# SELBERG'S TRACE-FORMULA

a generalization of the Poisson summation formula

*a lecture of Gábor Halász*

in the Bolyai College of the Eötvös Loránd University, Budapest

First talk:

## The Poisson summation formula

Let a function $k(x)$ be given, and define the function $K(x) = \sum\limits_{n=-\infty}^{\infty} k(x+n)$. (We disregard convergence problems in this heuristic discussion.) Clearly, $K(x+1) = K(x)$, and its Fourier series can be written as $K(x) = \sum\limits_{j=-\infty}^{\infty} a_j e^{2\pi ijx}$ with

$$\begin{aligned} a_j &= \int_0^1 K(x)e^{-2\pi ijx}\,dx = \int_0^1 \sum_{n=-\infty}^{\infty} k(x+n)e^{-2\pi ijx}\,dx \\ &= \sum_{n=-\infty}^{\infty} \int_0^1 k(x+n)e^{-2\pi ijx}\,dx = \int_{-\infty}^{\infty} k(x)e^{-2\pi ijx}\,dx = \kappa(2\pi j), \end{aligned}$$

where, $\kappa(t) = \int_{-\infty}^{\infty} k(x)e^{-itx}\,dx$ is the Fourier transform of the function $k(x)$.

In such a way one gets by expressing $K(0)$ in two different ways the relation

$$\sum_{n=-\infty}^{\infty} k(n) = \sum_{j=-\infty}^{\infty} \kappa(2\pi j)\ .$$

The main benefit of the last formula: The left-hand side is an approximating sum of the integral $\int_{-\infty}^{\infty} k(x)\,dx$ which equals $\kappa(0)$. The sum of the terms $j \neq 0$ expresses the error of this approximation. (In the two-dimensional version of this formula — in this case the vectors with integer coordinates replace the role of the integers —, if $k(x)$ is the indicator function of a large circle, the left-hand side equals the number of lattice points in the circle, while at the right-hand side the term corresponding to the origin equals the area of the circle, and this gives a possibility to handle the not completely solved circle problem about the order of the error term. A version of this problem will be discussed later.

The same subject in a high-brow style

We shall prove the Poisson summation formula in the following more complicated form:

$$\sum_{n=-\infty}^{\infty} k(n) = \text{sum of the eigenvalues of an integral operator},$$

where the integral operator is appropriately defined. The reason for this more complicated approach is that it is simpler to generalize this form of the result to more general spaces.

Consider the real line $\mathbb{R}^1$, together with the usual metric $|x-y|$ and the Lebesgue measure. Let $\mathbf{T}_u\colon x \to x+u$ denote the shift with $u$. These transformations form a group for which the above introduced metric and measure are invariant. Let us denote the operator $\mathbf{T}_u f(x) = f(x+u)$ acting on the space of functions $f(x)$, $-\infty < x < \infty$, by the same letter.

A general operator $L$ will be called invariant, if $L_x f(x+u) = Lf(x)|_{x\to x+u}$, i.e. if $L\mathbf{T}_u = \mathbf{T}_u L$, (The subscript $x$ in $L_x$ means that the operator $L$ maps the function as a function of the variable $x$ to a function of the variable $x$. The remaining variables of the function where $L$ is acting are fixed parameters.)

We consider the following question: When is the integral operator $Lf = \int\limits_{-\infty}^{\infty} k(x,y)f(y)dy$ invariant?

We have $L\mathbf{T}_u f = \int_{-\infty}^{\infty} k(x,y)f(y+u)\,du$ and $\mathbf{T}_u Lf = \int_{-\infty}^{\infty} k(x+y,y)f(y)\;dy = \int_{-\infty}^{\infty} k(x+u,y+u)f(y+u)\,dy$, which means that the integral operator with kernel $k(x,y)$ is invariant if $k(x,y) = k(x+u,y+u)$, i.e. if $k(x,y)$ depends only on $x-y$; $k(x,y) = k(x-y)$, $Lf = \int_{-\infty}^{\infty} k(x-y)f(y)\,dy$ is a convolution.

When is the differential operator $Lf = \sum\limits_{\nu=0}^{m} a_\nu(x)\dfrac{d^\nu f}{dx^\nu}$ invariant?

$L\mathbf{T}_u f = \sum\limits_{\nu=0}^{m} a_\nu(x) f^\nu(x+u)$, $\mathbf{T}_u Lf = \sum\limits_{\nu=0}^{m} a_\nu(x+u)f^\nu(x+u)$, i.e. it is invariant if $a_\nu(x)$ is constant, $L = \sum\limits_{\nu=0}^{m} a_\nu D^\nu$, where $D = \dfrac{d}{dx}$. In such a way we described all invariant differential operators.

The above defined invariant operators are exchangeable. For instance, if $Lf = \int_{-\infty}^{\infty} k(x-y)f(y)\,dy$, then $DLf = \left(\int_{-\infty}^{\infty} k(x-y)f(y)\,dy\right)' =$ (we do not have to differentiate immediately) $= \left(\int_{-\infty}^{\infty} k(y)f(x-y)\,dy\right)' = \int_{-\infty}^{\infty} k(x-y)f'(y)\,dy = LDf$.

In the non-trivial step we exploited the fact that $L$ is invariant: If $L^0 f \stackrel{\text{def}}{=} Lf|_{x=0}$, then because of the invariance $L_x f(x) = L^0_y \mathbf{T}_x f(y)$. (This means in our case that $\int_{\infty}^{\infty} k(x-y)f(y)\,dy = \int_{-\infty}^{\infty} k(0-y)f(y+x)\,dy$.)

**Claim.** *Any two invariant operators $L_1$ and $L_2$ are exchangeable.*

*Proof:* By the previous argument $L^0$ determines an invariant operator $L$, and since both $L_1L_2$ and $L_2L_1$ are invariant, it is enough to show that $(L_1L_2)^0 = (L_2L_1)^0$, i.e. that $L_1^0 L_2 = L_2^0 L_1$.

We know that $L_2 f = L^0_{2y} f(y+x)$, hence $L^0_{1x} L^0_{2y} f(y+x) =$ (We exploit the fact that $L^0_1$ and $L^0_2$ are acting on different arguments. In this proof we restrict our attention to differential and integral operators, where the statement of this step can be simply checked.) $= L^0_{2y} L^0_{1x} f(y+x) =$ (because $x+y = y+x$, the group of shifts is commutative) $= L^0_{2y} L^0_{1x} f(x+y) = L^0_2 L_1 f$.

The exchangeability is important, because this means that the spectral decomposition of the operators is the same. For instance, if $f(x)$ is an eigenfunction of the operator $D$ with an (arbitrary complex) eigenvalue $\lambda$, $f(x) = e^{\lambda x}$ (all functions $Ae^{\lambda x}$ are such eigenfunctions), then $DLf = LDf = L\lambda f = \lambda Lf$, that is $Lf$ is also an eigenfunction of $D$ with the same eigenvalue. Since these eigenfunctions constitute a one-dimensional space $Lf = \Lambda e^{\lambda x} = \Lambda f(x)$, where $\Lambda$ depends only on $\lambda$ and $L$.

Let us determine this $\Lambda$ for the operator $Lf = \int_{-\infty}^{\infty} k(x-y)f(y)\,dy$.

Since $f(0) = 1$, $\Lambda = L^0 f = \int_{-\infty}^{\infty} k(-y)e^{\lambda y}\,dy = \int_{-\infty}^{\infty} k(y)e^{-\lambda y}\,dy$.

A typical discrete subgroup of the group of shifts is the group $\mathbf{T}_n$, $n = 0, \pm 1, \pm 2, \ldots$. If $\mathbf{T}_n f = f$, i.e. $f(x+n) = f(x)$ for all $n$, then the function $f$ is called "automorph" (periodic) with respect to the subgroup. If $L$ is invariant, then $\mathbf{T}_n Lf = L\mathbf{T}_n f = Lf$, i.e. $L$ maps periodic functions into periodic functions. If $L$ is an integral operator, then

$$Lf = \int_{-\infty}^{\infty} k(x-y)f(y)\,dy = \sum_{n=-\infty}^{\infty} \int_0^1 k(x-(y+n))f(y)\,dy = \int_0^1 K(x,y)f(y)\,dy) \ ,$$

where $K(x,y) = \sum\limits_{n=-\infty}^{\infty} k(x-(y+n))$ is a periodic kernel function in both variables.

In the space of periodic functions the operator $iD$ is self-adjoint, because $(Df,g) = \int_0^1 f'(x)\bar{g}(x)\,dx =$ (integrating by parts) $= -\int_0^1 f(x)\bar{g}'(x)\,dx = -(f,Dg)$, i.e. $(iDf,g) = i(Df,g) = -i(f,Dg) = (f,iDg)$. Hence in this space the eigenvalues of $D$ are imaginary, (If $Df = \lambda f$, then $(Df,f) = \lambda(f,f)$, $(Df,f) = -(f,Df) = -\bar{\lambda}(f,f) \Rightarrow \lambda = -\bar{\lambda}$), and the eigenfunctions corresponding to different eigenvalues are orthogonal: $((Df,g) = \lambda(f,g)$, $(Df,g) = -(f,Dg) = -\bar{\nu}(f,g)$, and if $\lambda \neq \nu$, then $(f,g) = 0$.)

We know that the (normalized) eigenfunctions are the functions $f_j(x) = e^{\lambda_j x} = e^{2\pi i j x}$. If we know that they constitute a complete orthonormal system (this follows from general results) then we can express the function $K(x,y)$ as

$$K(x,y) = \sum_{j,l=-\infty}^{\infty} c_{j,l} f_j(x) \bar{f}_l(y) \ .$$

If $\Lambda_j$ denotes the eigenvalue of the operator $Lf = \int_{-\infty}^{\infty} k(x-y)f(y)\,dy$ corresponding to the eigenvalue $\lambda_j$ of $D$ about which we know that

$$\Lambda_j = \int_{-\infty}^{\infty} k(y)e^{-\lambda_j y}\,dy = \int_{-\infty}^{\infty} k(y)e^{-2\pi i j y}\,dy = \kappa(2\pi j) \ ,$$

and

$$\Lambda_j f_j(x) = Lf_j = \int_0^1 K(x,y)f_j(y)\,dy = \int_0^1 \sum_{i,l=-\infty}^{\infty} c_{i,l} f_i(x)\bar{f}_l(y) f_j(y)\,dy$$

$$= \sum_{i,l=-\infty}^{\infty} c_{i,l} f_i(x) \int_0^1 \bar{f}_l(y) f_j(y)\,dy = \sum_{i=-\infty}^{\infty} c_{i,j} f_i(x) \ .$$

From here

$$c_{i,j} = \begin{cases} 0, & \text{if } i \neq 0 \\ \Lambda_j, & \text{if } i = j \end{cases} .$$

Hence $K(x,y) = \sum\limits_{j=-\infty}^{\infty} \Lambda_j f_j(x) \bar{f}_j(y)$.

Thus,

$$K(0,0) = \sum_{j=-\infty}^{\infty} \Lambda_j = \sum_{j=-\infty}^{\infty} \kappa(2\pi j) .$$

On the other hand,

$$K(x,x) = \sum_{n=-\infty}^{\infty} k(x-(x+n)) = \sum_{n=-\infty}^{\infty} k(n) ,$$

or because the function $K(x,x)$ is constant, the trace of $L$ equals

$$\int_0^1 K(x,x)\,dx = K(0,0) = \sum_{j=-\infty}^{\infty} \Lambda_j = \sum_{j=-\infty}^{\infty} \kappa(2\pi j) ,$$

and

$$\int_0^1 K(x,x)\,dx = K(0,0) = \sum_{n=-\infty}^{\infty} k(n) .$$

A comparison of the last two relations yields the Poisson formula.

Our program is the following: We want to find the analog of the above formula in a general space with a given group of isometries (shifts). We want to determine the invariant differential and integral operators, to calculate the eigenfunctions of the integral operators and the eigenvalues corresponding to them. Then we restrict the operators to the functions automorph with respect to a discrete subgroup of these isometries. We want to determine the eigenfunctions and corresponding eigenvalues of the invariant differential operators in this space, to calculate the kernel function of the integral operator. In such a way we get an analog of the Poisson summation formula.

In a special case we carry out the above program. We shall discuss the hyperbolic plane, i.e. the Poincaré representation of the Bolyai–Lobachevski geometry. We write down the invariant differential and integral operators with respect to the group of motions in this space and determine, by using an argument suggested by the previous proof, the analog of the Poisson summation formula in this space. One has to overcome certain technical difficulties which are related to the fact that the group of motions in this space is non-commutative.

Second talk:

**Invariant transformations in the hyperbolic plane**

We recall the following two classical (equivalent) models of the hyperbolic plane.

*The space:*

The unit circle of the complex plane $\{z\colon |z| < 1\}$

The upper half plane of the complex plane $H = \{z\colon \operatorname{Im} z > 0\}$

*The metric (length) in the space:*

$$\frac{|dz|}{1 - |z|^2}$$

$$I(\gamma) = \int_\gamma \frac{|dz|}{1 - |z|^2}$$

$\dfrac{|dz|}{y}$ if $z = x + iy$.

$$I(\gamma) = \int_\gamma \frac{|dz|}{y}$$

$$d(z, w) = \min_{\widehat{z,w}} I(\widehat{z,w}) = \log \frac{1 + \left|\frac{z-w}{z-\bar{w}}\right|}{1 - \left|\frac{z-w}{z-\bar{w}}\right|}$$

This expression is the function of $\frac{4}{\left|\frac{z-\bar{w}}{z-w}\right|^2 - 1} = \frac{|z-w|^2}{yv}$ if $z = x + iy$ and $w = u + iv$.

*The measure (area) in the space:*

$$\frac{dxdy}{(1 - |z|^2)^2}$$

$$d\sigma \overset{\text{def}}{=} \frac{dxdy}{y^2}$$

*The length and measure preserving movements of the space:*

$$\mathbf{T}z = \rho \frac{z - \zeta}{1 - z\bar{\zeta}},$$

$|\rho| = 1$, $|\zeta| < 1$

$$\mathbf{T}z = \frac{az + b}{cz + d}$$

$a$, $b$, $c$, $d$ are real numbers, $ad - bc > 0$ $(= 1)$

(This group of transformations can be identified with the quotient group $SL_2(R)/\{\pm \mathrm{Id}\}$, where $SL_2(R)$ denotes the group of $2 \times 2$ matrices with real coefficients and determinant 1.)

The group of all isometries: The above transformations together with their compositions with a reflection through an axis.

The movements written down are the orientation preserving isometries, their composition with a reflection are the orientation changing transformations.

Given a function $f(z)$ and a movement $\mathbf{T}$ on the hyperbolic plane, define $\mathbf{T}f$ as $\mathbf{T}f(z) \stackrel{\text{def}}{=} f(\mathbf{T}z)$.

**Definition.** *An operator $L$ acting on the functions over a hyperbolic space is called invariant, if $\mathbf{T}L = L\mathbf{T}$ for all movements $\mathbf{T}$ of the space.*

*Question:* Which integral operators $Lf = \int_H k(z,w)f(w)\sigma(\,dw)$ are invariant?

$$\mathbf{T}Lf(z) = \int_H k(\mathbf{T}z, w)f(w)\sigma(\,dw) = \int_H k(\mathbf{T}z, \mathbf{T}w)f(\mathbf{T}w)\sigma(\,dw)\ ,$$

and

$$L\mathbf{T}f(z) = \int_H k(z,w)f(\mathbf{T}w)\sigma(\,dw)\ .$$

Hence $L$ is invariant if and only if $k(z,w) = k(\mathbf{T}z, \mathbf{T}w)$ for all $\mathbf{T}$. The group of the movements $\mathbf{T}$ is strongly transitive, i.e. for any pairs $(z_1, w_1)$ and $(z_2, w_2)$ such that $d(z_1, w_1) = d(z_2, w_2)$ there exists a transformation $\mathbf{T}$ such that $z_2 = \mathbf{T}z_1$ and $w_2 = \mathbf{T}w_1$. Hence the kernel function of an invariant integral operator depends only on $d(z,w)$. In the positive half-plane model of the hyperbolic plane, because of the special form of the distance in this space, the kernel function of an invariant integral operator can be given in the form

$$k(z,w) = k\left(\frac{|z-w|^2}{yv}\right)\ .$$

Which differential operators $Lf = \sum a_{i,j}(z)\dfrac{\partial^{i,j}}{\partial x^i \partial y^j}f(z)$ are invariant?

Here we follow an argument similar to that given in the Euclidean space. We fix a point $z_0$, define the functional $L^0$, $L^0 f = Lf(z_0)$ and want to reconstruct the invariant operator $L$ by means of $L^0$.

*Remark:* The functional $L^0$ cannot be arbitrary. Indeed, if the transformation $\mathbf{T}$ is such that $\mathbf{T}z_0 = z_0$, then $L^0\mathbf{T}f = L\mathbf{T}f(z_0) = \mathbf{T}Lf(z_0) = Lf(\mathbf{T}z_0) = Lf(z_0) = L^0 f$ for this $\mathbf{T}$, i.e. the functional $L^0$ is rotation invariant. On the other hand, if $L^0$ is such a functional, then it defines an invariant operator $L$ in a unique way by the formula $Lf(z) = L^0\mathbf{T}_z f = L^0_w f(\mathbf{T}_z w)$. We formulate this statement in the following lemma.

**Lemma.** *Let us fix some number $z_0$ in the hyperbolic plane. We call a functional $L^0$ on the space of functions in the hyperbolic plane rotation invariant if $L^0 f = L^0\mathbf{T}f$ for all such movements $\mathbf{T}$ of the hyperbolic plane for which $\mathbf{T}z_0 = z_0$. If $L_0$ is a rotation invariant functional, then we define the operator $L$ by the formula $Lf(z) = L^0\mathbf{T}_z f = L^0_w f(\mathbf{T}_z w)$, where $\mathbf{T}_z$ is a movement such that $\mathbf{T}_z z_0 = z$. This definition is meaningful,* *i.e. the value of* $Lf(z)$ *does not depend on the special choice of* $\mathbf{T}_z$*. This operator* $L$ *is invariant, and all invariant operators* $L$ *can be obtained in such a way through the choice of* $L_0 f = Lf(z_0)$*.*

*Proof:* The definition of $Lf$ is meaningful. Indeed, let $\mathbf{T}_z$ and $\bar{\mathbf{T}}_z$ be two movements such that $\mathbf{T}_z z_0 = z$ and $\bar{\mathbf{T}}_z z_0 = z$. We have to show that $L^0\bar{\mathbf{T}}_z f = L^0\mathbf{T}_z f$, or with the notation $\mathbf{T}_z f = g$, $L^0 g = L^0\bar{\mathbf{T}}_z\mathbf{T}_z^{-1} g$ or $L^0 g(w) = L^0_w g(\mathbf{T}_z^{-1}\bar{\mathbf{T}}_z w)$. This formula follows from the rotational invariance of $L^0$ and the relation $\mathbf{T}_z^{-1}\bar{\mathbf{T}}_z z_0 = z_0$.

Let us consider an arbitrary motion $\mathbf{T}$. Then

$$\mathbf{T}Lf(z) = h(\mathbf{T}z) = L^0 g$$

with $h(u) = Lf(u) = L^0\mathbf{T}_u f$ $(h(\mathbf{T}z) = L^0\mathbf{T}_{\mathbf{T}z} f)$ and $g(u) = \mathbf{T}_{\mathbf{T}z} f(u) = f(\mathbf{T}_{\mathbf{T}z} u)$, where $\mathbf{T}_{\mathbf{T}z}$ is a motion of the hyperbolic plane such that $\mathbf{T}_{\mathbf{T}z} z_0 = \mathbf{T}z$.

$$L\mathbf{T}f(z) = L^0\mathbf{T}_z\mathbf{T}f = L^0 p = L_0\mathbf{T}' g = L_0 g$$

with $p(u) = \mathbf{T}_z\mathbf{T}f(u) = f(\mathbf{T}\mathbf{T}_z u)$ and $\mathbf{T}' = \mathbf{T}_{\mathbf{T}z}^{-1}\mathbf{T}\mathbf{T}_z$. The last but one identity holds in this relation because $p(u) = f(\mathbf{T}_{\mathbf{T}z}\mathbf{T}_{\mathbf{T}z}^{-1}\mathbf{T}\mathbf{T}_z u) = g(\mathbf{T}'u)$, i.e. $p = \mathbf{T}'g$. Finally, the last identity holds in this relation, because of the rotational invariance of $L^0$ and the equality $\mathbf{T}'z_0 = \mathbf{T}_{\mathbf{T}z}^{-1}\mathbf{T}\mathbf{T}_z z_0 = \mathbf{T}_{\mathbf{T}z}^{-1}\mathbf{T}z = z_0$. A comparison of the expressions obtained for $\mathbf{T}Lf$ and $L\mathbf{T}f$ shows that the operator $L$ defined in the lemma is invariant.

Finally, let $L$ be an invariant operator, and define $L_0 f = Lf(z_0)$. Then $L_0$ is a rotation invariant functional, and it determines the original operator $L$, since $L^0\mathbf{T}_z f = L\mathbf{T}_z f(z_0) = \mathbf{T}_z Lf(z_0) = Lf(\mathbf{T}_z z_0) = Lf(z)$.

Thus the invariant differential operators can be found by first describing the rotation invariant differential functionals

$$L^0 f = \sum a_{i,j} \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(z)\bigg|_{z=z_0}$$

with coefficients $a_{i,j} = a_{i,j}(z_0)$.

The problem can be slightly simplified by working in the unit circle model $\{z\colon |z| < 1\}$ of the hyperbolic space with $z_0 = 0$ and by rewriting the differential operator with the replacement $\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} - i\frac{1}{2}\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} + i\frac{1}{2}\frac{\partial}{\partial y}$. In this (equivalent) reformulation our problem is to find the rotational invariant differential functionals

$$L^0 f = \sum b_{i,j} \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} f(z)\bigg|_{z=0}$$

with some new coefficients $b_{i,j}$. The rotational invariance means in this case that the expression

$$L^0 f = \sum b_{i,j} \rho^i \bar{\rho}^j \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} f(z)\bigg|_{z=0}$$

has the same value for all $\rho$ if $|\rho| = 1$. This relation holds if

$$\sum_{i-j=h} b_{i,j}\rho^i\bar{\rho}^j \left.\frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} f(z)\right|_{z=0} = 0$$

for all functions $f$ and $h \neq 0$. This statement holds if $b_{i,j} = 0$ for $i \neq j$. So

$$L^0 f = \sum_{i=0}^{m} b_{i,i} \left.\left(\rho^i\bar{\rho}^i \frac{\partial^2}{\partial z \partial \bar{z}}\right)^i f(z)\right|_{z=0} = \sum_{i=0}^{m} c_i \left.\Delta^i f(z)\right|_{z=0} , \qquad (*)$$

because

$$4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta .$$

Thus for instance, if $L^0 f = \Delta f|_{z=0}$, then the invariant operator $L$ corresponding to it equals $Lf(z) = L^0_w f(\mathbf{T}_z w) = \Delta f(\mathbf{T}_z w)|_{w=0}$. The operator $\mathbf{T}_z$ can be chosen as $\mathbf{T}_z w = \dfrac{w+z}{1+w\bar{z}}$, for which $\left.\dfrac{\partial \mathbf{T}_z w}{\partial w}\right|_{w=0} = 1 - |z|^2$. This gives that

$$\begin{aligned}\Delta f(\mathbf{T}_z w) &= 4\frac{\partial^2 f(\mathbf{T}_z w)}{\partial w \partial \bar{w}} = 4\frac{\partial}{\partial \bar{w}} \left.\frac{\partial f}{\partial w}\right|_{\mathbf{T}_z w} \frac{\partial \mathbf{T}_z w}{\partial w} \\ &= \left.\frac{\partial^2}{\partial \bar{w} \partial w} f\right|_{\mathbf{T}_z w} \left|\frac{\partial \mathbf{T}_z w}{\partial w}\right|^2 + 4\left.\frac{\partial f}{\partial w}\right|_{\mathbf{T}_z w} \frac{\partial}{\partial \bar{w}} \frac{\partial \mathbf{T}_z w}{\partial w} = \Delta f|_{\mathbf{T}_z w} \left|\frac{\partial \mathbf{T}_z w}{\partial w}\right|^2 .\end{aligned}$$

The above calculation yields with $w = 0$ that the invariant differential operator corresponding to $L^0 f = \dfrac{\partial^2}{\partial z \partial \bar{z}} f$ is $\Delta f(\mathbf{T}_z w)|_{w=0} = \left(1 - |z|^2\right)^2 \Delta f(z) \stackrel{\text{def}}{=} D^* f(z)$, which is the hyperbolic Laplace operator. Also the polynomials

$$Lf = \sum_{l=0}^{m} c_l {D^*}^l f \qquad (**)$$

are invariant differential operators. We claim that all invariant differential operators are given by this formula. Indeed, if $L^0$ is a rotational invariant functional of the form $(*)$, then the invariant differential operator $Lf$ is a polynomial of the derivatives of $f$. The coefficient of the leading term $\frac{\partial^{2m}}{\partial z^m \partial \bar{z}^m}$ is a non-vanishing function. (It is $c_m(1-|z|^2)^{2m}$.) Hence the class of invariant differential operators of order $m$ with $0 \leq m \leq 2N$ constitute an $N+1$-dimensional linear subspace of the differential operators of order less than or equal to $2N$. Since the operators of the form $(**)$ with $0 \leq m \leq N$ give an $N+1$-dimensional subspace of it, all invariant differential operators can be given in the form $(**)$. This means that in the hyperbolic plane the ring of invariant differential operators is generated by the single element $D^*$. This result can be rewritten to the half-plane model of the hyperbolic plane. In this case the invariant differential operators are of the form $Lf = \sum_{l=0}^{m} c_l D^m f$, with $Df(z) = y^2 \Delta f(z)$, where $z = x + iy$ and $\Delta$ is the Laplace operator.

**Theorem.** *Let $L_1$ and $L_2$ be two invariant differential or integral operators on the hyperbolic plane. Then they are exchangeable, i.e. $L_1 L_2 = L_2 L_1$.*

*Proof:* We work in the positive half-plane model of the hyperbolic plane. Since all invariant differential operators are the polynomials of the operator $D$, they are exchangeable. So we may assume that one of the operators, say $L_2$ is an integral operator with a kernel function $k_2(z, w)$, and $L_1$ is either the operator $D$ or an integral operator with a kernel function $k_1(z, w)$.

In the proof we try to adapt the proof of the Claim in the first talk. Since $L_1 L_2$ and $L_2 L_1$ are also invariant operators, by the lemma it is enough to show that $(L_1 L_2)^0 f = (L_2 L_1)^0 f$. We can write $(L_1 L_2)^0 f = L_1 g(z_0)$ with

$$g(z) = L_2 f(z) = \mathbf{T}_z L_2 f(z_0) = L_2 \mathbf{T}_z f(z_0) = \int k_2(z_0, w) f(\mathbf{T}_z w) \sigma(\, dw)$$

and $(L_2 L_1)^0 f = L_2 h(z_0)$ with $h(z) = L_1 f(z)$. Here

$$h(z) = \mathbf{T}_z L_1 f(z_0) = L_1 \mathbf{T}_z f(z_0) = \int k_1(z_0, w) f(\mathbf{T}_z w) \sigma(\, dw)$$

if $L_1$ is an integral operator, and

$$h(z) = \mathbf{T}_z L_1 f(z_0) = L_1 \mathbf{T}_z f(z_0) = D_w f(\mathbf{T}_z w)|_{w=z_0}$$

if $L_1 = D$. If both $L_1$ and $L_2$ are integral operators, then

$$(L_1 L_2)^0 f = L_1 g(z_0) = \iint k_1(z_0, z) k_2(z_0, w) f(\mathbf{T}_z w) \sigma(\, dz) \sigma(\, dw)$$

and

$$\begin{aligned}(L_2 L_1)^0 f &= \iint k_2(z_0, z) k_1(z_0, w) f(\mathbf{T}_z w) \sigma(\, dz) \sigma(\, dw) \\ &= \iint k_1(z_0, z) k_2(z_0, w) f(\mathbf{T}_w z) \sigma(\, dz) \sigma(\, dw)\, .\end{aligned}$$

If $L_1 = D$, then

$$\begin{aligned}(L_1 L_2)^0 f = L_1 g(z_0) = D_z\ g(z)|_{z=z_0} &= D \int k_2(z_0, w) f(\mathbf{T}_z w) \sigma(\, dw) \Big|_{z=z_0} \\ &= \int k_2(z_0, w)\ D_z f(\mathbf{T}_z w)|_{z=z_0}\, \sigma(\, dw)\end{aligned}$$

and

$$(L_2L_1)^0 f = L_2 h(z_0) = \int k_2(z_0,z)h(z)\sigma(\,dz) = \int k_2(z_0,z)\ D_w f(\mathbf{T}_z w)\big|_{w=z_0}\,\sigma(\,dz)$$
$$= \int k_2(z_0,w)\ D_z f(\mathbf{T}_w z)\big|_{z=z_0}\,\sigma(\,dw)\ .$$

We have certain freedom in the choice of the operators $\mathbf{T}_z$. If we could define these operators in such a way that $\mathbf{T}_z w = \mathbf{T}_z\mathbf{T}_w z_0$ and $\mathbf{T}_w z = \mathbf{T}_w\mathbf{T}_z z_0$ agreed, then a comparison of the expressions $(L_1L_2)^0)$ and $(L_1L_2)^0)$ would imply the Theorem just as it was done in the proof of the Claim in the first talk. But we cannot define these operators in such a way because of the lack of commutativity in the group of motions in this space. Hence some new ideas are needed, and Selberg found such ideas. He introduced a weak commutativity property defined below which is satisfied by the hyperbolic plane, and worked out a more refined argument which proves the result under this weaker property.

With an appropriate choice of the transformations $\mathbf{T}_z$ the following weak commutativity property holds:

$$d(z_0,\mathbf{T}_z w) = d(z_0,\mathbf{T}_w z)\quad \text{for all } w \text{ and } z.$$

Indeed, by choosing the motions $\mathbf{T}_z$ in such a way that $\mathbf{T}_z z_0 = z$ and $\mathbf{T}_z z = z_0$ we get that

$$d(z_0,\mathbf{T}_w z) = d(\mathbf{T}_w w,\mathbf{T}_w z) = d(w,z) = d(\mathbf{T}_z w,\mathbf{T}_z z) = d(\mathbf{T}_z w, z_0)\ .$$

Let us first consider rotation invariant functions $f$, i.e. such functions for which $f(z) = f(\mathbf{T}z)$ for any motion $T$ with the property $\mathbf{T}z_0 = z_0$. Then the value of $f(z)$ depends only on $d(z_0,z)$. The previous calculations yield that $(L_1L_2)^0 f = (L_2L_1)^0 f$ for rotation invariant functions. To consider a general function $f$ let us introduce the following "averaging" operator

$$f_0(z) = Mf(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{T}_\vartheta f(z)\,d\vartheta\ , \tag{+}$$

where $\mathbf{T}_\vartheta$, $0\le\vartheta<2\pi$, "the rotation with angle $\vartheta$", is defined in the following way: Let us consider an analytic automorphism $\mathbf{U}$ of the unit circle to the positive half-plane such that $\mathbf{U}$ maps the origin to $z_0$. If $\mathbf{S}_\vartheta z = e^{i\vartheta}z$ denotes the rotation with angle $\vartheta$ in the unit circle, then $\mathbf{T}_\vartheta$ is the transformation corresponding to it through this automorphism, i.e. $\mathbf{T}_\vartheta z = \mathbf{U}\mathbf{S}_\vartheta\mathbf{U}^{-1}z$. The function $f_0 = Mf$ is rotation invariant, since for all $\vartheta'$

$$f_0(\mathbf{T}_{\vartheta'}z) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathbf{T}_\vartheta\mathbf{T}_{\vartheta'}z)\,d\vartheta = \int_0^{2\pi} f(\mathbf{T}_{\vartheta+\vartheta'}z)\,d\vartheta = \int_0^{2\pi} f(\mathbf{T}_\vartheta z)\,d\vartheta = f_0(z_0)\ ,$$

and all movements of the hyperbolic plane with the fixed point $z_0$ equals some $\mathbf{T}_\vartheta$.

If $L$ is an invariant operator, then $LM = ML$, since $L$ is exchangeable with all $\mathbf{T}_\vartheta$. Hence

$$(L_1L_2)^0 f = L_1L_2 f(z_0) = ML_1L_2 f(z_0) = L_1L_2 M f(z_0) = L_1L_2 f_0(z_0)$$

and

$$(L_2L_1)^0 f = L_2L_1 f(z_0) = ML_2L_1 f(z_0) = L_2L_1 M f(z_0) = L_2L_1 f_0(z_0)\ .$$

Since $L_1L_2f_0 = L_2L_1f_0$ the above relations imply that $(L_1L_2)^0 f = (L_2L_1)^0 f$, and $L_1L_2 = L_2L_1$ by the lemma. The theorem is proved.

THIRD TALK:

**The eigenvalues of invariant integral operators**

We can find many eigenfunctions of the operator $D = y^2\Delta$ in the upper half-plane model of the hyperbolic space. Thus for instance $y^s$, $z = x + iy$, is an eigenfunction with the eigenvalue $\lambda = s(s-1)$, since $Dy^s = y^2\Delta y^s = y^2 s(s-1)y^{s-2} = s(s-1)y^s$. If $L$ is an arbitrary invariant differential or integral operator, then by the previous statement $DLy^s = LDy^s = \lambda Ly^s$, i.e. $Ly^s$ is an eigenfunction of $D$ with the same eigenvalue. Following the argument of the first talk we would like to deduce from this statement that $Ly^s = \Lambda y^s$ with some $\Lambda$. However, the argument of the first talk does not work in the present case. In the real line case, considered in the first talk, it was exploited that the eigenspace of the differential operator corresponding to an eigenvalue $\lambda$ is one-dimensional. In the hyperbolic space $H$ the analogous statement does not hold. The eigenspace of the operator $D$ corresponding to an eigenvalue $\lambda$ is very large. For instance, the functions $\mathbf{T}y^s = (\operatorname{Im}\mathbf{T}z)^s$ are eigenfunctions with eigenvalue $\lambda$ for all motions $\mathbf{T}$.

What can be saved from the uniqueness of eigenfunctions?

If $M$ is the "averaging operator" with some center $z_0$ defined in formula (+) in the second talk, and $Df = \lambda f$, then for $f_0 = Mf$

$$Df_0 = DMf = MDf = \lambda Mf = \lambda f_0 \ ,$$

and $f_0$ is rotation invariant around the center $z_0$.

**Proposition.** *Let $g(z) = g(z, z_0)$ satisfy the following properties: $Dg = \lambda g$, $g(z)$ depends only on the value of the distance $d(z, z_0)$, (with fixed $z_0$ and $\lambda$), and $g(z_0) = 1$. Then the function $g(z)$ is defined uniquely by these properties.*

*Proof:* Here again it is simpler to work in the unit square model $\{|z| < 1\}$ with $z_0 = 0$. If $z = re^{i\vartheta}$, then the function $g$ can be written in the form $g(z) = g(r)$ with $r = |z|$. The Laplace operator $\Delta$ can be written in polar coordinate system as $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \vartheta^2}$. Hence we can write because of the rotation invariance of $g$

$$D^* g = (1 - r^2)^2 g''(r) = \lambda g(r)$$

with the boundary conditions $g(0) = 1$ and $g'(0) = 0$. (The last relation holds, because $g(r) = g(-r)$.) Since the coefficient $(1 - r^2)^2$ in this differential equation is non-vanishing, it can be solved uniquely.

How does the function $g(z, z_0)$, the unique rotation invariant eigenfunction of $D$ with center $z_0$, with eigenvalue $\lambda$ and normalization $g(z_0, z_0) = 1$, depend on $z_0$?

If $\mathbf{T}z_0 = z_0'$, then $g(\mathbf{T}z, z_0')$ depends only on $d(\mathbf{T}z, z_0') = d(z, \mathbf{T}^{-1}z_0') = d(z, z_0)$, $g(\mathbf{T}z_0, z_0') = g(z_0', z_0') = 1$, and with the notation $h(u) = g(\mathbf{T}u, z_0')$, $p(u) = g(u, z_0')$

$$Dh(z) = D\mathbf{T}p(z) = \mathbf{T}Dp(z) = \lambda \mathbf{T}p(z) = \lambda g(\mathbf{T}z, z_0') = \lambda h(z) \ .$$

This means that $h(z) = g(\mathbf{T}z, z_0')$ is an eigenfunction of $D$ with the eigenvalue $\lambda$, it depends only on $d(z, z_0)$, i.e. $h(z) = h(z')$ if $d(z, z_0) = d(z', z_0)$, which means that it is rotation invariant with center $z_0$, and $h(z_0) = 1$. Hence the Proposition implies that $h(z) = g(\mathbf{T}z, z_0') = g(z, z_0)$, or in other words $g(\mathbf{T}z, \mathbf{T}z_0) = g(z, z_0)$ for all motions $\mathbf{T}$ of the hyperbolic plane. This means that $g(z, z_0)$ depends only on $d(z, z_0)$, i.e. it has the same properties as the kernel function of an invariant integral operator.

If $L$ is an arbitrary invariant differential or integral operator, then

$$D_z L_z g(z, z_0) = L_z D_z g(z, z_0) = \lambda L_z g(z, z_0) \ .$$

But $L_z g(z, z_0)$, as a function of $z$ and $z_0$, depends only on $d(z, z_0)$, since

$$L_z g(\mathbf{T}z, \mathbf{T}z_0) = Lp(\mathbf{T}z) = \mathbf{T}Lp(z) = L\mathbf{T}p(z) = Lq(z) = L_z g(z, z_0)$$

for all movements $\mathbf{T}$ with $p(u) = g(u, \mathbf{T}z_0)$ and $q(u) = g(\mathbf{T}u, \mathbf{T}z_0) = g(u, z_0)$. In particular, because of the rotational symmetry $L_z g(z, z_0) = \Lambda g(z, z_0)$ with some $\Lambda = \Lambda(z_0)$. Moreover, $\Lambda(z_0)$ does not depend on $z_0$, since $\Lambda(z_0) = L^0 g(z_0, z_0)$, and this function has the same value for all $z$.

If $f$ is an arbitrary function such that $Df = \lambda f$, and we fix some number $z_0$ as the center of the averaging operator $M$, then $Df_0 = \lambda f_0$ with $f_0 = Mf$, and $f_0$ is rotation invariant around the point $z_0$. Because of the uniqueness of rotational invariant eigenfunctions $f_0(z) = f_0(z_0) g(z, z_0)$, and this implies for all invariant integral or differential operators $L$ that

$$Lf_0(z) = f_0(z_0) L_z g(z, z_0) = f_0(z_0) \Lambda g(z, z_0) = \Lambda f_0(z)$$

with a $\Lambda$ independent of the function $f$ and $z_0$. From here

$$Lf(z_0) = MLf(z_0) = LMf(z_0) = Lf_0(z_0) = \Lambda f_0(z_0) = \Lambda f(z_0) \ .$$

Since $z_0$ can be chosen in an arbitrary way, and $\Lambda$ is independent of $z_0$, $Lf = \Lambda f$. This is a very important observation which we formulate in the following Theorem.

**Theorem.** *If $f$ is an eigenfunction of the operator $D$ with eigenvalue $\lambda$, i.e. $Df = \lambda f$, and $L$ in an invariant differential or integral operator, then $Lf = \Lambda f$ with a $\Lambda$ which depends only on the operator $L$ and the eigenvalue $\lambda$, but not on the eigenfunction $f$.*

If $f$ is an eigenfunction of $D$, then we can calculate the eigenvalue of $Lf$ by this theorem without the knowledge of $f$, if we know an arbitrary eigenfunction of $D$ with the same eigenvalue $\lambda$. We know such an eigenfunction, namely the function $y^s$ with the eigenvalue $\lambda$. This eigenvalue $\lambda$ can be arbitrary with the choice of $s = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda}$.

Let us calculate for instance the eigenvalue $\Lambda$ of the integral operator $Lf = \int_H k(z, w) f(w) w(\,d\sigma)$ if $k(z, w) = k\left(\frac{|z - w|^2}{yv}\right)$.

We know that

$$\int_H k(z,w)v^s\sigma(\,dv) = \Lambda y^s$$

with $w = u + iv$. Taking the point $z = i$ $(y = 1)$, we get

$$\begin{aligned}\Lambda = \int_H k(i,w)v^s\sigma(\,dv) &= \int_0^\infty \int_{-\infty}^\infty k\left(\frac{|i-w|^2}{v}\right) v^s \frac{du\,dv}{v^2} \\ &= \int_0^\infty \int_{-\infty}^\infty k\left(\frac{u^2+(v-1)^2}{v}\right) v^{s-2}\,du\,dv\ .\end{aligned}$$

Put $t \stackrel{\text{def}}{=} \dfrac{(v-1)^2}{v} = v + \dfrac{1}{v} - 2$. Then

$$\begin{aligned}\int_{-\infty}^\infty k\left(\frac{u^2+(v-1)^2}{v}\right) du = \int_{-\infty}^\infty k\left(\frac{u^2}{v}+t\right) du &= 2\int_0^\infty k(\tau+t)\frac{\sqrt{v}}{2\sqrt{\tau}}\,d\tau \\ &= \sqrt{v}\int_0^\infty \frac{k(\tau+t)}{\sqrt{\tau}}\,d\tau \stackrel{\text{def}}{=} vg(\log v)\ ,\end{aligned}$$

$$\Lambda = \int_0^\infty vg(\log v)v^{s-2}\,dv = \int_{-\infty}^\infty g(y)e^{sy}\,dy \stackrel{\text{def}}{=} G(s)\ ,$$

where $\lambda = s(s-1)$, $s = \dfrac{1}{2} \pm \sqrt{\dfrac{1}{4}+\lambda}$.

Fourth talk:

**Discrete subgroups of the group of motions in the hyperbolic plane**

**Definition.** *The group* $\Gamma \subset$ group of motions in $H$ *is called discrete if the set* $\{\gamma z\colon \gamma \in \Gamma\}$ *has no point of density for a fixed* $z \in H$.

**Definition.** *The function* $f(z)$, $z \in H$, *is automorph with respect to the group* $\Gamma$, *if* $f(\gamma z) = f(z)$ *for all* $\gamma \in \Gamma$.

From now on $f(z)$ will always denote an automorph function. We also introduce the notation $z_1 \sim z_2 \pmod \Gamma$ if there exists a motion $\gamma \in \Gamma$, such that $\gamma z_1 = z_2$.

Put $\mathcal{S} \stackrel{\text{def}}{=}$ equivalence classes of $z \in H$ with respect to $\sim$. A set $\mathcal{D} \subset H$ is called a fundamental domain if it contains exactly one point from all equivalence classes. In other words, the sets $\{\gamma\mathcal{D}\}$, $(\gamma \in \Gamma)$ produce a disjoint cover of the space $H$. (In the Euclidean plane the discrete subgroups of the group of motions have the form $\Gamma = \Gamma_{w_1,w_2} = \{\gamma_{m,n}\colon z \to z + nw_1 + mw_2\}$, with some non-colinear vectors $w_1$ and $w_2$. The automorph functions are the doubly periodic functions, $\mathcal{D}$ can be chosen as a lattice parallelogram, $S$ is topologically a torus. In the one-dimensional case $\mathcal{D} = [0,1)$, $\mathcal{S}$ is a circle.)

Let us also assume that $\Gamma$ is free of fixed points. This means that if $\gamma \in \Gamma$ has a fixed point, $(\gamma z_0 = z_0)$, then $\gamma$ is the identity. We also assume that $\Gamma$ has a "compact" fundamental domain $\mathcal{D}$ whose closure $\bar{\mathcal{D}}$ (in the compactified complex sphere) is a subset of $H$.

It is far from trivial that there are such subgroups $\Gamma$ of the motions of $H$ which satisfy all above conditions. It follows from the theory of Riemann surfaces that this is possible. The fundamental domains corresponding to the group $\Gamma$ show some similarities with the torus appearing as a fundamental domain in the Euclidean case. If it has genus $k$, $k > 1$, which heuristically means that the domain $\mathcal{D}$ is topologically a "sphere with $k$ ears", then there exists a discrete subgroup $\Gamma$ with a fundamental domain $\mathcal{D}$ which is a regular polygon with $4k$ edges and angles $\pi/2$. The area of this fundamental domain is determined by its genus $k$. There are such pairs of edges of the boundary which can be stuck together in opposite direction, and the points stuck together are equivalent with respect to $\Gamma$. The explanation of these results is not the subject of the present lecture.

Under these conditions $\mathcal{S}$ is a compact (closed) Riemann surface, topologically it is a sphere $\mathcal{G}$ with "ears", its genus $g > 1$. There exists a universal cover $H \to \mathcal{S}$. This universal cover together with the metric and measure on $H$ induce a finite metric and measure on $\mathcal{S}$, invariant with respect to $\Gamma$.

**The hyperbolic Laplace operator in a fundamental domain**

If $L$ is an invariant operator, $f(z)$ is an automorph function, then $Lf$ is again an automorph function, since $Lf(\gamma z)=\mathbf{T}_\gamma Lf(z)=L\mathbf{T}_\gamma f(z)=Lf(z)$ for all $z\in H$ and $\gamma\in\Gamma$, where the operator $\mathbf{T}_\gamma$ is defined by the relation $\mathbf{T}_\gamma f(z)=f(\gamma z)$. Hence the operator $L$ can be considered as an operator acting in the space of automorph functions. In particular, this restriction can be done for the operator $D=y^2\Delta$. The operator $D$ is a self-adjoint negative operator in the space $L_2(\mathcal{S},d\sigma)$. Indeed, the Green formula in the domain $\mathcal{D}$ gives that the operator $D$ is negative, i.e.

$$\begin{aligned}(Df,f)&=\int_{\mathcal{D}} Df(z)\overline{f(z)}\sigma(\,dz)=\iint_{\mathcal{D}} y^2\Delta f(z)\overline{f(z)}\frac{dx\,dy}{y^2}\\&=\underbrace{\int_{\partial D}\frac{\partial f}{\partial n}\bar f\,ds}_{=0}-\underbrace{\iint_{\mathcal{D}}(f_x\bar f_x+f_y\bar f_y)\,dx\,dy}_{\geq 0}\leq 0\ ,\end{aligned}$$

since, as it is proved in the theory of the Riemann surfaces, the boundary $\partial\mathcal{D}$ consists of finitely many pairs of curves $(\ell,\ell')$ such that an appropriate $\gamma\in\Gamma$ maps $\ell$ to $\ell'=\gamma\ell$, and

$$\left.\frac{\partial f}{\partial n}\right|_z=-\left.\frac{\partial f}{\partial n}\right|_{\gamma z}|\gamma'(z)|\ .$$

This relation implies that

$$\int_\ell\frac{\partial f}{\partial n}\bar f\,ds=-\int_\ell\left.\frac{\partial f}{\partial n}\right|_{\gamma z}|\gamma'(z)|\bar f\,ds=-\int_{\gamma\ell}\frac{\partial f}{\partial n}\bar f\,ds\ ,$$

and the first integral at the right-hand side of the expression for $(Df,f)$ equals zero. It can be proved similarly that $(Df,g)=(f,Dg)$, i.e. $D$ is a self-adjoint operator.

If $Df\equiv 0$, then $f$ is a harmonic function, and $f\equiv$ const. by the maximum principle. It can be proved that the set of functions $Df$ is everywhere dense in the space $L_0^2(\mathcal{D},\sigma)$, consisting of the functions $f$, $\int_{\mathcal{D}} f^2(z)\sigma(\,dz)<\infty$ and $\int_{\mathcal{D}} f(z)\sigma(\,dz)=0$. It is proved in the functional analysis that because of these properties the operator $D$ can be inverted, and the inverse of $D$ is an integral operator of the form

$$D^{-1}f(z)=\int_{\mathcal{D}} G(z,w)f(w)\sigma(\,dw)$$

with a kernel function $G(z,w)$, $G(z,w)=G(w,z)$, and $G(z,w)$ is a automorph function in both variables. This statement is equivalent to the classical result in the analysis by which the Laplace operator on the domain $\mathcal{D}$ (with periodic boundary conditions) in the space $L_0^2(\mathcal{D},\text{Lebesgue measure})$ is invertible, and the inverse is an integral operator with a kernel function $G(z,w)$. The kernel function $G(z,w)$ is the same in these two problems. Its symmetry properties follow from the properties of the operator $D$ (or $\Delta$).

We only give a very sketchy explanation why the inverse operator $D^{-1}$ can be expressed in such a way. If $G(z,w)$ is an automorph function in the variable $w$ which is of the form $-\log|z-w| + \log|z_0 - w| +$ analytic function in the variable $w$, ($z_0$ is a fixed point in $H$) then it is relatively simple to show by means of the Green formula that it can be taken as the kernel function of the integral operator which is the inverse of $D$. It follows from the general theory that such a function $G$ exits.

The integral operator is again a negative operator, and since the kernel function $G(z,w)$ in it is a nice function, (it has only logarithmical singularities) the inverse of the operator $D$ is a Hilbert–Schmidt operator. It follows from the general theory of such operators that there exists a system of complete orthonormal system $\{f_i(z)\}_{i=1}^{\infty}$ of eigenfunctions of the operator $D^{-1}$ with eigenvalue $\lambda_i$ in the space $L_0^2(\mathcal{D},\sigma)$. Moreover, $\lambda_i \to 0$, if $i \to \infty$, and only the eigenvalue zero can have infinite multiplicity. But in our case the zero eigenvalue does not appear, since $F \equiv 0 \Rightarrow \Delta F \equiv 0$. In such a way we get a complete orthonormal system of eigenfunctions $f_i(z)$ of $D$ in the space $L_0^2(\mathcal{D},\sigma)$, $Df_i = \lambda_i f_i$ with $\lambda_i < 0$, and $\lambda_i \to -\infty$ as $i \to \infty$. Let us adjust the number $\lambda_0 = 0$ and $f_0(z) = \dfrac{1}{\sqrt{\sigma(\mathcal{D})}}$ to this system. Then $\{f_i\}_{i=0}^{\infty}$ is a complete orthonormal system consisting of eigenfunctions of $D$ in the $L^2(\mathcal{D}, d\sigma)$ space.

### The trace of an invariant integral operator

For an invariant integral operator $L$ with kernel function $k(z,w)$

$$\begin{aligned} Lf(z) &= \int_H k(z,w)f(w)\sigma(\,dw) = \sum_{\gamma\in\Gamma}\int_{\gamma\mathcal{D}} k(z,w)f(w)\sigma(\,dw) \\ &= \sum_{\gamma\in\Gamma}\int_{\mathcal{D}} k(z,\gamma w)\, f(w)\sigma(\,dw) = \int_{\mathcal{D}} K(z,w)f(w)\,\sigma(\,dw)\ , \end{aligned}$$

where

$$K(z,w) = \sum_{\gamma\in\Gamma} k(z,\gamma w)\ .$$

The function $K(z,w)$ is automorph in both variables, hence it can be expanded by the complete orthonormal system $\{f_i(z)\overline{f_j(w)}\}$ in $\mathcal{D}\times\mathcal{D}$, just as in the real line model considered in the first talk. This expansion has the form

$$K(z,w) = \sum_{i=1}^{\infty} \Lambda_i f_i(z)\overline{f_i(w)}\ ,$$

where $Lf_i = \Lambda_i f_i$.

The number $\Lambda_i$ can be calculated, as a function of $\lambda_i$, without the knowledge of $f_i$, and this was already done in the third talk. If we take $z = w$, and integrate on the diagonal, i.e. we calculate the trace of the operator $L$ on $\mathcal{D}$, then the functions $f_i$ disappear from the formulas, and we get

$$\int_{\mathcal{D}} K(z,z)\sigma(\,dz) = \sum_{i=0}^{\infty}\Lambda_i \int_{\mathcal{D}} |f_i(z)|^2\sigma(\,dz) = \sum_{i=0}^{\infty}\Lambda_i\ ,$$

that is,

$$\sum_{\gamma\in\Gamma}\int_{\mathcal{D}} k(z,\gamma z)\sigma(\,dz) = \int_{\mathcal{D}}\sum_{\gamma\in\Gamma} k(z,\gamma z)\sigma(\,dz) = \sum_{i=0}^{\infty}\Lambda_i\ . \tag{++}$$

We want to rewrite the left-hand side of (++) in a more appropriate form. In the real line case the kernel function $k(z,\gamma z)$ $(= k(x-(x+n)))$ of the corresponding integral operator was in the variable $z$ $(x)$ constant. Let us look the corresponding question in the present case. When does the equation $k(w,\gamma_0 w) = k(z,\gamma z)$ hold?

If $w = \mathbf{T}z$, then

$$k(w,\gamma_0 w) = k(\mathbf{T}z,\gamma_0\mathbf{T}z) = k(z,\mathbf{T}^{-1}\gamma_0\mathbf{T}z)$$

hence $k(w,\gamma_0 w) = k(z,\gamma z)$ if $\gamma = \mathbf{T}^{-1}\gamma_0\mathbf{T}$. If $\gamma = \gamma_0$, then this condition means that $\gamma_0$ and $\mathbf{T}$ are exchangeable. (In the real line case they are always exchangeable, but in the present case this property may be violated). In the general case this equality holds if $\gamma$ and $\gamma_0$ are conjugate. Therefore, let us sum for different conjugate classes separately. Let us denote the conjugate classes in the form $\{\gamma_0\}$.

Let us conjugate by an element be $g$: $\gamma = g^{-1}\gamma_0 g$. (This conjugation would be meaningful also for $g\notin\Gamma$, but we shall consider only conjugations with elements from $\Gamma$.) Two such conjugates agree, i.e. $g_1^{-1}\gamma_0 g_1 = g^{-1}\gamma_0 g$, if and only if $g_1 g^{-1}\gamma_0 = \gamma_0 g_1 g^{-1}$, which means that $g_1 g^{-1}\in[\gamma_0]$, or $g_1\in[\gamma_0]g$, where $[\gamma_0]$ denotes the group of element in $\Gamma$ exchangeable with $\gamma_0$. This group $[\gamma_0]$ is called in algebra the centralizator group of $\gamma_0$. Thus the sum $\sum\limits_{\gamma\in\{\gamma_0\}}$ can also be written in the form $\sum\limits_{\substack{g\\ g\in\Gamma/[\gamma_0]}}$, where $g$ are fixed representation elements from the right cosets of the subgroup $[\gamma_0]$.

If $\gamma\in\{\gamma_0\}$, then $\gamma = g^{-1}\gamma_0 g$ with some $g$, and

$$\begin{aligned}\int_{\mathcal{D}} k(z,\gamma z)\sigma(\,dz) = \int_{\mathcal{D}} k(z,g^{-1}\gamma_0 gz)\sigma(\,dz) &= \int_{\mathcal{D}} k(gz,\gamma_0 gz)\sigma(\,dz)\\ &= \int_{g\mathcal{D}} k(z,\gamma_0 z)\sigma(\,dz)\ .\end{aligned}$$

$$\sum_{\gamma\in\{\gamma_0\}}\int_{\mathcal{D}} k(z,\gamma z)\sigma(\,dz) = \sum_{\substack{g\\ g\in\Gamma/[\gamma_0]}}\int_{g\mathcal{D}} k(z,\gamma_0 z)\sigma(\,dz) = \int\limits_{\substack{\cup g\mathcal{D}\\ g\in\Gamma/[\gamma_0]}} k(z,\gamma_0 z)\sigma(\,dz)\ .$$

If the representation elements $g\in\Gamma/[\gamma_0]$ are multiplied from the left with the elements of $[\gamma_0]$, then we get the elements of $\Gamma$, and all elements exactly once. Thus $\mathcal{D}(\gamma_0) \stackrel{\text{def}}{=} \bigcup\limits_{g\in\Gamma/[\gamma_0]} g\mathcal{D}$ is such a set that the sets $\gamma\mathcal{D}(\gamma_0)$, $\gamma\in[\gamma_0]$, produce a disjoint cover of $H$. In other words, $\mathcal{D}(\gamma_0)$ is a fundamental domain of the discrete subgroup $[\gamma_0]$. We know that $k(z,\gamma_0 z)$ is automorph with respect to $[\gamma_0]$, $(k(gz,\gamma_0 gz) = k(z,g^{-1}\gamma_0 gz) = k(z,\gamma_0 z)$ for

$g \in [\gamma_0]$), hence it has no importance how to choose the fundamental domain where we integrate. We formulate the result we obtained in the following lemma.

**Lemma.** *For all $\gamma_0 \in \Gamma$ the relation*

$$\sum_{\gamma \in \{\gamma_0\}} \int_{\mathcal{D}} k(z, \gamma z)\sigma(\,dz) = \int_{\mathcal{D}(\gamma_0)} k(z, \gamma_0 z)\sigma(\,dz)$$

*holds, where $\{\gamma_0\} = \{g^{-1}\gamma_0 g;\ g \in \Gamma\}$ denotes the class of elements in $\Gamma$ which are conjugated to $\gamma_0$, and $\mathcal{D}(\gamma_0)$ is a fundamental domain of the centralizator group $[\gamma_0] \subset \Gamma$ of $\gamma_0$. The function $k(z, \gamma_0 z)$ at the right-hand side of this formula is $[\gamma_0]$ automorph.*

*Remark:* It is possible that $\{\gamma_0\} = \{\gamma_1\}$ with some $\gamma_0 \neq \gamma_1$ and even $[\gamma_0] \neq [\gamma_1]$. But the right-hand side of the integral in the lemma agrees for such $\gamma_0$ and $\gamma_1$. Indeed, if $\gamma_1 = g^{-1}\gamma_0 g$, then

$$\int_{\mathcal{D}(\gamma_1)} k(z, \gamma_1 z)\sigma(\,dz) = \int_{g\mathcal{D}(\gamma_1)} k(z, \gamma_0 z)\sigma(\,dz)\ ,$$

and $g\mathcal{D}(\gamma_1)$ is a fundamental domain of $[\gamma_0]$. (If $h_j\mathcal{D}(\gamma_1)$, $j = 1, \dots,$ is a disjoint cover of $H$, and the transformations $h_j$ are the elements of $[\gamma_1]$, then the sets $gh_jg^{-1}(g\mathcal{D}(\gamma_1))$, $j = 1, \dots,$ also give a disjoint cover of $H$, and $gh_jg^{-1}$ are the elements of $[\gamma_0]$.)

We shall write the expression in (++) in a simpler form by means of the Lemma and a further investigation which will show that in the group $\Gamma$ the centralizators $[\gamma_0]$ have a very special form.

Fifth talk:

**The proof of Selberg's trace formula in the hyperbolic plane**

Let $\mathbf{T}$ be a motion of $H$. Its analytic extension to the complex sphere, which is a linear rational function, has exactly two fixed points (with multiplicity). If it is a double fixed point, which must be because of the relation $\mathbf{T}\bar{z} = \overline{\mathbf{T}z}$ either a real number or infinity, then the transformation $\mathbf{T}$ is called parabolic. If $\mathbf{T}$ has a fixed point in $H$ (its conjugate, which is not in $H$, is also a fixed point), then it is called an elliptic transformation. In the remaining case, the transformation $\mathbf{T}$ is called hyperbolic. Hyperbolic transformations have two different real or $(\infty)$ fixed points.

A discrete subgroup $\Gamma$ of the motions in $H$ cannot contain an elliptic transformation, since the non-unity elements of $\Gamma$ cannot have a fixed point in $H$. We also excluded the possibility of $\gamma \in \Gamma$ for a parabolic transformation $\gamma$ by the assumption that $\bar{\mathcal{D}}$ is compact. Indeed, for a parabolic transformation $d(z, \gamma z)$ can be taken arbitrary small with an appropriate choice of $z \in H$, (by a conjugation which takes the fixed point to infinity the proof of this statement can be reduced to the case when the double fixed point of $\gamma$ is infinity, i.e. $\gamma z = z + c$ with some real number $c$, and in this case it is easy to check this property) and we show that because of this property $\Gamma$ cannot contain a parabolic transformation. Otherwise, by mapping a sequence of points with the above properties into the domain $\mathcal{D}$ we could choose a sequence $z_n$ with $z_n \in \mathcal{D}$ and $\gamma_n \in \Gamma$ in such a way that $0 < d(z_n, \gamma_n z_n) \to 0$, as $n \to \infty$. Then, because of the compactness of $\bar{\mathcal{D}}$ we may assume, by considering a subsequence of the sequence $z_n$, that $z_n \to z_0 \in H$. Since $d(\gamma_n z_n, \gamma_n z_0) = d(z_0, z_n) \to 0$, $d(z_0, \gamma_n z_0) \le d(z_0, z_n) + d(z_n, \gamma_n z_n) + d(\gamma_n z_n, \gamma_n z_0) \to 0$, as $n \to 0$, and $\gamma_n \ne \mathrm{Id}$. But this contradicts to the fact, that since $\Gamma$ has no fixed point, and it is discreet, $\min\limits_{\substack{\gamma\in\Gamma\\ \gamma\ne\mathrm{Id}}} d(z_0, \gamma z_0) > 0$. So $\Gamma$ contains only hyperbolic transformations.

Let $\gamma$ be a hyperbolic transformation. If $\mathbf{T}$ is such a motion of $H$ (not necessarily in the group $\Gamma$) which maps the two fixed points of $\gamma$ into 0 and $\infty$, then the two fixed points of $\mathbf{T}^{-1}\gamma\mathbf{T}$ are 0 and $\infty$, hence $\mathbf{T}^{-1}\gamma\mathbf{T}z = \rho z$ with some $\rho > 0$, $\rho \ne 1$. Moreover, we may assume with a possible conjugation with $\dfrac{1}{z}$ that $\rho > 1$. This number $\rho$ is uniquely determined by $\gamma$ (and even by $\{\gamma\}$). We call it the norm of $\gamma$, and introduce the notation $\rho = N(\gamma) = N(\{\gamma\})$.

Two motions of $H$ are exchangeable if and only if their fixed points agree. For the motions exchangeable with $\gamma_0$ the same motion $\mathbf{T}$ can be chosen for writing it as the conjugate of a linear transformations, and we see that this group is isomorphic to the multiplicative group of the positive numbers $\rho > 0$. A discrete subgroup of this group is an infinite cyclic group, and we call a generator $\gamma^*$ of it a primitive element. $[\gamma_0] = \{\gamma^{*m}\}_{m=-\infty}^{\infty}$. Put $\rho^* = N(\gamma^*)$.

$$\int_{\mathcal{D}(\gamma_0)} k(z, \gamma_0 z)\sigma(\,dz) = \int_{\mathcal{D}(\gamma_0)} k(\mathbf{T}^{-1}z, \mathbf{T}^{-1}\gamma_0 z)\sigma(\,dz) = \int_{\mathbf{T}^{-1}\mathcal{D}(\gamma_0)} k(z, \mathbf{T}^{-1}\gamma_0\mathbf{T}z)\sigma(\,dz)$$

$$= \int_{\mathbf{T}^{-1}\mathcal{D}(\gamma_0)} k(z, \rho_0 z)\sigma(\,dz)\ , \quad \rho_0 = N(\gamma_0)\ ,$$

$\mathbf{T}^{-1}\mathcal{D}(\gamma_0)$ is a fundamental domain of the group $\mathbf{T}^{-1}[\gamma_0]\mathbf{T} = \{\rho^{*m} z\}_{m=-\infty}^{\infty}$, and since $k(z, \mathbf{T}^{-1}\gamma_0\mathbf{T}z)$ is automorph for $\mathbf{T}^{-1}[\gamma_0]\mathbf{T}$ we can choose an arbitrary fundamental domain of $\mathbf{T}^{-1}[\gamma_0]\mathbf{T}$ to calculate the last integral.

It is clear that the half-ring $\{1 \le |z| < \rho^*\} \cap H$ is such a domain. The function $k(z, \rho_0 z)$ is invariant for all transformations $\mathbf{T}z = \rho z$. Hence, it is constant along the rays, and we have to integrate only with respect to the angle $\vartheta$, where $z = re^{i\vartheta}$.

$$\begin{aligned}
I &= \int_{\mathbf{T}^{-1}\mathcal{D}(\gamma_0)} k(z, \rho_0 z)\sigma(\,dz) = \int_0^\pi \int_1^{\rho^*} k(e^{i\vartheta}, \rho_0 e^{i\vartheta}) \frac{r\,dr\,d\vartheta}{r^2 \sin^2\vartheta} \\
&= \int_0^\pi k(e^{i\vartheta}, \rho_0 e^{i\vartheta}) \frac{\log \rho^*}{\sin^2\vartheta}\,d\vartheta = 2\int_0^{\pi/2} k\left(\frac{\left|e^{i\vartheta} - \rho_0 e^{i\vartheta}\right|^2}{\sin\vartheta \rho_0 \sin\vartheta}\right) \frac{\log\rho^*}{\sin^2\vartheta}\,d\vartheta \\
&= 2\log\rho^* \int_0^{\pi/2} k\left(\frac{(\rho_0 - 1)^2}{\rho_0 \sin^2\vartheta}\right) \frac{d\vartheta}{\sin^2\vartheta}\ .
\end{aligned}$$

Put $t = \dfrac{(\rho_0 - 1)^2}{\rho_0} = \rho_0 + \dfrac{1}{\rho_0} - 2$ and $\tau = \dfrac{t}{\sin^2\vartheta}$, that is $\vartheta = \arcsin\sqrt{\dfrac{t}{\tau}}$. Then with this notation

$$\begin{aligned}
I &= 2\log\rho^* \int_t^\infty k(\theta) \frac{\tau}{t} \frac{1}{\sqrt{1 - \frac{t}{\tau}}} \frac{\sqrt{t}}{2\tau^{3/2}}\,d\tau \\
&= \frac{\log\rho^*}{\sqrt{t}} \int_t^\infty \frac{k(\tau)}{\sqrt{\tau - t}}\,d\tau = \frac{\log\rho^*}{\sqrt{t}} \int_0^\infty \frac{k(\tau + t)}{\sqrt{\tau}}\,d\tau \\
&= \frac{\log\rho^*}{\sqrt{t}} \sqrt{\rho_0} g(\log\rho_0) = \frac{\log\rho^*}{\frac{\rho_0 - 1}{\sqrt{\rho_0}}} \sqrt{\rho_0} g(\log\rho_0) = \frac{\log\rho^*}{1 - \frac{1}{\rho_0}} g(\log\rho_0)
\end{aligned}$$

by the definition of $g(\cdot)$.

Putting these results together

$$\sum_{\gamma \in \{\gamma_0\}} \int_{\mathcal{D}} k(z, \gamma z)\sigma(\,dz) = \frac{\log N(\gamma^*)}{1 - \frac{1}{N(\gamma_0)}} g(\log N(\gamma_0))\ .$$

This relation does not hold for the one element conjugate class containing the origin. But

$$\int_{\mathcal{D}} k(z, z)\,\sigma(\,dz) = k(0)\sigma(\mathcal{D}) = \frac{\sigma(\mathcal{D})}{4\pi} \int_{-\infty}^\infty r\tanh(\pi r) G\left(\frac{1}{2} + ir\right)\,dr$$

with the previous definition of $G(\cdot)$, what can be shown by a long but for us not so interesting calculation.

So the starting trace formula

$$\sum_{\gamma\in\Gamma}\int_{\mathcal{D}} K(z,\gamma z)\sigma(\,dz) = \sum_{i=0}^{\infty}\Lambda_i$$

has the following form:

$$\frac{\sigma(\mathcal{D})}{4\pi}\int_{-\infty}^{\infty} r\tanh(\pi r)G\left(\frac{1}{2}+ir\right)\,dr + \sum_{\{\gamma_0\}}\frac{\log N(\{\gamma^*\})}{1-\frac{1}{N(\{\gamma_0\})}}g(\log N(\{\gamma_0\})) = \sum_{i=0}^{\infty}G(s_i)\ .$$

Here

$$G(s) = \int_{-\infty}^{\infty} g(y)e^{sy}\,dy\ ,$$

(the functions $g(\cdot)$, $G(\cdot)$ and $k(\cdot)$ can be calculated from each others by means of simple formulas),

$$s_i = \frac{1}{2} + \sqrt{\frac{1}{4}+\lambda_i}\ ,$$

(one has to consider only one of the square roots), where $\lambda_0 = 0$, $\lambda_i < 0$, $(i = 1, 2, \dots)$ are the eigenvalues of $D$ with multiplicity,

$$\sigma(\mathcal{D}) = \text{the area of a fundamental domain} = 4\pi(g-1).$$

(The number $g$ is the genus of $\mathcal{S}$.) In the sum the $\{\gamma_0\}$ runs through the conjugate classes not containing the identity, $\{\gamma^*\}$ is the class of the corresponding primitive element, i.e. $\gamma^*$ is defined by the relations $\gamma^* \in \Gamma$, and ${\gamma^*}^m = \gamma_0$, with the largest possible $m$, $N(\cdot)$ denotes the norm of an element $\gamma \in \Gamma$ introduced in this talk.

This is the famous Selberg formula in the simplest case.