$$
$$\left. n-(x_2+\ldots+x_{k-1})<x_k<\left(n+\frac{1}{3}\right)-(x_2+\ldots+x_{k-1})\right\},\qquad n=1,2,\ldots$$

It is easy to see that the sets D_n are disjoint for different n, their Lebesgue measure $\lambda(D_n)>C_1 n^{k-2}$ and the integrand in (1.13) is bigger than $C_2(n^{\frac{1}{k}-1})^k$ on the set D_n with appropriate positive constants C_1, C_2.

Thus the integral in (1.13) can be estimated from below by

$$\sum_{n=1}^{\infty}\int_{D_n}\left|\frac{e^{i(x_1+\ldots+x_k)}-1}{i(x_1+\ldots+x_k)}\right|^2 |x_1|^{\frac{1}{k}-1}\ldots|x_k|^{\frac{1}{k}-1}\,dx_1\ldots dx_k$$
$$\geqq C_1 C_2\sum_{n=1}^{\infty}\frac{1}{n}=\infty.$$

We will show in a subsequent paper that in the case $\alpha=\frac{1}{k}$ the central limit theorem holds again, but the norming factor may be different from the usual $A_N=\sqrt{N}$.

We will obtain Theorem 1 as a consequence of a more general theorem, in which the parameter set of the $X-s$ is multi-dimensional. In order to formulate this result, we introduce some definitions and notations.

R^ν will denote the ν-dimensional Euclidian space, $\mathscr{B}^\nu$ the Borel σ-algebra on it. $(\cdot,\cdot)$ means scalar product, and $|\cdot|$ absolute value in R^ν. Z^ν is the set of points in R^ν with integer coordinates. Given an $x\in R^\nu$ or $x\in Z^\nu$ the superscripts $x^{(1)},\ldots,x^{(\nu)}$ denote its coordinates. If $x\in R^\nu$, $[x]$ denotes its integer part, i.e. $n=[x]\in Z^\nu$, and $x^{(j)}-1<n^{(j)}\leqq x^{(j)}$, $j=1,2,\ldots\nu$. $S^{\nu-1}$ is the unit sphere in R^ν; $S^{\nu-1}=\{x\,|\,|x|=1,\ x\in R^\nu\}$. Given a set $A\in\mathscr{B}^1$ A^ν denotes its ν-th power, i.e. $A^\nu=\{x\,|\,x\in R^\nu,\ x^{(j)}\in A, j=1,\ldots\nu\}$. Finally, if $A\in\mathscr{B}^\nu$, ∂A denotes its boundary. Let μ_N, $N=1,2,\ldots$ be a sequence of finite measures on $\mathscr{B}^\nu$. We say that the sequence μ_N

tends weakly to a finite measure μ_0 if $\int f(x)\,\mu_N(dx)\to\int f(x)\,\mu_0(dx)$ for every bounded continuous function f on R^ν. Let μ_N, $N=1,2,\ldots$ be a sequence of locally finite measures (i.e. $\mu_N(B)<\infty$ for every bounded $B\in\mathscr{B}^\nu$). We say that they tend to a locally finite measure μ_0 locally weakly, if $\int f(x)\,\mu_N(dx)\to\int f(x)\,\mu_0(dx)$ for every continuous function f with a bounded support. A sequence μ_N of finite measures tends weakly to a measure μ_0 (which is necessarily also finite), iff the sequence μ_N tends locally weakly to μ_0, and

$$\lim_{A\to\infty}\sup_N \mu_N(|x|>A)=0. \tag{1.14}$$

A sequence μ_N of bounded (locally bounded) measures tends weakly (locally weakly) to a measure μ_0 iff for every (every bounded) set with the property $\mu_0(\partial B)=0$ we have $\lim \mu_N(B)=\mu_0(B)$.

A set of random variables X_n, $n\in Z^\nu$ is called a ν-dimensional stationary Gaussian field, if the random variables $X_{n_1},\ldots X_{n_k}$ have a joint normal distribution for any $n_1,\ldots n_k\in Z^\nu$; $EX_n=EX_j$, and $EX_0X_j=EX_nX_{n+j}$ for any $j,n\in Z^\nu$. $r(n)=EX_0X_n$ is the correlation function of the field. We assume throughout this paper that $EX_0=0$, $EX_0^2=1$.

A stationary Gaussian field always has a unique spectral measure G, concentrated on the cube $(-\pi,\pi]^\nu$, such that

$$r(n)=\int e^{i(\lambda,n)}\,G(d\lambda). \tag{1.15}$$

Obviously we have

$$G((-\pi,\pi]\nu)=EX_0^2=1.$$

A stationary Gaussian random field can always be represented in the form

$$X_n=\int e^{i(n,x)}\,Z_G(dx),$$

where Z_G is the random spectral measure of the field (see e.g. [1]).

We are given a function $H(x)$ with the properties (1.3) and (1.4). We define

$$Y_n^N=\frac{1}{A_N}\sum_{j\in B_n^N} H(X_j),\qquad n\in Z^\nu,\ N=1,2,\ldots \tag{1.16}$$

with an appropriate norming factor A_N, where

$$B_n^N=\{j\,|\,j\in Z^\nu,\ n^{(l)}N\leqq j^{(l)}<(n^{(l)}+1)N,\ l=1,\ldots\nu\} \tag{1.17}$$

We denote $B^N=B_0^N$. We need the following

Proposition 1. *Let the stationary Gaussian random field X_n, $n\in Z^\nu$ have a correlation function*

$$r(n)\sim|n|^{-\alpha}L(|n|)\,a\left(\frac{n}{|n|}\right),\qquad n\to\infty \tag{1.18}$$

where $0<\alpha<\nu$, $L(t)$ is a slowly varying function of $t\in[0,\infty)$ and $a(t)$ is a continuous function on $S^{\nu-1}$.

Let G be the spectral measure of X_n, *and define*

$$G_N(A)=\frac{N^\alpha}{L(N)}G(N^{-1}A),\ A\in B^\nu,\ N=1,2,\dots \tag{1.19}$$

Then there exists a locally finite measure G_0 *such that*

$$\lim_{N\to\infty} G_N=G_0. \tag{1.20}$$

in the sense of locally weak convergence. G_0 *can be considered as the spectral measure of a generalized stationary random field on* R^ν. *It has the following self-similarity property:*

$$G_0(A)=t^{-\alpha}G_0(tA),\quad A\in B^\nu,\ t\in(0,\infty), \tag{1.21}$$

and it is determined by the relation

$$2^\nu\int_{R^\nu} e^{i(t,x)}\prod_{j=1}^{\nu}\frac{1-\cos x^{(j)}}{(x^{(j)})^2}G_0(dx)$$

$$=\int_{[-1,1]^\nu}(1-|x^{(1)}|)\dots(1-|x^{(\nu)}|)\frac{a\left(\frac{x+t}{|x+t|}\right)}{|x+t|^\alpha}dx\qquad t\in R^\nu. \tag{1.22}$$

This proposition is a variant of well-known Tauberian theorems. As the authors could not trace this variant in the literature they give its proof as a by-product of other constructions. The main result of the paper is

Theorem 1′. *Let the conditions of Proposition 1 be fulfilled, and let k be the smallest index in the expansion* (1.5) *such that* $c_k\neq 0$. *Assume that*

$$0<\alpha<\frac{\nu}{k}. \tag{1.23}$$

With the choice of

$$A_N=N^{\nu-\frac{k\alpha}{2}}[L(N)]^{\frac{k}{2}} \tag{1.24}$$

the finite dimensional distributions of the random fields defined in (1.7) *tend to those of the random field* Y_n^*, $n\in Z^\nu$, *given by the formula*

$$Y_n^*=c_k\int e^{i(n_1+\dots+x_k)}K_0(x_1\dots x_k)Z_{G_0}(dx_1)\dots Z_{G_0}(dx_k). \tag{1.25}$$

The last formula means multiple Wiener-Itô integral with respect to the random spectral measure determined by the measure G_0 *(* G_0 *is defined in Proposition 1), and*

$$K_0(x_1,\dots,x_k)=\prod_{j=1}^{\nu}\frac{e^{i(x_1^{(j)}+\dots+x_k^{(j)})}-1}{i(x_1^{(j)}+\dots+x_k^{(j)})}. \tag{1.26}$$

Remark 1.1. Theorem 1 is a consequence of Theorem 1′ and Proposition 1. Because of the formula for change of variables in Wiener-Itô integrals (see [1] Proposition 4.2) it is sufficient to show that under the conditions of Theorem 1 the spectral measure G_0 has the density function $D^{-1}|x|^{\alpha-1}$. If G has a spectral density $D^{-1}|x|^{\alpha-1}$, $-\pi<x\leqq\pi$ then $r(n)\sim n^{-\alpha}$ (see [9] §5.2) and G_0 has also a spectral density $D^{-1}|x|^{\alpha-1}$. Relation (1.22) implies that in the case $\nu=1$, G_0 depends only on α in (1.1).

Sections 2 and 3 of this paper contain the proof of Theorem 1′. In Sect. 4 and 5 relations to some earlier results are discussed. In Sect. 6 we investigate a generalization of Theorem 1′ to the case when H is a function of several variables.

2. Proof of the Main Theorem

Condition (1.18) implies that the measure G is nonatomic (i.e. there is no point with positive G measure). There is a well-known result (see e.g. [9] §3.9) which implies this fact in the one-dimensional case. Also the multi-dimensional case can be proved the same way. Indeed, given a ν-dimensional spectral measure G, one can consider its one-dimensional projection $\hat{G}(A)=G(A\times(-\pi,\pi]^{\nu-1})$, $A\in\mathscr{B}^1$. Its Fourier coefficients satisfy the relation $\hat{r}(k)=r(k,0,\dots,0)$. Thus the measure $\hat{G}$ and therefore also the measure G is nonatomic.

Thus the definition of Wiener-Itô integral given in [1] with respect to the random spectral measures Z_{G_N} and Z_{G_0} is maningful.

Let us first discuss the case $H(x)=H_k(x)$. By the formula expressing Wiener-Itô integrals in terms of Hermite polynomials we may write

$$\begin{aligned}H_k(X_n)&=H_k\left(\int e^{i(n,x)}Z_G(dx)\right)\\&=\int e^{i(n_1+\dots+x_k)}Z_G(dx_1)\dots Z_G(dx_k).\end{aligned}\tag{2.1}$$

Now applying the notations of Theorem 1′ and the formula for change of variables in Wiener-Itô integrals, mentioned before, we can see that the random variables (1.16) have the same joint distributions for fixed N as the following ones, which we identify with them for the sake of simplicity.

$$\begin{aligned}Y_n^N&=\frac{1}{N^\nu}\sum_{j\in B_n^N}\int e^{i\frac{1}{N}(j,x_1+\dots+x_k)}Z_{G_N}(dx_1)\dots Z_{G_N}(dx_k)\\&=\int e^{i(n,x_1+\dots+x_k)}K_N(x_1,\dots,x_k)\,Z_{G_N}(dx_1)\dots Z_{G_N}(dx_k)\end{aligned}\tag{2.2}$$

where Z_{G_N} is the random measure corresponding to G_N, and

$$\begin{aligned}K_N(x_1,\dots,x_k)&=\sum_{j\in B_0^N}\frac{1}{N^\nu}e^{i\frac{1}{N}(j,x_1+\dots+x_k)}\\&=\prod_{l=1}^{\nu}\frac{\exp x(i(x_1^{(l)}+\dots+x_k^{(l)}))-1}{\left[\exp\left(i\frac{1}{N}(x_1^{(l)}+\dots+x_k^{(l)})\right)-1\right]N}.\end{aligned}\tag{2.3}$$

Let us introduce the following piecewise constant modification of the Fourier transform:

$$\varphi_N(t_1,\dots,t_k)=\int e^{i\frac{1}{N}((j_1,x_1)+\dots+(j_k,x_k))}|K_N(x_1,\dots,x_k)|^2\, G_N(dx_1)\dots G_N(dx_k) \tag{2.4}$$

where $j_p=[t_p N]$, $p=1,2,\dots,k$. Using the middle term in (2.3) and (1.15) we can see that

$$\begin{aligned}\varphi_N(t_1,\dots,t_k)&\\ &=\frac{1}{N^{2\nu-k\alpha}L(N)^k}\sum_{p\in B_0^N}\sum_{q\in B_0^N} r(p-q+j_1)\dots r(p-q+j_k)\\ &=\frac{1}{N^{2\nu-k\alpha}L(N)^k}\sum_{p\in\tilde B^N}(N-|p^{(1)}|)\dots(N-|p^{(\nu)}|)\, r(p+j_1)\dots r(p+j_k),\end{aligned} \tag{2.5}$$

where

$$\tilde B^N=\{p\,|\,-N<p^{(j)}<N,\ j=1,2\dots\nu\}.$$

This formula enables us to investigate the asymptotic behaviour of φ_N.

In order to prove Theorem 1′ we need the following lemmas:

Lemma 1. $\lim_{N\to\infty}\varphi_N(t_1,\dots,t_k)=g(t_1,\dots,t_k)$ *uniformly in every bounded region, where*

$$\begin{aligned}&g(t_1,\dots,t_k)\\ &=\int_{[-1,1]^\nu}(1-|x^{(1)}|)\dots(1-|x^{(\nu)}|)\frac{a\left(\frac{x+t_1}{|x+t_1|}\right)}{|x+t_1|^\alpha}\dots\frac{a\left(\frac{x+t_k}{|x+t_k|}\right)}{|x+t_k|^\alpha}\,dx\end{aligned} \tag{2.6}$$

is a continuous function.

Lemma 2. *Let $\mu_1,\mu_2,\dots$ be a sequence of finite measures on R^l such that $\mu_N(R^l-[-C_N\pi,C_N\pi]^l)=0$, with some sequence $C_N\to\infty$. Define the function*

$$\varphi_N(t)=\int_{R^l} e^{i\left(\frac{j}{C_N},x\right)}\mu_N(dx), \tag{2.7}$$

where $j\in Z^l$ is $j=[t\,C_N]$. If for every $t\in R^l$ the sequence $\varphi_N(t)$ tends to a function $\varphi(t)$ continuous in the origin then μ_N weakly tends to a finite measure μ_0. $\varphi(t)$ is the Fourier transform of μ_0.

Lemma 3. *Let G_N be a sequence of non-atomic spectral measures on $\mathscr{B}^\nu$ tending locally weakly to a non-atomic spectral measure G_0, $\hat K_N(x_1,\dots,x_k)$ a sequence of measurable functions on $R^{k\nu}$ tending to a continuous function $\hat K_0(x_1,\dots,x_k)$ uniformly in any rectangle $[-A,A]^{k\nu}$. Moreover, let the functions $\hat K_N$ satisfy the relation*

$$\lim_{A\to\infty}\int_{R^{k\nu}-[-A,A]^{k\nu}}|\hat K_N(x_1,\dots,x_k)|^2\, G_N(dx_1)\dots G_N(dx_k)=0 \tag{2.8}$$

uniformly for $N=0,1,2,\ldots$. Then the Wiener-Itô integral

$$\int \hat{K}_0(x_1,\ldots,x_k)\, Z_{G_0}(dx_1)\ldots Z_{G_0}(dx_k) \tag{2.9}$$

exists and the sequence of Wiener-Itô integrals

$$\int \hat{K}_N(x_1,\ldots,x_k)\, Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k) \tag{2.10}$$

tends in distribution to the integral (2.9) *as $N\to\infty$.*

It is enough to prove that for any integer l, $n_1,\ldots,n_l \in Z^\nu$ and real numbers $\beta_1,\ldots,\beta_l$ the distribution of the random variable

$$\sum_{p=1}^{l} \beta_p Y_{n_p}^N = \sum_{p=1}^{l} \int \beta_p e^{i(n_p, x_1+\ldots+x_k)} K_N(x_1,\ldots,x_k)\, Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k)$$

(see (2.2)) tends to that of the random variable

$$\sum_{p=1}^{l} \beta_p Y_{n_p}^* = \sum_{p=1}^{l} \int \beta_p e^{i(n_p, x_1+\ldots+x_k)} K_0(x_1,\ldots,x_k)\, Z_{G_0}(dx_1)\ldots Z_{G_0}(dx_k).$$

We shall apply Lemma 3 with the choice

$$\hat{K}_N(x_1,\ldots,x_k) = \sum_{p=1}^{l} \beta_p e^{i(n_p, x_1+\ldots+x_k)} K_N(x_1,\ldots,x_k) \tag{2.11}$$

and

$$\hat{K}_0(x_1,\ldots,x_k) = \sum_{p=1}^{l} \beta_p e^{i(n_p, x_1+\ldots+x_k)} K_0(x_1,\ldots,x_k). \tag{2.12}$$

We have to check the validity of the conditions of Lemma 3.

A comparison of formulas (1.26) and (2.3) makes it clear that $\hat{K}_N \to \hat{K}_0$ uniformly in $[-A, A]^{k\nu}$. The convergence $G_N \to G_0$ is stated in Proposition 1.

The sequence of the measures μ_N,

$$\mu_N(A) = \int_A \left| \sum_{p=1}^{l} \beta_p e^{i(n, x_1+\ldots+x_k)} \right|^2 \cdot |K_N(x_1,\ldots,x_k)|^2\, G_N(dx_1)\ldots G_N(dx_k), \quad A \in \mathscr{B}^{k\nu} \tag{2.13}$$

tends locally weakly to the measure μ_0,

$$\mu_0(A) = \int_A \left| \sum_{p=1}^{l} \beta_p e^{i(n, x_1+\ldots+x_k)} \right|^2 \cdot |K_0(x_1,\ldots,x_k)|^2\, G_0(dx_1)\ldots G_0(dx_k), \quad A \in \mathscr{B}^{k\nu}. \tag{2.14}$$

The measure μ_N, $N=1,2,\ldots$ is finite and is concentrated to the rectangle $(-N\pi, N\pi]^{k\nu}$. The following identity holds:

$$\psi_N(t)=\int_{R^{k\nu}} e^{i\frac{1}{N}(j,x)}\mu_N(dx) = \sum_{r=1}^{l}\sum_{s=1}^{l}\beta_r\beta_s\varphi_N(t_1+n_r-n_s,\ldots,t_k+n_r-n_s) \tag{2.15}$$

where $t=(t_1,\ldots,t_k)$, $t_m\in R^\nu$, $m=1,2,\ldots,k$; $j=[tN]$.

Thus Lemmas 1 and 2 imply that the measure μ_0 is finite, and the sequence μ_N tends weakly to it. Thus the condition (1.14) is fulfilled and this implies (2.8). Thus Lemma 3 implies Theorem 1 in the special case $H(x)=H_k(x)$.

Let us now consider the case of a general $H(x)$. Define

$$Z_n^N=\sum_{s\in B_n^N}\sum_{j=k+1}^{\infty} c_j H_j(X_s), \qquad n\in Z^\nu,\ N=1,2,\ldots.$$

Relation (1.11) implies that

$$E(Z_n^N)^2=\sum_{j=k+1}^{\infty} c_j^2 j! \sum_{s,t\in B_n^N}[r(t-s)]^j.$$

It is easy to check with the help of (1.6) and (1.18) that

$$E(Z_n^N)^2=O(N^{2\nu-(k+1)\alpha}L(N)^{k+1})+O(N^\nu) \qquad \text{as } N\to\infty.$$

Thus $A_N^{-1}Z_n^N\to 0$ in probability as $N\to\infty$ for every n, and this implies that $H(x)$ can be replaced with $c_k H_k(x)$ in Theorem 1.

3. Proof of the Lemmas and of the Proposition

Proof of Lemma 1. Let us define the function

$$f_N(t_1,\ldots,t_k,x) = \left(1-\frac{[x^{(1)}N]}{N}\right)\ldots\left(1-\frac{[x^{(\nu)}N]}{N}\right)\frac{r([xN]+j_1)}{N^{-\alpha}L(N)}\cdots\frac{r([xN]+j_k)}{N^{-\alpha}L(N)}, \tag{3.1}$$

where again $j_p=[t_pN]$, $p=1,2,\ldots,k$, $x\in[-1,1]^\nu$.

Because of (2.5) we have

$$\varphi_N(t_1,\ldots,t_k)=\int_{[-1,1]^\nu} f_N(t_1,t_2,t_k,x)\,dx. \tag{3.2}$$

Define the set

$$A_\varepsilon^N(t_1,\ldots,t_k)=\{x\,|\,x\in[-1,1]^\nu,\ |x+t_l|<\varepsilon \text{ for some } l,\ l=1,\ldots,k\}.$$

The well-known Karamata theorem (see e.g. [5] Appendix 1) implies that for any $C>\varepsilon>0$

$$\lim_{N\to\infty} \sup_{\varepsilon N<m<CN} \left|\frac{L(m)}{L(N)}-1\right|=0. \tag{3.3}$$

Because of (1.18) and (3.3) we have for any $K>0$, $\varepsilon>0$

$$\lim_{N\to\infty} \sup_{\substack{|t_1|<K,\dots,|t_k|<K \\ x\in[-1,1]^\nu - A_\varepsilon^N(t_1,\dots,t_k)}} |f_N(t_1,\dots,t_k,x)-f(t_1,\dots,t_k,x)|=0 \tag{3.4}$$

where

$$f(t_1,t_2,\dots,t_k,x) = (1-|x^{(1)}|)\dots(1-|x^{(\nu)}|)\frac{a\left(\frac{x+t_1}{|x+t_1|}\right)}{|x+t_1|^\alpha}\dots\frac{a\left(\frac{x+t_k}{|x+t_x|}\right)}{|x+t_k|^\alpha}. \tag{3.5}$$

In order to complete the proof of Lemma 1 it is sufficient to show that

$$\int_{\{|x+t_l|<\varepsilon\}\cap[-1,1]^\nu} |f_N(t_1,t_2,\dots,t_k,t_k,x)|\,dx<C(\varepsilon) \tag{3.6}$$

and

$$\int_{\{|x+t_l|<\varepsilon\}\cap[-1,1]^\nu} |f(t_1,t_2,\dots,t_k,x)|\,dx<C(\varepsilon) \tag{3.7}$$

for every $l=1,2,\dots,k$ if $|t_1|<K,\dots,|t_k|<K$, where $C(\varepsilon)\to 0$ as $\varepsilon\to 0$. Relation (3.7) also implies the existence and the continuity of $g(t_1,\dots,t_k)$.

By Hölder's inequality

$$\int_{\{|x+t_l|<\varepsilon\}\cap[-1,1]^\nu} |f(t_1,t_2,\dots,t_k,x)|\,dx \le \left[\prod_{s=1}^k \int_{\{x+t_l|<\varepsilon\}\cap[-1,1]^\nu} (1-|x^{(s)}|)^k \frac{\left|a\left(\frac{x+t_s}{|x+t_s|}\right)\right|^k}{|x+t_s|^{k\alpha}}\,dx\right]^{\frac{1}{k}} \le K_1\,\varepsilon^{\nu-k\alpha}. \tag{3.8}$$

Here and in what follows $K_1, K_2, \dots$ denote some appropriate constants depending only k and the correlation function $r(\cdot)$.

Let us now turn to the proof of (3.6). Let $\gamma>0$ be so small that $\nu-k(\alpha+\gamma)>0$. It is easy to deduce from the Karamata theorem that a slowly varying function $L(t)$ satisfies

$$L(t)\le K_2\left(\frac{N}{t}\right)^\gamma L(N), \qquad 0\le t\le NK_3,\ N=1,\dots. \tag{3.9}$$

Thus formula (1.18) implies that

$$|r(n)|\le K_4|n|^{-\alpha}L(|n|)\,a\left(\frac{n}{|n|}\right)\le K_5\,N^{-\alpha}L(N)\left(\frac{|n|}{N}\right)^{-\alpha-\gamma}. \tag{3.10}$$

Thus for large enough N

$$\begin{aligned}
&\int_{\{|x+t_l|<\varepsilon\}} |f_N(t_1,\dots,t_k,x)|\,dx \\
&\leqq \frac{1}{N^{2\nu-k\alpha}L(N)^k}\cdot \sum_{\substack{p\in \bar{B}^N\\ |p+j_l|<2\varepsilon N}} (N-|p^{(1)}|)\dots(N-|p^{(\nu)}|)\,|r(p+j_1)\dots r(p+j_k)| \qquad (3.11)\\
&\leqq \frac{K_6}{N^{2\nu}} \sum_{\substack{p\in \bar{B}^N\\ |p+j_l|<2\varepsilon N}} (N-|p^{(1)}|)\dots(N-|p^{(\nu)}|)\left(\frac{|p+j_1|}{N}\right)^{-\alpha-\gamma}\dots\left(\frac{|p+j_k|}{N}\right)^{-\alpha-\gamma}\\
&\leqq \int_{[-1,1]^\nu\cap\{|x+t_l|<\varepsilon\}} \hat{f}(t_1,t_2,\dots,t_k,x)\,dx.
\end{aligned}$$

where

$$\hat{f}(t_1,\dots,t_k,x)=(1-|x^{(1)}|)\dots(1-|x^{(\nu)}|)\frac{1}{|x+t_1|^{\alpha+\gamma}}\dots\frac{1}{|x+t_k|^{\alpha+\gamma}}.$$

Now an estimation similar to (3.8) shows that the right-hand side of (3.11) is smaller than $K_7\,\varepsilon^{\nu-k(\alpha+\gamma)}$. Thus (3.6) also holds.

Proof of Lemma 2. Lemma 2 is an analogue of the well-known theorem about the equivalence of the weak convergence of measures and the convergence of their Fourier transforms. Their proofs are also very similar.

First we show that for any $\varepsilon>0$ there exists a $K>0$ such that

$$\mu_N(x\,|\,x\in R^l,\ |x^{(1)}|>K)<\varepsilon \qquad \text{for every } N\geqq 1. \tag{3.12}$$

As $\varphi(t)$ is continuous in the origin, we can find a $\delta>0$ such that

$$|\varphi(0,\dots,0)-\varphi(t,0,\dots,0)|<\frac{\varepsilon}{2} \qquad \text{if } |t|<\delta, \tag{3.13}$$

We have

$$0\leqq \operatorname{Re}[\varphi_N(0,\dots,0)-\varphi_N(t,\dots,0)]\leqq 2\,\varphi_N(0,\dots,0). \tag{3.14}$$

The sequence in the middle of (3.14) tends to $\operatorname{Re}[\varphi(0,\dots,0)-\varphi(t,\dots,0)]$. The right-hand side of (3.14) is bounded since it is convergent. Thus, because of the Lebesgue theorem and (3.13) we may write

$$\begin{aligned}
&\lim_{N\to\infty}\int_0^\delta \frac{1}{\delta}\operatorname{Re}[\varphi_N(0,\dots,0)-\varphi_N(t,\dots,0)]\,dt\\
&\quad=\int_0^\delta \frac{1}{\delta}\operatorname{Re}[\varphi(0,\dots,0)-\varphi(t,\dots,0)]\,dt<\frac{\varepsilon}{2}.
\end{aligned}$$

Applying this relation together with the inequality

$$|1-e^{iy}|>C|y|,\quad -\pi\leqq y\leqq \pi \qquad \text{with some } C>0$$

we obtain the following inequality for arbitrary $K>0$:

$$\begin{aligned}
\frac{\varepsilon}{2} &> \lim_{N\to\infty} \operatorname{Re}\frac{1}{\delta}\int_0^\delta [\varphi_N(0,\ldots,0)-\varphi_N(t,\ldots,0)]\,dt \\
&= \lim_{N\to\infty} \operatorname{Re} \int_{[-C_N\pi,\, C_N\pi]^l} \left[1-\frac{1}{\delta C_N}\sum_{j=0}^{[\delta C_N]} e^{i\frac{jx^{(1)}}{C_N}}\right] \mu_N(dx) \\
&\geqq \limsup_{N\to\infty} \operatorname{Re} \int_{K<|x^{(1)}|<C_N\pi} \left[1-\frac{1}{\delta C_N}\sum_{j=0}^{[\delta C_N]} e^{i\frac{jx^{(1)}}{C_N}}\right] \mu_N(dx) \\
&= \limsup_{N\to\infty} \operatorname{Re} \int_{K<|x^{(1)}|<C_N\pi} \left[1-\frac{1}{\delta C_N}\,\frac{1-e^{i([\delta C_N]+1)\frac{jx^{(1)}}{C_N}}}{1-e^{i\frac{x^{(1)}}{C_N}}}\right] \mu_N(dx) \\
&\geqq \limsup_{N\to\infty} \int_{\{|x^{(1)}|>K\}} \left[1-\frac{2}{\delta CK}\right] \mu_N(dx).
\end{aligned}$$

Choosing $K=4/\delta C$ we obtain that $\limsup_{N\to\infty} \mu_N(|x^{(1)}|>K)\leqq\varepsilon$ which implies (3.12).

Applying the same argument to the other coordinates we find that for any $\varepsilon>0$, some $K=K(\varepsilon)$ and all N

$$\mu_N(R^l-[-K,K]^l)<\varepsilon. \tag{3.15}$$

Define

$$\tilde{\varphi}_N(t)=\int_{R^l} e^{i(t,x)}\,\mu_N(dx), \qquad t\in R^l. \tag{3.16}$$

A comparison of (2.7) and (3.16) shows that (3.15) implies the relation $\tilde{\varphi}_N(t)-\varphi_N(t)\to 0$ as $N\to\infty$. Thus $\tilde{\varphi}_N(t)\to\varphi(t)$, and Lemma 2 follows from standard theorems on Fourier transforms.

Proof of Proposition 1 Put

$$\begin{aligned}
\bar{K}_N(x)&=|K_N(x)|^2=\prod_{j=1}^{\nu}\frac{1-\cos x^{(j)}}{N^2\left(1-\cos\frac{x^{(j)}}{N}\right)} \qquad x\in R^\nu \\
\bar{K}_0(x)&=|K_0(x)|^2=2^\nu\prod_{j=1}^{\nu}\frac{1-\cos x^{(j)}}{(x^{(j)})^2}
\end{aligned} \tag{3.17}$$

where K_N and K_0 are the functions introduced in (2.3) and (1.26) for $k=1$. It is clear that $\bar{K}_N(x)\to\bar{K}_0(x)$ uniformly in every bounded region as $N\to\infty$. Applying Lemmas 1 and 2 to the case $k=1$ we obtain that the sequence of the measures

$$\mu_N(B)=\int_B \bar{K}_N(x)\,G_N(dx), \qquad B\in\mathscr{B}^\nu \tag{3.18}$$

tends weakly to the finite measure μ_0 determined by its Fourier transform $g(t)$, given by formula (2.6) in case $k=1$. As $\bar{K}_N(x)$ is continuous and does not vanish in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^\nu$ we have for every $B \in \mathscr{B}^\nu$; $B \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^\nu$ and $\mu_0(\partial B)=0$

$$\lim_{N\to\infty} G_N(B) = \int_B [\bar{K}_0(x)]^{-1} \mu_0(dx) = G_0(B) \tag{3.19}$$

where $G_0(\cdot)$ is a measure on the measurable subsets of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^\nu$.

We show that for any $t>0$ and $B \in \mathscr{B}^\nu$, $B \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^\nu$, $G_0(\partial B)=0$ the relation

$$\lim_{N\to\infty} G_N(tB) = t^\alpha G_0(B) \tag{3.20}$$

holds. Let us choose $M = \left[\frac{N}{t}\right]$. Then $G_N(tB) = \left(\frac{N}{M}\right)^\alpha \frac{L(M)}{L(N)} G_M\left(t\frac{M}{N}B\right)$.

It is clear that $\left(\frac{N}{M}\right)^\alpha \frac{L(M)}{L(N)} \to t^\alpha$, and thus because of the condition $G_0(\partial B)=0$

$$G_M\left(t\frac{M}{N}B\right) \to G_0(B) \quad \text{as} \quad N\to\infty.$$

Thus (3.20) is proved.

Relation (3.20) implies that for every $B_1, B_2 \in \mathscr{B}^\nu$ $B_1, B_2 \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^\nu$, $B_1 = tB_2$ with some $t>0$ the identity

$$G_0(B_1) = t^\alpha G_0(B_2) \tag{3.21}$$

holds. Indeed, by (3.20) this identity holds if $G_0(\partial B_j)=0$, $(j=1,2)$, but then it must hold also without this restriction. Now we can define $G_0(B)$ for every bounded set $B \in \mathscr{B}^\nu$ in the following way: If $B \in \mathscr{B}^\nu$, $B \subset \left[-K\frac{\pi}{2}, K\frac{\pi}{2}\right]^\nu$ then $G_0(B) = K^\alpha G_0\left(\frac{1}{K}B\right)$. This definition does not depend on K, and if $B \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^\nu$ it agrees with the definition given before. Extending this set function G_0 to $\mathscr{B}^\nu$ we obtain a locally finite measure G_0. The relation (3.20) holds also without the condition $B \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^\nu$, and this implies that G_N tends locally weakly to G_0 if $N\to\infty$. It is evident that (1.21) holds.

The relation (3.18) and the fact that $\bar{K}_N \to \bar{K}_0$ uniformly in every bounded set imply the equation

$$\mu_0(B) = \int_B \bar{K}_0(x)\, G_0(dx) \tag{3.22}$$

for every bounded (and thus also for every unbounded) set $B \in \mathscr{B}^\nu$. As the Fourier transform of μ_0 is g, relations (3.22) and (2.6) imply (1.22).

Proof of Lemma 3. In this proof we shall use the notations of §4 in [1]. Let $\bar{H}^k_{G_0}$ be the subspace of the space $\hat{H}^k_{G_0}$, introduced in [1], consisting of those functions $h \in \hat{H}^k_{G_0}$ for which every level set, $\{h=c\}$ for some c, satisfies the relation $G_0(\partial\{h=c\})=0$. If $h \in \bar{H}^k_{G_0}$ the identity

$$\lim_{N\to\infty} \int h(x_1, \ldots, x_k)\, Z_{G_N}(dx_1) \ldots Z_{G_N}(dx_k)$$
$$= \int h(x_1, \ldots, x_k)\, Z_{G_0}(dx_1) \ldots Z_{G_0}(dx_k) \tag{3.23}$$

holds, where lim means convergence in distribution. Indeed, the integrals in (3.23) are the polynomials of the random variables $Z_{G_N}(B)$ (the $B-s$ are the level sets of the function h) with coefficients independent of N. Now, as the joint distributions of the variables $Z_{G_N}(B)$ tend to the joint distributions of the variables $Z_{G_0}(B)-s$, (3.23) holds true.

We claim that for any $\varepsilon>0$, there exists an $h \in \bar{H}^k_{G_0}$ such that

$$\int_{R^{k\nu}} |\hat{K}_N(x_1, \ldots, x_k) - h(x_1, \ldots, x_k)|^2\, G_N(dx_1) \ldots G_N(dx_k) < \varepsilon \tag{3.24}$$

if $N=0$ or $N>N(\varepsilon)$.

Indeed, because of (2.8) the $\hat{K}_N$ can be approximated by functions with compact support. Thus the continuity of $\hat{K}_0$ and the uniform convergence of the K_N-s to K_0 on bounded regions imply (3.24). (3.24) is equivalent to the statement

$$E\left|\int [\hat{K}_N(x_1, \ldots, x_k) - h(x_1, \ldots, x_k)]\, Z_{G_N}(dx_1) \ldots Z_{G_N}(dx_k)\right|^2 < \varepsilon \tag{3.25}$$

if $N=0$ or $N>N(\varepsilon)$.

The condition (2.8) implies that

$$\int |\hat{K}_0(x_1, \ldots, x_k)|^2\, G_0(dx_1) \ldots G_0(dx_k) < \infty.$$

so the Wiener-Itô integral (2.9) exists and the formulas (3.23), (3.25) imply Lemma 3.

4. Discussion on the Conditions of Theorem 1′

The most important condition of Theorem 1′ is formula (1.18) which describes the asymptotic behaviour of the correlation function. Now we discuss possibilities of weakening it.

Remark 4.1. In a paper of Dobrushin and Takahashi [4], devoted to the description of the Gaussian self-similar fields with discrete parameters, the following result is proved.

Let G_0 be a spectral measure satisfying the self-similarity property (1.21) with some α, $0<\alpha<\nu$, and let $H(x)=x$. The finite dimensional distributions of the field defined in (1.16) with an appropriate sequence A_N, tend to the finite dimensional distributions of the Gaussian self-similar field given in (1.25) with $k=1$, iff $A_N = N^{\nu-\frac{\alpha}{2}} L(N)$ with a slowly varying function $L(\cdot)$ and the relation

(1.20) holds. Thus in case of $k=1$ the main condition (1.18) can be replaced by the weaker condition (1.20) concerning the behaviour of the spectral measure.

Remark 4.2. One may ask whether condition (1.18) can be substituted with (1.20) also in the case $H(x)=H_k(x)$, $k\geqq 2$.

The answer is in the negative. First we give a heuristic explanation of the difference between the cases $k=1$ and $k\geqq 2$. Then we briefly discuss an example where the behaviour of the fields defined in (1.16) is completely different in cases $H(x)=x$ and $H(x)=H_2(x)=x^2-1$.

For the sake of simplicity we consider only the distribution of the random variable Y_0^N. Y_0^N is expressed in formula (2.2) as the integral of the function $K_N(x_1,\dots,x_k)$ with respect to the random orthogonal measure Z_G. In case of $k=1$, $K_N(x)$ is bounded outside of a neighbourhood of 0 over the rectangle $(-N\pi, N\pi]^\nu$, i.e. over the support of G_N. This fact may explain why a condition like (1.20) about the local behaviour of the spectral measure in the neighbourhoods of the origin is a sufficient condition of Theorem 1′ in case $k=1$.

On the other hand if $k\geqq 2$, $K_N(x_1,\dots,x_k)$ is unbounded in every neighbourhood of a point $(x_1,\dots,x_k)\in R^{k\nu}$ satisfying the relation $x_1+\dots+x_k=0$. Therefore it is natural to expect that in this case a big singularity of the spectral measure G outside of the origin may have an influence on the limit behaviour of Y_0^N. The following example shows that this is really the case.

Let $\nu=1$, and let the stationary Gaussian sequence X_n, $n=\dots-1,0,1,\dots$ have spectral measure with the spectral density

$$g(x)=C_1|x|^{-\alpha}+C_2(|x-a|^{-\beta}+|x+a|^{-\beta}),$$

where $0<\alpha<\beta<1$, $\beta>\frac{1}{2}$, $0<\alpha<\pi$, C_1, $C_2>0$. Let us consider the sequence of random variables Y_0^N defined in (1.7) in cases $H(x)=x$ and $H(x)=H_2(x)=x^2-1$. We claim that this sequence converges in distribution in both cases, but in the first case the good norming factor is $A_N=N^{\frac{1+\alpha}{2}}$, and in the second one $A_N=N^\beta$. This means that in the first case the norming factor depends only on α and in the second only on β. Moreover, the limit distribution is the normal distribution if $H(x)=x$, and another one if $H(x)=x^2-1$.

We briefly sketch the proof. It is not difficult to compute that

$$\begin{aligned} EX_k X_{k+n} &= \int_{-\pi}^{\pi} e^{inx} g(x)\,dx \\ &= K_1 n^{\alpha-1}\left(1+O\left(\frac{1}{n}\right)\right)+K_2 n^{\beta-1}\cos na\left(1+O\left(\frac{1}{n}\right)\right) \end{aligned} \tag{4.1}$$

for some positive K_1, K_2.

The following relation can be proved by means of (4.1):

$$E\left(X_1\sum_{k=1}^{j} X_k\right)=\frac{K_1}{\alpha} j^\alpha+O(1), \tag{4.2}$$

i.e. the second term on the right-hand side of (4.1) has a small effect in the expression (4.2) because of the factor $\cos na$. It is not difficult to see that (4.2) implies that in case of $H(x)=x$ the distribution of Y_0^N with norming $A_N=N^{\frac{1+\alpha}{2}}$ tends to a normal one. The case of $H(x)=H_2(x)$ is different.

We may compute $EH_2(X_k)\,H_2(X_{n+k})$ by means of (4.1) and (1.11), and we obtain that

$$EH_2(X_k)\,H_2(X_{n+k})=K_2^2\,n^{2(\beta-1)}(1+\cos 2na+o(1)).$$

It is not difficult to see by the help of this relation that the second moments of the random variables in the sequence

$$N^{-\beta}\sum_{k=0}^{N-1} H_2(X_k), \qquad N=1,2,\dots \tag{4.3}$$

have a limit as $N\to\infty$.

Applying the diagram formula (see e.g. [1] formula (4.23)) it can be shown that every moment of the expressions in (4.3) converges as $N\to\infty$, and these limits are the moments of a uniquely determined distribution. This fact proves that Y_0^N has a limit distribution with the given norming factor A_N also in the case $H(x)=H_2(x)$. The calculation shows that the third moment of the limit distribution is positive, and therefore the limit distribution cannot be normal. It would be also interesting to discuss more general situations than the case discussed before.

Remark 4.3. The condition about the continuity of the function $a(t)$ in Theorem 1′ can be weakened. Carrying out the estimations in Lemma 1 more carefully one can see that it is sufficient to assume that $a(t)$ is a Riemann integrable function.

Remark 4.4. In the proof of Theorem 1′ we used only condition (1.20) and the fact that φ_N tends to g. Because of formula (2.5) the last condition can be interpreted as a condition on the asymptotical behaviour on the correlation function which is weaker than (1.18).

5. Comparison with Previous Results

Rosenblatt [6] and Taqqu [8] formulated the problem in a different way. Now we reformulate our results in order to show their equivalence to those of Rosenblatt and Taqqu in the cases investigated by them.

First we give an "integral version" of Theorem 1′. We preserve the notations of § 1.

Define

$$Z^N(n)=\frac{1}{N^{\nu-\frac{k\alpha}{2}}L(N)^{\frac{k}{2}}}\sum_{j\in D_n} H(X_j);\ N=1,2,\dots,n\in Z^\nu,$$
$$n^{(s)}\leqq N, \qquad s=1,2,\dots \nu, \tag{5.1}$$

where

$$D_n = \{j \mid j \in Z^\nu,\ 0 \leqq j^{(s)} < n^{(s)},\ s = 1, 2, \ldots \nu\}.$$

Consider the random fields

$$Z_t^N = Z^N([tN]), \qquad N = 1, 2, \ldots;\ t \in [0, 1]^\nu \tag{5.2}$$

and

$$Z_t^0 = c_k \int \prod_{j=1}^{\nu} \frac{e^{it^{(j)}(x_1^{(j)} + \ldots + x_k^{(j)})} - 1}{i(x_1^{(j)} + \ldots + x_k^{(j)})} Z_{G_0}(dx_1) \ldots Z_{G_0}(dx_k). \tag{5.3}$$

The following result is a simple consequence of Theorem 1′. (Actually they are equivalent.)

Theorem 2. *Under the conditions of Theorem 1′ the joint distribution of* $Z_{t_1}^N, \ldots, Z_{t_l}^N$ *weakly tends to that of* $Z_{t_1}^0, \ldots, Z_{t_l}^0$ *as* $N \to \infty$ *for every* l *and* $t_1, \ldots, t_l \in [0, 1]^\nu$.

Proof. It is easy to check, using the properties of the Wiener-Itô integrals, that $E(Z_t^0 - Z_s^0)^2$ is a continuous function of the variables t and s. Applying the main condition (1.18) on the correlation function one can see that $E(Z_t^N - Z_s^N)^2$, $N = 1, 2, \ldots$ is a sequence of uniformly continuous functions of the variables t and s.

Therefore it is sufficient to show that given any integer $M > 0$, the statement of Theorem 2 holds for $t_1 = \frac{j_1}{M}, \ldots, t_l = \frac{j_l}{M}$ where $j_s \in Z^\nu$, and its coordinates are between 0 and M. But this fact is a straight consequence of Theorem 1′, applying it to the subsequence Y_n^{NM}, $n \in Z^\nu$, $N = 1, \ldots$

One of Taqqu's main results concerns Theorem 2 when $\nu = 1, k = 2$. He described the limit distribution as that of the random variable $\sum_{k=1}^{\infty} \lambda_k(\xi_k^2 - 1)$ where $\xi_1 \xi_2, \ldots$ are independent standard normal zero-one random variables, and $\lambda_k \geqq 0$, $k = 1, 2, \ldots$ is an appropriate sequence of positive numbers with $\sum \lambda_k^2 < \infty$. (He calls it the Rosenblatt distribution, because it first appeared in a paper of Rosenblatt [6].) In order to show that Taqqu's representation is equivalent to ours we express the double Wiener-Itô integral

$$\int H(x, y)\, Z_G(dx)\, Z_G(dy) \tag{5.4}$$

in another form.

Here G is the spectral measure of a generalized field. Therefore

$$G(A) = G(-A), \qquad A \in \mathscr{B}^\nu. \tag{5.5}$$

The function $H(\cdot, \cdot)$ is a complex-valued measurable function with the properties

$$H(x, y) = H(y, x) = \overline{H(-y, -x)},\ x, y \in R^\nu$$
$$\int_{R^{2\nu}} |H(x, y)|^2\, G(dx) \quad (dy) < \infty. \tag{5.6}$$

We shall consider the real Hilbert space L_G^2, consisting of the complex valued functions $f(x)$, $x \in R^\nu$ such that

$$f(x) = \overline{f(-x)}, \qquad x \in R^\nu \tag{5.7}$$
$$\int_{R^\nu} |f(x)|^2 \, G(dx) < \infty.$$

The scalar product of $f(x)$ and $g(x)$ is defined as

$$(f, g) = \int f(x) \overline{g(x)} \, G(dx)$$

We define the integral operator

$$A f(x) = \int_{R^\nu} H(x, -y) f(y) \, G(dy). \tag{5.8}$$

which, because of (5.5) and (5.6), maps L_G^2 into L_G^2. It is easy to see that A is a self-adjoint Hilbert-Schmidt operator, therefore it has a system of real eigenvalues $\lambda_1, \lambda_2, \ldots$ in such a way that

$$\sum \lambda_k^2 < \infty. \tag{5.9}$$

Proposition 2. *The distribution of the stochastic integral* (5.4) *agrees with that of the series*

$$\sum \lambda_k (\xi_k^2 - 1) \tag{5.10}$$

where $\xi_1, \xi_2, \ldots$ *are independent standard normal random variables. The series* (5.10) *converges both in the mean square sense and with probability 1 because of* (5.9).

Proof. Let $\varphi_1, \varphi_2, \ldots$ be a complete orthonormal system of eigenvectors of the operator A.[1]
One can write

$$H(x, -y) = \sum_{k=1}^{\infty} \lambda_k \varphi_k(x) \overline{\varphi_k(y)}, \tag{5.11}$$

where the convergence is meant in the $L_G^2 \otimes L_G^2$ sense. The properties of the Wiener-Itô integral imply that

$$\begin{aligned} &\int_{R^{2\nu}} H(x, y) Z_G(dx) Z_G(dy) \\ &\quad = \sum_k \int \lambda_k \varphi_k(x) \varphi_k(y) Z_G(dx) Z_G(dy) \\ &\quad = \sum_k \lambda_k H_2\left(\int \varphi_k(x) Z_G(dx)\right), \end{aligned} \tag{5.12}$$

[1] In most textbooks on functional analysis the existence of a complete orthonormal system of eigenvectors is proved only in complex Hilbert spaces. Nevertheless the changes needed for the proof in a real Hilbert space are trivial

where $H_2(x)=x^2-1$ is the second Hermite polynomial. The random variables $\int \varphi_k(x)\, Z_G(dx)$, $k=1,2,\ldots$ are jointly Gaussian and independent because of the orthogonality of the φ_k-s. Thus (5.12) implies the proposition. It is easy to check that the operator A corresponding to the integral in formula (5.3) (in case $k=2$) is a positive operator.

Indeed, in this case the kernel of the operator A is $H(x-y)$ with

$$H(x)=\prod_{j=1}^{\nu} \frac{e^{it(j)x(j)}-1}{ix^{(j)}}.$$

Since $H(x)$ is the Fourier transform of the uniform distribution on the cube $\mathop{\times}\limits_{j=1}^{\nu} [0, t_j)$, hence it is a positive definite function.

Therefore we have for any $f\in L_G^2$

$$(A f, f)=\iint H(x-y)\, f(x)\, \overline{f(y)}\, G(dx)\, G(dy)\geqq 0$$

as we claimed.

Thus in this case every eigenvalue λ_k is non-negative.

6. On More General Functions

We shall discuss here the generalization of the original problem to the case when H is an arbitrary square integrable functional of the Gaussian field.

Let X_n $n\in Z^\nu$ be the random field considered in Theorem 1′, and let $\mathscr{L}^2$ denote the real Hilbert space of all square integrable functionals of the field X_n, $n\in Z^\nu$ i.e. the space of all random variables with finite second moment, which are measurable with respect to the σ-algebra generated by the random field X_n, $n\in Z^\nu$. It is known, see e.g. [1], that any $\xi\in\mathscr{L}^2$, $E\xi=0$ can be represented in the form

$$\xi=\sum_{j=1}^{\infty} \frac{1}{j!} \int \alpha_j(x_1,\ldots,x_j)\, Z_G(dx_1)\ldots Z_G(dx_j) \tag{6.1}$$

where the α_j-s are complex valued functions in the space $L_G^2\otimes\ldots\otimes L_G^2$ with the properties

$$\alpha_j(x_1,\ldots,x_j)=\overline{\alpha_j(-x_1,\ldots,-x_j)},\ x_1,\ldots,x_j\in R^\nu \tag{6.2}$$

and

$$\sum_{j=1}^{\infty} \frac{1}{j!} \int_{R^{j\nu}} |\alpha_j(x_1,\ldots,x_j)|^2\, G(dx_1)\ldots G(dx_j)<\infty. \tag{6.3}$$

On the random field $X_n, n\in Z^\nu$ there exists a unique group of isometrical transformations T_n: $\mathscr{L}^2\to\mathscr{L}^2$, $n\in Z^\nu$ in such a way that

$$T_n(X_i)^s=(X_{i+n})^s, \quad i, n\in Z^\nu,\ s=0,1,2,\ldots \tag{6.4}$$

This group is called the shift group. We shall say that the random field

$$U_n = T_n \xi, \qquad n \in Z^\nu, \tag{6.5}$$

where $\xi \in \mathscr{L}^2$, $E\xi = 0$ is arbitrary, is a stationary field subordinated to the field X_n, $n \in Z^\nu$.

It is easy to see that U_n can be given in the form

$$U_n = \sum_{k=1}^{\infty} \frac{1}{k!} \int \exp[i(n, x_1 + \ldots + x_k)]\, \alpha_k(x_1, \ldots x_k) \cdot Z_G(dx_1) \ldots Z_G(dx_k), \qquad n \in Z^\nu. \tag{6.6}$$

(A similar result for generalized fields was proven in [1].) An important example of subordinate fields is the following one:

$$\begin{aligned} \xi &= H(X_{p_1}, \ldots, X_{p_s}) \\ U_n &= H(X_{p_1+n}, \ldots, X_{p_s+n}) \end{aligned} \tag{6.7}$$

where $p_1, \ldots, p_s \in Z^\nu$, $s = 1, 2, \ldots, H = H(x_1, \ldots, x_s)$ is such a function that $\xi \in \mathscr{L}^2$ and $E\xi = 0$. Let

$$Y_n^N = \frac{1}{A_N} \sum_{j \in B_n^N} U_j \tag{6.8}$$

where B_n^N is the same as in Theorem 1′, and A_N is an appropriate norming factor.

We have the following

Theorem 3. *Given a subordinated field U_n in the form* (6.6), *let k be the largest integer such that $\alpha_j = 0$ in $(L_G^2)^j$ for every $j < k$. Let $\alpha_k(x_1, \ldots, x_k)$ be a bounded function, continuous in the origin and such that*

$$\alpha_k(0, \ldots, 0) \neq 0. \tag{6.9}$$

Let us assume that (1.18) *holds with $0 < \alpha < \frac{\nu}{k}$. Moreover let the relation*

$$\sum_{j=k+1}^{\infty} \frac{1}{j!} \frac{N^{-(j-k)\alpha}}{L(N)^{j-k}} \int_{R^{j\nu}} \left| \alpha_j \left(\frac{x_1}{N}, \ldots, \frac{x_j}{N} \right) \right|^2 \cdot |K_N(x_1, \ldots, x_j)|^2\, G_N(dx_1) \ldots G_N(dx_j) \to 0 \tag{6.10}$$

be satisfied. Then, with the choice $A_N = N^{\frac{\nu - k\alpha}{2}} L(N)^{\frac{k}{2}}$ the finite dimensional distributions of the field Y_n^N tend to those of the field $\frac{1}{k!} \alpha_k(0, 0, \ldots, 0)\, Y_n^*$, $n \in Z^\nu$, where Y_n^* is defined in (1.25).

Proof. The proof is very similar to that of Theorem 1′. We can write Y_n^N in the form

$$
\begin{aligned}
Y_n^N &= \sum_{j=k}^{\infty} \frac{1}{j!} N^{-\frac{(j-k)\alpha}{2}} L(N)^{\frac{k-j}{2}} S_j^{(N)} \\
&= \sum_{j=k}^{\infty} \frac{1}{j!} \frac{N^{-\frac{j-k}{2}\alpha}}{L(N)^{\frac{j-k}{2}}} \int \exp\left[i(n, x_1+\ldots+x_j)\right] \\
&\quad \cdot \alpha_j\left(\frac{x_1}{N}, \ldots, \frac{x_j}{N}\right) K_N(x_1, \ldots, x_j)\, Z_{G_N}(dx_1) \ldots Z_{G_N}(dx_j).
\end{aligned} \tag{6.11}
$$

(More precisely, the field defined in (6.11) has the same distribution as Y_n^N.)

The term, corresponding to $j=k$ in the sum (6.11) tends to $\frac{1}{k!}\alpha_k(0, \ldots, 0)\, Y_n^*$.

This can be proved just the same way as Theorem 1′. The only difference is that we have to replace the function $\hat{K}_N(x_1, \ldots, x_k)$ defined in (2.11) by

$$\frac{1}{k!}\alpha_k\left(\frac{x_1}{N}, \ldots, \frac{x_k}{N}\right) \hat{K}_N(x_1, \ldots, x_k) \text{ in the proof.}$$

The sum of the other terms tends to 0 in the mean square sense because of condition (6.10).

As to the main problems in connection with Theorem 3, we have to write the subordinated field U_n in the form (6.6) and check the conditions (6.9) and (6.10). We make some remarks regarding them.

Remark 6.1. If ξ is of the form $\xi=(X_{n_1})^{j_1} \ldots (X_{n_s})^{j_s}$ then ξ can be written as

$$\xi = \left(\int e^{i(n_1, x)} Z_G(dx)\right)^{j_1} \ldots \left(\int e^{i(n_s, x)} Z_G(dx)\right)^{j_s}. \tag{6.12}$$

Applying the diagram formula for product of Wiener-Itô integrals (see e.g. [1] Proposition 4.1) this product can be written as the sum of multiple Wiener-Itô integrals. This transformation enables us to write the field U_n in the desired form (6.6). If ξ is the sum of some terms given in the form (6.11), then the above mentioned method can be applied for each term.

If ξ is given in the form

$$\xi = \sum_{\substack{i_1, \ldots, i_s \\ j_1, \ldots, j_s \\ s=1,2,\ldots}} c_{i_1, \ldots, i_s}^{j_1, \ldots, j_s} H_{j_1}(Y_{i_1}) \ldots H_{j_s}(Y_{i_s}), \tag{6.13}$$

where $Y_j = \int h_j(x)\, Z_G(dx)$ and $h_1, h_2, \ldots \in L_G^2$ are orthonormal elements in L_G^2, a well-known formula (see e.g. formulas 4.14 and 4.15 in [1]) can be applied. This result yields that

$$
\begin{aligned}
&H_{j_1}(Y_{i_1}) \ldots H_{j_s}(Y_{i_s}) \\
&= \int \frac{1}{n!}\Big(\sum_{(i_1, \ldots, i_n) \in \pi_n} g_{i_1}(x_1) \ldots g_{i_n}(x_n)\Big) Z_G(dx_1) \ldots Z_G(dx_n)
\end{aligned}
$$

where $g_i = h_r$ for $j_1+\ldots+j_{r-1} < i \leqq j_1+\ldots+j_r$, $n=j_1+\ldots+j_r$, and π_n denotes the set of all permutations of the numbers $1, 2, \ldots, n$.

Generally it is not easy to write ξ in the form (6.13). If ξ is given in the form (6.7) the following algorithm makes it possible to arrive at it. By orthogonalization we can find some linear combinations

$$Y_j = \sum_{k=1}^{s} c_{j,k} X_{p_k}, \qquad j = 1, 2, \ldots, s',$$

with $s \leqq s'$, of the vectors $X_{p_1}, \ldots, X_{p_s}$ and a function $\tilde{H}(u_1, \ldots, u_{s'})$ in such a way that the Y_j-s are orthogonal, $EY_j^2 = 1$ and

$$H(X_{p_1}, \ldots, X_{p_s}) = \tilde{H}(Y_1, \ldots, Y_{s'}).$$

Defining $h_j(x) = \sum_{k=1}^{s} c_{j,k} \exp[i(p_k, x)]$, $j = 1, 2, \ldots, s'$ we get an orthonormal system $h_1, \ldots, h_{s'}$ in L_G^2, and $Y_j = \int h_j(x) Z_G(dx)$, $j = 1, 2, \ldots, s'$. Expanding the function $\tilde{H}(u_1, \ldots, u_{s'})$ by the product of Hermite polynomials $H_{j_1}(u_1) \ldots H_{j_{s'}}(u_{s'})$ we obtain the desired expansion. In this case all functions $\alpha_j(x_1, \ldots, x_j)$ appearing in (6.6) are bounded and continuous.

Remark 6.2. We show that if ξ is given by formula (6.7) then the relation (6.10) is always satisfied. In the proof we apply the second algorithm of Remark (6.1) preserving the notations. By means of this algorithm we can write

$$\xi = \sum_{l=k}^{\infty} V_l$$

where

$$V_l = \frac{1}{l!} \int \beta_l(x_1, \ldots, x_l) Z_G(dx_1) \ldots Z_G(dx_l),$$

and the function $\beta_l \in H_G^l$ is of the form

$$\beta_l(x_1, \ldots, x_l) = \sum_{j_1, \ldots, j_l = 1}^{s'} d_{j_1, \ldots, j_l} h_{j_1}(x_1) \ldots h_{j_l}(x_l)$$

with appropriate constants $d_{j_1, \ldots, j_l}$.

It is not difficult to see that

$$\sum_{j_1, \ldots, j_l = 1}^{s'} d_{j_1, \ldots, j_l}^2 = EV_l^2$$

Let us also remark that

$$\sum_{l=k}^{\infty} EV_l^2 = E\xi^2 < \infty \tag{6.14}$$

On the other hand $E(T^n V_{l_1})(T^m V_{l_2}) = 0$ if $l_1 \neq l_2$ and condition (6.10) is equivalent to the relation

$$\sum_{l=k+1}^{\infty} \frac{1}{N^{2\nu-k\alpha} L(N)^k} E W_{N,l}^2 \to 0 \tag{6.15}$$

if $N \to \infty$, where

$$W_{N,l} = \sum_{j \in B_0^N} T^j V_l.$$

In order to see these relations one has to observe that the variable $S_j^{(N)}$ in (6.11) agrees with $W_{N,j}$ in the case discussed now.

We introduce the notations

$$T^n h_j(x) = \sum_{k=1}^{s} c_{j,k} \exp[i(p_k + n, x)], \qquad n \in Z^\nu, j = 1, 2, \ldots, s'.$$

Condition (1.18) implies that

$$|\int T^p h_{j_1}(x) \overline{T^{p+m} h_{j_2}(x)} G(dx)$$
$$= E(T^p Y_{j_1}) \overline{(T^{p+m} Y_{j_2})} \leqq \frac{K L(|m|)}{|m|^\alpha}$$

for every $j_1, j_2 = 1, 2, \ldots, s'$ and $p, m \in Z^\nu$, where K is an appropriate constant. This formula and the definition of V_l together imply that

$$|E(T^{m_1} V_l)(T^{m_2} V_l)| \leqq \frac{K^l L(|m_1 - m_2|)^l}{|m_1 - m_2|^{\alpha l}} \left(\sum_{j_1, \ldots, j_l = 1}^{s'} |d_{j_1, \ldots, j_l}| \right)^2. \tag{6.16}$$

It is not difficult to prove by the help of (6.16) that

$$\frac{E W_{N,l}^2}{N^{2\nu-k\alpha} L(N)^k} \to 0 \tag{6.17}$$

for every $l > k$, as $N \to \infty$.

In order to prove (6.15) we have to make a better estimate on $E W_{N,l}^2$ for large l. To this end let us fix a sufficiently large positive integer C, to be chosen later, and let us write

$$E W_{N,l}^2 = \sum_{\substack{|m_1 - m_2| < C \\ m_1, m_2 \in B_0^N}} E(T^{m_1} V_l)(T^{m_2} V_l) + \sum_{\substack{|m_1 - m_2| \geqq C \\ m_1, m_2 \in B_0^N}} E(T^{m_1} V_l)(T^{m_2} V_l).$$

The absolute value of the first sum in the last formula is less than $(2C)^\nu N^\nu E V_l^2$. For $|m_1 - m_2| > C$ we apply the estimate (6.16) together with the observation that

$$\left(\sum_{j_1, \ldots, j_l = 1}^{s'} |d_{j_1, \ldots, j_l}| \right)^2 \leqq (s')^l \sum_{j_1, \ldots, j_l = 1}^{s'} d_{j_1, \ldots, j_l}^2 = (s')^l E V_l^2.$$

The latter inequality is a consequence of the inequality between the arithmetic and the quadratic mean. These estimates give that

$$E W_{N,l}^2 \leqq (2C)^\nu N^\nu E V_l^2 + (Ks')^l E V_l^2 \sum_{\substack{|m_1-m_2| \geqq C \\ m_1, m_2 \in B_0^N}} \frac{L(|m_1-m_2|)^l}{|m_1-m_2|^{\alpha l}}$$
$$\leqq N^\nu E V_l^2 \left[(2C)^\nu + (Ks')^l \sum_{j=C}^{N} 2\nu j^{\nu-1-\alpha l} L(j)^l\right]. \tag{6.18}$$

If C is sufficiently large, independently of l, we can write

$$\sum_{j=C}^{N} 2\nu j^{\nu-1-\alpha l} L(j)^l \leqq \sum_{j=C}^{N} j^{\nu-1-\frac{\alpha l}{2}}. \tag{6.19}$$

Let us consider the case $l \geqq L = 4\nu/\alpha$. Choosing a $C > (Ks')^{4/\alpha}$ we get that

$$\sum_{j=C}^{N} j^{\nu-1-\frac{\alpha l}{2}} \leqq C^{-\frac{\alpha l}{4}} \sum_{j=C}^{N} j^{\nu-1-\frac{\alpha l}{4}} \leqq \frac{K_1}{(Ks')^l} \tag{6.20}$$

where K_1 does not depend on l.

Substituting (6.19) and (6.20) in (6.18) we obtain that the inequality

$$E W_{N,l}^2 \leqq K_2 N^\nu E V_l^2$$

holds for $l \geqq L$ with an appropriate K_2. Adding up the last inequality for every $l \geqq L$ we get that

$$\sum_{l=L}^{\infty} E W_{N,l}^2 = O(N^\nu).$$

This relation together with (6.17) implies (6.15).

Remark 6.3. The value k in Theorem 3 can be found as the largest integer such that $E(X_n)^j \xi = 0$ for all $j < k$ and $n \in Z^\nu$. (See e.g. [3] Sect. 5).

Remark 6.4. The condition $\alpha_k(0, \ldots, 0) \neq 0$ in Theorem 3 is essential. If ξ is given in the form (6.13) (this can always be achieved if ξ is given by formula (6.7)) then condition (6.9) is equivalent to the relation

$$\sum_{\substack{j_1+\ldots+j_s=k \\ s=1,2,\ldots}} (j_1!)(j_2!)\ldots(j_s!)\, c_{i_1,\ldots,i_s}^{j_1,\ldots,j_s} \neq 0.$$

If $\alpha_k(0, \ldots, 0) = 0$ Theorem 3 yields no more than the convergence of Y_n^N to 0. Thus one would like to try to get a limit theorem with a different norming factor. In this case it turns out that the k-th term is not the only one which has a role in the limiting behaviour of Y_n^N. The description of such cases is an interesting open problem.

References

1. Dobrushin, R.L.: Gaussian and their subordinated self-similar random fields. Ann. Probability **7**. No. 1, 1–28 (1979)
2. Dobrushin, R.L.: Automodel generalized random fields and their renorm group. In volume: Multi-component stochastic systems (in Russian) p. 179–213. Moscow: Nauka, 1978. English edition: New York: Marcel Dekker Ed. [in preparation]
3. Dobrushin, R.L., Minlos, R.A.: Polynomials of random functions. (in Russian) Achievements, Uspeschi, Math. Sci. XXXII. No 2 **194**, 67–122 (1977)
4. Dobrushin, R.L., Takahashi, J.: Self-similar Gaussian fields. [To appear]
5. Ibragimov, I.A., Linnik, J.V.: Independent and stationary sequences of random variables. Groningen: Walters-Noordhoff 1971
6. Rosenblatt, M.,: Independence and dependence. Proc. 4th Sympos. Math. Statist. Probability pp. 411–443. Univ. California: Berkeley University Press 1961
7. Simon, B.: The $P(\phi)_2$ Euclidean (Quantum) field theory. Princeton: Princeton University Press 1974
8. Taqqu, M.S.: Weak convergence to Fractional Brownian Motion and the Rosenblatt Process. Z. Wahrscheinlichkeitstheorie verw. Gebiete **31**, 287–302 (1975)
9. Zygmund, A.: Trigonometric series. Cambridge: Cambridge University Press 1959

Received July 25, 1978