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On a Class of Self-Similar Fields

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In [2] the behavior of some self-similar fields was investigated where their self-similarity parameter is near a critical value. There is another critical value at which these fields become meaningless. In this paper we investigate those fields in the vicinity of the other critical parameter. We also generalize the results of [2] by considering a more general class of self-similar fields.

1. Introduction

In [2] the asymptotic behavior of some self-similar fields was investigated in the case where their self-similarity parameter is near a critical value. These fields, denoted by H_{k,ϵ,a_k}, were defined by the formula

$$H_{k,\epsilon,a_k}(\varphi) = \frac{1}{k!} \int \tilde{\varphi}(x_1 + \cdots + x_k) Z_{G_{k,\epsilon,a_k}}(dx_1) \ldots Z_{G_{k,\epsilon,a_k}}(dx_k) \qquad \varphi \in \mathcal{S}. \tag{1.1}$$

Here $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions in the ν-dimensional Euclidean space $\mathcal{R}^\nu$, G_{k,ϵ,a_k} is the spectral measure with density function

$$\begin{aligned} g_{k,\epsilon,a_k}(x) &= |x|^{-\alpha(k,\epsilon)} a_k\left(\frac{x}{|x|}\right), \qquad x \in \mathcal{R}^\nu, \\ \alpha(k,\epsilon) &= \nu\left(1 - \frac{1}{k}\right) + \frac{\epsilon}{k}, \qquad 0 < \epsilon < \nu, \end{aligned} \tag{1.2}$$

$a_k(v)$ is a real-valued even $]\nu/k[-1$ times differentiable function on the unit sphere $S^{\nu-1}$ of $\mathcal{R}^\nu$, where $]u[$ is the smallest integer not smaller than u, and $\sim$ denotes the Fourier transform. The random spectral measure $Z_{G_{k,\epsilon,a_k}}$ corresponding to the spectral measure G_{k,ϵ,a_k} and the Wiener-Itô integral with respect to it is defined as in [1].

We have also considered the discrete time version of the field H_{k,ϵ,a_k}

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defined by the formula

$$\bar{H}_{k,\epsilon,a_k}(n) = \frac{1}{k!}\int \tilde{\varphi}_n(x_1 + \cdots + x_k) Z_{G_{k,\epsilon,a_k}}(dx_1) \ldots Z_{G_{k,\epsilon,a_k}}(dx_k), \quad n \in \mathfrak{Z}^\nu, \tag{1.3}$$

where $\mathfrak{Z}^\nu$ denotes the ν-dimensional integer lattice and φ_n, $n \in \mathfrak{Z}^\nu$, is the indicator function of the rectangle

$$\times_{j=1}^{\nu}\left[n^{(j)}, n^{(j)} + 1\right), \qquad n = (n^1, \ldots, n^\nu).$$

(Here and in the following we denote the coordinates of a vector by superscripts.)

In [2] it was shown that the fields defined by formulas (1.1) and (1.3) are meaningful for $0 < \epsilon < \nu$. Their asymptotic behavior was investigated in the case when $\epsilon \to 0$. The first result of this paper describes their asymptotic behavior in the case when $\epsilon \to \nu$.

Let us fix a standard normal variable η, and define the generalized fields L_k, $k = 1, 2, \ldots$ by the formula

$$L_k(\varphi) = \tilde{\varphi}(0) H_k(\eta), \qquad \varphi \in \mathcal{S}, \tag{1.4}$$

where H_k denotes the kth Hermite polynomial with leading coefficient 1. Their discrete counterparts are the fields $\bar{L}_k$, $k = 1, 2, \ldots$ defined by the formula

$$\bar{L}_k(n) = H_k(\eta), \qquad n \in \mathfrak{Z}^\nu. \tag{1.5}$$

We emphasize that in the definition of the fields L_k the same random variable η is used for all $\varphi \in \mathcal{S}$ and k. An analogous statement holds for the fields $\bar{L}_k$. Now we formulate the following:

Theorem 1. *For all positive integers N the joint distributions of the fields $k!(\nu - \epsilon)^{k/2} D_k^{-k/2} H_{k,\epsilon,a_k}$, $1 \leqslant k \leqslant N$, tend to the joint distributions of the fields L_k defined in (1.4) if $\epsilon \to \nu$, where $D_k = \int_{S^{\nu-1}} a_k(v)\,dv$. The joint distributions of the discrete fields $k!(\nu - \epsilon)^{k/2} D_k^{-k/2} \bar{H}_{k,\epsilon,a_k}$ tend to the joint distributions of the fields $\bar{L}_k$ defined in (1.5) if $\epsilon \to \nu$.*

Now we consider the following class of self-similar fields H_{k,ϵ,a_k,p,u_k}.

$$H_{k,\epsilon,a_k,p,u_k}(\varphi) = \frac{1}{k!}\int \tilde{\varphi}(x_1 + \cdots + x_k)|x_1 + \cdots + x_k|^p \times u_k\left(\frac{x_1 + \cdots + x_k}{|x_1 + \cdots + x_k|}\right) Z_{G_{k,\epsilon,a_k}}(dx_1) \ldots Z_{G_{k,\epsilon,a_k}}(dx_k) \qquad \varphi \in \mathcal{S}, \tag{1.6}$$

where p is a real number and $u_k(x)$ is a complex-valued continuous function over $S^{\nu-1}$, $u_k(-x) = \overline{u_k(x)}$. Naturally we consider these fields only if the stochastic integrals in (1.6) are meaningful. The discrete versions of fields are the fields $\overline{H}_{k,\epsilon,a_k,p,u_k}$ defined by

$$\overline{H}_{k,\epsilon,a_k,p,u_k}(n) = \frac{1}{k!} \int \tilde{\varphi}_n(x_1 + \cdots + x_k)|x_1 + \cdots + x_k|^p \times u_k\left(\frac{x_1 + \cdots + x_k}{|x_1 + \cdots + x_k|} \right) \times Z_{G_{k,\epsilon,a_k}}(dx_1) \ldots Z_{G_{k,\epsilon,a_k}}(dx_k), \qquad n \in \mathfrak{Z}^\nu. \tag{1.7}$$

The above-defined fields contain the fields defined by formulas (1.1) and (1.3) as special cases. They appeared in some limit theorems in [3]. The next theorem is a generalization of the results in [2].

Let us consider the Gaussian fields H^p_{k,u_k} whose distributions are determined by the formulas

$$EH^p_{k,u_k}(\varphi) = 0, \qquad \varphi \in \mathcal{S} \tag{1.8}$$

and

$$EH^p_{k,u_k}(\varphi)H^p_{k,u_k}(\psi) = \int \tilde{\varphi}(x)\bar{\tilde{\psi}}(x)|x|^{2p}\left|u_k\left(\frac{x}{|x|}\right)\right|^2 dx \qquad \varphi,\psi \in \mathcal{S}. \tag{1.8'}$$

Their discrete versions are the fields $\overline{H}^p_{k,u_k}$ whose distributions are defined by the formulas

$$EH^p_{k,u_k}(n) = 0 \tag{1.9}$$

and

$$EH^p_{k,u_k}(n)H^p_{k,u_k}(m) = \int \tilde{\varphi}_n(x)\bar{\tilde{\varphi}}_m(x)|x|^{2p}\left|u_k\left(\frac{x}{|x|}\right)\right|^2 dx, \qquad n, m \in \mathfrak{Z}^\nu. \tag{1.9'}$$

Theorem 2. *The fields H_{k,ϵ,a_k,p,u_k}, $k \geq 2$, are meaningful if $0 < \epsilon < \nu$ and $2p + \nu - \epsilon > 0$. Assume that these inequalities hold, and let, moreover, $2p + \nu > 0$. Then for all integers N, $N \geq 2$, the joint distributions of the fields $(\epsilon B_k^{-1})^{1/2} H_{k,\epsilon,a_k,p,u_k}$, where $2 \leq k \leq N$, and B_k is as defined in (1.10) of [2], tend to the joint distributions of the independent Gaussian fields H^p_{k,u_k}, $2 \leq k \leq N$, defined in (1.8) and (1.8′), if $\epsilon \to 0$. (We define the independence and the convergence of joint distributions of generalized fields as in [2] in the section before the formulation of Theorem 2.)*

Theorem 2′. *The fields $\overline{H}_{k,\epsilon,a_k,p,u_k}$, $k \geq 2$ are meaningful if $0 < \epsilon < \nu$, $2p + \nu - \epsilon > 0$ and $2p - \epsilon < 0$. Assume that these inequalities hold, and let, moreover, $-\nu < 2p < 1$. Then for all integers N, $N \geq 2$, the joint distributions of the fields $(\epsilon B_k^{-1})^{1/2}\overline{H}_{k,\epsilon,a_k,p,u_k}$, $2 \leq k \leq N$, tend to the joint distributions of the independent Gaussian fields $\overline{H}^p_{k,u_k}$, $2 \leq k \leq N$, defined in (1.9) and (1.9′) if $\epsilon \to 0$.*

In the special case $p=0$, $u_k(x)\equiv 1$, Theorem 2 of this paper coincides with Theorem 2 of [2]. We could not find the counterpart of Theorem 1 if the more general class of fields defined in (1.6) and (1.7) is considered. We remark that in Theorem 2′ more restrictive conditions had to be imposed for the existence of the random fields than in Theorem 2. This difference is due to the fact that the functions φ_n, $n\in\mathfrak{Z}^\nu$, unlike the functions $\varphi\in\mathcal{S}$, tend relatively slowly to zero at infinity.

This paper consists of three sections. Section 2 contains the proof of Theorem 1 and Section 3 the proof of Theorems 2 and 2′.

2. Proof of Theorem 1

Let us define the measures $\mu^\epsilon_{k,\varphi}$, $\varphi\in\mathcal{S}$, on $\mathfrak{R}^{k\nu}$ by the formula

$$\mu^\epsilon_{k,\varphi}(A)=\int_A|\tilde\varphi(x_1+\cdots+x_k)|^2G_{k,\epsilon}(dx_1)\dots G_{k,\epsilon}(dx_k),\qquad A\in\mathfrak{B}^{k\nu}.$$

(Here and in the following we omit the index a_k if it leads to no ambiguity. The same will be done with the index u_k.) The main step of the proof is to show that the measure $\mu^\epsilon_{k,\varphi}$ is essentially concentrated in a small neighborhood of the origin if $\epsilon\approx\nu$. First we prove that for all $\varphi\in\mathcal{S}$ and $d>0$

$$\int|\tilde\varphi(x_1+\cdots+x_k)|^2I(|x_1|>d)G_{k,\epsilon}(dx_k)\dots G_{k,\epsilon}(dx_k)$$
$$=O\big((\nu-\epsilon)^{-k+1/3}\big).\qquad(2.1)$$

(We denote by $I(A)$ the indicator function of the set A.) To this end, we first show that for $\nu-1/2<\epsilon<\nu$,

$$I=\left|\int e^{i(t,x)}|x|^{-\epsilon}a\Big(\frac{x}{|x|}\Big)\exp\Big(-\frac{|x|^2}{2A}\Big)I(|x|>d)\,dx\right|$$
$$<K|t|^{-2/3}(\nu-\epsilon)^{-2/3},\qquad(2.2)$$

where the constant K may depend on d but not on A. First we prove (2.2) in the case $\nu=1$. Because of the monotonicity of the function

$$|x|^{-\epsilon}\exp(-|x|^2/2A)$$

and the periodicity of the trigonometric functions, we get that

$$I\leqslant C\left|\int_{-\infty}^{\infty}\cos(tx)\exp\Big(-\frac{x^2}{2A}\Big)|x|^{-\epsilon}\,dx\right|\leqslant 2C\int_0^{2\pi/|t|}|x|^{-\epsilon}\,dx$$
$$=\frac{C'}{1-\epsilon}|t|^{1-\epsilon}<\frac{C'}{1-\epsilon}<K|t|^{-2/3}(1-\epsilon)^{-2/3}\qquad\text{if}\quad|t|<(1-\epsilon)^2,$$

and

$$I\leqslant 4\int_d^{d+(2\pi/|t|)}|x|^{-\epsilon}\,dx\leqslant\frac{C|d|^{-\epsilon}}{|t|}\leqslant K|t|^{-2/3}(1-\epsilon)^{-2/3}\quad\text{if}\quad|t|\geqslant(1-\epsilon)^2.$$

In the case $\nu > 1$ we get, by rewriting the expression in (2.2) in polar coordinates and by applying (2.2) for the case $\nu = 1$, that

$$I = \int_{S^{\nu-1}} a(s) \int_{-\infty}^{\infty} |r|^{\nu-1-\epsilon} \exp\left(-\frac{r^2}{2A}\right) I(|r| > d) \exp\left[i\left(\frac{t}{|t|}, s\right)|t|r\right] dr\, ds$$

$$\leq K(\nu-\alpha)^{-2/3}|t|^{-2/3} \int_{S^{\nu-1}} a(s) \left|\left(\frac{t}{|t|}, s\right)\right|^{-2/3} ds \leq \bar{K}|t|^{-2/3}(\nu-\alpha)^{-2/3}.$$

Now we can estimate the integral in (2.1) by the method used in the proof of Lemma 2 in [2]. By the monotone convergence theorem,

$$\int |\tilde{\varphi}(x_1 + \cdots + x_k)|^2 I(|x_1| > d) G_{k,\epsilon}(dx_1) \ldots G_{k,\epsilon}(dx_k)$$

$$= \lim_{A\to\infty} \int |\tilde{\varphi}(x_1 + \cdots + x_k)|^2 I(|x_1| > d) \exp\left(-\frac{|x_1|^2}{2A}\right) \cdots$$

$$\exp\left(-\frac{|x_k|^2}{2A}\right) G_{k,\epsilon}(dx_1) \ldots G_{k,\epsilon}(dx_k) = \lim_{A\to\infty} I_A(\epsilon). \tag{2.3}$$

By applying the same argument for the estimation of $I_A(\alpha)$ as in Lemma 2 of [2], we get, denoting by $\psi(u)$ the function whose Fourier transform is $|\tilde{\varphi}(u)|^2$,

$$I_A(\epsilon) = \int \psi(u) \left[\int e^{i(x,u)} |x|^{-\epsilon} \exp\left(-\frac{|x|^2}{2A}\right) dx\right]^{k-1}$$

$$\times \left[\int e^{i(x,u)} |x|^{-\epsilon} \exp\left(-\frac{|x|^2}{2A}\right) I(|x| > d)\, dx\right] du.$$

We can estimate the first inner integral in the last formula by formula (2.6) of Lemma 1 of [2] and the second inner integral by formula (2.2). Let us observe that $C_\alpha < K(\nu-\alpha)^{-1}$ for $\nu - 1/2 < \alpha < \nu$ in Lemma 1 of [2], as can be seen from (2.7) and the last formula in the proof of Lemma 1 of that paper. Hence we obtain

$$|I_A(\epsilon)| \leq \bar{K}(\nu-\alpha)^{-k+1/3} \int |\psi(u)|\, |u|^{-(k-1)(\nu-\epsilon)-2/3} du.$$

The last integral is convergent for $(k-2)/(k-1)^{\nu} + 2/3(k-1) < \epsilon < \nu$; hence the last relation, together with (2.3), implies relation (2.1).

Obviously we may replace $I(|x_1| > d)$ by $I(|x_j| > d)$, $j = 1, \ldots, k$, in (2.1). Hence, by exploiting the L_2 isomorphism property of Wiener-Itô integrals, we obtain

$$E\Big[(\nu-\epsilon)^{k/2}\Big(\int \tilde{\varphi}(x_1+\cdots+x_k)Z_{G_{k,\epsilon}}(dx_1)\ldots Z_{G_{k,\epsilon}}(dx_k)$$
$$-\tilde{\varphi}(0)\int I(|x_1|<d)\ldots I(|x_k|<d)Z_{G_{k,\epsilon}}(dx_1)\ldots Z_{G_{k,\epsilon}}(dx_k)\Big)\Big]^2$$
$$=\frac{1}{k!}(\nu-\epsilon)^k\int\big[\tilde{\varphi}(x_1+\cdots+x_k)-\tilde{\varphi}(0)I(|x_1|<d)\ldots$$
$$I(|x_k|<d)\big]^2 G_{k,\epsilon}(dx_1)\ldots G_{k,\epsilon}(dx_k)$$
$$\leqslant K_1(\nu-\epsilon)^{1/3}+K_2\sup_{|x|<kd}|\tilde{\varphi}(x)-\tilde{\varphi}(0)|^2. \tag{2.4}$$

By Itô's formula,

$$(\nu-\epsilon)^{k/2}\tilde{\varphi}(0)\int I(|x_1|<d)\ldots I(|x_k|<d)Z_{G_{k,\epsilon}}(dx_1)\ldots Z_{G_{k,\epsilon}}(dx_k)$$
$$=\tilde{\varphi}(0)\Big[(\nu-\epsilon)\int I(|x|<d)G_{k,\epsilon}(dx)\Big]^{k/2}H_k(\eta), \tag{2.5}$$

where $\eta=[\int I(|x|<d)G_{k,\epsilon}(dx)]^{-1/2}\int I(|x|<d)Z_{G_{k,\epsilon}}(dx)$ is a standard normal random variable. On the other hand, it is not difficult to see that

$$(\nu-\epsilon)\int I(|x|<d)G_{k,\epsilon}(dx)\to\int_{S^{\nu-1}}a_k(v)\,dv \qquad \text{as } \epsilon\to\nu. \tag{2.6}$$

Let $\nu-\epsilon$ and d tend to zero. If d tends to zero sufficiently slowly, then the right side of (2.4) tends to zero (the constant K_1 depends on d), and relation (2.6) remains valid. Hence formulas (2.4), (2.5), and (2.6) imply that $(\nu-\epsilon)^{k/2}D_k^{-k/2}H_{k,\epsilon}(\varphi)$ tends in distribution to $L_k(\varphi)$ as $\epsilon\to\nu$. The multidimensional convergence of these random variables and the discrete field version of this result can be proved in the same way.

3. Proof of Theorem 2

In [2] two different methods were presented for proving the results of that paper. Both methods could be adapted for proving Theorem 2 of this paper. We have chosen the second method since fewer changes are needed in the proof.

Let us define the measures $\mu_{k,\epsilon}$, $0<\epsilon<\nu$, on $\mathfrak{R}^\nu$ by the formula

$$\mu_{k,\epsilon}(A)=\int_{B(A)}G_{k,\epsilon}(dx_1)\ldots G_{k,\epsilon}(dx_k), \tag{3.1}$$

where

$$B(A)=\{x=(x_1,\ldots,x_k)\in\mathfrak{R}^{k\nu},\,x_1+\cdots+x_k\in A\}.$$

We need some properties of the measures $\mu_{k,\epsilon}$. We recall that a sequence of locally finite measures G_n (i.e., $G_n(A) < \infty$ for all bounded sets $A \in \mathcal{B}^\nu$) is said to tend *vaguely* to a locally finite measure G_0 if

$$\int f(x) G_n(dx) \to \int f(x) G_0(dx)$$

for all continuous functions with bounded support.

Lemma 1. *For all $k = 2, 3, \ldots$ the measures $\epsilon D_k^{-1} \mu_{k,\epsilon}$ tend vaguely to the Lebesgue measure λ on $\mathcal{R}^\nu$ as $\epsilon \to 0$, where $D_k = (2\pi)^{-\nu} \int_{S^{\nu-1}} b_{\alpha_0}(v)\,dv$, $\alpha_0 = ((k-1)/k)\nu$, and b_α is as defined in Lemma* 1 *of* [2].

Proof of Lemma 1. The proof of Lemma 2 in [2], with some small modifications, gives

$$\epsilon D_k \int \varphi(x) \mu_{k,\epsilon}(dx) \to \int \varphi(x)\,dx$$

for all $\varphi \in \mathcal{S}$. Since all infinitely differentiable functions with compact support belong to $\mathcal{S}$, and this class of functions is everywhere dense in the class of continuous functions with compact support (with respect to the supremum norm), this relation implies Lemma 1.

We need the next result in order to control the behavior of the measures $\mu_{k,\epsilon}$ at infinity and in a neighborhood of the origin.

Lemma 2. *For all $A \geqslant 1$, $\delta > 0$, $0 < \epsilon < \nu - \delta$ and $\varphi \in \mathcal{S}$,*

a.
$$\epsilon \left| \int_{\{|x| > A\}} \varphi(x) |x|^{2p} \mu_{k,\epsilon}(dx) \right| < C_k A^{-\eta}$$

$$\epsilon \left| \int_{\{|x| < 1/A\}} \varphi(x) |x|^{2p} \mu_{k,\epsilon}(dx) \right| < C_k A^{-\eta}$$

with $\eta = 2p + \nu - \epsilon$ if $\eta > \delta$. (The constant C_k may depend on φ, p and δ.)

b.
$$\epsilon \int_{\{|x| > A\}} \prod_{j=1}^{\nu} \frac{1}{1 + (x^{(j)})^2} |x|^{2p} \mu_{k,\epsilon}(dx) < C_k A^{-\zeta},$$

$$\epsilon \int_{\{|x| < 1/A\}} \prod_{j=1}^{\nu} \frac{1}{1 + (x^{(j)})^2} |x|^{2p} \mu_{k,\epsilon}(dx) < C_k A^{-\eta}$$

with $\zeta = 1 - 2p + \epsilon$ and η the same as in part (a) *if $\eta > \delta$ and $\zeta > \delta$. (Here C_k may depend on δ and p.)*

Proof of Lemma 2. Define the measures $\bar{\mu}_{k,\epsilon}$,

$$\bar{\mu}_{k,\epsilon}(A) = \int_{B(A)} |x_1|^{-\alpha(k,\epsilon)} \ldots |x_k|^{-\alpha(k,\epsilon)}\,dx_1 \ldots dx_k, \qquad A \in \mathcal{B}^\nu.$$

Obviously $\mu_{k,\epsilon}(A) \leqslant [\sup_{v \in S^{\nu-1}} a_k(v)]^k\, \bar{\mu}_{k,\epsilon}(A)$ for all $A \in \mathcal{B}^\nu$, hence $\mu_{k,\epsilon}$ can be replaced by $\bar{\mu}_{k,\epsilon}$ in Lemma 2. We claim that

$$\bar{\mu}_{k,\epsilon}(B) = C(k,\epsilon)\int_B |x|^{-\epsilon}\,dx \text{ for all } B \in \mathfrak{B}^\nu \text{ and } C(k,\epsilon) < C(k)\epsilon^{-1}. \tag{3.2}$$

The first relation in (3.2) follows from the homogeneity property $\bar{\mu}_{k,\epsilon}(\lambda B) = \lambda^{-\epsilon+\nu}\,\bar{\mu}_{k,\epsilon}(B)$ for all $\lambda > 0$ and $B \in \mathfrak{B}^\nu$, and the invariance of $\bar{\mu}_{k,\epsilon}$ under all rotations of $\mathfrak{R}^\nu$. The second relation in (3.2) can be read from Lemmas 1 and 2 in [2]. Indeed, to check this relation it is enough to check, for example, that the right side of (2.12) in [2] is less than $C(k)\epsilon^{-1}$ if $\alpha = \nu(1-1/k) + \epsilon/k$. But in our case $a(v) \equiv 1$, therefore

$$\tilde{f}_\alpha(u) = C(\alpha)|u|^{\alpha-\nu} \quad \text{with } |C(\alpha)| < C \quad \text{for } \nu(1-1/k) < \alpha < \nu - \delta/k$$

by Lemma 1 of [2], and these facts imply the desired relation. Because of formula (3.2) the following estimates imply part (a) of Lemma 2:

$$\left|\int_{\{|x|>A\}} \varphi(x)|x|^{2p-\epsilon}\,dx\right| < C\int_{\{|x|>A\}} |x|^{-\eta-\nu}\,dx < C'A^{-\eta}$$

since $|\varphi(x)| < C(\alpha)|x|^{-\alpha}$ for all $|x| > 1$ and $\alpha > 0$ if $\varphi \in \mathcal{S}$.

$$\left|\int_{\{|x|<1/A\}} \varphi(x)|x|^{2p-\epsilon}\,dx\right| < C\int |x|^{2p-\epsilon}\,dx < C'A^{-\eta}.$$

The second inequality of part (b) can be proved in the same way. To prove the first inequality we have to show that

$$I = \int_{\{|x|>A\}} \prod_{j=1}^{\nu} \frac{1}{1+(x^{(j)})^2}\,|x|^{2p-\epsilon}\,dx < C_k A^{-\zeta}.$$

Obviously,

$$I \leqslant \nu \int_{\{|x|>A,\ \max_{1\leqslant j\leqslant \nu}|x^{(j)}| = |x^{(1)}|\}} |x|^{2p-\epsilon} \prod_{j=1}^{\nu} \frac{1}{1+(x^{(j)})^2}\,dx.$$

Observe that $\max_{1\leqslant j\leqslant\nu}|x^{(j)}| \leqslant |x| \leqslant \nu \max_{1\leqslant j\leqslant\nu}|x^{(j)}|$. Hence $|x|^{2p-\epsilon} \leqslant \max(1, (\nu|x^{(1)}|)^{2p-\epsilon})$ on the set $\{|x^{(1)}| = \max_{1\leqslant j\leqslant\nu}|x^{(j)}|,\ |x| > A\}$, and that this set is contained in the set $\{|x^{(1)}| > A/\nu\}$. Therefore

$$\begin{aligned} I &\leqslant C\int_{\{|x_1|>A/\nu\}} |x^{(1)}|^{2p-\epsilon} \prod_{j=2}^{\nu} \frac{1}{1+(x^{(j)})^2}\,dx \\ &= 2C\left[\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx\right]^{\nu-1} \int_{A/\nu}^{\infty} \frac{x^{2p-\epsilon}}{1+x^2}\,dx \\ &\leqslant C'A^{-\zeta}. \end{aligned}$$

Lemma 2 is proved.

Now we can prove that the variance of $(\epsilon B_k^{-1})^{1/2} H_{k,\epsilon,p}(\varphi)$ tends to the variance of $H_p^k(\varphi)$, $\varphi \in \mathcal{S}$ as $\epsilon \to 0$. One has to observe that

$$D_k \epsilon \int_{\{1/A < |x_1 + \cdots + x_k| < A\}} |\tilde{\varphi}(x_1 + \cdots + x_k)|^2 |x_1 + \cdots + x_k|^{2p} \times \left| u_k\left(\frac{x_1 + \cdots + x_k}{|x_1 + \cdots + x_k|} \right) \right|^2 G_{k,\epsilon}(dx_1) \ldots G_{k,\epsilon}(dx_k)$$

$$= \epsilon D_k \int_{\{1/A < |x| < A\}} |\tilde{\varphi}(x)|^2 |x|^{2p} \left| u_k\left(\frac{x}{|x|} \right) \right|^2 \mu_{k,\epsilon}(dx)$$

$$\to \int_{\{1/A < |x| < A\}} |\tilde{\varphi}(x)|^2 |x|^{2p} \left| u_k\left(\frac{x}{|x|} \right) \right|^2 dx \qquad \text{as} \quad \epsilon \to 0$$

for all $A > 1$ because of Lemma 1. The corresponding integrals on the sets $|x_1 + \cdots + x_k| > A$ and $|x_1 + \cdots + x_k| < 1/A$ are negligible for large A because of Lemma 2. These facts imply the desired convergence. By exploiting the orthogonality of Wiener-Itô integrals of different multiplicity, one can see immediately that the linear combinations of the variables $(\epsilon B_k^{-1})^{1/2} H_{k,\epsilon,p}(\varphi_k)$ also have the prescribed asymptotic variance. Since $|\tilde{\varphi}_n(x)|^2 \leqslant C \prod_{j=1}^{\nu} 1/(1 + |x^{(j)}|^2)$, $n \in \mathfrak{Z}^{\nu}$, the asymptotic variance in the discrete field case can be determined in the same way. It is not difficult to see, by using Lemma 2, that the fields $H_{k,\epsilon,p}$ and $\overline{H}_{k,\epsilon,p}$ exist under the conditions of Theorems 2 and 2′.

We complete the proof of Theorems 2 and 2′ by slightly modifying the arguments of Section 4 in [2].

Let us define the measures $\mu_{k,\epsilon,\varphi}^p$ on $R^{k\nu}$ by the formula

$$\mu_{k,\epsilon,\varphi}^p(B) = \int_B |\tilde{\varphi}(x_1 + \cdots + x_k)|^2 |x_1 + \cdots + x_k|^2 G_{k,\epsilon}(dx_1) \ldots G_{k,\epsilon}(dx_k),$$
$$B \in \mathfrak{B}^{k\nu}, \tag{3.3}$$

where either $\varphi \in \mathcal{S}$ or $\varphi = \varphi_n$, $n \in \mathfrak{Z}^n$. It follows from Lemma 2 that

$$I_1^{\epsilon}(A) = \mu_{k,\epsilon,\varphi}^p\left(|x_1 + \cdots + x_k| > A \text{ or } |x_1 + \cdots + x_k| < \frac{1}{A} \right)$$
$$\leqslant C_k \epsilon^{-1} A^{-\eta^*} \tag{3.4}$$

with $\eta^* = \eta$ if $\varphi \in \mathcal{S}$ and $\eta^* = \min(\eta, \zeta)$ if $\varphi = \varphi_n$, $n \in \mathfrak{Z}^{\nu}$. Here η and ζ are the same as in Lemma 2. Inequality (3.4) corresponds to inequality (4.4) in [2]. Inequalities (4.5) and (4.6) in [2] also remain valid after the following changes. In the definitions of I_2^{ϵ} and I_3^{ϵ} we replace the measure $\mu_{k,\epsilon,\varphi}$ by $\mu_{k,\epsilon,\varphi}^p$, defined in (4.3), the event $|x_1 + \cdots + x_k| < A$ by the event $1/A < |x_1 + \cdots + x_k| < A$, and we write $A^{\nu + 2|p|}$ instead of A^{ν} on the right side of these inequalities. Now we can complete the proof of Theorems 2 and 2′ by making small changes in the proof of Section 4 in [2]. The sequences $a_j = a_j(\epsilon)$ and the sets D_k^j, $\overline{D}^k$ can be defined in the same way as in [2]. It can be seen, by choosing $A = \epsilon^{-\delta}$ with a sufficiently small $\delta > 0$, that relations (4.1) and (4.2) of [2] remain valid if $\mu_{k,\epsilon,\varphi}$ is replaced by $\mu^{P}_{k,\epsilon,\varphi}$. The estimates needed for the justification of these relations are almost the same as the estimates at the end of Section 4 in [2]. These relations imply that the central limit theorem holds for the elements of the classes $(\epsilon B_k^{-1})^{1/2} H_{k,\epsilon}$ and $(\epsilon B_k^{-1})^{1/2} \overline{H}_{k,\epsilon}$. Since the asymptotic variance of the random variables of these fields is already determined, Theorems 2 and 2′ are proved.

References

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