LIMIT THEOREMS ABOUT THE DISTRIBUTION OF ALMOST PERIODIC FUNCTIONS

Péter Major

Mathematical Institute of the
Hungarian Academy of Sciences
and
Bolyai College of the Eötvös
Loránd University, Budapest

Abstract. We prove a limit theorem about the distribution of an almost periodic function $F(R)=\sum\limits_{n=1}^\infty a_ne^{2\pi i\lambda_nR}$, $\sum\limits_{n=1}^\infty |a_n|^2<\infty$, when R is uniformly distributed in an interval $[0,T]$, and $T\to\infty$. Also a limit theorem is proved about the distribution of the random vector $\bigl(F(R),F(R+w(R,T))\bigr)$, $R\in[0,T]$, if the function $w(R,T)$ is appropriately defined. Similar results were proved also in other papers. (See [2] and [3].) The proofs in this paper are essentially different from the previous ones, and they may give some new insight to this problem. Previous proofs were based on ergod theoretical arguments, while in this paper some standard methods of Fourier analysis are applied. These investigations were motivated by the study of the limit behavior of the number of lattice points in a randomly magnified strip in the plane.

1. Introduction

In this paper the following problem is discussed: Let us consider a function

$$F(R)=\sum_{n=1}^{\infty} a_n e^{2\pi i\lambda_n R} \tag{1.1}$$

with

$$\sum_{n=1}^{\infty} |a_n|^2<\infty\,, \tag{1.2}$$

where $\lambda_1,\ \lambda_2,\ \dots$ are different non-zero real numbers. We also assume that $\lambda_{2n}=-\lambda_{2n-1}$ and $a_{2n}=\bar a_{2n-1}$ for $n=1,\,2,\,\dots$. This restriction is not essential, we only impose it to work with real valued functions. Define the distribution μ_T of the function $F(R)$ with respect to the uniform distribution in the interval $[0,T]$ by the formula

$$\mu_T(\mathbf{A})=\frac1T\lambda\{R\colon 0\le R\le T,\ F(R)\in\mathbf{A}\} \tag{1.3}$$

for any measurable set $\mathbf{A}\subset\mathbb{R}^1$, where λ denotes the Lebesgue measure. We want to prove that the measures μ_T have a weak limit. We also want to prove a generalization of this result in the case when the joint distribution of the functions $F(R)$ and $F(R+w(R,T))$ are investigated with a nice function $w(R,T)$. We shall study the limit distribution of this vector if R is uniformly distributed in an interval $[aT,bT]$, $0<a<b\le1$, and $T\to\infty$. Choose some constants $0<a<b\le1$, consider a function $w(R,T)$, $aT\le R\le bT$, and define the joint distribution of the functions $F(R)$ and $F(R+w(R,T))$ by the formula

$$\mu_{T,w,(a,b)}(\mathbf{A})=\frac{1}{(b-a)T}\lambda\{R\colon aT\le R\le bT,\ (F(R),F(R+w(R,T)))\in\mathbf{A}\}\,, \tag{1.4}$$

for any measurable set $\mathbf{A}\subset\mathbb{R}^2$. We want to prove that under appropriate conditions the measures $\mu_{T,w,(a,b)}$ with fixed numbers $0<a<b\le1$ converge weakly to a probability measure as $T\to\infty$.

Problems of such type arose in the investigation of the number of lattice points in a randomly magnified domain $R\mathbf{C}$, where $\mathbf{C}$ is a convex set with a smooth boundary, and R is a randomly chosen magnifying constant. It is proved, (see [2]), that the number of lattice points $N(R)$ in the domain $R\mathbf{C}$ after an appropriate normalization $\chi(R)=\dfrac{N(R)-\text{Area}\,(R\mathbf{C})}{\sqrt R}$ can be written in the form (1.1). We are interested in the limit behavior of the number of lattice points in a randomly enlarged domain $R\mathbf{C}$ or in a randomly defined strip $(R+\alpha(R))\mathbf{C}\setminus R\mathbf{C}$, with an appropriately defined function $\alpha(R)$, when the number R is randomly chosen. This can be described by means of the representation of $\chi(\mathbf{C})$ in the form of a series (1.1) and the above indicated limit theorems.

Actually the results of the present paper are only slight generalizations of earlier papers (see [2], [3]), where similar results were proved because of the same motivation. Nevertheless, we think that it is useful to revisit this problem for the following reason:

Our approach is different from that of the above mentioned papers, and we think that it has some interesting features. In previous papers the proofs were based on the ergod theorem. Because of this approach some measure theoretical problems arose whose solution seems to be hard. We want to show that these problems can be avoided by replacing the ergod theorem by a multi-dimensional continuous time version of the following well-known number theoretical result: For an irrational number α the sequence $n\alpha$ (mod 1), $n = 1, 2, \dots$, is asymptotically uniformly distributed in the interval $[0, 1]$.

Before formulating the results of this paper we have to explain the content of formula (1.1). The function $F(R)$ in this formula is considered as an element of the Besicovitch space, i.e. we assume that it has the following property: For all $\varepsilon > 0$ there exists an index $p_0 = p_0(\varepsilon)$ in such a way that

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| F(R) - \sum_{n=1}^{p} a_n e^{2\pi i \lambda_n R} \right|^2 dR < \varepsilon \tag{1.5}$$

for $p > p_0$. The theory of Besicovitch spaces can be found in [1], but in the present paper we do not need its fine details. Here we only use relation (1.5). Let us remark that the definition in (1.5) does not define the function $F(R)$ in a unique way. Indeed, if $F(R)$ and $\bar{F}(R)$ are two functions such that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(R) - \bar{F}(R)|^2 \, dR = 0 \,,$$

then the functions $F(R)$ and $\bar{F}(R)$ simultaneously satisfy or do not satisfy relation (1.5). Thus the theorems formulated below state in particular that the limit distribution appearing in them do not depend on which function $F(R)$ we take from those satisfying formula (1.5). Our first result is the following

Theorem 1. *For all functions $F(R)$ satisfying (1.1) and (1.2) the probability measures μ_T defined in (1.3) converge weakly to a probability measure μ as $T \to \infty$. Moreover, for all continuous functions $g(u)$ such that $|g(u)| < Au^2 + B$ with some appropriate numbers $A > 0$ and $B > 0$, the relation*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(F(R)) \, dR = \lim_{T\to\infty} \int g(u) \mu_T(\, du) = \int g(u) \mu(\, du) \tag{1.6}$$

holds. In particular,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(R) \, dR = 0 \,, \tag{1.7}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(R)^2 \, dR = \sum_{n=1}^{\infty} |a_n|^2 \,. \tag{1.7'}$$

To formulate Theorem 2 first we have to clarify how to define the class of "width" functions $w(R, T)$ in it. We consider two different cases. In case a) this width has constant order, and in a point $R = uT$, $a \le u \le b$, the value of the "width" function $w(R, T)$ is close to a monotone function $K(u)$ for large T, and in case b) it tends to infinity as $T \to \infty$ in a regular way. First we formulate Theorem 2, and then show that it contains the results of Section 2 in [3] as a special case.

Theorem 2. *Let a function $F(R)$ be given, which satisfies relations (1.1) and (1.2), and let $K(x)$ be a continuously differentiable, monotone (increasing or decreasing) function with non-vanishing derivative in an interval $[a,b]$, $0<a<b\le 1$, and $K(x)>0$ for all $x\in[a,b]$. Assume that the function $w(R,T)$, $T>0$, $aT<R<bT$, satisfies one of the following conditions:*

a) $w(R,T)=K\left(\frac{R}{T}\right)+o(1)$ *and* $\frac{\partial}{\partial R}w(R,T)=\frac{1}{T}K'\left(\frac{R}{T}\right)(1+o(1))$, $aT\le R\le bT$.

b) There exists some function $L(T)$, $L(T)\to\infty$ and $\frac{L(T)}{T}\to 0$ as $T\to\infty$, such that $w(R,T)=L(T)K\left(\frac{R}{T}\right)(1+o(1))$, *and* $\frac{\partial}{\partial R}w(R,T)=\frac{L(T)}{T}K'\left(\frac{R}{T}\right)(1+o(1))$, $aT\le R\le bT$.

The term $o(1)$ is uniformly small for $aT\le R\le bT$ as $T\to\infty$ in both cases a) and b).

Then the measures $\mu_{T,w,(a,b)}$ defined in (1.4) with these functions $w(R,T)$ have a weak limit $\bar\mu$ on $\mathbb{R}^2$ as $T\to\infty$. In case a) $\bar\mu$ equals some probability measure $\mu^{K(x)}_{(a,b)}$, i.e. it depends only on the function $K(x)$ and not on the special form of the function $w(R,t)$. The relation

$$\bar\mu=\mu^\infty_{(a,b)}=\mu\times\mu \tag{1.8}$$

holds, if $w(R,T)$ satisfies the conditions of case b), where μ is the probability measure defined in Theorem 1, and $\times$ denotes direct product. In particular, the limit measures $\mu^\infty_{(a,b)}$ do not depend on the parameters a and b.

The statement about the weak convergence of the measures $\mu_{T,w,(a,b)}$ can be strengthened in the following way: If $w(R,T)$ satisfies condition a) or b), and $g(u,v)$ is a continuous function such that $|g(u,v)|<A(u^2+v^2)+B$ with some appropriate $A>0$ and $B>0$, then

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{(b-a)T}\int_{aT}^{bT}g(F(R),F(R+w(R,T)))\,dR&=\lim_{T\to\infty}\int g(u,v)\,\mu_{T,w,(a,b)}(du,dv)\\&=\int g(u,v)\,\bar\mu(du,dv)\end{aligned} \tag{1.9}$$

with $\bar\mu=\mu^{K(x)}_{(a,b)}$ in case a) and $\bar\mu=\mu^\infty_{(a,b)}=\mu\times\mu$ in case b).

Let us fix some function $K(x)$ which satisfies the conditions imposed on it in Theorem 2. Then, the probability measures $\mu^{zK(x)}_{(a,b)}$ depend continuously on z in the weak topology for $0<z<\infty$, and

$$\lim_{z\to\infty}\mu^{zK(x)}_{(a,b)}=\mu^\infty_{(a,b)}\ , \tag{1.10}$$

Let us recall that the probability measures $\mu^{zK(x)}$ are called continuous in the weak topology if for all bounded and continuous functions g the integrals $\int g\,d\mu^{zK(x)}$ are continuous functions of z, and they converge to a measure μ^∞ as $z\to\infty$ if $\lim\limits_{z\to\infty}\int g\,d\mu^{zK(x)}=\int g\,d\mu^\infty$ for all bounded and continuous functions g.

In paper [3] the following problem is investigated. Let the function $F(R)$ be equal to the normalized number of lattice points $\chi(R)$ in a domain $R\mathbf{C}$. We are interested in the asymptotic behavior of the number of lattice points in a strip $(R+w(R,T))\mathbf{C}\setminus R\mathbf{C}$, where R is uniformly distributed in an interval $aT<R<bT$. In such an investigation the knowledge of the limit distribution of the vector $\big(F(R),F(R+w(R,T))\big)$, $aT\le R\le bT$, as $T\to\infty$, can be useful. The most interesting choice of the function $w(R,T)$ is that when for fixed T the area of the set $(R+w(R,T))\mathbf{C}\setminus R\mathbf{C}$ equals a constant $S(T)$, and the function $S(T)$ satisfies the relation

$$\lim_{T\to\infty}\frac{S(T)}{2T}=z,\quad 0<z\le\infty,\qquad\text{and}\quad \lim_{T\to\infty}\frac{S(T)}{T^2}=0\,. \tag{1.11}$$

This case is considered in paper [3]. If the area of the set $\mathbf{C}$ equals one, and the area of the strip $(R+w(R,T))\mathbf{C}\setminus R\mathbf{C}$ is $S(T)$, then the function $w(R,T)$ satisfies the equality

$$w(R,T)^2+2Rw(R,T)=S(T)\,. \tag{1.12}$$

It is not difficult to see that the function $w(R,T)$ defined by formulas (1.11) and (1.12) satisfies the conditions of Theorem 2. If the number z is finite, then case a) of Theorem 2 holds with $K(x)=z/x$, and if it equals infinity, then case b) holds with $K(x)=1/x$ and $L(T)=S(T)/2T$. Hence the results of Section 2 of [3] are consequences of Theorem 2 with the above choice of the function $K(x)$ and $L(T)$.

One would like to give an explicit description of the limit measures appearing in Theorems 1 and 2. We return to this question at the end of this paper. Here we formulate a result which gives a decomposition of the measures $\mu_{(a,b)}^{K(x)}$. Let us define the distribution of the vector $(F(R),F(R+x))$ in the interval $[0,T]$ with a fixed number $0\le x<\infty$ by the formula:

$$\nu_T^x(\mathbf{A})=\frac{1}{T}\lambda\{R\colon 0\le R\le T,\quad (F(R),F(R+x))\in\mathbf{A}\} \tag{1.13}$$

for any measurable set $\mathbf{A}\subset\mathbb{R}^2$. Now we formulate the following

Theorem 3. *For fixed $0<x<\infty$ the measures ν_T^x converge weakly to a probability measure ν^x, and also the relation*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g(F(R),F(R+x))\,dR=\int g(u,v)\,\nu^x(du,dv)\,, \tag{1.14}$$

holds if $g(u,v)$ is a continuous function, and $|g(u,v)|<A(u^2+v^2)+B$ with some constants $A>0$ and $B>0$. For a fixed function $g(u,v)$ the integral at the right-hand side of (1.14) is a continuous and bounded function of x.

The identity

$$\mu_{(a,b)}^{K(x)}=\frac{1}{(b-a)}\int_a^b\nu^{K(x)}\,dx=\frac{1}{(b-a)}\int_{K(a)}^{K(b)}\frac{\nu^x}{K'(K^{-1}(x))}\,dx \tag{1.15}$$

holds for the function $\mu_{a,b}^{K(x)}$ defined in Theorem 2.

We shall prove the following corollary of the above results:

Corollary. *Let $h(x)$ be an integrable function on an interval $[a,b]$, $0<a<b\le 1$. Let the function $w(R,T)$ satisfy the conditions of case a) of Theorem 2. Then the relation*

$$\lim_{T\to\infty}\frac{1}{T}\int_{aT}^{bT} g\bigl(F(R),F(R+w(R,T))\bigr)h\left(\frac{R}{T}\right)\,dR=\int_a^b h(x)\int g(u,v)\,\nu^{K(x)}(du,dv)\,dx \tag{1.16}$$

holds for all continuous functions $g(u,v)$ if one of the following conditions is satisfied: Either $g(u,v)$ is bounded or $h(u)$ is square integrable and $|g(u,v)|<A(u^2+v^2)+B$ with some appropriate numbers $A>0$ and $B>0$.

This paper consists of four sections. In Section 2 we prove the Theorems in the special case when the sum (1.1) defining the function $F(R)$ contains finitely many terms. In Section 3 we carry out a limiting procedure which proves the results of the paper by means of Section 2. In Section 4 we make some comments and prove some generalizations.

Let us make a short comparison of the method of this paper with previous ones. The main difference of the proof of Theorem 1 in this paper and in [2] is that we replace the application of the ergod theorem by a number theoretical distribution theorem. The formulation of Theorems 2 and 3 are very close to the results of Section 2 in [3]. The proofs are essentially different. In paper [3] these results were deduced from Theorem 1 by a tricky ergod theoretical argument. Here we show that a slight modification of the proof of Theorem 1 supplies a direct proof for them.

2. The proof of the results in a special case

Let us consider the case when the function $F(R)$ is defined by a finite sum

$$F(R)=F_p(R)=\sum_{n=1}^{p} a_n e^{2\pi i\lambda_n R} \tag{2.1}$$

with an even number p, real non-zero numbers λ_n such that $\lambda_{2n}=-\lambda_{2n-1}$, $a_{2n}=\bar a_{2n-1}$, and the measures μ_T, $\mu_{T,w,(a,b)}$ and ν_T^x are defined in formulas (1.3), (1.4) and (1.13) by means of this function. We prove in this Section Theorems 1, 2 and 3 in the case when these measures are determined by a function of the form (2.1). In the next Section we prove the result for general functions $F(R)$ defined in (1.1) by approximating them with the functions F_p appearing in (2.1). We shall indicate the dependence of these measures on the functions F_p by denoting them as $\mu_T(F_p)$, $\mu_{T,w,(a,b)}(F_p)$ and $\nu_T^x(F_p)$ when necessary. The main results of this Section are the following

Propositions 1., 2. and 3. *Let the function $F(R)$ be defined by the finite trigonometrical sum (2.1), and let the measures μ_T, $\mu_{T,w,(a,b)}$ and ν_T^x be defined in formulas (1.3), (1.4), and (1.13) by means of this function $F(R)$. If the function $w(R,T)$ and $S(T)$ satisfies the conditions of Theorem 2, then Theorems 1, 2, and 3 hold with this choice of the corresponding measures.*

Proof of Proposition 1. We have to investigate the asymptotic behavior of the expression

$$\frac{1}{T}\int_0^T g(F(R))\,dR \tag{2.2}$$

as $T \to \infty$ in the case when $g(u)$ is a bounded continuous function. We shall rewrite, following the argument of [2] and [4], the expression in (2.2) as an integral on a torus with respect to an appropriate measure. It is useful to work, when handling the function $F(R)$, with frequencies linearly independent over the rational numbers. Since the frequencies λ_n may not have this property we express them as a linear combination of some numbers $\tau_1, \dots, \tau_s$ linearly independent over the rational numbers

$$\lambda_n = T_n(\tau_1, \dots, \tau_s) = \sum_{k=1}^{s} A(n,k)\tau_k \ , \quad n = 1, 2, \dots, p, \ k = 1, \dots, s \tag{2.3}$$

with integer coefficients $A(n,k)$. Let V denote the unit interval with the group action addition modulo 1. Introduce its s-fold and p-fold direct products

$$\mathcal{V} = \underbrace{V \times \cdots \times V}_{s \text{ times}} \tag{2.4}$$

and

$$\mathcal{V}' = \underbrace{V \times \cdots \times V}_{p \text{ times}} \ . \tag{2.4$'$}$$

Define the maps $U \colon \mathbb{R}^1 \to \mathcal{V}$

$$U(R) = \{R\tau_k \ (\mathrm{mod}\, 1), \quad k = 1, \dots, s\} \ . \tag{2.5}$$

$V \colon \mathcal{V} \to \mathcal{V}'$

$$V(u_1, \dots, u_s) = \left\{ \sum_{n=1}^{s} A(n,k)u_k \ (\mathrm{mod}\, 1), \quad n = 1, \dots, p \right\} \tag{2.6}$$

for $(u_1, \dots, u_s) \in \mathcal{V}$ with the integer coefficients $A(n,k)$ appearing in (2.3) and $G \colon \mathcal{V}' \to \mathbb{R}^1$

$$G(u_1, \dots, u_p) = \sum_{n=1}^{p} a_n e^{2\pi i u_n} \ . \tag{2.7}$$

Clearly, $F(R) = G\big(V(U(R))\big)$. Define the probability measure ρ_T on $\mathcal{V}$ induced by the map U by the formula

$$\rho_T(\mathbf{A}) = \frac{1}{T}\lambda\{R \colon 0 \le R \le T, \ U(R) \in \mathbf{A}\} \tag{2.8}$$

for all measurable sets $\mathbf{A} \subset \mathcal{V}$.

Then the integral (2.2) can be rewritten as

$$\frac{1}{T}\int_0^T g(F(R))\, dR = \int_{\mathcal{V}} g(G(V(u)))\, \rho_T(du) \ . \tag{2.9}$$

The relation

$$\rho_T \Rightarrow \rho \quad \text{as } T \to \infty \tag{2.10}$$

holds, where ρ denotes the Haar measure on $\mathcal{V}$, and $\Rightarrow$ means weak convergence of probability measures. Relation (2.10) is a known result. Nevertheless, we give its proof, because it is short, and we need its modification in the proof of Proposition 2. By Weil's lemma (or by the characteristic function method on commutative compact groups) to prove (2.10) it is enough to check that, with the notation $(u_1,\dots,u_s)=u\in\mathcal{V}$,

$$\begin{aligned}\lim_{T\to\infty}\int\exp\left\{2\pi i\sum_{k=1}^s m_k u_k\right\}\rho_T(\,du) &= \lim_{T\to\infty}\frac1T\int_0^T \exp\left\{2\pi i\sum_{k=1}^s m_k\tau_k R\right\}dR\\ &= \lim_{T\to\infty}\frac{\exp\left\{2\pi i T\sum\limits_{k=1}^s m_k\tau_k\right\}-1}{2\pi i T\sum\limits_{k=1}^s m_k\tau_k}=0\end{aligned}$$

if $m_1,m_2,\dots,m_k$ are integers, and not all of them equal zero. Relations (2.9) and (2.10) imply that

$$\lim_{T\to\infty}\frac1T\int_0^T g(F(R))\,dR=\lim_{T\to\infty}\int g(G(V(u))\rho_T(du)=\int g(G(V(u))\rho(du)\ ,$$

since $g\big(G(V(u))\big)$ is a bounded, continuous function. The last relation implies that

$$\lim_{T\to\infty}\frac1T\int_0^T g(F(R))\,dR=\int g(u)\,\mu(du) \tag{2.11}$$

with the measure μ defined on $\mathbb{R}^1$ by the relation

$$\mu(\mathbf{A})=\rho\{u\colon u\in\mathcal{V},\quad G(V(u))\in\mathbf{A}\} \tag{2.12}$$

for all measurable sets $\mathbf{A}\in\mathbb{R}^1$. Relations (2.11) and (2.12) imply that the measures μ_T converge weakly to the measure μ defined in (2.12). To complete the proof of Proposition 1 observe that the function $|F(R)|$ is bounded by $C_p=\sum_{n=1}^p|a_n|$ for all $R\in\mathbb{R}^1$. Hence all measures μ_T and μ are concentrated in the interval $[-C_p,C_p]$, and relation (1.6) follows for all continuous functions $g(u)$, since they can be replaced by their truncation at $\pm C_p$, which are bounded continuous functions. Finally, relations (1.7) and (1.7$'$) follow from the observation that $F(R)$ is a finite sum, and the relations

$$\lim_{T\to\infty}\frac1T\int_0^T e^{i\lambda_n R}\,dR=0$$

$$\lim_{T\to\infty}\frac1T\int_0^T e^{i(\lambda_n-\lambda_{n'})R}\,dR=\delta(n,n')$$

hold. □

Proof of Proposition 2. The proof of Proposition 2 is very similar to that of Proposition 1. The main difference is that we have to carry out the integral transformations by means of different functions G, U and V, and the statement formulated in (2.10) has to be generalized.

Define the maps $U\colon \mathbb{R}^1 \to \mathcal{V}\times\mathcal{V}$

$$\begin{aligned} U(R) = U(R,w,T) = \{R\tau_k \ (\mathrm{mod}\, 1), \quad k=1,\dots,s\ ,\\ (R+w(R,T))\tau_k \ (\mathrm{mod}\, 1), \quad k=1,\dots,s\}\ , \end{aligned} \tag{2.13}$$

$V\colon \mathcal{V}\times\mathcal{V} \to \mathcal{V}'\times\mathcal{V}'$

$$\begin{aligned} V(u_1,\dots,u_{2s}) = \Bigg\{ \sum_{k=1}^{s} A(n,k)u_k \ (\mathrm{mod}\, 1), \quad n=1,\dots,p\ ,\\ \sum_{k=1}^{s} A(n,k)u_{k+s} \ (\mathrm{mod}\, 1), \quad n=1,\dots,p \Bigg\} \end{aligned} \tag{2.14}$$

for $(u_1,\dots,u_{2s}) \in \mathcal{V}\times\mathcal{V}$ with the integer coefficients $A(n,k)$ appearing in (2.3) and $G\colon \mathcal{V}'\times\mathcal{V}' \to \mathbb{R}^2$

$$G(u_1,\dots,u_{2p}) = \left(\sum_{n=1}^{p} a_n e^{2\pi i u_n},\ \sum_{n=1}^{p} a_n e^{2\pi i u_{n+p}} \right)\ . \tag{2.15}$$

Then $\big(F(R), F(R+w(R,T))\big) = G\big(V(U(R))\big)$. Define the probability measure $\bar\rho_T = \bar\rho_{T,w,(a,b)}$ on $\mathcal{V}\times\mathcal{V}$ induced by the map U by the formula

$$\bar\rho_{T,w,(a,b)}(\mathbf{A}) = \frac{1}{(b-a)T}\lambda\{R\colon aT \le R \le bT, \quad U(R,w,T)) \in \mathbf{A}\} \tag{2.16}$$

for any measurable set $\mathbf{A} \subset \mathcal{V}\times\mathcal{V}$.

We claim that if $w(R,T)$ satisfies the conditions of Theorem 2, then the limit relation

$$\bar\rho_{T,w,(a,b)} \Rightarrow \bar\rho \quad \text{as } T\to\infty \tag{2.17}$$

holds with a probability measure $\bar\rho$ on $\mathcal{V}\times\mathcal{V}$. Moreover, we claim that

$$\bar\rho = \bar\rho_{(a,b)}^{K(x)} \tag{2.17$'$}$$

if $w(R,T)$ satisfies the conditions of case a) of Theorem 2 with $K(x)$, i.e. the limit depends only on the function $K(x)$ in this case, and

$$\bar\rho = \bar\rho_{a,b}^{\infty} = \text{the Haar measure } \rho\times\rho \text{ on } \mathcal{V}\times\mathcal{V} \tag{2.17$''$}$$

if $w(R,T)$ satisfies the conditions of case b) of Theorem 2.

To prove (2.17) we show that the Fourier coefficients

$$L_{w,T}(m_1,\dots,m_{2s}) = \int_{\mathcal{V}\times\mathcal{V}} \exp\left\{2\pi i \sum_{k=1}^{2s} m_k u_k\right\} \bar\rho_{T,w,(a,b)}(\,du)$$

with $u=(u_1,\dots,u_{2s})\in\mathcal{V}\times\mathcal{V}$ have a limit

$$\lim_{T\to\infty} L_{w,T}(m_1,\dots,m_{2s})=L(m_1,\dots,m_{2s}) \tag{2.18}$$

for all integers $m_1,\dots,m_{2s}$. These Fourier coefficients can be rewritten by an integral transformation as

$$L_{w,T}(m_1,\dots,m_{2s})=\frac{1}{(b-a)T}\int_{aT}^{bT} e^{i(A(m_1,\dots,m_{2s})R+B(m_1,\dots,m_{2s})w(R,T))}\,dR \tag{2.19}$$

with

$$\begin{aligned} A(m_1,\dots,m_{2s})&=2\pi\sum_{k=1}^{s}(m_k+m_{k+s})\tau_k\ ,\\ B(m_1,\dots,m_{2s})&=2\pi\sum_{k=1}^{s}m_{k+s}\tau_k\ . \end{aligned} \tag{2.19$'$}$$

Because of the linear independence of the numbers τ_k both expressions $A(m_1,\dots,m_{2s})$ $B(m_1,\dots,m_{2s})$ can disappear simultaneously only if all coefficients m_k are zero, which is a trivial case. Otherwise we claim that

$$\lim_{T\to\infty} L_{w,T}(m_1,\dots,m_{2s})=\begin{cases} 0 & \text{if } A(m_1,\dots,m_{2s})\neq 0\\ \frac{1}{(b-a)}\int_a^b e^{iBK(u)}\,du & \text{if } A(m_1,\dots,m_{2s})=0 \text{ and}\\ & \quad w(R,T) \text{ satisfies case a)}\\ 0 & \text{if } A(m_1,\dots,m_{2s})=0 \text{ and}\\ & \quad w(R,T) \text{ satisfies case b)} \end{cases} \cdot \tag{2.20}$$

The first line in relation (2.20) can be proved by means of relation (2.19) with the change of variables $AR+Bw(R,T)=u$. Let us observe that because of the conditions of Theorem 2 $\frac{du}{dR}=A+o(1)$ uniformly for $aT\le R\le bT$, and the boundaries of the domain of integration after the change of variables are $aAT(1+o(1))$ and $bAT(1+o(1))$. Hence we get that

$$\lim_{T\to\infty} L_{w,T}(m_1,\dots,m_{2s})=\lim_{T\to\infty}\frac{(1+o(1))}{(b-a)A(m_1,\dots,m_{2s})T}\int_{aA(m_1,\dots,m_{2s})T}^{bA(m_1,\dots,m_{2s})T} e^{iu}\,du=0$$

in this case. If the conditions of the second line of (2.20) hold, i.e. when the conditions of case a) of Theorem 2 hold, and $A=0$, then we can calculate the expression (2.19) with the change of variables $u=\dfrac{R}{T}$. Simple calculation shows that

$$\frac{1}{(b-a)T}\int_{aT}^{bT} e^{iBw(R,T)}\,dR\to\frac{1}{(b-a)}\int_a^b e^{iBK(u)}\,du\ ,$$

and the second relation of (2.20) holds. The third line of (2.20) (this case holds when $A=0$ and condition b) is satisfied.) can be proved similarly with the change of variables

$u=\dfrac{R}{T}$ and $v=\dfrac{1}{L(T)}w(uT,T)$. Some calculation shows that $v=K(u)(1+o(1))$, $\dfrac{\partial v}{\partial u}=K'(u)(1+o(1))$, and

$$\begin{aligned}\frac{1}{(b-a)T}\int_{aT}^{bT} e^{iBw(R,T)}\,dR&=\frac{1}{(b-a)}\int_a^b e^{iBw(uT,T)}\,du\\&=\frac{1}{(b-a)}\int_{K(a)}^{K(b)} e^{iBL(T)v}\frac{1}{K'(K^{-1}(v))}(1+o(1))\,dv\to 0\end{aligned}$$

by the Riemann lemma. The convergence of the Fourier coefficients formulated in relation (2.20) implies formulas (2.17), (2.17′) and (2.17″). In particular, relation (2.17″) holds, since in the case when $w(R,T)$ satisfies case b) of Theorem 2, then all non-trivial Fourier coefficients of $\bar\rho$ equal zero.

Let us also show that the Fourier coefficients in the second line of formula (2.20) corresponding to the function $zK(x)$ tend to zero as $z\to\infty$. This relation holds, because $B\neq 0$ in this case, and the Riemann lemma yields that

$$\frac{1}{(b-a)}\int_a^b e^{izB(m_1,\dots,m_{2s})K(u)}\,du=\frac{1}{(b-a)}\int_{K_1(a)}^{K_1(b)}\frac{e^{izB(m_1,\dots,m_{2s})u}}{K'(K^{-1}(u))}\,du\to 0\quad\text{as } z\to\infty \tag{2.21.}$$

Let $g(u,v)$ be a bounded continuous function. We get similarly to the argument of Proposition 1 from relations (2.17) (2.17′) and (2.17″) that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{(b-a)T}\int_{aT}^{bT} g(F(R),F(R+w(R,T))\,dR&=\lim_{T\to\infty}\frac{1}{(b-a)T}\int_{aT}^{bT} g\big(G\big(V(U(R))\big)\big)\,dR\\&=\lim_{T\to\infty}\int g(G(V(u))\bar\rho_{T,w,(a,b)}\,(du)=\int g(G(V(u))\bar\rho\,(du)\end{aligned}\tag{2.22}$$

with $\bar\rho=\bar\rho^{K(x)}_{(a,b)}$ if case a) and $\bar\rho=\bar\rho^{\infty}_{(a,b)}$ if case b) of Theorem 2 holds. Hence

$$\lim_{T\to\infty}\frac{1}{(b-a)T}\int_{aT}^{bT} g(F(R),F(R+w(R,T))\,dR=\int g(u,v)\bar\mu(du,dv) \tag{2.23}$$

with

$$\bar\mu(\mathbf{A})=\bar\rho\{u\colon u\in\mathcal{V}\times\mathcal{V},\quad G(V(u))\in\mathbf{A}\} \tag{2.24}$$

for all measurable sets $\mathbf{A}\in\mathbb{R}^2$.

Relation (2.23) implies the weak convergence of the measures $\mu_{T,w,(a,b)}$ to $\bar\mu$. Formula (2.24) together with the form of the measures $\bar\rho$ imply that the measure μ has the prescribed form if cases a) of Theorem 2 holds, i.e. it depends only on the function $K(x)$. If case b) of Theorem 2 holds, then a comparison of formulas (2.12) and (2.24) together with relation (2.17″) and the product form of the functions G and V in formulas (2.14) and (2.15) imply formula (1.8). Relation (1.9) can be deduced from the weak convergence in the same way as the analogous result in Proposition 1. To prove formula

(1.10) and continuity of the measures $\mu^{zK(x)}_{(a,b)}$ (in the variable z) it is enough to prove the continuity of the expression $\int g(G(V(u))\bar\rho^{zK(x)}_{(a,b)}\,(du)$. Because of the Weierstrass approximation theorem it is enough to check the continuity of the Fourier coefficients. This follows from relations (2.20) and (2.21). Proposition 2 is proved. □

Proof of Proposition 3. The proof is based on a representation similar to Proposition 2. Define the maps $U\colon \mathbb{R}^1 \to \mathcal{V}\times\mathcal{V}$

$$U(R)=U(R,x)=\{R\tau_k \pmod 1,\quad k=1,\dots,s\ ,\\ (R+x)\tau_k \pmod 1,\quad k=1,\dots,s\}\ . \tag{2.25}$$

and the maps $V(u_1,\dots,u_{2s})$ and $G(u_1,\dots,u_{2p})$ by formulas (2.14) and (2.15) as in the proof of Proposition 2. Introduce the measures

$$\hat\rho_{T,x}(\mathbf{A})=\frac{1}{T}\lambda\{R\colon 0\le R\le T,\quad U(R,x)\in\mathbf{A}\} \tag{2.26}$$

for any measurable set $\mathbf{A}\subset\mathcal{V}\times\mathcal{V}$. Then

$$\hat\rho_{T,x}\Rightarrow\hat\rho^x\quad \text{as } T\to\infty \tag{2.27}$$

with Fourier coefficients

$$L^x(m_1,\dots,m_{2s})=\begin{cases}0 & \text{if } A(m_1,\dots,m_{2s})\ne 0\\ \exp\{iB(m_1,\dots,m_{2s})x\} & \text{if } A(m_1,\dots,m_{2s})=0\end{cases}\ . \tag{2.28}$$

with the functions $A(m_1,\dots,m_{2s})$ and $B(m_1,\dots,m_{2s})$ defined in (2.19′). Then we get the proof of the identity (1.14) as in the proof of Proposition 2 with the limit measure

$$\nu^x(\mathbf{A})=\hat\rho^x\{u\colon u\in\mathcal{V}\times\mathcal{V},\quad G(V(u))\in\mathbf{A}\} \tag{2.29}$$

for all measurable $\mathbf{A}\in\mathbb{R}^2$. The expression at the right-hand side of (1.14) is clearly a bounded function, and it is a continuous function of x, because the Fourier coefficients of $\hat\rho^x$ are continuous functions of x. A comparison of the Fourier coefficients in (2.20) and (2.28) yields that

$$\bar\rho^{K(x)}_{(a,b)}=\frac{1}{(b-a)}\int_a^b\hat\rho^{K(x)}\,dx=\frac{z}{(b-a)}\int_{K(a)}^{K(b)}\frac{1}{K'K^{-1}(x)}\hat\rho^x\,dx\ .$$

This relation together with (2.24) and (2.29) imply relation (1.15). Proposition 3 is proved. □

3. Proof of the Theorems

First we formulate an estimate which enables us to generalize the results of Section 2 to functions satisfying (1.1) and (1.2).

Let $g(u,v)$ be a continuous function such that $|g(u,v)| < A(u^2+v^2)+B$ with some $A>0$ and $B>0$, $F(R)$ a function satisfying (1.1) and (1.2), $F_p(R)$ the trigonometrical series containing the first p terms of $F(R)$, and let some numbers $0<a<b\le 1$ and functions $w(R,T)$, satisfying either case a) or case b) of Theorem 2. We claim that for all $\varepsilon>0$ there are some thresholds $p_0=p_0(\varepsilon)$ and $T_0=T_0(p,\varepsilon)$ for $p>p_0$ such that

$$\begin{aligned}\Bigg| \frac{1}{(b-a)T}\int_{aT}^{bT} & g\big(F(R),F(R+w(R,T))\big)\,dR \\ & -\frac{1}{(b-a)T}\int_{aT}^{bT} g\big(F_p(R),F_p(R+w(R,T))\big)\,dR\Bigg| < \varepsilon \end{aligned} \tag{3.1}$$

for any $p>p_0$ and $T>T_0(p,\varepsilon)$. Moreover, the threshold p_0 can be chosen depending only on ε, a, b and $g(u,v)$, but not depending on the choice of the function $w(R,T)$ of which we only require that it satisfied the conditions of Theorem 2.

To prove relation (3.1) first we make the following observation: For any $(u,v)\in\mathbb{R}^2$ and $(u_0,v_0)\in\mathbb{R}^2$ and $1>\eta>0$ there exist some constants $K=K(\eta)$ and K_0 depending only on the function $g(u,v)$ such that

$$\begin{aligned}|g(u,v)-g(u_0,v_0)| < \eta + K((u-u_0)^2+(v-v_0)^2) + K_0(u_0^2+v_0^2)I(\{u_0^2+v_0^2>\eta^{-1}\}) \\ + K_0(u^2+v^2)I(\{u^2+v^2>\eta^{-1}\})\,, \end{aligned} \tag{3.2}$$

where $I(A)$ denotes the indicator function of the set A.

Indeed, relation (3.2) holds with

$$K_0 = 2\sup_{u^2+v^2\ge 1}\frac{|A(u^2+v^2)+B|}{u^2+v^2} \le 2(A+B)$$

if $u_0^2+v_0^2>\eta^{-1}$ or $u^2+v^2>\eta^{-1}$. On the complementary set this inequality holds if $(u_0-u)^2+(v_0-v)^2<\delta$ with some $\delta=\delta(\eta)$ because of the uniform continuity of the function $g(u,v)$ on this set. Finally, relation (3.2) holds on the remaining set if $K=K(\eta)$ is chosen sufficiently large. Relation (3.2) can be rewritten in a simpler form. We can apply the inequality

$$(u^2+v^2)I(\{u^2+v^2>\eta^{-1}\}) \le 2u^2 I(\{u^2>(2\eta)^{-1}\} + 2v^2 I(\{v^2>(2\eta)^{-1}\}\,,$$

and write with the help of this relation that

$$\begin{aligned}|g(u,v)-g(u_0,v_0)| < \eta + K\big((u-u_0)^2+(v-v_0)^2\big) \\ + K_0\big[(u_0^2 I(\{u_0^2>\eta^{-1}\} + v_0^2 I\{v_0^2>\eta^{-1}\}\big] \\ + K_0\big[(u^2 I(\{u^2>\eta^{-1}\} + v^2 I\{v^2>\eta^{-1}\}\big]\,. \end{aligned} \tag{3.2$'$}$$

with a new constant K which corresponds to the bound $\eta/2$ and with a new constant K_0 which is the double of the original one.

We shall prove (3.1) from (3.2′) with the choice $(u_0,v_0)=(F_p(R),F_p(R+w(R,T)))$, $(u,v)=(F(R),F(R+w(R,T)))$ with an appropriate $\eta>0$ and $p=p(\eta)$ and then by integration with respect to R.

By relation (3.2′)

$$\begin{aligned}&\bigl|g\bigl(F(R),F(R+w(R,T))\bigr)-g\bigl(F_p(R),F_p(R+w(R,T))\bigr)\bigr|\\ &\qquad<\eta+K\bigl[(F(R)-F_p(R))^2+(F(R+w(R,T))-F_p(R+w(R,T)))^2\bigr]\\ &\qquad+K_0\bigl[F_p(R)^2I\{F_p(R)^2>\eta^{-1}\}+F_p(R+w(R,T))^2I\{F_p(R+w(R,T))^2>\eta^{-1}\}\bigr]\\ &\qquad+K_0\bigl[F(R)^2I\{F(R)^2>\eta^{-1}\}+F(R+w(R,T))^2I\{F(R+w(R,T))^2>\eta^{-1}\}\bigr].\end{aligned}$$

The inequality

$$\left|\frac{K_0}{(b-a)T}\int_{aT}^{bT}F(R)^2I\{F(R)^2>\eta^{-1}\}\,dR\right|<\frac{\varepsilon}{8} \tag{3.3}$$

holds, if $\eta<\eta(\varepsilon)$ and $T>T(\varepsilon)$. Indeed, by relation (1.5) there is some $\bar p=\bar p(\varepsilon)$, and $T(\varepsilon)$ in such a way that

$$\frac{K_0}{(b-a)T}\int_{aT}^{bT}|F(R)-F_{\bar p}(R)|^2\,dR<\frac{\varepsilon}{32}\,.$$

for $T>T(\varepsilon)$. The function $|F_{\bar p}(R)|$ is bounded. Put $\eta=\inf\limits_R 4|F_{\bar p}(R)|^{-2}$. Then the last inequality implies (3.3), since $F(R)^2<4|F(R)-F_{\bar p}(R)|^2$ on the set $\{F(R)^2>\eta^{-1}\}$. We also claim that

$$\left|\frac{K_0}{(b-a)T}\int_{aT}^{bT}F(R+w(R,T))^2I\{F(R+w(R,T))^2>\eta^{-1}\}\,dR\right|<\frac{\varepsilon}{8} \tag{3.3′}$$

if $\eta<\eta(\varepsilon)$ and $T>T(\varepsilon)$. This can be proved similarly to (3.3) with some modification. Apply the change of variables $u=R+w(R,T)$ in the integral in (3.3′). Since $w(R,T)$ satisfies the conditions Theorem 2, $\frac{du}{dR}\to1$ uniformly for $aT\le R\le bT$, as $T\to\infty$. The domain of integration after this change of variable is the interval $[aT(1+o(1)),bT(1+o(1))]$. Hence after this change of variables the integral in (3.3′) can be estimated in the same way as in (3.3). Relations (3.3) and (3.3′) remain valid if the function $F(R)$ is replaced by $F_p(R)$ with $p>\bar p$, and $T>T(\varepsilon,p)$.

Choose η so that relations (3.3) and (3.3′) and their variants for the function $F_p(R)$ hold and $\eta<\varepsilon/4$. Then, because of relation (1.5) and the argument in the proof of (3.3′) some thresholds $p_0=p(\eta)$ and $T_0=T_0(\eta,p)$ can be chosen in such a way that for $p>p_0$ and $T>T_0$

$$\frac{K}{(b-a)T}\int_{aT}^{bT}(F(R)-F_p(R))^2\,dR<\frac{\varepsilon}{8} \tag{3.4}$$

and

$$\frac{K}{(b-a)T}\int_{aT}^{bT}(F(R+w(R,T))-F_p(R+w(R,T)))^2\,dR<\frac{\varepsilon}{8} \tag{3.4′}$$

with the constant $K=K(\eta)$ appearing in formula (3.2′). Formulas (3.3), (3.3′), their variants for the function $F_p(R)$, (3.4) and (3.4′) together with the relation $\eta<\varepsilon/4$ imply (3.1).

It follows from (3.1) and relation (1.9) already proved for the function F_p that

$$\limsup_{T\to\infty}\left|\frac{1}{(b-a)T}\int_{aT}^{bT} g\big(F(R),F(R+w(R,T))\big)\,dR - \int g(u,v)\bar\mu(F_p)(\,du,\,dv)\right|<\varepsilon$$

for all $p>p(\varepsilon)$ with $\bar\mu(F_p)=\mu_{(a,b)}^{K(x)}(F_p)$ if $w(R,T)$ satisfies the conditions of case a) and with $\bar\mu(F_p)=\mu_{(a,b)}^{\infty}(F_p)$ if it satisfies the conditions of case b) of Theorem 2. This relation implies that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{(b-a)T}\int_{aT}^{bT} g\big(F(R),F(R+w(R,T))\big)\,dR &= \lim_{T\to\infty}\int g(u,v)\mu_{T,w,(a,b)}(\,du,\,dv)\\ &=\lim_{p\to\infty}\int g(u,v)\,\bar\mu(F_p)(du,\,dv)\ .\end{aligned} \tag{3.5}$$

with the same choice of the measure $\bar\mu(F_p)$ as in the previous formula. The last relation also means that all limits in this formula exist. Since relation (3.5) also holds for $g(u,v)=u^2+v^2$, hence the measures $\mu_{T,w,(a,b)}$ are uniformly tight, and we get by applying relation (3.5) that there exists the limit

$$\bar\mu=\lim_{T\to\infty}\mu_{T,w,(a,b)}=\lim_{p\to\infty}\bar\mu(F_p)\ , \tag{3.6}$$

with the same measures $\bar\mu(F_p)$ as in (3.5) and in the previous formula, and also relation (1.9) holds with the function $F(R)$. Moreover, for a fixed bounded continuous function $g(u,v)$ the limit

$$\lim_{p\to\infty}\int g(u,v)\,\mu_{(a,b)}^{zK(x)}(F_p)(du,\,dv)=\int g(u,v)\,\mu_{(a,b)}^{zK(x)}(du,\,dv)$$

is uniform in z, and this fact together with the continuity properties of the measures $\mu_{(a,b)}^{zK(x)}(F_p)$ imply the continuity of the measures $\mu_{a,b}^{zK(x)}$ for $0<z<\infty$ and relation (1.10). This completes the proof of Theorem 2 with the exception of formula (1.8).

A similar, but simpler argument shows that relation (3.1) holds if the pair of functions $g\big(F(R),F(R+w(R,T))\big)$ and $g\big(F_p(R),F_p(R+w(R,T))\big)$ are replaced by the pairs of functions $g\big(F(R),F(R+x)\big)$ and $g\big(F_p(R),F_p(R+x)\big)$ or $g\big(F(R)\big)$ and $g\big(F_p(R)\big)$.

The same argument as in the proof of Theorem 2 with the first replacement yields the existence of the limit

$$\lim_{p\to\infty}\nu^x(F_p)=\nu^x$$

together with relation (1.14) and the continuity of the integral in (1.14) as a function of x. If $|g(u,v)|<A(u^2+v^2)+B$, then the bound

$$\left|\int g(u,v)\nu^x(\,du,\,dv)\right|\le\limsup_{T\to\infty}\frac{1}{T}\int_0^T A\big[(F(R)^2+F(R+x)^2)+B\big]\,dR\le\text{ const.}$$

holds with a constant independent of x, as we claimed. The identity (1.15) follows by a simple limiting procedure when $F(R)$ is approximated by the functions $F_p(R)$. This completes the proof of Theorem 3.

The second replacement in formula (3.1) supplies the proof of Theorem 1 in the same way. Finally, since the measures $\mu(F_p)$ tends to μ and $\mu^\infty_{(a,b)}(F_p)=\mu(F_p)\times\mu(F_p)$ tend to $\mu^\infty_{(a,b)}$ as $p\to\infty$, a limiting procedure implies the second relation in (1.10). The proof of the Theorems is completed.

Proof of the Corollary. Let us first consider the case when a simple function $h(u)=h_k(u)$ is chosen which is the linear combination of the indicator function of certain intervals. In this case formula (1.15) implies (1.16). A general density function $h(u)$ can be approximated by a sequence $h_k(u)$ in such a way that

$$\lim_{k\to\infty}\int_a^b |h_k(u)-h(u)|^p\,du=0\;,\quad p=1,\;2.$$

Then a simple limiting procedure $h_k\to h$ gives the proof of the Corollary. □

4. Some comments and generalizations

The limit distributions in Theorems 1, 2 and 3 were given as the limit of a sequence of probability measures $\mu(F_p)$, $\mu^{K(x)}_{(a,b)}(F_p)$, $\mu^\infty_{(a,b)}$ and $\nu^x(F_p)$ which appeared as the solution of the corresponding problems when the function F was replaced by finite trigonometrical series. To describe these approximating measures we had to express the frequencies $\lambda_1,\dots,\lambda_p$ as the linear combination of some numbers $\tau_1,\dots,\tau_s$ linearly independent over the rational numbers with integer coefficients. This is possible for all finite subsets of the frequencies λ_n appearing in (1.1), but may be not possible for all λ_n simultaneously. We shall say that the function $F(R)$ has almost independent frequencies if all frequencies $\{\lambda_n,\ n=1,2,\dots\}$ in formula (1.1) can be expressed simultaneously as the finite linear combination of some numbers $\tau_1,\ \tau_2,\ \dots$ linearly independent over the rational numbers with integer coefficients. In this case the limit distributions in Theorems 1 and 3 can be described directly. If the function $F(R)$ arises as the Fourier expansion of a randomly magnified convex domain with a nice boundary, then it has almost periodic frequencies in the generic case, but not always. The case when it has almost independent frequencies is discussed in detail in paper [3]. This property holds for instance if the function $F(R)$ gives the Fourier expansion of the number of lattice points in concentrical circles of radius R.

If the function $F(R)$ in (1.1) is a finite trigonometrical series, then the limit distributions appearing in Theorems 1 and 3 have a relatively simple form. They are the distribution of a random variable of the form $\sum a_je^{2\pi iT_j}$, where all (finitely many) T_j are linear combinations of independent on the interval $[0,1]$ uniformly distributed random variables with integer coefficients.

This can be seen by following the construction of the limit measures in the proofs of Section 2. Indeed, to understand the structure of the limit measure μ appearing in Theorem 1 let us express the frequencies λ_n in the form (2.3), and define the functions V and G by means of this formula as it was done in (2.6) and (2.7). Then formula (2.12) states that the measure μ is equal to the distribution of the random variable

$G(V(\xi))$, where $\xi = (\xi_1, \dots, \xi_s)$ is a uniformly distributed random variable on the torus $\mathcal{V}$ defined in vector (2.4). The coordinates of the random vector $V(\xi_1, \dots, \xi_s)$ are linear combinations of independent, uniformly distributed random variables in $[0,1]$ with integer coefficients, and this fact together with the form of the function G gives a representation of $G(V(\xi))$ in the above described form.

The measures ν^x can also be represented in a similar way. Here again, the measure ν^x is the limit distribution of the random variable $G(V(\xi))$, but now the functions G and V are defined in (2.14) and (2.15), and $\xi = (\xi_1, \dots, \xi_{2s})$ is a $\hat\rho^x$ distributed random vector, where $\hat\rho^x$ is the probability measure on $\mathcal{V} \times \mathcal{V}$ with Fourier coefficients (2.28). Actually $\hat\rho^x$ distributed random vector has a very simple representation. Indeed, let $\eta_1, \dots, \eta_s$ be independent uniformly distributed random variables on the unit interval $[0,1]$, and let $\eta_{s+k} = \eta_k + \tau_k$, (mod 1), $k = 1, \dots, s$. Then relation (2.19′) and the expression for the Fourier coefficients (2.28) imply that $(\eta_1, \dots, \eta_{2s})$ is a $\bar\rho^x$ distributed random vector. Then, since the vectors $(\eta_k, \eta_{k+s}) = (\eta_k, \eta_k + \tau_k \pmod 1)$ are independent, and η_k is uniformly distributed in $[0,1]$, the same argument works as in the case of the measure μ.

If the function $F(R)$ has almost independent frequencies, then the set of frequencies $\{\lambda_n,\ n = 1, \dots, p\}$ can be expressed in (2.3) with numbers τ_k and coefficients $A(n,k)$ independent of p. In the polynomials whose distribution equal $\mu(F_p)$ and $\nu^x(F_p)$ the same independent random variables can be used for different p. Then the limit distribution μ and ν^x are the distribution of the limit of the random variables constructed for the representation of $\mu(F_p)$ and $\nu^x(F_p)$. Let us observe that the random variables constructed in such a way converge in L_2 norm as $p \to \infty$, and not only their distribution is convergent. This convergence holds, because of (1.2) and the orthogonality of the terms $e^{2\pi i T_j}$ appearing in these expressions. (Actually this representation could be proved by working directly with the function $F(R)$ instead of its approximation by the functions $F_p(R)$.)

In certain cases the above representation is even simpler. So e.g. if $F(R)$ is the Fourier expansion of the number of lattice points in a circle of radius R, then $F(R)$ has almost independent periods. Moreover, each λ_n can be expressed as a single τ_k multiplied by an integer. In this case the above argument yields a representation of μ and ν^x as the distribution of sums of independent random variables. The measures $\mu_{(a,b)}^{K(x)}$ appearing in Theorem 2 do not have such a simple representation as μ or ν^x. On the other hand, they can be expressed as the mixture of the measures ν^x as it is done in (1.15). This relation together with the continuity of the measures ν^x also implies that

$$\lim_{b \to a} \mu_{(a,b)}^{K(x)} = \nu^{K(a)} .$$

Theorems 2 and 3 can be generalized in a natural way. The vectors

$$\left(F\Big(R + \sum_{j=1}^{l} w_j(R,T)\Big), \quad l = 1, \dots, m \right), \qquad aT < R < bT$$

or

$$\left(F\Big(R + \sum_{j=1}^{l} x_j\Big), \quad l = 1, \dots, m \right), \qquad 0 < R < T$$

have a limit distribution as $T \to \infty$ if all $w_j(R,T)$ satisfy the conditions of Theorem 2. They also have the continuity properties analogous to Theorem 2 and 3. In particular, the limit of the first vector equals the m-fold direct product of the measure μ if all $w(_j(R,T)$ satisfy the conditions of case a) in Theorem 2 with functions $z_jK(x)$, and $z_j \to \infty$. The proofs can be done by slightly modifying the method of the present paper. We omit the details.

In Theorem 2 we assumed that $aT \le R \le bT$ with some $a > 0$. Some of the results follow automatically also for $a = 0$ from our results, but to generalize all statements of Theorem 2 to the case when the parameter a can take also the value zero some additional conditions must be imposed. To carry out all required limiting procedures we must know that

$$\lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \int_0^{\varepsilon T} F(R + w(R,T))\, dR = 0$$

To guarantee the last relation some additional properties should be imposed on the function $w(R,T)$. Since the restriction $a > 0$ is not essential in applications we proved Theorem 2 only under the condition $a > 0$.

The results of this paper were proved originally for finite trigonometrical sums in Section 2, and then in Section 3 these results were generalized to functions which can be well approximated by finite trigonometrical sums. The content of formulas (1.1) and (1.2) was the possibility of such a good approximation. In applications this condition can be checked. On the other hand, the weak convergence of the random variables $F(R)$, $F\big((R), F(R + w(R,T))\big)$ or $\big(F(R), F(R + x)\big)$ in Theorems 1, 2 and 3 also hold if formulas (1.1) and (1.2) are replaced by the following weaker condition: There exists a sequence of finite trigonometrical sums $F_p(R)$, $p = 1, 2, \dots$, such that

$$\lim_{p\to\infty} \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \min\{1, |F_p(R) - F(R)|\}\, dR = 0\ . \tag{4.1}$$

Similar conditions were formulated in paper [2] or [4].

We only briefly explain why formula (4.1) implies the weak convergence in Theorems 1, 2 and 3. If we consider functions of the form $g(u,v) = g_{s,t}(u,v) = e^{i(su+tv)}$, then one can show by means of condition (4.1) and the relation $\frac{\partial}{\partial R}(R + w(R,T)) = 1 + o(1)$ that the under the conditions of Theorem 2 the limits

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{(b-a)T} \int_{aT}^{bT} g\big(F(R), F(R + w(R,T))\big)\, dR \\ &\qquad = \lim_{T\to\infty} \lim_{p\to\infty} \frac{1}{(b-a)T} \int_{aT}^{bT} g\big(F_p(R), F_p(R + w(R,T))\big)\, dR \end{aligned} \tag{4.2}$$

exist. Hence to prove the weak convergence of the distribution of the random vectors $\big(g(F(R), g(F(R + w(R,T)))\big)$, $aR \le T \le bT$, it is enough to check the compactness of these distributions in the weak topology. To do this it is enough to show that for any $\varepsilon > 0$ there exits a constant $K = K(\varepsilon)$ such that the following relation holds. The function $h(u,v) = h_K(u,v) = H_K(u^2 + v^2)$, where $H_K(u)$ is defined by the relations

$H_K(u) = 0$ for $|u| \le K$, $H_K(u) = 1$ for $|u| \ge 2K$, and $H_K(u)$ is given by linear interpolation for $K < |u| < 2K$, satisfies the inequality

$$\limsup_{T\to\infty} \frac{1}{(b-a)T} \int_{aT}^{bT} h\big(F_p(R), F_p(R+w(R,T))\big)\, du\, dv < \varepsilon \ .$$

To prove this relation, choose a number $\bar p$ such that for $T > T(\bar p)$

$$\frac{1}{2T}\int_{-T}^{T} \min\{1, |F_{\bar p}(R) - F(R)|\}\, dR < \frac{(b-a)}{2}\varepsilon \ .$$

Then, since $F_{\bar p}(R)$ is a bounded function we can choose $K = 1 + 2\sup\limits_R F_{\bar p}(R)$. It is not difficult to see that $\int h\big(F_{\bar p}(R), F_{\bar p}(R+w(R,T))\big)\, dR = 0$, and relation (4.2) holds with this choice of the function $h_K(u,v)$. The analogue of Theorem 2 under condition (4.1) can be proved by working out the details. The modified version of Theorems 1 and 3 can be proved similarly.

In this paper we did not discuss such functions $w(R,T)$ which satisfy the relation

$$\lim_{T\to 0} \sup_{0\le R\le T} w(R,T) = 0 \ .$$

The reason for this omission is not our disinterest for this case. Actually, the description of this case is a very exciting problem. This is related to the investigation of the limit behavior of the number of lattice points in randomly chosen thin strips. This is a very interesting problem with many unsolved conjectures and few rigorous results. The methods of the present paper are not sufficient to study such problems. Here some essentially new ideas are needed.

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