Z. Wahrscheinlichkeitstheorie verw. Geb. 27, 79–86 (1973)

A Limit Theorem for the Robbins-Monro Approximation

P. Major and P. Révész

1. Introduction

Let $M(x)$ $(-\infty < x < +\infty)$ be an unknown monotonically increasing function with

$$M(\theta)=0 \quad \text{and} \quad M(x)\neq 0 \qquad \text{if } x\neq\theta.$$

Suppose that we can measure the value of $M(x)$ only with some random error Y_x, i.e. the value $M(x)+Y_x$ for any x can be obtained by an experiment. Our aim is to find the root θ.

Robbins and Monro ([1]) constructed the following sequence: let X_1 be an arbitrary real number and define the sequence $\{X_n\}$ by the recursion

$$X_{n+1}=X_n-\frac{1}{n} Z_n \qquad (n=1,2,\ldots) \tag{1}$$

where $Z_n=M(X_n)+Y_{X_n}$.

Blum ([2]) under some simple conditions proved that $P(X_n\to\theta)=1$.

Chung ([3]) investigated the behaviour of the sequence $\{(X_n-\theta)\}$. Under some further conditions he proved that if

$$M'(\theta)=\alpha_1=\alpha>\tfrac{1}{2} \quad \text{and} \quad D^2(Y_x)=\sigma^2(x)\to\sigma^2$$

$(x\to\theta)$ then

$$P\{\sqrt{n}(X_n-\theta)<t\}\to\frac{1}{\sqrt{2\pi}\,s}\int_{-\infty}^{t} e^{-\frac{u^2}{2s^2}}\,du=\mathcal{N}(0,s)$$

where $s^2=\sigma^2/(2\alpha-1)$. Some further results in this direction are given in [4, 5, 6] (among others).

In this paper we intend to investigate the sequence $\{(X_n-\theta)\}$ in the case $0<M'(\theta)\leq\frac{1}{2}$.

2. Results

From now on the following conditions of Blum ([2]) will be assumed:

Condition 1.

$$P(Y_{X_n}<t\mid Y_{X_1}, Y_{X_2},\ldots,Y_{X_{n-1}})=P(Y_{X_n}<t\mid X_1, Y_{X_1},\ldots,Y_{X_{n-1}},X_n)$$
$$=P(Y_{X_n}<t\mid X_n)=H(t\mid X_n).$$

Condition 2.

$$E(Y_{X_n}\mid X_n)=\int t\,dH(t\mid X_n)=0.$$

Condition 3. There exist positive constants c and d such that $|M(x)| \leqq c + d|x|$.

Condition 4.

$$E(Y_{X_n}^2 | X_n) = \int t^2 \, dH(t | X_n) \leqq K^2 < +\infty .$$

Condition 5. $M(x) < 0$ if $x < 0$ and $M(x) > 0$ if $x > 0$ (i.e. we assume: $\theta = 0$).

Condition 6. $\inf\limits_{\delta_1 \leqq |x| \leqq \delta_2} |M(x)| > 0$ for every pair of positive numbers δ_1, δ_2.

Conditions 5 and 6 are able to replace the condition of monotonicity, so the condition of monotonicity will not be assumed. Further we assume:

Condition 7. $M(x)$ is twice differentiable at 0.

Our first theorem corresponds to the case $\alpha = \frac{1}{2}$.

Theorem 1. *Suppose that* $M'(0) = \alpha = \frac{1}{2}$,

$$D^2(Y_x) = \sigma^2(x) \to \sigma^2 \qquad (x \to 0) \tag{2}$$

and

$$\lim_{A \to \infty} \lim_{\varepsilon \searrow 0} \sup_{|x| \leqq \varepsilon} \int_{|Y_x| \geqq A} Y_x^2 \, dP = 0. \tag{3}$$

Then

$$P\left\{\sqrt{\frac{n}{\log n}} X_n < t\right\} \to \mathcal{N}(0, \sigma^2).$$

Now we turn to the case $0 < M'(0) = \alpha < \frac{1}{2}$. Our second theorem is a strong law for this case.

Theorem 2. *Suppose that* $0 < M'(0) = \alpha = \alpha_1 < \frac{1}{2}$ *and* $M''(0) = \alpha_2$. *Then there exists a random variable* $Z = Z_1$ *such that*

$$P(n^\alpha X_n \to Z) = 1.$$

It is natural to ask: how can we characterize the behaviour of the sequence $n^\alpha X_n - Z$? An answer to this question is given in

Theorem 3. *Suppose that* $\frac{1}{4} < M'(0) = \alpha < \frac{1}{2}$ *and* (2) *and* (3) *hold. Then*

$$P\left(n^{\frac{1}{2}-\alpha}(n^\alpha X_n - Z) < t\right) = P\left(\sqrt{n} X_n - n^{\frac{1}{2}-\alpha} Z < t\right) \to \mathcal{N}\left(0, \frac{\sigma^2}{1-2\alpha}\right)$$

where Z was defined in Theorem 2.

Let us mention that the statement that the limit distribution of $\sqrt{n} X_n - n^{\frac{1}{2}-\alpha} Z$ is normal is clearly correct in the case $\alpha > \frac{1}{2}$.

Now we turn to the investigation of $n^\alpha X_n - Z$ in the case $\alpha = \frac{1}{4}$. One can prove

Theorem 4. *Suppose that* $M'(0) = \alpha_1 = \alpha = \frac{1}{4}$, $M''(0) = \alpha_2$ *and* (2) *and* (3) *hold. Then*

$$P\left(n^\alpha(n^\alpha X_n - Z) - \frac{\alpha_2}{2\alpha} Z^2 < t\right) \to \mathcal{N}(0, 2\sigma^2).$$

The case $\alpha < \frac{1}{4}$ is characterized in

Theorem 5. *Suppose that* $0 < M'(0) = \alpha < \frac{1}{4}$ and $M''(0) = \alpha_2$. *Then*

$$n^\alpha(n^\alpha X_n - Z) \to \frac{\alpha_2}{2\alpha} Z^2 = Z_2$$

with probability 1.

The sequence $n^\alpha(n^\alpha X_n - Z) - Z_2$ can be characterized by

Theorem 6. *Suppose that* $\frac{1}{6} < M'(0) = \alpha < \frac{1}{4}$ *further* (2) *and* (3) *hold. Then*

$$P\left(n^{\frac{1}{2}-2\alpha}[n^\alpha(n^\alpha X_n - Z) - Z_2] < t\right) = P\left(\sqrt{n}\,X_n - n^{\frac{1}{2}-\alpha}Z - n^{\frac{1}{2}-2\alpha}Z_2 < t\right) \to \mathcal{N}\left(0, \frac{\sigma^2}{1-2\alpha}\right).$$

In fact, in Theorem 6 the condition $\frac{1}{6} < \alpha < \frac{1}{4}$ can be replaced by the condition $\frac{1}{6} < \alpha \leq \frac{1}{4}$ and in this form Theorem 4 becomes a special case of Theorem 6. Theorem 4 has been formulated because its proof is slightly different.

Continuing this process one can get our

Theorem $3k+1$. *Suppose that*

$$M'(0) = \alpha = \alpha_1 = \frac{1}{2(k+1)}, \qquad M''(0) = \alpha_2, \ldots, M^{(k+1)}(0) = \alpha_{k+1}$$

and (2) *and* (3) *hold. Then*

$$P\left(n^{(k+1)\alpha}X_n - n^{k\alpha}Z - n^{(k-1)\alpha}\frac{\alpha_2}{2\alpha}Z^2 - n^{(k-2)\alpha}c_3 Z^3 - \cdots - c_{k+1}Z^{k+1} < t\right)$$
$$\to \mathcal{N}\left(0, \frac{\sigma^2}{1-2\alpha}\right) = \mathcal{N}\left(0, \frac{k+1}{k}\sigma^2\right)$$

where $c_3, c_4, \ldots, c_{k+1}$ *are constants depending on* $\alpha_1, a_2, \ldots, a_{k+1}$.

Theorem $3k+2$. *Suppose that*

$$0 < M'(0) = \alpha < \frac{1}{2(k+1)} \quad \textit{and} \quad M''(0) = \alpha_2, \ldots, M^{(k+1)}(0) = \alpha_{k+1}.$$

Then

$$n^{(k+1)\alpha}X_n - n^{k\alpha}Z - n^{(k-1)\alpha}\frac{\alpha_2}{2\alpha}Z^2 - \cdots - c_k Z^k \to c_{k+1}Z_{k+1}$$

with probability 1 *where* $c_3, c_4, \ldots, c_{k+1}$ *are constants depending on* $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$.

Theorem $3k+3$. *Suppose that*

$$\frac{1}{2(k+2)} < M'(0) = \alpha = \alpha_1 < \frac{1}{2(k+1)} \quad \textit{and} \quad M''(0) = \alpha_2, \ldots, M^{(k+1)}(0) = \alpha_{k+1}$$

and (2) *and* (3) *hold. Then*

$$P\left(n^{\frac{1}{2}-(k+1)\alpha}(n^{(k+1)\alpha}X_n - n^{k\alpha}Z - \cdots - c_{k+1}Z^{k+1}) < t\right) \to \mathcal{N}(0, s^2),$$
$$s^2 = \frac{\sigma^2}{1-2\alpha}$$

where $c_3, c_4, \ldots, c_{k+1}$ *are constants depending on* $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$.

The authors are indebted to Prof. L. Schmetterer for his valuable remarks, especially the present forms of the proofs of Lemmas 3 and 4 are due to him.

3. Lemmas

Our first two lemmas are known:

Lemma 1 ([8]). *Let $\{b_n^2\}$ be a sequence of real numbers for which*

$$b_{n+1}^2 \leqq \left(1-\frac{\rho}{n}\right) b_n^2 + \frac{A}{n^2} \qquad (n=1,2,\dots)$$

where $0<\rho<1$ and $A>0$. Then

$$b_n^2 \leqq \frac{B}{n^\rho}$$

where B is constant depending on A, ρ and b_1.

Lemma 2 ([4] p. 377). *Let U_{nk} $(k, n=1,2,\dots)$ be a double array such that*

$$E(U_{nk} \mid U_{n1}, U_{n2}, \dots, U_{n,k-1})=0$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} E\,|E(U_{nk}^2 \mid U_{n1}, U_{n2}, \dots, U_{n,k-1}) - E(U_{nk}^2)|=0$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} E(U_{nk}^2)=s^2$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} E(U_{nk}^2 \cdot \chi_{\{|U_{nk}|>\varepsilon\}})=0$$

where χ_A is the indicator function of A. Then $S_n=\sum_{k=1}^{\infty} U_{nk}$ is asymptotically normal with mean 0 *and variance s^2.*

Lemma 3. *Let $X_n=X_n(\omega)$ $(\omega\in\Omega)$ be the Robbins-Monro process (obeying Conditions* 1–6*). Then for any $\varepsilon>0$, $\delta>0$ there exists a measurable set $F\subset\Omega$ such that*

$$P(F)>1-\delta$$

and

$$\int_F X_n^2\, dP \leqq \frac{1}{n^{2(\alpha-\varepsilon)}}$$

if n is big enough, where $M'(0)=\alpha\leqq\frac{1}{2}$.

Proof. By Blum's theorem ([2]) our conditions imply that $X_n\to 0$ with probability 1. Choose an $\varepsilon'>0$ and an $\eta>0$ such that $M(x)=\alpha x+\varepsilon(x)x$ where $|\varepsilon(x)|<\varepsilon'$ whenever $|x|<\eta$. Define $F=\bigcap_{n\geqq n_0}\{\omega\colon |X_n(\omega)|<\eta\}$. If n_0 is large enough then $P(F)\geqq 1-\delta$.

Let

$$F_{n_0}=\{\omega\colon |X_{n_0}|<\eta\} \quad \text{and} \quad F_{n_0+k}=\bigcap_{n=n_0}^{n_0+k}\{\omega\colon |X_n|<\eta\}.$$

It follows that

$$\int_{F_n} X_{n+1}^2 \leqq \int_{F_n} X_n^2 - \frac{2}{n}(\alpha-\varepsilon')\int_{F_n} X_n^2 + \frac{2}{n^2}\int_{F_n} X_n^2 + \frac{K^2}{n^2}$$

for $n \geqq n_0$. Taking into account that $\int_{F_{n+1}} X_{n+1}^2 \leqq \int_{F_n} X_n^2$ we get

$$b_{n+1} \leqq b_n \left(1 - \frac{2}{n}(\alpha - \varepsilon)\right) + \frac{K^2}{n^2}$$

where

$$b_n = \int_{F_n} X_n^2.$$

Since $\varepsilon' + 2/n^2 \leqq \varepsilon$ if n is large enough, it follows that $b_n = O\left(\frac{1}{n^{2(\alpha-\varepsilon)}}\right)$ (if n is large enough) and therefore $\int_F X_n^2 = O\left(\frac{1}{n^{2(\alpha-\varepsilon)}}\right)$.

Lemma 4. *Let $\eta_1, \eta_2, \ldots$ be a sequence of random variables for which*

$$E(\eta_{n+1} | \eta_n, \ldots, \eta_1) = 0 \qquad (n = 1, 2, \ldots), \tag{4}$$

$$E(\eta_{n+1}^2 | \eta_n, \ldots, \eta_1) \leqq K^2 \qquad (n = 1, 2, \ldots). \tag{5}$$

Then

$$T_n = n^{k\alpha} \sum_{j=n}^{\infty} \frac{1}{j^{1-\alpha}} \left(1 + O\left(\frac{1}{j}\right)\right) \eta_j \qquad \left(0 < \alpha < \frac{1}{2(k+1)}\right)$$

converges to 0 *with probability* 1 ($n \to \infty$).

Proof. Let ε be a positive number for which $2(k+1)\alpha < 1 - 2\varepsilon$ and put

$$S_l = \sum_{j=1}^{l} \frac{\eta_j}{j^{\frac{1}{2}+\varepsilon}}.$$

Then by Kolmogorov's inequality

$$P\{\max_{1 \leqq l \leqq n} |S_l| \geqq L\} \leqq \frac{1}{L^2} \sum_{l=1}^{n} \frac{K^2}{l^{1+2\varepsilon}} \leqq \frac{1}{L^2} \sum_{l=1}^{\infty} \frac{K^2}{l^{1+2\varepsilon}}$$

that is

$$P\{\omega \colon |S_l(\omega)| \text{ is uniformly bounded}\} = P(G) \geqq 1 - \delta$$

for any $\delta > 0$ if m is large enough. It follows from Abel's theorem that

$$T_n(\omega) = O\left(n^{k\alpha} \frac{1}{n^{\frac{1}{2}-\alpha-\varepsilon}}\right) = o(1)$$

whenever $\omega \in G$.

4. Proofs

The following simple formulas will be frequently used in all proofs. By the iteration of (1) one can obtain

$$X_{n+m+1} = \prod_{k=0}^{m} \left(1 - \frac{\alpha}{n+k}\right) X_n - \sum_{i=0}^{m} \frac{1}{n+i} \prod_{j=i+1}^{m} \left(1 - \frac{\alpha}{n+j}\right) (U(X_j) + Y_j)$$

and applying the relations

$$1-\frac{\alpha}{i}=e^{-\frac{\alpha}{i}+O\left(\frac{1}{i^2}\right)},\qquad \sum_{i=n}^{m}\frac{1}{i}=\log\frac{m}{n}+O\left(\frac{1}{n}\right)$$

$$\sum_{i=n}^{\infty}\frac{1}{i^2}=O\left(\frac{1}{n}\right)$$

we get

$$(n+m)^{\alpha}X_{n+m+1}=n^{\alpha}\left(1+O\left(\frac{1}{n}\right)\right)X_n-\sum_{j=n}^{n+m}\frac{1}{j^{1-\alpha}}\left(1+O\left(\frac{1}{j}\right)\right)(U(X_j)+Y_j). \tag{6}$$

Introduce the notations

$$A_{\alpha}(u,v)=\sum_{j=u}^{v}\frac{1}{j^{1-\alpha}}\left(1+O\left(\frac{1}{j}\right)\right)U(X_j),$$

$$B_{\alpha}(u,v)=\sum_{j=u}^{v}\frac{1}{j^{1-\alpha}}\left(1+O\left(\frac{1}{j}\right)\right)Y_j.$$

Now we can turn to the

Proof of Theorem 1. Theorem 1 clearly follows from (6) and the following two statements:

(i) for any $\varepsilon>0$ there exists a measurable set $F\subset\Omega$ such that $P(F)>1-\varepsilon$ and $A_{\frac{1}{2}}(1,n)/\sqrt{\log n}\to 0$ in probability as $n\to\infty$.

(ii) $B_{\frac{1}{2}}(1,n)/\sqrt{\log n}$ tends to $\mathcal{N}(0,\sigma^2)$ in law as $n\to\infty$.

(i) follows from Lemma 3.

(ii) can be obtained as a consequence of Lemma 2, making use of Blum's theorem. The details will not be given because we could repeat the method used in Sacks' paper ([4]).

Proof of Theorem 2. It is clearly enough to prove that for any $\varepsilon>0$ there exists a measurable set $F\subset\Omega$ such that $P(F)>1-\varepsilon$ and the series $A_{\alpha}(1,\infty)$, $B_{\alpha}(1,\infty)$ are convergent on F.

Since

$$E(Y_j\,|\,Y_{j-1},\dots,Y_1)=0,\qquad E(Y_j^2\,|\,Y_{j-1},\dots,Y_1)\leqq K^2\quad (j=1,2,\dots)$$

with probability 1. The almost everywhere convergence of $B_{\alpha}(1,\infty)$ on Ω follows from Kolmogorov's three series theorem (see e.g. [7] 387).

Now choose the set F as it was chosen in Lemma 4. Since

$$|U(X_j)|\leqq DX_j^2\quad\text{and}\quad \sum_{j=1}^{\infty}\frac{1}{j^{1-\alpha}}\left(1+O\left(\frac{1}{j}\right)\right)X_j^2$$

is convergent (on F, by the Beppo-Levi theorem), the series $A_{\alpha}(1,\infty)$ is really convergent.

Proof of Theorem 3. Letting m tend to infinity in (6) we get

$$n^{\alpha}X_n-Z=A_{\alpha}(n,\infty)+B_{\alpha}(n,\infty).$$

In order to prove our Theorem it is enough to show that

$$n^{\frac{1}{2}-\alpha} A_\alpha(n, \infty)$$

tends to 0 in probability $(n \to \infty)$ and

$$n^{\frac{1}{2}-\alpha} B_\alpha(n, \infty)$$

tends to $\mathcal{N}(0, \sigma^2/(1-2\alpha))$ in law as $n \to \infty$.

Our first statement follows from Theorem 2 if $\frac{1}{4}<\alpha<\frac{1}{2}$ and the second one from Lemma 3 (making use of Sacks' ideas) if $\alpha<\frac{1}{2}$.

Proof of Theorem 4. As a first step we prove that:

$$n^\alpha A_\alpha(n, \infty) \to \frac{\alpha_2}{2\alpha} Z^2 \qquad (n \to \infty, \alpha \leqq \tfrac{1}{4})$$

with probability 1. To see this, set

$$U(X_j)=U_1(X_j)=\frac{\alpha_2}{2} X_j^2+U_2(X_j)$$

where $U_2(X_j)=o(X_j^2)$.

Then

$$\begin{aligned} n^\alpha A_\alpha(n, \infty) &= n^\alpha \sum_{j=n}^{\infty} \frac{1}{j^{1-\alpha}}\left(1+O\left(\frac{1}{j}\right)\right)\left(\frac{\alpha_2}{2} X_j^2+U_2(X_j)\right) \\ &= n^\alpha \sum_{j=n}^{\infty} \frac{\alpha_2}{2 j^{1-\alpha}} \frac{Z^2}{j^{2\alpha}}+o(1) \to \frac{\alpha_2}{2\alpha_1} Z^2 . \end{aligned} \tag{7}$$

Theorem 4 follows from this fact and from the fact that

$$\sqrt[4]{n}\, B_{\frac{1}{4}}(n, \infty)$$

tends to $\mathcal{N}(0, 2\sigma^2)$ in law as $n \to \infty$. This fact was already stated in the Proof of Theorem 3.

Proof of Theorem 5. In the Proof of Theorem 4 we have already seen that

$$n^\alpha A_\alpha(n, \infty) \to \frac{\alpha_2}{2\alpha} Z^2 .$$

In order to prove Theorem 5 it is enough to show that

$$n^\alpha B_\alpha(n, \infty) \to 0 \qquad (\text{with probability } 1;\, n \to \infty).$$

This statement is a straight consequence of Lemma 4.

Proof of Theorem 6. As a first step we prove that

$$n^\alpha A_\alpha(n, \infty)=\frac{\alpha_2}{2\alpha_1} Z^2+\frac{O(1)}{n^\alpha} . \tag{8}$$

Since by Theorem 5

$$X_n = \frac{Z}{n^\alpha} + \frac{\alpha_2}{2\alpha} \frac{Z^2}{n^{2\alpha}} + \frac{O(1)}{n^{2\alpha}}$$

and

$$X_n^2 = \frac{Z^2}{n^{2\alpha}} + \frac{O(1)}{n^{3\alpha}}$$

making use of (7) we get (8).

Now one can get the Theorem as follows:

$$n^\alpha (n^\alpha X_n - Z) = \frac{\alpha_2}{2\alpha} Z^2 + \frac{O(1)}{n^\alpha} + n^\alpha B_\alpha(n, \infty)$$

and

$$n^{\frac{1}{2}-2\alpha} \left[n^\alpha (n^\alpha X_n - Z) - \frac{\alpha_2}{2\alpha} Z^2 \right] = \frac{n^{\frac{1}{2}-2\alpha}}{n^\alpha} O(1) + n^{\frac{1}{2}-\alpha} B_\alpha(n, \infty)$$

where the first member of the right hand side tends to 0 with probability 1 and the second one is asymptotically normal (see the Proof of Theorem 3).

Theorems $3k+1$, $3k+2$, and $3k+3$ can be proved by induction, using the Taylor expansion of $U(X)$ up to $(k+1)$ terms.

References

1. Robbins, H., Monro, S.: A stochastic approximation method. Ann. Math. Statist. **22**, 400–407 (1951)
2. Blum, J.R.: Approximation methods which converge with probability 1. Ann. Math. Statist. **25**, 382–386 (1954)
3. Chung, K.L.: On a stochastic approximation method. Ann. Math. Statist. **25**, 463–483 (1954)
4. Sacks, J.: Asymptotic distribution of stochastic approximation procedures. Ann. Math. Statist. **29**, 373–405 (1958)
5. Fabian, V.: On asymptotic normality in stochastic approximation. Ann. Math. Statist. **39**, 1327–1332 (1968)
6. Woodroofe, M.: Normal approximation and large deviations for the Robbins-Monro process. Z. Wahrscheinlichkeitstheorie verw. Gebiete **21**, 329–338 (1972)
7. Loéve, M.: Probability Theory. Princeton: Van Nostrand 1960
8. Venter, J.H.: On Dvoretzky Stochastic approximation theorems. Ann. Math. Statist. **37**, 1534–1544 (1966)

P. Major
P. Révész
Magyar Tudományos Akadémia Matematikai
Kutató Intézete
Réaltanoda U. 13–15
Budapest V, Hungary

(Received October 27, 1972; in revised form May 10, 1973)