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AN IMPROVEMENT OF STRASSEN'S INVARIANCE PRINCIPLE

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Let a distribution function $F(x)$, $\int x\,dF(x) = 0$, $\int x^2 dF(x) = 1$ be given. Strassen constructed two sequences $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ of independent, identically distributed random variables, the X_i with distribution function $F(x)$ and the Y_i with standard normal distribution, in such a way that the partial sums $S_n = \sum_{i=1}^n X_i$ and $T_n = \sum_{i=1}^n Y_i$ satisfy the relation $|S_n - T_n| = O((n \log\log n)^{\frac{1}{2}})$ with probability 1. Earlier we proved that this result cannot be improved. Now we show however that an approximation $|S_n - T_n| = O(n^{\frac{1}{2}})$ can be achieved, if the Y_i are independent normal variables whose variances are appropriately chosen.

Introduction. The main result of this paper is the following.

THEOREM. *Let a distribution function* $F(x)$, $\int x\,dF(x) = 0$, $\int x^2 dF(x) = 1$ *be given. Define*

$$\sigma_k^2 = \int_{-2^{\frac{n}{2}}}^{2^{\frac{n}{2}}} x^2 dF(x) - \left[\int_{-2^{\frac{n}{2}}}^{2^{\frac{n}{2}}} x\,dF(x)\right]^2 \quad \text{if} \quad 2^n \le k < 2^{n+1}.$$

A sequence of i.i.d. rv's $X_1, X_2, \cdots$ *with distribution function* $F(x)$ *and a sequence of independent normal random variables* $Y_1, Y_2, \cdots$ $EY_k = 0$, $EY_k^2 = \sigma_k^2$ *can be constructed in such a way that the partial sums* $S_n = \sum_{k=1}^n X_k$, $T_n = \sum_{k=1}^n Y_k$, $n = 1, 2, \cdots$ *satisfy the relation*

$$|S_n - T_n| = O(n^{\frac{1}{2}}) \qquad \text{w.p. } 1. \tag{1.1}$$

Both the functional limit theorem for i.i.d. rv's and Strassen's strong invariance principle are easy consequences of this theorem.

Let us make some remarks to this theorem. We investigated the following problem in several papers [3], [4], [5], [6]: Let a sequence of i.i.d. rv's $X_1, X_2, \cdots$ $EX_1 = 0$, $EX_1^2 = 1$ and a function $f(x)$ be given. Is it possible to construct a sequence of i.i.d rv's with standard normal distribution in such a way that the partial sums $S_n = \sum_{k=1}^n X_k$, $T_n = \sum_{k=1}^n Y_k$ satisfy the relation

$$\frac{S_n - T_n}{f(n)} \to 0 \qquad \text{w.p. } 1? \tag{1.2}$$

Let us assume that $x^{\frac{1}{2}-\varepsilon} > f(x)$ with some $\varepsilon > 0$, $\log x / f(x) \to 0$, and $f(x)$ is a sufficiently smooth function. The answer is affirmative if

$$\sum_{n=1}^\infty P(|X_1| > f(n)) < \infty.$$

On the other hand the answer is negative if

$$\sum_{n=1}^\infty P(|X_1| > f(n)) = \infty \tag{1.3}$$

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(see also the remarks after Theorem 3 in [4]), as the following simple argument of Breiman [1] shows.

If relation (1.3) holds then by the Borel-Cantelli lemma

$$|S_n - S_{n-1}| = |X_n| > f(n) \qquad \text{i.o.} \quad \text{w.p.} \ 1.$$

On the other hand

$$|T_n - T_{n-1}| < \tfrac{1}{2}f(n) \qquad \text{for} \quad n > n_0(\omega) \quad \text{w.p.} \ 1.$$

Therefore,

$$|S_n - T_n| > \tfrac{1}{4}f(n) \qquad \text{i.o.} \quad \text{w.p.} \ 1.$$

Now we are interested in the following question. How can $f(x)$ be chosen in (1.2), if no more than the existence of EX_1^2 is assumed?

Strassen [8] proved that $f(x)$ can be chosen as $(x \log \log x)^{\frac{1}{2}}$. We proved in [6] that Strassen's result cannot be improved. Since $EX_1^2 < \infty$ is equivalent to

$$(1.4) \qquad \Sigma_{n=1}^{\infty} P\big(|X_1| > n^{\frac{1}{2}}\big) < \infty,$$

Breiman's argument gives only that $f(x)$ cannot be smaller than $x^{\frac{1}{2}}$. Our present result means that if the EY_n^2's are appropriately chosen then an approximation satisfying (1.2) with $f(x) = x^{\frac{1}{2}}$ is possible. Thus Breiman's argument yields a sharp lower bound in this case too.

Let us explain why a better approximation can be expected if the variances of the Y_n's are changed. Define

$$X_n' = X_n \quad \text{if} \quad |X_n| < n^{\frac{1}{2}}$$
$$= 0 \qquad \text{otherwise}$$

and

$$S_n' = \Sigma_{k=1}^{n} X_k'.$$

Then relation (1.4) implies $X_n = X_n'$ for $n > n(\omega)$ w.p. 1.

Thus

$$|S_n - S_n'| < K(\omega) \qquad \text{w.p. } 1;$$

therefore we may substitute S_n with S_n' in (1.2). But doing so, it is natural to couple X_n' with a Y_n whose first two moments agree with those of X_n'. We are going to show that this coupling can be done with an accuracy defined by (1.1). It is not difficult to see that EX_n' is very small, therefore Y_n can be substituted with $Y_n - EY_n$ without violating (1.1). But if we change the variance so that $EY_n^2 = 1$, then (1.1) may not hold any longer.

D^2X_n slightly differs from σ_n^2. But having proved the theorem we can easily prove the following

COROLLARY. *Let a sequence $a_n > 0$, $n = 1, 2, \cdots$ be given that satisfies $An^{\frac{1}{2}} < a_n < Bn^{\frac{1}{2}}$ with some $B > A > 0$. Define*

$$\sigma_n'^2 = \textstyle\int_{-a_n}^{a_n} x^2 dF(x) - \left[\int_{-a_n}^{a_n} x dF(x)\right]^2.$$

Then a sequence of independent normal random variables $Y_1', Y_2', \cdots, EY_n' = 0$, $EY_n'^2 = \sigma_n'^2$ *can be constructed in such a way that* S_n *and* $T_n' = \Sigma_{k=1}^n Y_k'$, $n = 1, 2, \cdots$ *satisfy* (1.1).

2. Proof of the theorem. The proof will be a refinement of that given in [6]. The main difference is that in [6] we applied a theorem of Heyde which is actually a consequence of the Berry-Esseen inequality. Now we use the following strengthened version of the Berry-Esseen inequality:

THEOREM A (see [6]). *Let* $X_1, X_2, \cdots, X_n$ *be* i.i.d rv's. $EX_1 = 0$, $EX_1^2 = 1$, $E|X_1|^3 = \mu_3$. *Define*

$$F_n(x) = P\left(n^{-\frac{1}{2}}\Sigma_{k=1}^n X_k < x\right).$$

Then the inequality

$$|F_n(x) - \phi(x)| \leqslant \frac{C\mu_3}{n^{\frac{1}{2}}(1 + |x|^3)}$$

holds true, where C *is a universal constant.* ($\phi(x)$ *is the standard normal distribution function.*)

Let X be a rv with distribution function $F(x)$. Set

$$X^n = X \quad \text{if} \quad |X_n| < 2^{\frac{n}{2}}$$
$$= 0 \quad \text{otherwise}.$$

Let us consider a sequence $X_1^{(n)}, X_2^{(n)}, \cdots, X_{2^n}^{(n)}$ of i.i.d. rv's where $X_1^{(n)}$ has the same distribution as $X^{(n)}$. Define

$$S_k^{(n)} = \Sigma_{j=1}^k X_j^{(n)}, \qquad k = 1, 2, \cdots, 2^n.$$

We consider another sequence $Y_1^{(n)}, \cdots, Y_{2^n}^{(n)}$ of i.i.d. rv's where $Y_1^{(n)}$ is normally distributed, $EY_1^{(n)} = EX^{(n)}$, and $EY_1^{(n)^2} = EX^{(n)^2}$. Put $T_k^{(n)} = \Sigma_{j=1}^k Y_j^{(n)}$, $k = 1, 2, \cdots, 2^n$. First we show that in order to prove our theorem it is enough to construct two sequences $X_k^{(n)}, Y_k^{(n)}$, $k = 1, \cdots, 2^n$ satisfying the relation

$$P\left(\sup_{k \leqslant 2^n} |S_k^{(n)} - T_k^{(n)}| > \varepsilon_n 2^{\frac{n}{2}}\right) < \delta_n, \tag{2.1}$$

where $\varepsilon_n > 0$, $\delta_n > 0$, $n = 1, 2, \cdots$ are appropriately chosen numerical sequences with the property $\varepsilon_n \to 0$, $\Sigma\delta_n < \infty$. In fact, if (2.1) holds we can construct sequences of rv's $X_k^{(n)}, Y_k^{(n)}$, $n = 1, 2, \cdots, k = 1, 2, \cdots, 2^n$ which are independent for different n and which satisfy (2.1) for every n. Then we define two infinite sequences of rv's as follows. If $k = 2^n + j$, $1 \leqslant j \leqslant 2^n$, then $\tilde{X}_k = X_j^{(n)}$ and $\tilde{Y}_k = Y_j^{(n)}$. The partial sums $\tilde{S}_k = \Sigma_{l=1}^k \tilde{X}_l$, $\tilde{T}_k = \Sigma_{l=1}^k Y_l$ can be expressed as

$$\tilde{S}_k = \Sigma_{l=1}^{n-1} S_{2^l}^{(l)} + S_j^{(n)}$$

$$\tilde{T}_k = \Sigma_{l=1}^{n-1} T_{2^l}^{(l)} + T_j^{(n)}.$$

Relation (2.1) implies that

$$\Sigma_{n=1}^{\infty} P\Big(\sup_{0\leqslant j\leqslant 2^n}|\big(\tilde{S}_{2^n+j} - \tilde{S}_{2^n}\big) - \big(\tilde{T}_{2^n+j} - \tilde{T}_{2^n}\big)| > \varepsilon_n(2^n)^{\frac{1}{2}}\Big) < \Sigma\,\delta_n < \infty.$$

Therefore for $2^n < k \leqslant 2^{n+1}$

$$|\tilde{S}_k - \tilde{T}_k| \leqslant K(\omega) + \Sigma_{j=1}^{n}\varepsilon_j\, 2^{\frac{j}{2}};$$

that gives

$$|\tilde{S}_n - \tilde{T}_n| = o\big(n^{\frac{1}{2}}\big). \tag{2.2}$$

If the probability space, where the random variables $\tilde{X}_k - s$ and $\tilde{Y}_k - s$ have been defined, is large enough, one can define a sequence X_k, $k = 1, 2, \cdots$ of i.i.d. rv's with distribution function $F(x)$ in such a way that $X_k = \tilde{X}_k$ on the set $\{|X_k| < 2^{\frac{n}{2}}\}$; $k = 2^n + j$; $1 \leqslant j \leqslant 2^n$. These X_k and $\tilde{X}_k$ satisfy the relation

$$\Sigma P(X_k \neq \tilde{X}_k) = \Sigma_{n=1}^{\infty} 2^n P(X^2 > 2^n) < \infty,$$

therefore

$$|S_n - \tilde{S}_n| \leqslant K(\omega) \qquad \text{w.p. } 1 \tag{2.3}$$

where $S_n = \Sigma_{k=1}^{n} X_k$.

Set $Y_n = \tilde{Y}_n - E\tilde{Y}_n$ and $T_n = \Sigma_{k=1}^{n} Y_k$. If we show that

$$|T_n - \tilde{T}_n| = o\big(n^{\frac{1}{2}}\big) \tag{2.4}$$

then formulas (2.2), (2.3) and (2.4) prove that (2.1) implies the theorem.

But

$$|T_n - \tilde{T}_n| = |\Sigma_{k=1}^{n} E\tilde{X}_k| \leqslant \Sigma_{k=1}^{n} |E\tilde{X}_k|.$$

Thus in order to prove (2.4) it is enough to show that

$$n^{-\frac{1}{2}}\Sigma_{k=1}^{n}|E\tilde{X}_k| \to 0.$$

Now this is a consequence of the following estimation and the Kronecker lemma

$$\begin{aligned}\Sigma_{k=1}^{\infty} k^{-\frac{1}{2}}|E\tilde{X}_k| &\leqslant \Sigma_{n=1}^{\infty}(2)^{\frac{n}{2}}\Big|\int_{-2^{\frac{n}{2}}}^{2^{\frac{n}{2}}} x\, dF(x)\Big| \\ &\leqslant \Sigma_{n=1}^{\infty}(2)^{\frac{n}{2}}\int_{|x|>2^{\frac{n}{2}}} |x|\, dF(x) \\ &\leqslant \big(2^{\frac{1}{2}} + 1\big)\Sigma_{n=1}^{\infty}\int_{2^{\frac{n}{2}} < |x| < 2^{\frac{n+1}{2}}} x^2\, dF(x) < \infty.\end{aligned}$$

Now we will reduce formula (2.1) to a weaker statement. Define

$$\alpha_n = (2)^{-\frac{n}{2}}\int_{-2^{\frac{n}{2}}}^{2^{\frac{n}{2}}} |x|^3\, dF(x).$$

We claim that

$$\Sigma_{n=1}^{\infty}\alpha_n < \infty. \tag{2.5}$$

Indeed,

$$\begin{aligned}\Sigma_n \alpha_n &= \Sigma\,(2)^{-\frac{n}{2}}\int_{-2^{\frac{n}{2}}}^{2^{\frac{n}{2}}} |x|^3\, dF(x) \\ &\leqslant \Sigma\,\frac{\big(2 + 2^{\frac{1}{2}}\big)}{2^{\frac{n}{2}}}\int_{2^{\frac{n}{2}} \leqslant |x| < 2^{\frac{n+1}{2}}} |x|^3\, dF(x) \leqslant \big(4 + 2(2^{\frac{1}{2}})\big)\int_{-\infty}^{\infty} x^2\, dF(x) < \infty.\end{aligned}$$

One can define a sequence u_n, $u_n > 0$, $u_n \to \infty$, $2^n(\log(1/\alpha_n))^{-1} > u_n > (\log(1/\alpha_n))^{-1}$ in such a way that even

$$(2.5)' \qquad \Sigma u_n^{\frac{3}{4}} \alpha_n < \infty$$

would hold true. (We may assume that $n > n_0$ for some n_0.)

Now we reduce (2.1) to the following formula

$$(2.1)' \qquad P\left(\sup_{0 \leqslant k \leqslant 2^n/m} |S_{km}^{(n)} - T_{km}^{(n)}| > \varepsilon_n 2^{\frac{n}{2}}\right) < \delta_n,$$

where $\varepsilon_n > 0$, $\delta_n > 0$, $\Sigma \delta_n < \infty$, $\varepsilon_n \to 0$, m is a power of 2 satisfying the inequality $2^n(u_n \log(1/\alpha_n))^{-1} \leqslant m < 2^{n+1}(u_n \log(1/\alpha_n))^{-1}$. To show that (2.1)′ implies (2.1) we need some estimation.

Applying Theorem A and Lemma 3.21 in [2] we have for any $k \leqslant 2^n/m$

$$\begin{aligned}
P\Big(\sup_{km \leqslant j < (k+1)m} \big|\big(S_j^{(n)} - S_{km}^{(n)}\big) - \big(ES_j^{(n)} - ES_{km}^{(n)}\big)\big| > 2^{\frac{n}{2}} u_n^{-\frac{1}{4}}\Big) \\
&\leqslant 4P\Big(S_m^{(n)} - ES_m^{(n)} > \tfrac{1}{2} 2^{\frac{n}{2}} u_n^{-\frac{1}{4}}\Big) \\
&\leqslant 4P\Big(T_m^{(n)} - ET_m^{(n)} > \tfrac{1}{2} 2^{\frac{n}{2}} u_n^{-\frac{1}{4}}\Big) + \frac{K\alpha_n}{u_n^{\frac{1}{4}} \log 1/\alpha_n} \\
&\leqslant \exp\Big(\tfrac{1}{32}(u_n)^{\frac{1}{2}} \log \alpha_n\Big) + K \frac{\alpha_n}{u_n^{\frac{1}{4}} \log 1/\alpha_n} \leqslant K' \frac{\alpha_n}{u_n^{\frac{1}{4}} \log 1/\alpha_n},
\end{aligned}$$

where K and K' are appropriate constants.

Therefore we have

$$\begin{aligned}
P\Big(\sup_{0 \leqslant k \leqslant 2^n/m} \sup_{km \leqslant j \leqslant (k+1)m} \big|S_j^{(n)} - S_{km}^{(n)} \\
- \big(ES_j^{(n)} - ES_{km}^{(n)}\big)\big| > 2^{\frac{n}{2}} u_n^{-\frac{1}{4}}\Big) \leqslant K'_{u_n}{}^{\frac{3}{4}} \alpha_n.
\end{aligned}$$

A similar inequality holds for the $T_n - ET^{(n)}$'s, too. $\Sigma u_n^{\frac{3}{4}} \alpha_n < \infty$ because of (2.5)′, therefore these inequalities imply that if (2.1)′ holds with some ε_n and δ_n, then (2.1) holds with $\varepsilon_n' = \varepsilon_n + u_n^{-\frac{1}{4}}$ and $\delta_n' = \delta_n + 2K' u_n^{\frac{3}{4}} \alpha_n$.

Now we turn to the construction of the sequences $X_k^{(n)}$, $Y_k^{(n)}$, $k = 1, 2, \cdots, 2^n$ satisfying (2.1′).

Let Z_1, Z_2, Z_l, $l = 2^n/m$ be i.i.d. rv's with standard normal distribution. Put

$$T_{km}^{(n)} - T_{(k-1)m}^{(n)} = m^{\frac{1}{2}} \sigma_n Z_k + mEX^{(n)},$$

and

$$S_{km}^{(n)} - S_{(k-1)m}^{(n)} = m^{\frac{1}{2}} \sigma_n F_m^{-1}(\phi(Z_k)) + mEX^{(n)}, \quad k = 1, 2, \cdots, l$$

where $F_m(x) = P((S_m^{(n)} - ES_m^{(n)})/m^{\frac{1}{2}}\sigma_n < x)$, $F^{-1}(x) = \sup(y; F(y) \leqslant x)$ and $\sigma_n^2 = D^2 X^{(n)}$.

These sequences have the prescribed joint distributions, therefore we may construct the sequences $X_j^{(n)}$, $Y_j^{(n)}$, $j = 1, 2, \cdots, 2^n$ in such a way that the variables $\Sigma_{j=1}^{km} X_j^{(n)}$, $\Sigma_{j=1}^{km} Y_j^{(n)}$ agree with these $S_{km}^{(n)}$ and $T_{km}^{(n)}$. These sequences will satisfy (2.1)'. To prove it we estimate the first two moments of the rv's

$$U_k = \left[\left(S_{km}^{(n)} - S_{(k-1)m}^{(n)}\right) - \left(T_{km}^{(n)} - T_{(k-1)m}^{(n)}\right)\right] I(|Z_k| > x), \qquad k = 1, 2, \cdots, l$$

where x is the bigger solution of the equation $x^2 \exp(-x^2/2) = K(u_n)^{\frac{1}{2}}\alpha_n(\log 1/\alpha_n)^{\frac{1}{2}}$ (K will be chosen later), and $I(A)$ denotes the indicator function of the set A. The relation

$$U_k = m^{\frac{1}{2}}\sigma_n\left[F_m^{-1}(\phi(Z_k)) - Z_k\right] \qquad \text{if} \quad |Z_k| \leqslant x$$

holds true.

We claim that

$$\text{(2.6)} \qquad \left|F_m^{-1}(\phi(Z_k)) - Z_k\right| < V_k \qquad \text{if} \quad |Z_k| \leqslant x,$$

where

$$V_k = K \exp\left(Z_k^2/2\right)\frac{u_n^{\frac{1}{2}}\alpha_n(\log 1/\alpha_n)^{\frac{1}{2}}}{1 + |Z_k|^3}.$$

(2.6) is equivalent to

$$\text{(2.7)} \qquad F_m(Z_k - V_k) < \phi(Z_k) < F_m(Z_k + V_k).$$

Theorem A gives that

$$F_m(Z_k - V_k) - \phi(Z_k - V_k) \leqslant C\frac{\alpha_n(u_n \log 1/\alpha_n)^{\frac{1}{2}}}{1 + |Z_k - V_k|^3}$$

with an appropriate constant C.

On the other hand, on the set $(|Z_k| \leqslant x)$ we have

$$\phi(Z_k) - \phi(Z_k - V_k) > \tfrac{1}{5} V_k \exp\left(-Z_k^2/2\right),$$

since $|Z_k V_k| \leqslant 1$.

The last two inequalities give that if K is sufficiently large then the first inequality in (2.7) holds true. The second inequality can be proved similarly.

Now (2.6) gives that

$$\begin{aligned} EU_k^2 &\leqslant K_1 m \textstyle\int_{-x}^{x} u_n\alpha_n^2 (\log 1/\alpha_n) \exp(y^2/2)(1 + |y|^6)^{-1}\, dy \\ &\leqslant K_2 \cdot 2^n\alpha_n^2 x^{-7} \exp(x^2/2) \leqslant K_3 \cdot 2^n\alpha_n u_n^{-\frac{1}{2}}(\log 1/\alpha_n)^{-\frac{1}{2}}x^{-5} \\ &\leqslant K_4 2^n\alpha_n u_n^{-\frac{1}{2}}(\log 1/\alpha_n)^{-3}, \end{aligned}$$

and

$$\begin{aligned} |EU_k| &\leqslant K m^{\frac{1}{2}} \textstyle\int_{-x}^{x} \alpha_n u_n^{\frac{1}{2}}(\log 1/\alpha_n)^{\frac{1}{2}}(1 + |y|^3)^{-1}\, dy \\ &\leqslant K_5 2^{\frac{n}{2}}\alpha_n \leqslant K_6 2^{\frac{n}{2}}\alpha_n. \end{aligned}$$

Applying the Kolmogorov inequality we obtain

$$P\left(\sup_{0 \leqslant k \leqslant l}\left|\Sigma_{j=1}^{k} U_j - EU_j\right| > 2^{\frac{n}{2}}(\log 1/\alpha_n)^{-1}\right) \leqslant K_7\alpha_n u_n^{\frac{1}{2}},$$

therefore

$$P\left(\sup_{k\leqslant l}\left|\Sigma_{j=1}^{k}U_j\right| > 2^{\frac{n}{2}}\left((\log 1/\alpha_n)^{-1} + K\alpha_n u_n \log 1/\alpha_n\right)\right) \leqslant K_7\alpha_n u_n^{\frac{1}{2}}.$$

Further we remark that

$$P(|Z_k| > x) \leqslant \left(2^{\frac{1}{2}}/\pi^{\frac{1}{2}}x\right)\exp(-x^2/2)$$
$$\leqslant K u_n^{\frac{1}{2}}\alpha_n(\log 1/\alpha_n)^{\frac{1}{2}}x^{-3} \leqslant K_8 u_n^{\frac{1}{2}}\alpha_n(\log 1/\alpha_n)^{-1}.$$

Therefore we have

$$P\left(\sup_{0\leqslant k\leqslant l}\left|S_{km}^{(n)} - T_{km}^{(n)}\right| > 2^{\frac{n}{2}}\left((\log 1/\alpha_n)^{-1} + K_6\alpha_n u_n \log 1/\alpha_n\right)\right)$$
$$\leqslant P\left(\sup_{0\leqslant k\leqslant l}\left|\Sigma_{j=1}^{k}u_j\right| > 2^{\frac{n}{2}}\left((\log 1/\alpha_n)^{-1} + K_6\alpha_n u_n \log 1/\alpha_n\right)\right)$$
$$+\Sigma_{k=1}^{l}P(|Z_k| > x) \leqslant K_g\alpha_n u_n^{\frac{3}{2}}.$$

This means that (2.1)′ holds with $\varepsilon_n = K_6 u_n \log 1/\alpha_n + (\log 1/\alpha_n)^{-1}$ and $\delta_n = K_g\alpha_n u_n^{\frac{3}{2}}$. Thus the proof is completed.

PROOF OF THE COROLLARY. Y_n' can be chosen as $Y_n' = (\sigma_n'/\sigma_n)Y_n$. Then it is enough to show that $n^{-\frac{1}{2}}\Sigma_{k=1}^{n}(Y_k - Y_k') \to 0$ if $n \to \infty$ w.p. 1. The estimate

$$\Sigma n^{-1}D^2(Y_n - Y_n') < \infty \tag{2.8}$$

implies this relation. But

$$\Sigma n^{-1}D^2(Y_n - Y_n') \leqslant \Sigma n^{-1}(\sigma_n' - \sigma_n)^2$$
$$\leqslant \Sigma n^{-1}\left[\max\left(\sigma_n^2, \sigma_n'^2\right) - \min\left(\sigma_n^2, \sigma_n'^2\right)\right]$$

and an easy calculation shows that the last sum is finite.

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