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An estimate on the maximum of a nice class of stochastic integrals.

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Summary: Let a sequence of iid. random variables ξ_1, \dots, ξ_n be given on a space (X, \mathcal{X}) with distribution μ together with a nice class \mathcal{F} of functions $f(x_1, \dots, x_k)$ of k variables on the product space (X^k, \mathcal{X}^k) . For all $f \in \mathcal{F}$ we consider the random integral $J_{n,k}(f)$ of the function f with respect to the k -fold product of the normalized signed measure $\sqrt{n}(\mu_n - \mu)$, where μ_n denotes the empirical measure defined by the random variables ξ_1, \dots, ξ_n and investigate the probabilities $P\left(\sup_{f \in \mathcal{F}} |J_{n,k}(f)| > x\right)$ for all $x > 0$. We show that for nice classes of functions, for instance if \mathcal{F} is a Vapnik–Červonenkis class, an almost as good bound can be given for these probabilities as in the case when only the random integral of one function is considered.

1. Introduction. Formulation of the main results

The following problem is studied in this paper: Let a probability measure μ be given on a measure space (X, \mathcal{X}) , take a sequence ξ_1, \dots, ξ_n of independent, identically distributed (X, \mathcal{X}) valued random variables with distribution μ , and define the empirical measure μ_n ,

$$\mu_n(A) = \frac{1}{n} \#\{j : \xi_j \in A, 1 \leq j \leq n\}, \quad A \in \mathcal{X},$$

of the sample ξ_1, \dots, ξ_n . Let us take a nice set \mathcal{F} of measurable functions $f(x_1, \dots, x_k)$ on the k -fold product space (X^k, \mathcal{X}^k) and define the integrals $J_{n,k}(f)$ of the functions $f \in \mathcal{F}$ with respect to the k -fold product of the normalized empirical measure μ_n by the formula

$$J_{n,k}(f) = \frac{n^{k/2}}{k!} \int' f(u_1, \dots, u_k) (\mu_n(du_1) - \mu(du_1)) \dots (\mu_n(du_k) - \mu(du_k)),$$

where the prime in \int' means that the diagonals $u_j = u_l, 1 \leq j < l \leq k$,
are omitted from the domain of integration. (1.1)

In this work I try to give a good estimate on the probabilities $P\left(\sup_{f \in \mathcal{F}} |J_{n,k}(f)| > x\right)$ for all $x > 0$. To formulate the main result of the paper first I introduce the following definition.

Definition of L_p -dense classes of functions. *Let us have a measure space (Y, \mathcal{Y}) and a set \mathcal{G} of \mathcal{Y} measurable functions on this space. We call \mathcal{G} an L_p -dense class with parameter D and exponent L if for all numbers $1 \geq \varepsilon > 0$ and probability measures ν on the space (Y, \mathcal{Y}) there exists a finite ε -dense subset $\mathcal{G}_{\varepsilon, \nu} = \{g_1, \dots, g_m\} \subset \mathcal{G}$ in the*

space $L_p(Y, \mathcal{Y}, \nu)$ consisting of $m \leq D\varepsilon^{-L}$ elements, i.e. such a set $\mathcal{G}_{\varepsilon, \nu} \subset \mathcal{G}$ for which $\inf_{g_j \in \mathcal{G}_{\varepsilon, \nu}} \int |g - g_j|^p d\nu < \varepsilon^p$ for all functions $g \in \mathcal{G}$. (Here the set $\mathcal{G}_{\varepsilon, \nu}$ may depend on the measure ν , but its cardinality is bounded by a number depending only on ε .)

In this paper we shall work with such classes of functions \mathcal{F} which contain only functions with absolute value less than or equal to 1. In this case \mathcal{F} is an L_p -dense class of functions for all $1 \leq p < \infty$ (with an exponent and a parameter depending on p) if there is a number $1 \leq p < \infty$ for which it is L_p -dense. We shall formulate our statements mainly for L_p -dense classes of functions with the parameter $p = 2$, since this seems to be the most convenient choice. Our main result is the following

Theorem. *Let us have a non-atomic measure μ on the space (X, \mathcal{X}) together with an L_2 -dense class \mathcal{F} of functions $f = f(x_1, \dots, x_k)$ of k variables with some parameter D and exponent L on the product space (X^k, \mathcal{X}^k) which consists of at most countably many functions, and satisfies the conditions*

$$\|f\|_\infty = \sup_{x_j \in X, 1 \leq j \leq k} |f(x_1, \dots, x_k)| \leq 1, \quad \text{for all } f \in \mathcal{F} \quad (1.2)$$

and

$$\|f\|_2^2 = E f^2(\xi_1, \dots, \xi_k) = \int f^2(x_1, \dots, x_k) \mu(dx_1) \dots \mu(dx_k) \leq \sigma^2 \quad \text{for all } f \in \mathcal{F} \quad (1.3)$$

with some constant $\sigma > 0$. Let us also assume that the parameter D of the L_2 -dense class \mathcal{F} satisfies the condition

$$D \leq n^\beta \quad \text{with some } \beta \geq 0. \quad (1.4)$$

Then there exist some constants $C = C(k) > 0$, $\alpha = \alpha(k) > 0$ and $M = M(k) > 0$ depending only on the parameter k such that the supremum of the random integrals $J_{n,k}(f)$, $f \in \mathcal{F}$, defined by formula (1.1) satisfies the inequality

$$P \left(\sup_{f \in \mathcal{F}} |J_{n,k}(f)| \geq x \right) \leq CD \exp \left\{ -\alpha \left(\frac{x}{\sigma} \right)^{2/k} \right\} \quad (1.5)$$

if $n\sigma^2 \geq \left(\frac{x}{\sigma} \right)^{2/k} \geq M(L + \beta + 1)^{3/2} \log \frac{2}{\sigma}$,

where β is the number in (1.4), and the number D in formula (1.5) agrees with the parameter of the L_2 -dense class \mathcal{F} .

The condition that \mathcal{F} is a countable class of functions can be weakened. To formulate such a result the following definition will be introduced.

Definition of countable majorizability. *A class of functions \mathcal{F} is countably majorizable in the space $(X^k, \mathcal{X}^k, \mu^k)$ if there exists a countable subset $\mathcal{F}' \subset \mathcal{F}$ such*

that for all numbers $x > 0$ the sets $A(x) = \{\omega: \sup_{f \in \mathcal{F}} |J_{n,k}(f)(\omega)| \geq x\}$ and $B(x) = \{\omega: \sup_{f \in \mathcal{F}'} |J_{n,k}(f)(\omega)| \geq x\}$ satisfy the identity $P(A(x) \setminus B(x)) = 0$.

Clearly, $B(x) \subset A(x)$. In the above definition we demanded that for all $x > 0$ the set $B(x)$ is almost as large as $A(x)$. Now the following corollary of the Theorem is given.

Corollary 1 of the Theorem. *Let a class of functions \mathcal{F} satisfy the conditions of the Theorem with the only exception that instead of the condition about the countable cardinality of \mathcal{F} it is assumed that \mathcal{F} is countably majorizable in the space $(X^k, \mathcal{X}^k, \mu^k)$. Then \mathcal{F} satisfies the Theorem.*

The condition that the class of functions \mathcal{F} is countable was imposed to avoid some unpleasant measure theoretical difficulties which would arise if we had to work with possibly non-measurable sets. On the other hand, I have the impression that Corollary 1 can be applied in all investigations where an estimate about the supremum of multiple random integrals with respect to a normalized empirical measure is needed. It is not difficult to prove that Corollary 1 follows from the Theorem. To do this we have to show that if \mathcal{F} is an L_2 -dense class with some parameter D and exponent L , and $\mathcal{F}' \subset \mathcal{F}$, then \mathcal{F}' is also an L_2 -dense class with the same exponent L , only with a possibly different parameter D' .

To prove this statement let us choose for all numbers $1 \geq \varepsilon > 0$ and probability measures ν on (Y, \mathcal{Y}) some functions $f_1, \dots, f_m \in \mathcal{F}$ with $m \leq D \left(\frac{\varepsilon}{2}\right)^{-L}$ elements, such that the sets $\mathcal{D}_j = \left\{f: \int |f - f_j|^2 d\nu \leq \left(\frac{\varepsilon}{2}\right)^2\right\}$ satisfy the relation $\bigcup_{j=1}^m \mathcal{D}_j = Y$. For all sets \mathcal{D}_j for which $\mathcal{D}_j \cap \mathcal{F}'$ is non-empty choose a function $f'_j \in \mathcal{D}_j \cap \mathcal{F}'$. In such a way we get a collection of functions f'_j from the class \mathcal{F}' containing at most $2^L D \varepsilon^{-L}$ elements which satisfies the condition imposed for L_2 -dense classes with exponent L and parameter $2^L D$ for this number ε and measure ν .

The following Corollary of the Theorem may be of special interest. It is similar to some results of paper [2] or Theorem 5.3.14 in [4].

Corollary 2 of the Theorem. *Let us consider a non-atomic probability measure μ on a measure space (X, \mathcal{X}) and an L_2 -dense class \mathcal{F} of functions on the k -fold product space (X^k, \mathcal{X}^k) with some exponent L and parameter D which is either countable or countably majorizable. Let us also assume that $\sup_{x_j \in X, 1 \leq j \leq k} |f(x_1, \dots, x_k)| \leq 1$ for all*

$f \in \mathcal{F}$. Then the supremum of the random stochastic integrals $J_{n,k}(f)$, $f \in \mathcal{F}$, satisfies the inequality

$$P\left(\sup_{f \in \mathcal{F}} |J_{n,k}(f)| \geq x\right) \leq C e^{-\alpha x^{2/k}} \quad (1.6)$$

for all $x > 0$ with some constants $\alpha = \alpha(k) > 0$ and $C = C(k, D, L)$ depending on the parameter k on the exponent and parameter of the L_2 -dense class \mathcal{F} .

Proof of Corollary 2. Let us first assume that $D \leq n$, and apply the result of the Theorem with $\sigma = 1$. Then conditions (1.3) and (1.4) of the Theorem hold. Also the first part of the condition in (1.5) holds if $x \leq n^{k/2}$, and $P(|J_{n,k}(f)| > x) = 0$ if $x > n^{k/2}$. The second condition of (1.5) is satisfied if $x^{2/k} \geq M((L + \beta + 1)^{3/2} \log 2$, hence relation (1.6) holds if $x \geq \text{const.}$ with an appropriate constant. If the number C in (1.6) is chosen sufficiently large, then the right-hand side of (1.6) is greater than 1 for $x \leq \text{const.}$ In the case $n \leq D$ the random integral $|J_{n,k}(f)|$ is less than $\frac{2^k D^{k/2}}{k!}$. Hence the statement of Corollary 2 holds for all $x > 0$ with an appropriate choice of the parameter C .

In the Theorem we have considered the supremum of multiple random integrals for a nice class of functions of k variables with respect to the k -fold product of a normalized empirical measure. It was shown that if the variances of the random integrals we have considered are less than some number $\sigma^2 > 0$, then under some additional conditions this supremum takes a value larger than x with a probability less than $P(C\sigma\eta > x)$, where η is a standard normal random variable, and $C = C(k)$ is a universal constant depending only on the multiplicity k of the random integrals. This is the sharpest estimate we can expect. Moreover, this estimate seems to be sharp also in that respect that the conditions imposed for its validity cannot be considerably weakened. If condition (1.2) does not hold or $n\sigma^2 < (\frac{x}{\sigma})^{2/k}$, then the estimate of the Theorem may not hold any longer even if the class of functions \mathcal{F} contains only one function. In such cases there exist examples for which the probability $P(J_{n,k}(f) > x)$ is too large. Indeed, in such cases it may happen that the value of relatively few members of the sample take the random integral larger than x with relatively large probability, and the remaining part of the sample does not diminish it. Here I do not work out the details of such examples.

If $(\frac{x}{\sigma})^{2/k} < M \log \frac{2}{\sigma}$ with a not too large number $M > 0$, then the estimate of the Theorem may be violated again, but in this case the reason for it is that the supremum of many small random variables may be large. To understand this let us consider the following analogous problem. Take a Wiener process $W(t)$, $0 \leq t \leq 1$, and consider the supremum of the expressions $W(t) - W(s) = \int f_{s,t}(u)W(du) = \bar{J}(f_{s,t})$, with the functions $f_{s,t}(\cdot)$ on the interval $[0, 1]$ defined by the formula $f_{s,t}(u) = 1$ if $s \leq u \leq t$, $f_{s,t}(u) = 0$ if $0 \leq u < s$ or $t < u \leq 1$. If we consider the class of functions $\mathcal{F}_\sigma = \{f_{s,t}: \int f_{s,t}^2(u) du = t - s \leq \sigma^2\}$, then it is natural to expect that

$$P\left(\sup_{f_{s,t} \in \mathcal{F}_\sigma} \bar{J}(f_{s,t}) > x\right) \leq e^{-\text{const.} \cdot (x/\sigma)^2}.$$
 However, this relation does not hold if $x =$

$x(\sigma) < (1-\varepsilon)\sqrt{2 \log \frac{1}{\sigma}}\sigma$ with some $\varepsilon > 0$. In such cases $P\left(\sup_{f_{s,t} \in \mathcal{F}_\sigma} \bar{J}(f_{s,t}) > x\right) \rightarrow 1$, as

$\sigma \rightarrow 0$. This can be proved relatively simply with the help of the estimate $P(\bar{J}(f_{s,t}) > x(\sigma)) \geq \text{const.} \cdot \sigma^{1-\varepsilon}$ if $|t-s| = \sigma^2$ and the independence of the random integrals $\bar{J}(f_{s,t})$ if the functions $f_{s,t}$ are indexed by such pairs (s, t) for which the intervals (s, t) are disjoint.

Some additional work would show that a similar picture arises if we integrate with respect to the normalized empirical measure of a sample with uniform distribution on

the interval $[0, 1]$ instead of a Wiener process. This yields an example for an L_2 -dense class of functions in the case $k = 1$ for which the estimate of the Theorem does not hold any longer if $(\frac{x}{\sigma})^{2/k} < M \log \frac{2}{\sigma}$ with some $M < \sqrt{2}$. At a heuristic level it is clear that such an example can be given also for $k > 1$, and the number M in condition (1.5) has to be chosen larger if we want that the Theorem hold also for an L_2 -dense class of functions \mathcal{F} with a large exponent L . (In this paper I did not try to find the best possible condition of the Theorem in the right-hand side inequality of (1.5).)

One would like to see some interesting examples when the Theorem is applicable and to have some methods to check the conditions of the Theorem. It is useful to know that if \mathcal{F} is a Vapnik–Červonenkis class of functions whose absolute values are bounded by 1, then \mathcal{F} is an L_2 -dense class.

To formulate the above statement more explicitly let us recall that a class of subsets \mathcal{D} of a set S is a Vapnik–Červonenkis class if there exist some constants $B > 0$ and $K > 0$ such that for all integers n and sets $S_0(n) = \{x_1, \dots, x_n\} \subset S$ of cardinality n the collection of sets of the form $S_0(n) \cap D$, $D \in \mathcal{D}$, contains no more than Bn^K subsets of $S_0(n)$. A class of real valued functions \mathcal{F} on a space (Y, \mathcal{Y}) is a Vapnik–Červonenkis class if the graphs of these functions is a Vapnik–Červonenkis class, i.e. if the sets $A(f) = \{(y, t) : y \in Y, \min(0, f(y)) \leq t \leq \max(0, f(y))\}$, $f \in \mathcal{F}$, constitute a Vapnik–Červonenkis class of sets on the product space $Y \times R^1$.

An important result of Dudley states that a Vapnik–Červonenkis class of functions whose absolute values are bounded by 1 is an L_1 -dense class. The parameter and exponent of this L_1 -dense class can be bounded by means of the constants B and K appearing in the definition of Vapnik–Červonenkis classes. On the other hand, an L_1 -dense class of functions bounded by 1 is also an L_2 -dense class (with possibly different exponent and parameter), since $\int |f - g|^2 d\nu \leq 2 \int |f - g| d\nu$ in this case. Dudley's result, whose proof can be found e.g. in Chapter II of Pollard's book [9] (the 25° approximation lemma contains this result in a slightly more general form) is useful for us, because there are results which enable us to prove that certain classes of functions constitute a Vapnik–Červonenkis class.

This work is a continuation of my paper [8], where this question was discussed in the special case when \mathcal{F} contains only one function. Here Theorem 1 (or its equivalent version Theorem 1') of [8] will be applied, but no additional argument of that work is needed. As I have mentioned in [8], the investigation of this paper was motivated by some non-parametric maximum likelihood estimate problems. Earlier I could only prove a much weaker version of this result in [7].

I found some results similar to that of this paper in the work of Arcones and Gine [2], where the tail-behaviour of the supremum of degenerated U -statistics was investigated if the kernel functions of these U -statistics constitute a Vapnik–Červonenkis class. But the bounds of that paper do not give a better estimate if we have the additional information that the variances of the U -statistics we consider are small. On the other hand, one of the main goal of the present paper was to prove such estimates. (Let me remark that formula (1.3) imposes a condition on the variances of the random integrals we consider in this paper. See Lemma 3 in [8].) I know of one work where the dependence

of the estimate on the variance was investigated in a similar case. This is Alexander's paper [1], where the problem of the present paper was studied in the special case $k = 1$. Alexander proved in this case a sharper result. He also studied the case of non-identically distributed random variables and gave an upper bound for the distribution function of the supremum of random integrals with almost as good constants as in the case of a single random integral. Probably a similar result also holds for multiple stochastic integrals, but the proof requires a more careful analysis. Alexander's paper was interesting for me first of all, because I learned some ideas from it which I strongly needed in the present work. On the other hand, I also needed some new arguments, because in the study of multiple stochastic integrals some new difficulties had to be overcome.

This paper consists of six sections and an appendix. In Section 2 the Theorem is reduced to a simpler statement formulated in Proposition 3. Section 3 contains some important results needed in the proof, and the main ideas of the proof are explained there. In particular, the proof of Proposition 3 is reduced to another statement formulated in Proposition 4. Proposition 4 is proved simultaneously with another result described in Proposition 5. To make the proof more transparent first I give it in the special case $k = 1$ in Section 4. Sections 5 and 6 contain the proof of Propositions 4 and 5 in the general case. In Section 5 it is shown how a symmetrization argument can be applied to prove these results, and finally the proof is completed in Section 6. The Appendix contains the proof of an estimate about the tail behaviour of the distribution of homogeneous polynomials of Rademacher functions.

2. Reduction of the Theorem to a simpler result

I shall prove with the help of a natural argument, called the Chaining argument in the literature, and the result Theorem 1' in paper [9] the following result.

Proposition 1. *Let us fix some number $\bar{A} \geq 2^k$, and assume that a class of functions \mathcal{F} satisfies the conditions of the Theorem with a number M in these conditions which may depend also on \bar{A} . Then a number $0 \leq \bar{\sigma} \leq \sigma \leq 1$ and a collection of functions $\mathcal{F}_{\bar{\sigma}} = \{f_1, \dots, f_m\} \subset \mathcal{F}$ with $m \leq D\bar{\sigma}^{-L}$ elements can be chosen in such a way that the sets $\mathcal{D}_j = \{f: f \in \mathcal{F}, \int |f - f_j|^2 d\mu \leq \bar{\sigma}^2\}$, $1 \leq j \leq m$, satisfy the relation $\bigcup_{j=1}^m \mathcal{D}_j = \mathcal{F}$, and*

$$P \left(\sup_{f \in \mathcal{F}_{\bar{\sigma}}} |J_{n,k}(f)| \geq \frac{x}{\bar{A}} \right) \leq 2CD \exp \left\{ -\alpha \left(\frac{x}{4\bar{A}\sigma} \right)^{2/k} \right\} \quad (2.1)$$

$$\text{if } n\sigma^2 \geq \left(\frac{x}{\sigma} \right)^{2/k} \geq ML \log \frac{2}{\sigma}$$

with the constants $\alpha = \alpha(k)$, $C = C(k)$ appearing in Theorem 1' of [8] and the exponent L and parameter D of the L_2 -dense class \mathcal{F} if the constant $M = M(k, \bar{A})$ is chosen sufficiently large. Beside this, also the inequalities $64 \left(\frac{x}{\bar{A}\bar{\sigma}} \right)^{2/k} \geq n\bar{\sigma}^2 \geq \left(\frac{x}{A\sigma} \right)^{2/k}$ and $n\bar{\sigma}^2 \geq \frac{M^{2/3}(L+\beta+1)\log n}{50A^{4/3}}$ hold, provided that $n\sigma^2 \geq \left(\frac{x}{\sigma} \right)^{2/k} \geq M(L+\beta+1)^{3/2} \log \frac{2}{\sigma}$.

Remark: The introduction of the number $\bar{A} \geq 2$ in Proposition 1 may seem a bit artificial. Its role is to guarantee that such a number $\bar{\sigma}$ could be defined in Proposition 1

which satisfies the inequality $\left(\frac{x}{\bar{\sigma}}\right)^{2/k} \geq An\bar{\sigma}^2$ with a sufficiently large previously fixed constant $A = A(k)$.

Proof of Proposition 1. For all $p = 0, 1, 2, \dots$ choose a set $\mathcal{F}_p = \{f_{p,1}, \dots, f_{p,m_p}\} \subset \mathcal{F}$ with $m_p \leq D4^{pL}\sigma^{-L}$ elements in such a way that $\inf_{1 \leq j \leq m_p} \int (f - f_{p,j})^2 d\mu \leq 16^{-p}\sigma^2$ for all $f \in \mathcal{F}$. For all pairs (j, p) , $p = 1, 2, \dots$, $1 \leq j \leq m_p$, choose a precursor $(j', p-1)$, $j' = j'(j, p)$, $1 \leq j' \leq m_{p-1}$, in such a way that the functions $f_{j,p}$ and $f_{j',p-1}$ satisfy the relation $\int |f_{j,p} - f_{j',p-1}|^2 d\mu \leq \sigma^2 16^{-p}$. Then we have $\int \left(\frac{f_{j,p} - f_{j',p-1}}{2}\right)^2 d\mu \leq \frac{1}{4}\sigma^2 16^{-p}$ and $\sup_{x_j \in X, 1 \leq j \leq k} \left| \frac{f_{j,p}(x_1, \dots, x_k) - f_{j',p-1}(x_1, \dots, x_k)}{2} \right| \leq 1$. Theorem 1' of [8] yields that

$$P(A(j, p)) = P\left(|J_{n,k}(f_{j,p} - f_{j',p-1})| \geq \frac{2^{-(1+p)x}}{A}\right) \leq C \exp\left\{-\alpha \left(\frac{2^{p-1}x}{A\sigma}\right)^{2/k}\right\}$$

$$\text{if } \frac{n\sigma^2 2^{-4p}}{4} \geq \left(\frac{2^{p-1}x}{A\sigma}\right)^{2/k}, \quad 1 \leq j \leq m_p, \quad p = 1, 2, \dots, \quad (2.2)$$

and

$$P(B(s)) = P\left(|J_{n,k}(f_{0,s})| \geq \frac{x}{2A}\right) \leq C \exp\left\{-\alpha \left(\frac{x}{2A\sigma}\right)^{2/k}\right\}, \quad 1 \leq s \leq m,$$

$$\text{if } n\sigma^2 \geq \left(\frac{x}{2A\sigma}\right)^{2/k}.$$

Choose the integer number R , $R \geq 0$, in such a way that $2^{(4+2/k)(R+1)} \left(\frac{x}{A\sigma}\right)^{2/k} \geq \frac{n\sigma^2}{2^{2-2/k}} \geq 2^{(4+2/k)R} \left(\frac{x}{A\sigma}\right)^{2/k}$, define $\bar{\sigma}^2 = 16^{-R}\sigma^2$ and $\mathcal{F}_{\bar{\sigma}} = \mathcal{F}_R$. (As $n\sigma^2 \geq \left(\frac{x}{\bar{\sigma}}\right)^{2/k}$ and $\bar{A} \geq 2^k$ by our conditions, there exists such a non-negative number R .) Then the cardinality m of the set $\mathcal{F}_{\bar{\sigma}}$ is clearly not greater than $D\bar{\sigma}^{-L}$, and $\bigcup_{j=1}^m \mathcal{D}_j = \mathcal{F}$. Beside this, the number R was chosen in such a way that the inequalities (2.2) and (2.3) can be applied for $1 \leq p \leq R$. Hence the definition of the precursor of a pair (j, p) implies that

$$P\left(\sup_{f \in \mathcal{F}_{\bar{\sigma}}} |J_{n,k}(f)| \geq \frac{x}{\bar{A}}\right) \leq P\left(\bigcup_{p=1}^R \bigcup_{j=1}^{m_p} A(j, p) \cup \bigcup_{s=1}^m B(s)\right)$$

$$\leq \sum_{p=1}^R \sum_{j=1}^{m_p} P(A(j, p)) + \sum_{s=1}^m P(B(s)) \leq \sum_{p=1}^{\infty} CD4^{pL}\sigma^{-L} \exp\left\{-\alpha \left(\frac{2^{p-1}x}{A\sigma}\right)^{2/k}\right\}$$

$$+ CD\sigma^{-L} \exp\left\{-\alpha \left(\frac{x}{2A\sigma}\right)^{2/k}\right\}.$$

If the condition $\left(\frac{x}{\bar{\sigma}}\right)^{2/k} \geq M(L+1)^{3/2} \log \frac{2}{\sigma}$ holds with a sufficiently large constant M (depending on \bar{A}), then the inequalities

$$4^{pL}\sigma^{-L} \exp\left\{-\alpha \left(\frac{2^{p-1}x}{A\sigma}\right)^{2/k}\right\} \leq 2^{-p} \exp\left\{-\alpha \left(\frac{2^p x}{4A\sigma}\right)^{2/k}\right\}$$

hold for all $p = 1, 2, \dots$, and

$$\sigma^{-L} \exp \left\{ -\alpha \left(\frac{x}{2\bar{A}\sigma} \right)^{2/k} \right\} \leq \exp \left\{ -\alpha \left(\frac{x}{4\bar{A}\sigma} \right)^{2/k} \right\}.$$

Hence the previous estimate implies that

$$\begin{aligned} P \left(\sup_{f \in \mathcal{F}_{\bar{\sigma}}} |J_{n,k}(f)| \geq \frac{x}{\bar{A}} \right) &\leq \sum_{p=1}^{\infty} CD 2^{-p} \exp \left\{ -\alpha \left(\frac{2^p x}{4\bar{A}\sigma} \right)^{2/k} \right\} \\ &+ CD \exp \left\{ -\alpha \left(\frac{x}{4\bar{A}\sigma} \right)^{2/k} \right\} \leq 2CD \exp \left\{ -\alpha \left(\frac{x}{4\bar{A}\sigma} \right)^{2/k} \right\}, \end{aligned}$$

and relation (2.1) holds. We have

$$\begin{aligned} n\bar{\sigma}^2 &= 2^{-4R} n\sigma^2 \leq 2^{-4R} \cdot 2^{(4+2/k)(R+1)+2-2/k} \left(\frac{x}{\bar{A}\sigma} \right)^{2/k} = 2^6 \cdot 2^{2R/k} \left(\frac{x}{\bar{A}\sigma} \right)^{2/k} \\ &= 2^6 \cdot \left(\frac{\sigma}{\bar{\sigma}} \right)^{1/k} \left(\frac{x}{\bar{A}\sigma} \right)^{2/k} = 2^6 \cdot \left(\frac{\bar{\sigma}}{\sigma} \right)^{1/k} \left(\frac{x}{\bar{A}\bar{\sigma}} \right)^{2/k}, \end{aligned}$$

hence $n\bar{\sigma}^2 \leq 2^6 \left(\frac{x}{\bar{A}\bar{\sigma}} \right)^{2/k}$. Beside this,

$$n\bar{\sigma}^2 = 2^{-4R} n\sigma^2 \geq 2^{2-2/k} \cdot 2^{-4R} \cdot 2^{4R+2R/k} \left(\frac{x}{\bar{A}\sigma} \right)^{2/k} \geq \left(\frac{x}{\bar{A}\sigma} \right)^{2/k}.$$

It remained to show that $n\bar{\sigma}^2 \geq \frac{M^{2/3}((L+\beta)\log n+1)}{50A^{4/3}}$.

This inequality clearly holds under the conditions of Proposition 1 if $\sigma \leq n^{-1/3}$, since in this case $\log \frac{2}{\sigma} \geq \frac{\log n}{3}$, and $n\bar{\sigma}^2 \geq \left(\frac{x}{\bar{A}\sigma} \right)^{2/k} \geq \bar{A}^{-2/k} M(L+\beta+1)^{3/2} \log \frac{2}{\sigma} \geq \frac{1}{3} \bar{A}^{-2/k} M(L+\beta+1) \log n$. If $\sigma \geq n^{-1/3}$, then the inequality $2^{(4+2/k)R} \left(\frac{x}{\bar{A}\sigma} \right)^{2/k} \leq \frac{n\sigma^2}{2^{2-2/k}}$

holds. Hence $2^{-4R} \geq 2^{(2-2/k)/(4+2/k)} \left[\frac{\left(\frac{x}{\bar{A}\sigma} \right)^{2/k}}{n\sigma^2} \right]^{4/(4+2/k)}$, and

$$n\bar{\sigma}^2 = 2^{-4R} n\sigma^2 \geq \frac{1}{\bar{A}^{4/3}} (n\sigma^2)^{1-\gamma} \left[\left(\frac{x}{\bar{A}\sigma} \right)^{2/k} \right]^{\gamma} \quad \text{with } \gamma = \frac{4}{4 + \frac{2}{k}} \geq \frac{2}{3}.$$

Since $n\sigma^2 \geq \left(\frac{x}{\sigma} \right)^{2/k} \geq \frac{M}{3} (L+\beta+1)^{3/2}$, and $n\sigma^2 \geq n^{1/3}$, the above estimates yield that $n\bar{\sigma}^2 \geq \bar{A}^{-4/3} (n\sigma^2)^{1/3} \left[\left(\frac{x}{\bar{A}\sigma} \right)^{2/k} \right]^{2/3} \geq \bar{A}^{-4/3} n^{1/9} \left(\frac{M}{3} \right)^{2/3} (L+\beta+1) \geq \frac{M^{2/3}(L+\beta+1)\log n}{50A^{4/3}}$.

Now I formulate Proposition 2 and show that the Theorem follows from Propositions 1 and 2.

Proposition 2. *Let us have a non-atomic measure μ on the space (X, \mathcal{X}) together with a sequence of independent and μ distributed random variables ξ_1, \dots, ξ_n and an*

L_2 -dense class of functions $f = f(x_1, \dots, x_k)$ of k variables with some parameter D and exponent L on the product space (X^k, \mathcal{X}^k) which consists of at most countably many functions and satisfies conditions (1.2), (1.3) and (1.4) with some $\sigma > 0$, and $n\sigma^2 > K((L + \beta) \log n + 1)$ with a sufficiently large number $K = K(k)$. Then there exists some number $\gamma = \gamma(k) > 0$ and threshold index $A_0 = A_0(k) > 0$ depending only on the order k of the stochastic integrals we consider such that

$$P \left(\sup_{f \in \mathcal{F}} |J_{n,k}(f)| \geq An^{k/2}\sigma^{k+1} \right) \leq e^{-\gamma A^{1/2k} n\sigma^2} \quad \text{if } A \geq A_0.$$

In the proof of the Theorem we exploit our freedom in the choice of the parameters in Propositions 1 and 2. Let us choose a number \bar{A}_0 such that $\bar{A}_0 \geq A_0$ and $\gamma \bar{A}_0^{1/2k} \geq \frac{1}{K}$ with the numbers A_0 , K and γ in Proposition 2. Proposition 1 will be applied with such a number \bar{A} for which the inequalities $(\frac{x}{\bar{\sigma}})^{2/k} \geq \frac{\bar{A}^{2/k}}{64} n\bar{\sigma}^2 \geq (4\bar{A}_0)^{2/k} n\bar{\sigma}^2$ hold with the above fixed parameter \bar{A}_0 and the number $\bar{\sigma}$ defined in the proof of Proposition 1. (Here and in the sequel we shall assume that the number x satisfies the condition $n\sigma^2 \geq (\frac{x}{\sigma})^{2/k} \geq M(L + \beta + 1)^{3/2} \log \frac{x}{\sigma}$ imposed both in Proposition 1 and in the Theorem.) Choose such a number M in Proposition 1 (and as a consequence in the Theorem too) for which also the inequality $n\bar{\sigma}^2 \geq \frac{M^{2/3}(L+\beta+1)\log n}{50A^{4/3}} \geq K((L + \beta) \log n + 1)$ holds with the number K appearing in the conditions of Proposition 2. Proposition 1 will be applied with the class of functions \mathcal{F} , the numbers σ and M considered in the Theorem and a number \bar{A} satisfying the above property while Proposition 2 with the above chosen number \bar{A}_0 , the number $\bar{\sigma}$ and the sets of functions \mathcal{D}_j defined in Proposition 1. More precisely, we apply Proposition 2 for the sets of functions $\frac{g-f_j}{2}$ where $g \in \mathcal{D}_j$ and f_j is the ‘center’ of the set \mathcal{D}_j appearing in the definition of the set \mathcal{D}_j in Proposition 1. Observe that these functions constitute an L_2 -dense class of functions with exponent L and parameter D .

Since $(1 - \frac{1}{A})x \geq \frac{x}{2} \geq 2\bar{A}_0 n^{k/2} \bar{\sigma}^{k+1}$ Propositions 1 and 2 with the above parameters yield that

$$\begin{aligned} P \left(\sup_{f \in \mathcal{F}} |J_{n,k}(f)| \geq x \right) &\leq P \left(\sup_{f \in \mathcal{F}_{\bar{\sigma}}} |J_{n,k}(f)| \geq \frac{x}{\bar{A}} \right) \\ &\quad + \sum_{j=1}^m P \left(\sup_{g \in \mathcal{D}_j} \left| J_{n,k} \left(\frac{f_j - g}{2} \right) \right| \geq \bar{A}_0 n^{k/2} \bar{\sigma}^{k+1} \right) \quad (2.4) \\ &\leq 2CD \exp \left\{ -\alpha \left(\frac{x}{4\bar{A}\sigma} \right)^{2/k} \right\} + D\bar{\sigma}^{-L} e^{-\gamma \bar{A}_0^{1/2k} n\bar{\sigma}^2}. \end{aligned}$$

Let us understand how the second term at the right-hand side of (2.4) can be estimated. The condition $n\bar{\sigma}^2 \geq K((L + \beta) \log n + 1)$ implies that $\bar{\sigma} \geq n^{-1/2}$, and by our choice of \bar{A}_0 we have $\gamma \bar{A}_0^{1/2k} n\bar{\sigma}^2 \geq \frac{1}{K} n\bar{\sigma}^2 \geq L \log n \geq 2L \log \frac{1}{\bar{\sigma}}$, i.e. $\bar{\sigma}^{-L} \leq e^{\gamma \bar{A}_0^{1/2k} n\bar{\sigma}^2/2}$.

As we have seen in Proposition 1 $n\bar{\sigma}^2 \geq \left(\frac{x}{A\bar{\sigma}}\right)^{2/k}$. The above relations imply that $\bar{\sigma}^{-L} e^{-\gamma\bar{A}_0^{1/2k} n\bar{\sigma}^2} \leq e^{-\gamma\bar{A}_0^{1/2k} n\bar{\sigma}^2/2} \leq \exp\left\{-\frac{\gamma}{2}\bar{A}_0^{1/2k}\bar{A}^{-2/k}\left(\frac{x}{\bar{\sigma}}\right)^{2/k}\right\}$. Then relation (2.4) gives that

$$P\left(\sup_{f \in \mathcal{F}} |J_{n,k}(f)| \geq x\right) \leq 2CD \exp\left\{-\frac{\alpha}{(4\bar{A})^{2/k}}\left(\frac{x}{\bar{\sigma}}\right)^{2/k}\right\} + D \exp\left\{-\frac{\gamma}{2}\bar{A}_0^{1/2k}\bar{A}^{-2/k}\left(\frac{x}{\bar{\sigma}}\right)^{2/k}\right\}.$$

The last formula means that under the conditions of the Theorem formula (1.5) holds (with some new appropriately defined constant $\alpha > 0$), and this is what we had to prove.

It remained to prove Proposition 2. Its proof requires some new ideas, and the remaining part of the paper deals with this problem. There is a counterpart of this result about so-called degenerate U -statistics. The study of degenerate U -statistics is technically simpler. Hence I formulate this result about U -statistics in Proposition 3 and show that it implies Proposition 2.

First I recall some notions we need to formulate Proposition 3. Let us have a sequence of independent and identically distributed random variables ξ_1, ξ_2, \dots with distribution μ on a measurable space (X, \mathcal{X}) together with a function $f = f(x_1, \dots, x_k)$ on the k -th power (X^k, \mathcal{X}^k) of the space (X, \mathcal{X}) . We define with their help the U -statistic $I_{n,k}(f)$ of order k , as

$$I_{n,k}(f) = \frac{1}{k!} \sum_{\substack{1 \leq j_s \leq n, s=1, \dots, k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} f(\xi_{j_1}, \dots, \xi_{j_k}). \quad (2.5)$$

(The function f in this formula will be called the kernel function of the U -statistic.)

A real valued function $f = f(x_1, \dots, x_k)$ on the k -th power (X^k, \mathcal{X}^k) of a space (X, \mathcal{X}) is called a canonical kernel function (with respect to the probability measure μ on the space (X, \mathcal{X})) if

$$\int f(x_1, \dots, x_{j-1}, u, x_{j+1}, \dots, x_k) \mu(du) = 0 \quad \text{for all } 1 \leq j \leq k \text{ and } x_s \in X, s \neq j.$$

Let me also introduce the notion of canonical functions in a more general case, because this notion appears later in Proposition 5. We call a function $f(x_1, \dots, x_k)$ on the k -fold product $(X_1 \times \dots \times X_k, \mathcal{X}_1 \times \dots \times \mathcal{X}_k, \mu_1 \times \dots \times \mu_k)$ of k not necessarily identical probability spaces $(X_j, \mathcal{X}_j, \mu_j)$, $1 \leq j \leq k$, if

$$\int f(x_1, \dots, x_{j-1}, u, x_{j+1}, \dots, x_k) \mu_j(du) = 0 \quad \text{for all } 1 \leq j \leq k \text{ and } x_s \in X_s, s \neq j.$$

A U -statistic with a canonical kernel function is called degenerate. Now I formulate Proposition 3.

Proposition 3. *Let us have a probability measure μ on a space (X, \mathcal{X}) together with a sequence of independent and μ distributed random variables ξ_1, \dots, ξ_n and an L_2 -dense class \mathcal{F} of canonical kernel functions $f = f(x_1, \dots, x_k)$ (with respect to the measure μ) with some parameter D and exponent L on the product space (X^k, \mathcal{X}^k) which consists of at most countably many functions, and satisfies conditions (1.2), (1.3) and (1.4) with some $\sigma > 0$. Let $n\sigma^2 > K((L + \beta) \log n + 1)$ with a sufficiently large constant $K = K(k)$. Then there exist some numbers $C = C(k) > 0$, $\gamma = \gamma(k) > 0$ and threshold index $A_0 = A_0(k) > 0$ depending only on the order k of the U -statistics we consider such that the degenerate U -statistics $I_{n,k}(f)$, $f \in \mathcal{F}$, defined in (2.5) satisfy the inequality*

$$P \left(\sup_{f \in \mathcal{F}} |n^{-k/2} I_{n,k}(f)| \geq A n^{k/2} \sigma^{k+1} \right) \leq C e^{-\gamma A^{1/2k} n \sigma^2} \quad \text{if } A \geq A_0.$$

(The constants in Propositions 2 and 3 may be different.) Before deducing Proposition 2 from Proposition 3 I formulate a simple lemma which will be useful also in the subsequent part of the paper. To formulate it let us introduce the following notations.

Let some measure spaces (Y_1, \mathcal{Y}_1) , (Y_2, \mathcal{Y}_2) and (Z, \mathcal{Z}) be given together with a probability measure μ on the space (Z, \mathcal{Z}) . Consider a function $f(y_1, z, y_2)$ on the product space $(Y_1 \times Z \times Y_2, \mathcal{Y}_1 \times \mathcal{Z} \times \mathcal{Y}_2)$, $y_1 \in \mathcal{Y}_1$, $z \in \mathcal{Z}$, $y_2 \in \mathcal{Y}_2$, and define their projection

$$P_\mu f(y_1, y_2) = \int f(y_1, z, y_2) \mu(dz), \quad y_1 \in Y_1, y_2 \in Y_2, \quad (2.6)$$

$$\bar{P}_\mu f(y_1, z, y_2) = P_\mu f(y_1, y_2), \quad y_1 \in Y_1, z \in Z, y_2 \in Y_2, \quad (2.6')$$

and

$$\begin{aligned} Q_\mu f(y_1, z, y_2) &= (I - \bar{P}_\mu) f(y_1, z, y_2) \\ &= f(y_1, z, y_2) - \bar{P}_\mu f(y_1, z, y_2), \quad y_1 \in Y_1, z \in Z, y_2 \in Y_2. \end{aligned} \quad (2.6'')$$

(The difference between the operators P_μ and \bar{P}_μ is that in the definition of the function $\bar{P}_\mu f$ we introduced a fictive argument z , i.e. $P_\mu f$ is defined on the space $Y_1 \times Y_2$ and \bar{P}_μ on the space $Y_1 \times Z \times Y_2$.)

Lemma 1. *Let us have some measure spaces (Y_1, \mathcal{Y}_1) , (Y_2, \mathcal{Y}_2) and (Z, \mathcal{Z}) , a probability measure μ on the space (Z, \mathcal{Z}) and a probability measure ρ on the product space $(Y_1 \times Y_2, \mathcal{Y}_1 \times \mathcal{Y}_2)$. The transformations P_μ , \bar{P}_μ and Q_μ defined in (2.6)–(2.6'') are contractions from the space $L_2(Y_1 \times Z \times Y_2, \rho \times \mu)$ to the spaces $L_2(Y_1 \times Y_2, \rho)$ and $L_2(Y_1 \times Z \times Y_2, \rho \times \mu)$ respectively, i.e.*

$$\begin{aligned} \|P_\mu f\|_{L_2, \rho}^2 &= \int P_\mu f(y_1, z, y_2)^2 \rho(dy_1, dy_2) \\ &= \|\bar{P}_\mu f\|_{L_2, \rho \times \mu}^2 = \int \bar{P}_\mu f(y_1, z, y_2)^2 \rho(dy_1, dy_2) \mu(dz) \\ &\leq \|f\|_{L_2, \rho \times \mu}^2 = \int f(y_1, z, y_2)^2 \rho(dy_1, dy_2) \mu(dz), \end{aligned} \quad (2.7)$$

and

$$\begin{aligned}
\|Q_\mu f\|_{L_2, \rho}^2 &= \int Q_\mu f(y_1, z, y_2)^2 \rho(dy_1, dy_2) \\
&= \int (f(y_1, z, y_2) - \bar{P}_\mu f(y_1, z, y_2))^2 \rho(dy_1, dy_2) \mu(dz) \quad (2.7') \\
&\leq \|f\|_{L_2, \rho \times \mu}^2 = \int f(y_1, z, y_2)^2 \rho(dy_1, dy_2) \mu(dz).
\end{aligned}$$

If \mathcal{F} is an L_2 -dense class of functions $f(y_1, z, y_2)$ on the product space $(Y_1 \times Z \times Y_2, \mathcal{Y}_1 \times \mathcal{Z} \times \mathcal{Y}_2)$, $y_1 \in \mathcal{Y}_1$, $z \in \mathcal{Z}$, $y_2 \in \mathcal{Y}_2$ with parameter D and exponent L , then also the classes $\mathcal{F}_\mu = \{P_\mu f, : f \in \mathcal{F}\}$ and $\bar{\mathcal{F}}_\mu = \{\bar{P}_\mu f, : f \in \mathcal{F}\}$ with the functions $P_\mu f$ and $\bar{P}_\mu f$ defined in formulas (2.6) and (2.6') are L_2 -dense classes with parameter D and exponent L in the spaces $(Y_1 \times Y_2, \mathcal{Y}_1 \times \mathcal{Y}_2)$ and $(Y_1 \times Z \times Y_2, \mathcal{Y}_1 \times \mathcal{Z} \times \mathcal{Y}_2)$ respectively, and the space $\mathcal{G}_\mu = \{f - \bar{P}_\mu f, f \in \mathcal{F}\}$ defined in (2.6'') is an L_2 -dense class with parameter $2^L D$ and exponent L in the space $(Y_1 \times Z \times Y_2, \mathcal{Y}_1 \times \mathcal{Z} \times \mathcal{Y}_2)$. Moreover, the class of functions $\mathcal{G}'_\mu = \{\frac{1}{2}(f - \bar{P}_\mu f), f \in \mathcal{F}\}$ is an L_2 -dense class with exponent L and parameter D .

Proof of Lemma 1. The Schwarz inequality yields that $P_\mu(f)^2 \leq \int f(y_1, z, y_2)^2 \mu(dz)$, and the inequality $\int [f(y_1, z, y_2) - \bar{P}_\mu f(y_1, z, y_2)]^2 \mu(dz) \leq \int f(y_1, z, y_2)^2 \mu(dz)$ also holds. Integrating these inequalities with respect to the probability measure $\rho(dy_1, dy_2)$ we get formulas (2.7) and (2.7').

Let us consider an arbitrary probability measure ρ on the space $(Y_1 \times Y_2, \mathcal{Y}_1 \times \mathcal{Y}_2)$. To prove that \mathcal{F}_μ is an L_2 -dense class we have to find $m \leq D\varepsilon^L$ functions $f_j \in \mathcal{F}_\mu$, $1 \leq j \leq m$, such that $\inf_{1 \leq j \leq m} \int (f_j - f)^2 d\rho \leq \varepsilon^2$ for all $f \in \mathcal{F}_\mu$. But a similar property holds in the space $Y_1 \times Z \times Y_2$ with the probability measure $\rho \times \mu$. This property together with the L_2 contraction property of P_μ formulated in (2.7) imply that \mathcal{F}_μ is an L_2 -dense class. The analogous property for $\bar{\mathcal{F}}_\mu$ follows from the already proved L_2 -density property of \mathcal{F}_μ and the fact that by replacing a measure ρ on $Y_1 \times Z \times Y_2$ by the measure $\bar{\rho} \times \mu$, where $\bar{\rho}$ is the projection of the measure ρ to the space $Y_1 \times Y$, i.e. $\bar{\rho}(B) = \rho(B \times Z)$ for $B \in \mathcal{Y}_1 \times \mathcal{Y}_2$ we do not change the L_2 norm of a difference $\bar{P}_\mu f - \bar{P}_\mu g$, $f, g \in \mathcal{F}$. Moreover, it equals to the L_2 norm of the difference $P_\mu f - P_\mu g$ with respect to the measure $\bar{\rho}$. Finally, the desired L_2 -density property of the set \mathcal{G}_μ can be deduced from the following observation. For any probability measure ρ on the space $Y_1 \times Z \times Y_2$ and pair of functions f and g such that $\int (f - g)^2 \frac{1}{2} (d\rho + d\bar{\rho} \times d\mu) \leq \frac{\varepsilon^2}{4}$, where $\bar{\rho}$ is the projection of the measure ρ to the space $Y_1 \times Y_2$, $\int ((f - \bar{P}_\mu f) - (g - \bar{P}_\mu g))^2 d\rho \leq 2 \int (f - g)^2 d\rho + 2 \int (\bar{P}_\mu f - \bar{P}_\mu g)^2 d\rho \leq 2 \int (f - g)^2 d\rho + 2 \int (f - g)^2 d\bar{\rho} \times d\mu \leq \varepsilon^2$. This means that if $\{f_1, \dots, f_m\}$ is an $\frac{\varepsilon}{2}$ -dense subset of \mathcal{F} in the space $L_2(Y_1 \times Z \times Y_2, \mathcal{Y}_1 \times \mathcal{Z} \times \mathcal{Y}_2, \bar{\rho})$ with $\bar{\rho} = \frac{1}{2}(\rho + \bar{\rho} \times \mu)$, then $\{Q_\mu f_1, \dots, Q_\mu f_m\}$ is an ε -dense subset of \mathcal{G}_μ in the space $L_2(Y_1 \times Z \times Y_2, \mathcal{Y}_1 \times \mathcal{Z} \times \mathcal{Y}_2, \rho)$. Moreover, if $\{f_1, \dots, f_m\}$ is an ε -dense subset with respect to the measure $\frac{1}{2}(\rho + \bar{\rho} \times \mu)$, then $\{\frac{1}{2}Q_\mu f_1, \dots, \frac{1}{2}Q_\mu f_m\}$ is an ε -dense subset of \mathcal{G}'_μ in the space $L_2(Y_1 \times Z \times Y_2, \mathcal{Y}_1 \times \mathcal{Z} \times \mathcal{Y}_2, \rho)$.

To deduce Proposition 2 from Proposition 3 let us first introduce the (random) probability measures $\mu^{(j)}$, $1 \leq j \leq n$, concentrated in the sample points ξ_j , i.e. let

$\mu^{(j)}(A) = 1$ if $\xi_j \in A$, and $\mu^{(j)}(A) = 0$ if $\xi_j \notin A$, $A \in \mathcal{A}$. Then we can write $\mu_n - \mu = \frac{1}{n} \left(\sum_{j=1}^n (\mu^{(j)} - \mu) \right)$, and formula (1.1) can be rewritten as

$$J_{n,k}(f) = \frac{1}{n^{k/2} k!} \sum_{l=1}^k \sum_{j_l=1}^n \int' f(u_1, \dots, u_k) \left(\mu^{(j_1)}(du_1) - \mu(du_1) \right) \dots \left(\mu^{(j_k)}(du_1) - \mu(du_1) \right).$$

To rearrange the above sum in a way more appropriate for us let us introduce the following notations: Let $\mathcal{P} = \mathcal{P}_k$ denote the set of partitions of the set $\{1, 2, \dots, k\}$, and given a sequence (j_1, \dots, j_k) , $1 \leq j_s \leq n$, $1 \leq s \leq k$, of length k let $H(j_1, \dots, j_k)$ denote that partition of \mathcal{P}_k in which two points s and t , $1 \leq s, t \leq k$, belong the same element of the partition if $j_s = j_t$. Given a set A , let $|A|$ denote its cardinality.

Let us rewrite the above expression for $J_{n,k}(f)$ in the form

$$J_{n,k}(f) = \frac{1}{n^{k/2} k!} \sum_{P \in \mathcal{P}} \sum_{\substack{(j_1, \dots, j_k), \\ 1 \leq j_l \leq n, 1 \leq l \leq k \\ H(j_1, \dots, j_k) = P}} \int' f(u_1, \dots, u_k) \left(\mu^{(j_1)}(du_1) - \mu(du_1) \right) \dots \left(\mu^{(j_k)}(du_1) - \mu(du_1) \right). \quad (2.8)$$

Let us remember that the diagonals $u_s = u_t$, $s \neq t$, were omitted from the domain of integration in the formula defining $J_{n,k}(f)$. This implies that in the case $j_s = j_t$ the measure $\mu^{(j_s)}(du_s) \mu^{(j_t)}(du_t)$ has zero measure in the domain of integration. We have to understand the cancellation effects caused by this relation. I want to show that because of these cancellations the expression in formula (2.8) can be rewritten as a linear combination of degenerated U -statistics with not too large coefficients. The U -statistics taking part in this linear combination can be bounded by means of Proposition 3, and this yields an estimate sufficient for our purposes. This seems to be a natural approach, but the detailed proof demands some rather unpleasant calculations.

Let us fix some $P \in \mathcal{P}$ and investigate the inner sum at the right-hand side of (2.8) corresponding to this partition P . For the sake of simplicity let us first consider such a sum that corresponds to a partition $P \in \mathcal{P}$ which contains a set of the form $\{1, \dots, s\}$ with some $s \geq 2$. The products of measures corresponding to the terms in the sum determined by such a partition contain a part of length s which has the form $(\mu^{(j)}(du_1) - \mu(du_1)) \dots (\mu^{(j)}(du_s) - \mu(du_s))$ with some $1 \leq j \leq n$. This part of the product can be rewritten in the domain of integration as

$$\sum_{l=1}^s (-1)^{s-1} \mu(du_1) \dots \mu(du_{l-1}) (\mu^{(j)}(du_l) - \mu(du_l)) \mu(du_{l+1}) \dots \mu(du_s) \\ + (-1)^{s-1} (s-1) \mu(du_1) \dots \mu(du_s).$$

Here we exploit that all other terms of this product disappears in the domain of integration. Let us also observe that the term $(-1)^{s-1}(s-1)\mu(du_1)\dots\mu(du_l)$ appears n -times as we sum up for $1 \leq j \leq n$. Similar calculation can be made for all partitions $P \in \mathcal{P}$ and all sets contained in the partitions, only the notation of the indices will be more complicated. By carrying out such a calculation the quantity $J_{n,k}(f)$ can be rewritten as the linear combination of integrals of the function $f(u_1, \dots, u_k)$ with respect to some product measures. The components of these products of measures have either the form $(\mu^{(j_s)}(du_s) - \mu(du_s))$ or the form $\mu(du_s)$, and all indices j_s in a product are different. Let us observe that to integrate a function f with respect to $(\mu^{(j_s)}(du_s) - \mu(du_s))$ is the same as to apply the operator $Q_{\mu,s} = I - \bar{P}_{\mu,s}$ for it and then to put $\xi_s = u_s$ in the s -th argument of the function $Q_{\mu,s}f$, and to integrate a function f with respect to $\mu(du_s)$ is the same as to apply the operator $P_{\mu,s}$ for it. Here $Q_{\mu,s}$ and $P_{\mu,s}$ are the operators Q_μ and P_μ defined in formulas (2.6'') and (2.6) if we choose in these formulas Y_1 as the product of the first $s-1$ components, Z as the s -th component and Y_2 as the product of the last $k-s$ components of the k -fold product X^k .

Let us work out the details of the above indicated calculations and for all sets $V \subset \{1, \dots, k\}$ let us gather in an internal sum depending on V those integrals for which the product of the measures contain a component of the form $\mu^{(j_s)}(du_s) - \mu(du_s)$, $1 \leq j_s \leq n$, if $s \in V$ and a term $\mu(du_s)$ if $s \notin V$. In such a way we get the identity

$$J_{n,k}(f) = \sum_{V \subset \{1,2,\dots,k\}} C(n,k,|V|) n^{-|V|/2} \frac{1}{k!} \sum_{\substack{1 \leq j_s \leq n, \\ j_s \neq j_{s'} \text{ if } s \neq s' \\ \text{for } s \in V}} f_V(\xi_{j_s}, s \in V) \quad (2.9)$$

with the functions

$$f_V(u_s, s \in V) = \prod_{s \in V} Q_{\mu,s} \prod_{t \in \{1,\dots,k\} \setminus V} P_{\mu,t} f(u_1, u_2, \dots, u_k) \quad \text{for all } V \subset \{1, \dots, k\} \quad (2.10)$$

and some coefficients $C(n,k,|V|)$ which satisfy the inequality $|C(n,k,|V|)| \leq G(k)$ with some constant $G(k) > 0$. The explicit formula for $C(n,k,|V|)$ is rather complicated, but the above estimate about the magnitude of this coefficient is sufficient for our purposes. This estimate of $C(k,n,|V|)$ is sharp, because those partitions $P \in \mathcal{P}$ which contain the $|V|$ one-point subsets of a set V and $(k-|V|)/2$ subsets of cardinality 2 of $\{1, \dots, k\} \setminus V$ yield a contribution of order $n^{-k/2} n^{k/2-|V|/2}$ to the coefficient $C(n,k,|V|) n^{-|V|/2}$.

Let us observe that the inner sum corresponding to a set V at the right-hand side of (2.9) is a U -statistic with the kernel function f_V defined in (2.10). Hence to carry out our program we have to understand the properties of this function f_V . It follows from Lemma 1 that under the conditions of Proposition 1 the set of functions $f_V, f \in \mathcal{F}$, is an L_2 -dense class with exponent L and parameter $2^{kL}D$, and $\left| \int f_V^2(u_s, s \in V) \prod_{s \in V} \mu(du_s) \right| \leq \sigma^2$ for all $V \in \{1, \dots, k\}$. Let me remark that this estimate states in particular that the constant term f_\emptyset defined in (2.10) with the choice $V = \emptyset$ satisfies the inequality $|f_\emptyset| \leq \sigma$. This estimate follows directly from the Schwarz

inequality, because

$$f_\emptyset^2 = \left(\int f(u_1, \dots, u_k) \mu(du_1) \dots \mu(du_k) \right)^2 \leq \int f^2(u_1, \dots, u_k) \mu(du_1) \dots \mu(du_k) \leq \sigma^2.$$

Another important observation is that the functions f_V are canonical kernel functions with respect to the measure μ . To prove this statement let us observe that the canonical property of a kernel function f_V can be reformulated as $P_{\mu,s} f_V(u_s, s \in V) = 0$ for all $s \in V$ and sets of parameters $u_t \in X, t \in V \setminus \{s\}$. This relation follows from the observation that the operators $\bar{P}_{\mu,u}, 1 \leq u \leq k$ are exchangeable, and $\bar{P}_{\mu,s}^2 = \bar{P}_{\mu,s}$ which implies that $\bar{P}_{\mu,s} Q_{\mu,s} = \bar{P}_{\mu,s} (I - \bar{P}_{\mu,s}) = 0$. (Actually, here we adapted the proof of the Hoeffding decomposition of U -statistics to our case.)

Formula (2.9) yields that

$$J_{n,k}(f) = \sum_{V \subset \{1,2,\dots,n\}} C(n,k,|V|) n^{-|V|/2} I_{n,|V|}(f_V(\xi_{j_s}, s \in V)),$$

and

$$\begin{aligned} P \left(\sup_{f \in \mathcal{F}} |J_{n,k}(f)| \geq A n^{k/2} \sigma^{k+1} \right) \\ \leq \sum_{V \subset \{1,2,\dots,n\}} P \left(\sup_{f \in \mathcal{F}} |n^{-|V|/2} I_{n,|V|}(f_V)| \geq \frac{A}{T} n^{k/2} \sigma^{k+1} \right) \end{aligned}$$

with some appropriate constant $T = T(k)$. Observe that under the Conditions of Proposition 2 $n\sigma^2 \geq 1$, hence $n^{k/2} \sigma^{k+1} \geq n^{|V|/2} \sigma^{|V|+1}$. This means that if the parameters A_0 and K are sufficiently large in the conditions of Propositions 2, then this conditions allow the application of Proposition 3 to bound the probability $P(n^{-|V|/2} |I_{n,|V|}(f_V)| \geq \frac{A}{T} n^{k/2} \sigma^{k+1}) \leq P(n^{-|V|/2} |I_{n,|V|}(f_V)| \geq \frac{A}{T} n^{|V|/2} \sigma^{|V|+1})$ for all functions f_V . Thus we get that the inequality

$$P \left(\sup_{f \in \mathcal{F}} |J_{n,k}(f)| \geq A n^{k/2} \sigma^{k+1} \right) \leq C 2^K e^{-\gamma(A/T)^{1/2k} n \sigma^2} \leq e^{-\gamma(A/2T)^{1/2k} n \sigma^2}$$

holds for $A \geq A_0$ with some $T = T(k)$ if first the constant K and then the constant A_0 are chosen sufficiently large in the conditions of Proposition 2. This means that Proposition 3 implies Proposition 2.

3. Some basic tools of the proof

First I formulate three results we apply in the proof of Proposition 3. The first of them helps us to carry out some symmetrization arguments, the second one yields a good estimate for the distribution of a homogeneous polynomial of independent random variables which take values ± 1 with probability $\frac{1}{2}$. Finally, the third result enables us to reduce Proposition 3 to a simpler statement.

The first result, formulated in Lemma 2 is a slight generalization of a simple lemma which can be found for instance in Pollard's book [9] (8° Symmetrization Lemma). I made this generalization, because it is more appropriate for our purposes.

Lemma 2. (Symmetrization Lemma) *Let $Z(n)$ and $\bar{Z}(n)$, $n = 1, 2, \dots$, be two sequences of random variables on a probability space (Ω, \mathcal{A}, P) . Let a σ -algebra $\mathcal{B} \subset \mathcal{A}$ be given on the probability space (Ω, \mathcal{A}, P) together with a \mathcal{B} measurable set B and two numbers $\alpha > 0$ and $\beta > 0$ such that the random variables Z_n , $n = 1, 2, \dots$ are \mathcal{B} measurable, and the inequality*

$$P(|\bar{Z}_n| \leq \alpha | \mathcal{B})(\omega) \geq \beta \quad \text{for all } n = 1, 2, \dots \text{ if } \omega \in B \quad (3.1)$$

holds. Then

$$P\left(\sup_{1 \leq n < \infty} |Z_n| > \alpha + x\right) \leq \frac{1}{\beta} P\left(\sup_{1 \leq n < \infty} |Z_n - \bar{Z}_n| > x\right) + (1 - P(B)) \quad \text{for all } x > 0. \quad (3.2)$$

In particular, if the sequences Z_n , $n = 1, 2, \dots$, and \bar{Z}_n , $n = 1, 2, \dots$, are two independent sequences of random variables, and $P(|Z_n| \leq \alpha) \geq \beta$ for all $n = 1, 2, \dots$, then

$$P\left(\sup_{1 \leq n < \infty} |Z_n| > \alpha + x\right) \leq \frac{1}{\beta} P\left(\sup_{1 \leq n < \infty} |Z_n - \bar{Z}_n| > x\right). \quad (3.2')$$

Proof of Lemma 2. Put $\tau = \min\{n: |Z_n| > \alpha + x\}$ if there exists such an n , and $\tau = 0$ otherwise. Then

$$\begin{aligned} P(\{\tau = n\} \cap B) &\leq \frac{1}{\beta} \int_{\{\tau = n\} \cap B} P(|\bar{Z}_n| \leq \alpha | \mathcal{B}) dP = \frac{1}{\beta} P(\{\tau = n\} \cap \{|\bar{Z}_n| \leq \alpha\} \cap B) \\ &\leq \frac{1}{\beta} P(\{\tau = n\} \cap \{|Z_n - \bar{Z}_n| > x\}) \quad \text{for all } n = 1, 2, \dots \end{aligned}$$

Hence

$$\begin{aligned} P\left(\sup_{1 \leq n < \infty} |Z_n| > \alpha + x\right) - (1 - P(B)) &\leq P\left(\left\{\sup_{1 \leq n < \infty} |Z_n| > \alpha + x\right\} \cap B\right) \\ &= \sum_{n=1}^{\infty} P(\{\tau = n\} \cap B) \leq \frac{1}{\beta} \sum_{n=1}^{\infty} P(\{\tau = n\} \cap \{|Z_n - \bar{Z}_n| > x\}) \\ &\leq \frac{1}{\beta} P\left(\sup_{1 \leq n < \infty} |Z_n - \bar{Z}_n| > x\right). \end{aligned}$$

Thus formula 3.2 is proved. If Z_n and \bar{Z}_n are two independent sequences, and $P(|Z_n| \leq \alpha) \geq \beta$ for all $n = 1, 2, \dots$, and we define \mathcal{B} as the σ -algebra generated by the random variables Z_n , $n = 1, 2, \dots$, then the condition (3.1) is satisfied also with $B = \Omega$. Hence relation (3.2') holds in this case. Lemma 2 is proved.

The second result we need is a multi-dimensional version of Hoeffding's inequality formulated in Proposition A:

Proposition A. *Let $\varepsilon_1, \dots, \varepsilon_n$ be independent random variables, $P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = \frac{1}{2}$, $1 \leq j \leq n$. Fix a positive integer k and define the random variable*

$$Z = \sum_{\substack{(j_1, \dots, j_k): 1 \leq j_l \leq n \text{ for all } 1 \leq l \leq k \\ j_l \neq j_{l'} \text{ if } l \neq l'}} a(j_1, \dots, j_k) \varepsilon_{j_1} \cdots \varepsilon_{j_k} \quad (3.3)$$

with the help of some real numbers $a(j_1, \dots, j_k)$ which are given for all sets of indices such that $1 \leq j_l \leq n$, $1 \leq l \leq k$, and $j_l \neq j_{l'}$ if $l \neq l'$. Put

$$S^2 = \sum_{\substack{(j_1, \dots, j_k): 1 \leq j_l \leq n \text{ for all } 1 \leq l \leq k \\ j_l \neq j_{l'} \text{ if } l \neq l'}} a^2(j_1, \dots, j_k) \quad (3.4)$$

Then

$$P(|Z| > x) \leq C \exp \left\{ -B \left(\frac{x}{S} \right)^{2/k} \right\} \quad \text{for all } x \geq 0 \quad (3.5)$$

with some constants $B > 0$ and $C > 0$ depending only on the parameter k . Relation (3.5) holds for instance with the choice $B = \frac{k}{2e(k!)^{1/k}}$ and $C = e^k$.

Proposition A is a relatively simple consequence of a famous and important result of the probability theory, the so-called hypercontractive inequality for Rademacher functions (see e.g. [3] or [6]). The hypercontractive inequality yields some moment inequalities that imply Proposition A. Nevertheless, I did not find this result in the literature. Therefore I explain in the Appendix how it follows from the hypercontractive inequality.

Remark: The parameter B given in Proposition A is not sharp. This is because the moment estimates I could prove are not sharp enough. They are sufficient to give the right order of the term in the exponent at the right-hand side of (3.5) but do not give the best possible constant B in this estimate.

Finally I formulate a decoupling type result which enables us to reduce Proposition 3 to a similar but simpler statement. This result compares the distribution of U -statistics with the distribution of such systems whose coordinates are chosen independently from each other. To make a clear distinction between this object and usual U -statistics I shall call it independent U -statistics. It is defined in the following way:

Definition of independent U -statistics. Let us have k independent copies $\xi_{1,s}, \dots, \xi_{n,s}, 1 \leq s \leq k$, of a sequence of independent and identically distributed random variables ξ_1, \dots, ξ_n with distribution μ on a measurable space (X, \mathcal{X}) together with a function $f = f(x_1, \dots, x_k)$ on the k -th power (X^k, \mathcal{X}^k) of the space (X, \mathcal{X}) . We define with their help the independent U -statistic $\bar{I}_{n,k}(f)$ by the formula

$$\bar{I}_{n,k}(f) = \frac{1}{k!} \sum_{\substack{1 \leq j_s \leq n, s=1, \dots, k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} f(\xi_{j_1,1}, \dots, \xi_{j_k,k}). \quad (3.6)$$

The following Proposition B holds.

Proposition B. Let us consider a countable sequence $f_l(x_1, \dots, x_k), l = 1, 2, \dots$, of functions on the k -fold product (X^k, \mathcal{X}^k) of some space (X, \mathcal{X}) together with some probability measure μ on the space (X, \mathcal{X}) . Given a sequence of independent and identically distributed random variables ξ_1, ξ_2, \dots with distribution μ on (X, \mathcal{X}) together with k independent copies $\xi_{1,s}, \xi_{2,s}, \dots, 1 \leq s \leq k$, of it we can define the U -statistics $I_{n,k}(f_l)$ and independent U -statistics $\bar{I}_{n,k}(f_l)$ for all $l = 1, 2, \dots$ and $n = 1, 2, \dots$. They satisfy the inequality

$$P \left(\sup_{1 \leq l < \infty} |I_{n,k}(f_l)| > x \right) \leq AP \left(\sup_{1 \leq l < \infty} |\bar{I}_{n,k}(f_l)| > \gamma x \right) \quad (3.7)$$

for all $x \geq 0$ with some constants $A = A(k) > 0$ and $\gamma = \gamma(k) > 0$ depending only on the order k of the U -statistics.

I shall deduce Proposition B from the result of paper [5] of de la Peña and Montgomery-Smith. At first sight one would think that this result is not sufficient for our purposes, since it compares the distribution function of a single U -statistic with its independent U -statistic counterpart, i.e. the supremum with respect to a class of functions is missing there. But this result is proved for general Banach space valued random variables. Therefore, as I show below, its application for an appropriate L_∞ space yields the desired result.

The proof of Proposition B (with the help of paper [5].) Let us apply the first part of Theorem 1 of [5] in the Banach space ℓ_∞ consisting of the infinite sequences $x = (x_1, x_2, \dots)$ of real numbers with norm $\|x\| = \sup_{1 \leq l < \infty} |x_l|$ for the kernel functions $f_{j_1, \dots, j_k}(x_1, \dots, x_k) = \bar{f}(x_1, \dots, x_k), \bar{f} = (f_1, f_2, \dots)$, mapping the space (X^k, \mathcal{X}^k) into the space ℓ_∞ . (Here we do not exploit that in the result of [5] the kernel functions may depend on the indices (j_1, \dots, j_k) .) Then the result in [5] states that

$$P \left(\left\| \sum_{\substack{1 \leq j_s \leq n, s=1, \dots, k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} \bar{f}(\xi_{j_1,1}, \dots, \xi_{j_k,k}) \right\| > x \right)$$

$$\leq AP \left(\left\| \sum_{\substack{1 \leq j_s \leq n, s=1, \dots, k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} \bar{f}(\xi_{j_1,1}, \dots, \xi_{j_k,k}) \right\| > \gamma x \right)$$

with some universal constants $A = A(k) > 0$ and $\gamma = \gamma(k) > 0$, and this statement is equivalent to relation (3.7).

Remark: Actually it would be enough to prove Proposition B only for the supremum of finitely many U -statistics with kernel functions f_1, \dots, f_N and then letting $N \rightarrow \infty$. In such a way we can avoid the work with infinite dimensional Banach spaces. Such an approach makes the proof simpler, in particular because some measure-theoretical problems arise if we are working with Banach spaces, where not all continuous linear functionals are measurable. Such a difficulty really occurs if we are working with $L_\infty(X)$ spaces with a set X of large cardinality. If we want to apply the result of [5] in the space ℓ_∞ , then we have to check that it is applicable in this case.

Now I formulate the following Proposition 3'.

Proposition 3'. *Let us have a probability measure μ on a space (X, \mathcal{X}) together with k independent copies $\xi_{1,s}, \dots, \xi_{n,s}$, $1 \leq s \leq k$, of a sequence of independent and μ distributed random variables ξ_1, \dots, ξ_n and a countable L_2 -dense class \mathcal{F} of canonical kernel functions $f = f(x_1, \dots, x_k)$ (with respect to the measure μ) with some parameter D and exponent L on the product space (X^k, \mathcal{X}^k) which satisfies conditions (1.2), (1.3) and (1.4) with some $\sigma > 0$. Let $n\sigma^2 > K((L + \beta) \log n + 1)$ with a sufficiently large constant $K = K(k)$. Then there exists some threshold index $A_0 = A_0(k) > 0$ such that the independent U -statistics $\bar{I}_{n,k}(f)$, $f \in \mathcal{F}$, defined in (3.6) satisfy the inequality*

$$P \left(\sup_{f \in \mathcal{F}} |n^{-k/2} \bar{I}_{n,k}(f)| \geq An^{k/2} \sigma^{k+1} \right) \leq e^{-A^{1/2k} n \sigma^2} \quad \text{if } A \geq A_0.$$

Proposition 3' and Proposition B imply Proposition 3. The proof of Proposition 3' applies some ideas of a paper of Alexander [1]. Let me briefly explain them.

Let us restrict our attention to the case $k = 1$. In this case a probability of the form $P \left(n^{-1/2} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n f(\xi_j) \right| > x \right)$ has to be estimated. By taking an independent copy of the sequence ξ_n (which disappears at the end of the of the calculation) a symmetrization argument can be applied which reduces the problem to the estimation of the probability $P \left(n^{-1/2} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n \varepsilon_j f(\xi_j) \right| > \bar{x} \right)$, where the random variables ε_j , $P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = \frac{1}{2}$, $j = 1, \dots, n$, are independent, and they are independent also of the random variables ξ_j . Beside this, the number \bar{x} is only slightly smaller than the number $x/2$. Let us bound the conditional probability of the event we have just introduced if the values

random variables ξ_j are prescribed in it. This conditional probability can be bounded by means of the one-dimensional version of Proposition A, and the estimate we get in such a way is useful if the conditional variance of the random variable we have to handle has a good upper bound. Such a bound exists, and some calculation reduces the original problem to the estimation of the probability $P\left(n^{-1/2} \sup_{f \in \mathcal{F}'} \left| \sum_{j=1}^n f(\xi_j) \right| > x^{1+\alpha}\right)$ with some new nice class of functions \mathcal{F}' and number $\alpha > 0$. This problem is very similar to the original one, but it is simpler, since the number x is replaced by a larger number $x^{1+\alpha}$ in it. By repeating this argument successively, in finitely many steps we get to an inequality that clearly holds.

The above sketched argument suggests a backward induction procedure to prove Proposition 3'. To carry out such a program I shall prove a result formulated in Proposition 4. To do this first I introduce the following notion.

Definition of good tail behaviour for a class of U -statistics. *Let us have some measurable space (X, \mathcal{X}) and a probability measure μ on it. Let us consider some class \mathcal{F} of functions $f(x_1, \dots, x_k)$ on the k -fold product (X^k, \mathcal{X}^k) of the space (X, \mathcal{X}) . Fix some positive integer n and positive number $\sigma > 0$, and take k independent copies $\xi_{1,s}, \dots, \xi_{n,s}$, $1 \leq s \leq k$, of a sequence of independent μ -distributed random variables ξ_1, \dots, ξ_n . Let us introduce with the help of these random variables the independent U -statistics $\bar{I}_{n,k}(f)$, $f \in \mathcal{F}$. Given some real number $T > 0$ we say that the set of independent U -statistics determined by the class of functions \mathcal{F} has a good tail behaviour at level T if the inequality*

$$P\left(\sup_{f \in \mathcal{F}} |n^{-k/2} \bar{I}_{n,k}(f)| \geq An^{k/2} \sigma^{k+1}\right) \leq \exp\left\{-A^{1/2k} n \sigma^2\right\} \quad \text{for all } A \geq T. \quad (3.8)$$

holds.

Now I formulate Proposition 4.

Proposition 4. *Let us fix a positive integer n , real number $\sigma > 0$ and a probability measure μ on a measurable space (X, \mathcal{X}) together with a countable L_2 -dense class \mathcal{F} of canonical kernel functions $f = f(x_1, \dots, x_k)$ (with respect to the measure μ) on the k -fold product space (X^k, \mathcal{X}^k) which has exponent L and parameter D , and the number D satisfies condition (1.4). Let us also assume that all functions $f \in \mathcal{F}$ satisfy the conditions $\sup_{x_j \in X, 1 \leq j \leq k} |f(x_1, \dots, x_k)| \leq 2^{-(k+1)}$, $\int f^2(x_1, \dots, x_k) \mu(dx_1) \dots \mu(dx_k) \leq \sigma^2$, and $n\sigma^2 > K((L + \beta) \log n + 1)$ with a sufficiently large fixed number $K = K(k)$. There exists some real number $A_0 = A_0(k) > 1$ such that for all classes of functions \mathcal{F} which satisfy the conditions of Proposition 4 the sets of U -statistics determined by the functions $f \in \mathcal{F}$ have a good tail behaviour at level T for some $T \geq A_0$, provided that they have a good tail behaviour at level $T^{4/3}$.*

It is not difficult to deduce Proposition 3' from Proposition 4. Indeed, let us observe that the set of U -statistics determined by a class of functions \mathcal{F} satisfying the conditions

of Proposition 4 has a good tail-behaviour at level $n^{k/2}$, since the probability at the left-hand side of (3.8) equals zero for $u \geq n^{k/2}$. Then we get from Proposition 4 by induction with respect to the number j , that this set of U -statistics has a good tail-behaviour also for $T = n^{-(4/3)^j k/2}$ for $j = 1, 2, \dots$ if $n^{-(4/3)^j k/2} \geq A_0$. This implies that if a class of functions \mathcal{F} satisfies the conditions of Proposition 4, then the set of U -statistics determined by this class of functions has a good tail-behaviour at level $T = A_0^{4/3}$, i.e. at a level which depends only on the order k of the (independent) U -statistics. This result implies Proposition 3', only we have to apply it not directly for the class of functions \mathcal{F} appearing in Proposition 3', but these functions have to be multiplied by a sufficiently small positive number depending only on k .

Thus to complete the proof of the Theorem it is enough to prove Proposition 4. I describe its proof in the special case $k = 1$ in the next section. This case is considered separately, because it may help to understand the ideas of the proof in the general case.

The main difficulty in the proof of Proposition 4 is related to a symmetrization procedure which is an essential part of the proof. We want to apply some randomization with the help of a symmetrization argument, and this requires a special justification. This is not a difficult problem in the case $k = 1$, where it is enough to calculate the variance of a U -statistic, but it becomes hard for $k \geq 2$. In this case we have to give a good estimate on certain conditional variances of some (independent) U -statistics with respect to some appropriate conditions. To overcome this difficulty we formulate a result in Proposition 5 and prove Propositions 4 and 5 simultaneously. Their proof follows the following line. First Proposition 4 and Proposition 5 are proved for $k = 1$. Then, if Propositions 4 and 5 are already proven for all $k' < k$, then first we prove Proposition 4 for k , and then Proposition 5 for the same k . Proposition 5 has a similar structure to Proposition 4. Before its formulation I introduce the following definition.

Definition of good tail behaviour for a class of integrals of U -statistics. *Let us have a product space $(X^k \times Y, \mathcal{X}^k \times \mathcal{Y})$ with some product measure $\mu^k \times \rho$, where $(X^k, \mathcal{X}^k, \mu^k)$ is the k -fold product of some probability space (X, \mathcal{X}, μ) , and (Y, \mathcal{Y}, ρ) is some other probability space. Fix some positive integer n and positive number $\sigma > 0$, and consider some class \mathcal{F} of functions $f(x_1, \dots, x_k, y)$ on the product space $(X^k \times Y, \mathcal{X}^k \times \mathcal{Y}, \mu^k \times \rho)$. Take k independent copies $\xi_{1,s}, \dots, \xi_{n,s}$, $1 \leq s \leq k$, of a sequence of independent, μ -distributed random variables ξ_1, \dots, ξ_n . For all $f \in \mathcal{F}$ and $y \in Y$ let us define the independent U -statistics $\bar{I}_{n,k}(f, y)$ by means of these random variables $\xi_{1,s}, \dots, \xi_{n,s}$, $1 \leq s \leq k$, and formula (3.6). Define with the help of these U -statistics $\bar{I}_{n,k}(f, y)$ the random integrals*

$$H_{n,k}(f) = \int \bar{I}_{n,k}(f, y)^2 \rho(dy), \quad f \in \mathcal{F}. \quad (3.9)$$

Choose some real number $T > 0$. We say that the set of random integrals $H_{n,k}(f)$, $f \in \mathcal{F}$, have a good tail behaviour at level T if

$$P \left(\sup_{f \in \mathcal{F}} n^{-k} H_{n,k}(f) \geq A^2 n^k \sigma^{2k+2} \right) \leq \exp \left\{ -A^{1/(2k+1)} n \sigma^2 \right\} \quad \text{for } A \geq T.$$

Proposition 5. Fix some positive integer n and real number $\sigma > 0$, and let us have a product space $(X^k \times Y, \mathcal{X}^k \times \mathcal{Y})$ with some product measure $\mu^k \times \rho$, where $(X^k, \mathcal{X}^k, \mu^k)$ is the k -fold product of some probability space (X, \mathcal{X}, μ) , and (Y, \mathcal{Y}, ρ) is some another probability space. Let us have a countable L_2 -dense class \mathcal{F} of canonical functions $f(x_1, \dots, x_k, y)$ on the product space $(X^k \times Y, \mathcal{X}^k \times \mathcal{Y}, \mu^k \times \rho)$ with some exponent L and parameter D which satisfies condition (1.4). Let us also assume that the functions $f \in \mathcal{F}$ satisfy the conditions

$$\sup_{x_j \in X, 1 \leq j \leq k, y \in Y} |f(x_1, \dots, x_k, y)| \leq 2^{-(k+1)}$$

and

$$\int f^2(x_1, \dots, x_k, y) \mu(dx_1) \dots \mu(dx_k) \rho(dy) \leq \sigma^2 \quad \text{for all } f \in \mathcal{F}.$$

Let the inequality $n\sigma^2 > K((L + \beta) \log n + 1)$ hold with a sufficiently large fixed number $K = K(k)$.

There exists some number $A_0 = A_0(k) > 1$ such that for all classes of functions \mathcal{F} which satisfy the conditions of Proposition 5 the random integrals $H_{n,k}(f)$, $f \in \mathcal{F}$, defined in (3.9) have a good tail behaviour at level T , provided that they have a good tail behaviour at level $T^{(2k+1)/2k}$.

Similarly to the argument formulated after Proposition 4 an inductive procedure yields the following corollary of Proposition 5.

Corollary of Proposition 5. If the class of functions \mathcal{F} satisfies the conditions of Proposition 5, then there exists a constant $\bar{A}_0 = \bar{A}_0(k) > 0$ depending only on k such that the integrals $H_{n,k}(f)$ determined by the class of functions \mathcal{F} have a good tail behaviour at level \bar{A}_0 .

4. The proof of Proposition 4 in the case $k = 1$

In this section Proposition 4 is proved in the special case $k = 1$. In this case we have to show that

$$P \left(\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n f(\xi_j) \right| \geq An^{1/2} \sigma^2 \right) \leq e^{-A^{1/2} n \sigma^2} \quad \text{if } A \geq T \quad (4.1)$$

if we know the same estimate for $A > T^{4/3}$ and all classes of functions satisfying the conditions of Proposition 4. This statement will be proved by means of the following symmetrization argument.

Lemma 3. Let the class of functions \mathcal{F} satisfy the conditions of Proposition 4 for $k = 1$. Let $\varepsilon_1, \dots, \varepsilon_n$ be a sequence of independent random variables, $P(\varepsilon_j = 1) =$

$P(\varepsilon_j = -1) = \frac{1}{2}$, independent also of the μ distributed random variables ξ_1, \dots, ξ_n .
Then

$$\begin{aligned} P \left(\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n f(\xi_j) \right| \geq An^{1/2}\sigma^2 \right) \\ \leq 4P \left(\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n \varepsilon_j f(\xi_j) \right| \geq \frac{A}{3}n^{1/2}\sigma^2 \right) \quad \text{if } A \geq T. \end{aligned} \quad (4.2)$$

Proof of Lemma 3. Let us construct an independent copy $\bar{\xi}_1, \dots, \bar{\xi}_n$ of the sequence ξ_1, \dots, ξ_n in such a way that all three sequences ξ_1, \dots, ξ_n , $\bar{\xi}_1, \dots, \bar{\xi}_n$ and $\varepsilon_1, \dots, \varepsilon_n$ are independent. Define the random variables $Z_n(f) = \frac{1}{\sqrt{n}} \sum_{h=1}^n f(\xi_h)$ and $\bar{Z}_n(f) = \frac{1}{\sqrt{n}} \sum_{h=1}^n f(\bar{\xi}_h)$ for all $f \in \mathcal{F}$. I claim that

$$P \left(\sup_{f \in \mathcal{F}} |Z_n(f)| > A\sqrt{n}\sigma^2 \right) \leq 2P \left(\sup_{f \in \mathcal{F}} |Z_n(f) - \bar{Z}_n(f)| > \frac{2}{3}A\sqrt{n}\sigma^2 \right). \quad (4.3)$$

This relation follows from Lemma 2 (the symmetrization lemma) applied for the countable sets $Z_n(f)$ and $\bar{Z}_n(f)$, $f \in \mathcal{F}$, with $x = \frac{2}{3}A\sqrt{n}\sigma^2$ and $\alpha = \frac{1}{3}A\sqrt{n}\sigma^2$, since the fields $Z_n(f)$ and $\bar{Z}_n(f)$ are independent, and $P(|Z_n(f)| \leq \alpha) > \frac{1}{2}$ for all $f \in \mathcal{F}$. Indeed, $E\bar{Z}_n(f)^2 \leq \sigma^2$, thus Chebishev's inequality implies that $P(|\bar{Z}_n(f)| \leq \sqrt{2}\sigma) \geq \frac{1}{2}$ for all $f \in \mathcal{F}$. On the other hand, we have assumed that $n\sigma^2 \geq K$ with some sufficiently large constant $K > 0$. Hence $\sigma \leq \frac{1}{\sqrt{K}}\sqrt{n}\sigma^2$, and $\sqrt{2}\sigma \leq \alpha = \frac{1}{3}A\sqrt{n}\sigma^2$ if the constant K is chosen sufficiently large.

Let us observe that the random field

$$Z_n(f) - \bar{Z}_n(f) = \frac{1}{\sqrt{n}} \sum_{j=1}^n (f(\xi_j) - f(\bar{\xi}_j)), \quad f \in \mathcal{F}, \quad (4.4)$$

and its randomization

$$\frac{1}{\sqrt{n}} \sum_{j=1}^n \varepsilon_j (f(\xi_j) - f(\bar{\xi}_j)), \quad f \in \mathcal{F}, \quad (4.4')$$

have the same distribution. Indeed, even the conditional distribution of (4.4') under the condition that the values of the ε_j -s are prescribed agrees with the distribution of (4.4) for all possible values of the ε_j -s. This follows from the observation that the distribution of the field (4.4) does not change if we exchange the random variables ξ_j and $\bar{\xi}_j$ for certain indices j , and this corresponds to considering the conditional distribution of the field in (4.4') under the condition that $\varepsilon_j = -1$ for these indices j , and $\varepsilon_j = 1$ for the remaining ones.

The above relation together with formula (4.3) imply that

$$\begin{aligned}
& P \left(\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n f(\xi_j) \right| \geq An^{1/2}\sigma^2 \right) \\
& \leq 2P \left(\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n \varepsilon_j [f(\xi_j) - \bar{f}(\xi_j)] \right| \geq \frac{2}{3}An^{1/2}\sigma^2 \right) \\
& \leq 2P \left(\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n \varepsilon_j f(\xi_j) \right| \geq \frac{A}{3}n^{1/2}\sigma^2 \right) \\
& \quad + 2P \left(\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n \varepsilon_j f(\bar{\xi}_j) \right| \geq \frac{A}{3}n^{1/2}\sigma^2 \right) \\
& = 4P \left(\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n \varepsilon_j f(\xi_j) \right| \geq \frac{A}{3}n^{1/2}\sigma^2 \right)
\end{aligned}$$

Lemma 3 is proved.

To prove Proposition 4 for $k = 1$ let us investigate the conditional probability

$$P(f, A | \xi_1, \dots, \xi_n) = P \left(\frac{1}{\sqrt{n}} \left| \sum_{j=1}^n \varepsilon_j f(\xi_j) \right| \geq \frac{A}{6} \sqrt{n} \sigma^2 \middle| \xi_1, \dots, \xi_n \right)$$

for all functions $f \in \mathcal{F}$, $A \geq T$ and values (ξ_1, \dots, ξ_n) . By Proposition A (with $k = 1$) we can write

$$P(f, A | \xi_1, \dots, \xi_n) \leq C \exp \left\{ -\frac{\frac{B}{36} A^2 n \sigma^4}{S^2(f, \xi_1, \dots, \xi_n)} \right\} \quad (4.5)$$

with

$$S^2(f, x_1, \dots, x_n) = \frac{1}{n} \sum_{j=1}^n f^2(x_j), \quad f \in \mathcal{F}.$$

Let us introduce the set

$$H = H(A) = \left\{ (x_1, \dots, x_n) : \sup_{f \in \mathcal{F}} S^2(f, x_1, \dots, x_n) \geq (1 + A^{4/3}) \sigma^2 \right\}. \quad (4.6)$$

I claim that

$$P((\xi_1, \dots, \xi_n) \in H) \leq e^{-A^{2/3} n \sigma^2} \quad \text{if } A \geq T. \quad (4.6')$$

To prove relation (4.6') let us consider the functions $\bar{f} = \bar{f}(f)$ for all $f \in \mathcal{F}$ defined by the formula $\bar{f}(x) = f^2(x) - \int f^2(x) \mu(dx)$, and introduce the class of functions

$\mathcal{F}' = \{\bar{f}(f) : f \in \mathcal{F}\}$. Let us show that the class of functions \mathcal{F}' satisfies the conditions of Proposition 4, hence the estimate (4.1) holds for the class of functions \mathcal{F}' if $A \geq T^{4/3}$.

The relation $\int \bar{f}(x)\mu(dx) = 0$ clearly holds. (In the case $k = 1$ this means that \bar{f} is a canonical function.) The condition $\sup |\bar{f}(x)| \leq \frac{1}{8} < \frac{1}{4}$ also holds if $\sup |f(x)| \leq \frac{1}{4}$, and $\int \bar{f}^2(x)\mu(dx) \leq \int f^4(x)\mu(dx) \leq \frac{1}{4} \int f^2(x)\mu(dx) \leq \frac{\sigma^2}{4} < \sigma^2$ if $f \in \mathcal{F}$. It remained to show that \mathcal{F}' is an L_2 -dense class with exponent L and parameter D .

To show this observe that $\int (\bar{f}(x) - \bar{g}(x))^2 \rho(dx) \leq 2 \int (f^2(x) - g^2(x))^2 \rho(dx) + 2 \int (f^2(x) - g^2(x))^2 \mu(dx) \leq 2(\sup(|f(x)| + |g(x)|))^2 (\int (f(x) - g(x))^2 (\rho(dx) + \mu(dx))) \leq \int (f(x) - g(x))^2 \bar{\rho}(dx)$ for all $f, g \in \mathcal{F}$, $\bar{f} = \bar{f}(f)$, $\bar{g} = \bar{g}(g)$ and probability measure ρ , where $\bar{\rho} = \frac{\rho + \mu}{2}$. This means that if $\{f_1, \dots, f_m\}$ is an ε -dense subset of \mathcal{F} in the space $L_2(X, \mathcal{X}, \bar{\rho})$, then $\{\bar{f}_1, \dots, \bar{f}_m\}$ is an ε -dense subset of \mathcal{F}' in the space $L_2(X, \mathcal{X}, \rho)$, and not only \mathcal{F} , but also \mathcal{F}' is an L_2 -dense class with exponent L and parameter D .

We get, by applying formula (4.1) for the number $A^{4/3} \geq T^{4/3}$ and the class of functions \mathcal{F}' that

$$\begin{aligned} P((\xi_1, \dots, \xi_n) \in H) &= P\left(\sup_{f \in \mathcal{F}'} \left(\frac{1}{n} \sum_{h=1}^n \bar{f}(\xi_j) + \frac{1}{n} \sum_{h=1}^n E f^2(\xi_j)\right) \geq (1 + A^{4/3}) \sigma^2\right) \\ &\leq P\left(\sup_{f \in \mathcal{F}'} \frac{1}{\sqrt{n}} \sum_{h=1}^n \bar{f}(\xi_j) \geq A^{4/3} n^{1/2} \sigma^2\right) \leq e^{-A^{2/3} n \sigma^2}, \end{aligned}$$

i.e. relation (4.6') holds.

Formula (4.5) and the definition (4.6) of the set H yield the estimate

$$P(f, A | \xi_1, \dots, \xi_n) \leq C e^{-BA^{2/3} n \sigma^2 / 40} \quad \text{if } (\xi_1, \dots, \xi_n) \notin H \quad (4.7)$$

for all $f \in \mathcal{F}$ and $A \geq T$ for the conditional probability $P(f, A | \xi_1, \dots, \xi_n)$. Let us introduce the conditional probability

$$P(\mathcal{F}, A | \xi_1, \dots, \xi_n) = P\left(\sup_{f \in \mathcal{F}} \frac{1}{\sqrt{n}} \left| \sum_{j=1}^n \varepsilon_j f(\xi_j) \right| \geq \frac{A}{3} \sqrt{n} \sigma^2 \mid \xi_1, \dots, \xi_n\right)$$

for all (ξ_1, \dots, ξ_n) and $A \geq T$. We shall estimate this conditional probability with the help of relation (4.7) if $(\xi_1, \dots, \xi_n) \notin H$. Given some set of n points (x_1, \dots, x_n) in the space (X, \mathcal{X}) let us introduce the measure $\nu = \nu(x_1, \dots, x_n)$ on (X, \mathcal{X}) in such a way that ν is concentrated in the points x_1, \dots, x_n , and $\nu(\{x_j\}) = \frac{1}{n}$. If $\int f^2(u)\nu(du) \leq \delta^2$

for a function f , then $\left| \frac{1}{\sqrt{n}} \sum_{j=1}^n \varepsilon_j f(x_j) \right| \leq n^{1/2} \int |f(u)|\nu(du) \leq n^{1/2} \delta$. Since we have

assumed that $n\sigma^2 \geq 1$, this estimate implies that if f and g are two functions such that $\int (f - g)^2 \nu(dx) \leq \delta^2$ with $\delta = \frac{A}{6n}$, then $\left| \frac{1}{\sqrt{n}} \sum_{j=1}^n \varepsilon_j f(x_j) - \frac{1}{\sqrt{n}} \sum_{j=1}^n \varepsilon_j g(x_j) \right| \leq \frac{A}{6\sqrt{n}} \leq \frac{A}{6} \sqrt{n} \sigma^2$.

Given some (random) point $(\xi_1, \dots, \xi_n) \in H$ let us consider the measure $\nu = \nu(\xi_1, \dots, \xi_n)$ corresponding to it, and choose a $\bar{\delta}$ -dense subset $\{f_1, \dots, f_m\}$ of \mathcal{F} in the space $L_2(X, \mathcal{X}, \nu)$ with $\bar{\delta} = \frac{1}{6n} \leq \delta = \frac{A}{6n}$, whose cardinality m satisfies the inequality $m \leq D\bar{\delta}^{-L}$. This is possible because of the L_2 -dense property of the class \mathcal{F} . (This is the point where the L_2 -dense property of the class of functions \mathcal{F} is exploited in its full strength.) The above facts imply that $P(\mathcal{F}, A|\xi_1, \dots, \xi_n) \leq \sum_{l=1}^m P(f_l, A|\xi_1, \dots, \xi_n)$ with these functions f_1, \dots, f_m . Hence relation (4.7) yields that

$$P(\mathcal{F}, A|\xi_1, \dots, \xi_n) \leq CD(6n)^L e^{-BA^{2/3}n\sigma^2/40} \quad \text{if } (\xi_1, \dots, \xi_n) \notin H \text{ and } A \geq T.$$

This inequality together with Lemma 3 and estimate (4.6') imply that

$$\begin{aligned} P\left(\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n f(\xi_j) \right| \geq An^{1/2}\sigma^2\right) &\leq 4P\left(\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n \varepsilon_j f(\xi_j) \right| \geq \frac{A}{3}n^{1/2}\sigma^2\right) \\ &\leq 4CD(6n)^L e^{-BA^{2/3}n\sigma^2/40} + 4e^{-A^{2/3}n\sigma^2} \quad \text{if } A \geq T. \end{aligned} \quad (4.8)$$

Since we have a better power of A in the exponent at the right-hand side of formula (4.8) than we need, the relation $n\sigma^2 \geq K((L+\beta)\log n + 1)$ holds, and we have the right to choose the constants K and A_0 , $A \geq A_0$, sufficiently large, it is not difficult to deduce relation (3.8) from relation (4.8). Indeed, the expression in the exponent at the right-hand side of (4.8) satisfies the inequality $\frac{B}{40}A^{2/3}n\sigma^2 \geq A^{1/2}n\sigma^2 + K((L+\beta)\log n + 1)$ if A_0 is sufficiently large, and

$$\begin{aligned} P\left(\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{j=1}^n f(\xi_j) \right| \geq An^{1/2}\sigma^2\right) \\ \leq 4C(6n)^{\beta+L} e^{-Kn} n^{-K(L+\beta)} e^{-A^{1/2}n\sigma^2} + 4e^{-A^{2/3}n\sigma^2} \leq e^{-A^{1/2}n\sigma^2} \end{aligned}$$

if $A \geq T$, and the constants A_0 and K are chosen sufficiently large.

5. The symmetrization argument

In the proof of Propositions 4 and 5 we need two symmetrization results for all $k \geq 1$ which play the same role as Lemma 3 in the case $k = 1$. These results are described in Lemmas 4A and 4B. In this section these results are formulated and proved. The proofs go by induction with respect to k . During the proof of Propositions 4 and 5 for k we may assume that they hold for $k' < k$.

Lemma 4A. *Let \mathcal{F} be a class of functions on the space (X^k, \mathcal{X}^k) which satisfies the conditions of Proposition 4 with some probability measure μ . Let us have k independent copies $\xi_{1,s}, \dots, \xi_{n,s}$, $1 \leq s \leq k$, of a sequence of independent μ distributed random variables ξ_1, \dots, ξ_n , and a sequence of independent random variables $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $P(\varepsilon_s = 1) = P(\varepsilon_s = -1) = \frac{1}{2}$, which is independent also of the random variables $\xi_{j,s}$,*

$1 \leq j \leq n, 1 \leq s \leq k$. Consider the independent U -statistics $\bar{I}_{n,k}(f)$, $f \in \mathcal{F}$, defined from these random variables by formula (3.6) and their randomized version

$$\bar{I}_{n,k}^\varepsilon(f) = \frac{1}{k!} \sum_{\substack{1 \leq j_s \leq n, s=1, \dots, k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} \varepsilon_{j_1} \cdots \varepsilon_{j_k} f(\xi_{j_1,1}, \dots, \xi_{j_k,k}), \quad f \in \mathcal{F}. \quad (5.1)$$

There exists some constant $A_0 = A_0(k)$ such that the inequality

$$P \left(\sup_{f \in \mathcal{F}} n^{-k/2} |\bar{I}_{n,k}(f)| > An^{k/2} \sigma^{k+1} \right) < 2^{k+1} P \left(\sup_{f \in \mathcal{F}} |\bar{I}_{n,k}^\varepsilon(f)| > 2^{-(k+1)} An^k \sigma^{k+1} \right) + 2^k n^{k-1} e^{-A^{1/(2k-1)} n \sigma^2 / k} \quad (5.2)$$

holds for all $A \geq A_0$.

Before formulating Lemma 4B needed in the proof of Proposition 5 I introduce some notations. Some of them will be needed later.

Let us consider a set of functions \mathcal{F} of functions $f(x_1, \dots, x_k, y) \in \mathcal{F}$ on a space $(X^k \times Y, \mathcal{X}^k \times \mathcal{Y}, \mu^k \times \rho)$ which satisfies the conditions of Proposition 5. Let us choose $2k$ independent copies $\xi_{1,s}^{(1)}, \dots, \xi_{n,s}^{(1)}, \xi_{1,s}^{(-1)}, \dots, \xi_{n,s}^{(-1)}$, $1 \leq s \leq k$, of a sequence of independent μ distributed random variables ξ_1, \dots, ξ_k together with a sequence of independent random variables $(\varepsilon_1, \dots, \varepsilon_n)$, $P(\varepsilon_s = 1) = P(\varepsilon_s = -1) = \frac{1}{2}$, $1 \leq s \leq n$ which are independent of them. For all subsets $V \subset \{1, \dots, k\}$ of the set $\{1, \dots, k\}$ let $|V|$ denote the cardinality of this set, and define for all functions $f(x_1, \dots, x_k, y) \in \mathcal{F}$ and $V \subset \{1, \dots, k\}$ the independent U -statistics

$$\bar{I}_{n,k}^V(f, y) = \frac{1}{k!} \sum_{\substack{1 \leq j_s \leq n, s=1, \dots, k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} f(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}, y), \quad f \in \mathcal{F}, \quad (5.3)$$

where $\delta_s = \pm 1$, $1 \leq s \leq k$, $\delta_s = 1$ if $s \in V$, and $\delta_s = -1$ if $s \notin V$, together with the random variables

$$H_{n,k}^V(f) = \int \bar{I}_{n,k}^V(f, y)^2 \rho(dy), \quad f \in \mathcal{F}. \quad (5.3')$$

Put

$$\bar{I}_{n,k}(f, y) = \bar{I}_{n,k}^{\{1, \dots, k\}}(f, y), \quad H_{n,k}(f) = H_{n,k}^{\{1, \dots, k\}}(f), \quad (5.3'')$$

i.e. these random variables appear if $V = \{1, \dots, k\}$ is taken in the previous definitions, and the random variables $\xi_{j,s}^{(1)}$, $1 \leq j \leq n$, $1 \leq s \leq k$ are inserted in the formulas defining these random variables.

Let us also define the ‘randomized version’ of the random variables $\bar{I}_{n,k}^V(f, y)$ and $H_{n,k}^V(f)$ as

$$\bar{I}_{n,k}^{(V, \varepsilon)}(f, y) = \frac{1}{k!} \sum_{\substack{1 \leq j_s \leq n, s=1, \dots, k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} \varepsilon_{j_1} \cdots \varepsilon_{j_k} f(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}, y), \quad f \in \mathcal{F}, \quad (5.4)$$

where $\delta_s = 1$ if $s \in V$, and $\delta_s = -1$ if $s \notin V$, and

$$H_{n,k}^{(V,\varepsilon)}(f) = \int \bar{I}_{n,k}^{(V,\varepsilon)}(f, y)^2 \rho(dy), \quad f \in \mathcal{F}. \quad (5.4')$$

Let us also introduce the random variables

$$\bar{W}(f) = \int \left[\sum_{V \subset \{1, \dots, k\}} (-1)^{|V|} \bar{I}_{n,k}^{(V,\varepsilon)}(f, y) \right]^2 \rho(dy), \quad f \in \mathcal{F}. \quad (5.5)$$

Now I formulate the symmetrization result applied in the proof of Proposition 5.

Lemma 4B. *Let \mathcal{F} be a set of functions on $(X^k \times Y, \mathcal{X}^k \times \mathcal{Y})$ which satisfies the conditions of Proposition 5 with some probability measure $\mu^k \times \rho$. Let us have $2k$ independent copies $\xi_{1,s}^{\pm 1}, \dots, \xi_{n,s}^{\pm 1}$, $1 \leq s \leq k$, of a sequence of independent μ distributed random variables ξ_1, \dots, ξ_n together with a sequence of independent random variables $\varepsilon_1, \dots, \varepsilon_n$, $P(e_s = 1) = P(\varepsilon_s = -1) = \frac{1}{2}$, $1 \leq s \leq n$, which is independent also of the previously considered sequences.*

There exists some $A_0 = A_0(k)$ such that if the integrals $H_{n,k}(f)$, $f \in \mathcal{F}$, determined by this class of functions \mathcal{F} have a good tail behaviour at level $T^{(2k+1)/2k}$ for some $T \geq A_0$, (this property was defined at the end of Section 3), then the inequality

$$P \left(\sup_{f \in \mathcal{F}} H_{n,k}(f) > A^2 n^{2k} \sigma^{2(k+1)} \right) < 2P \left(\sup_{f \in \mathcal{F}} |\bar{W}(f)| > \frac{A^2}{2} n^{2k} \sigma^{2(k+1)} \right) + 2^{2k+1} n^{k-1} e^{-A^{1/2k} n \sigma^2 / k} \quad (5.6)$$

holds with the random variables $H_{n,k}(f)$ and $\bar{W}(f)$ defined in formulas (5.3'') and (5.5) for all $A \geq T$.

Let us observe that in the symmetrization argument of Lemma 4B we have applied the symmetrization $\bar{I}_{n,k}^{(V,\varepsilon)}(f, y)$ of $\bar{I}_{n,k}^{(V)}(f, y)$, (compare formulas (5.3) and (5.4)), and compared the integral of the square of the random function $\bar{I}_{n,k}(f, y)$ with the integral of the square of a linear combination of the random functions $\bar{I}_{n,k}^{(V,\varepsilon)}(f, y)$. After this integration the effect of the ‘randomizing factors’ ε_j will be weaker. Nevertheless, also such an estimate will be sufficient for us. But the effect of this symmetrization procedure has to be followed more carefully. Hence a corollary of Lemma 4B will be presented which can be better applied than the original lemma. We get it by rewriting the random variable $\bar{W}(f)$ defined in (5.5) in another form with the help of some diagrams introduced below.

Let $\mathcal{G} = \mathcal{G}(k)$ denote the set of all diagrams consisting of two rows such that both rows are the set $\{1, \dots, k\}$ and the diagrams of \mathcal{G} contain some edges $(l_1, l'_1), \dots, (l_s, l'_s)$, $0 \leq s \leq k$ connecting some points (vertices) of the first row with some point (vertex) of the second row. The vertices l_1, \dots, l_s in the first row are all different, and the same

relation holds also for the vertices l'_1, \dots, l'_s in the second row. For each diagram $G \in \mathcal{G}$ let us define $e(G) = \{(l_1, l'_1) \dots, (l_s, l'_s)\}$, the set of its edges, $v_1(G) = \{l_1, \dots, l_s\}$, the set of its vertices in the first row and $v_2(G) = \{l'_1, \dots, l'_s\}$, the set of its vertices in the second row.

Given some diagram $G \in \mathcal{G}$ and two sets $V_1, V_2 \subset \{1, \dots, k\}$, we define with the help of the random variables $\xi_{s,1}^{(1)}, \dots, \xi_{s,n}^{(1)}, \xi_{s,1}^{(-1)}, \dots, \xi_{s,n}^{(-1)}$, $1 \leq s \leq k$, and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ taking part in the definition of the expressions $\bar{W}(f)$, $f \in \mathcal{F}$, the random variables $H_{n,k}(f|G, V_1, V_2)$:

$$H_{n,k}(f|G, V_1, V_2) = \sum_{\substack{(j_1, \dots, j_k, j'_1, \dots, j'_k) \\ 1 \leq j_s \leq n, j_s \neq j_{s'} \text{ if } s \neq s', 1 \leq s, s' \leq k, \\ 1 \leq j'_s \leq n, j'_s \neq j'_{s'} \text{ if } s \neq s', 1 \leq s, s' \leq k, \\ j_s = j'_{s'} \text{ if } (s, s') \in e(G), j_s \neq j'_{s'} \text{ if } (s, s') \notin e(G)}} \prod_{s \notin v_1(G)} \varepsilon_{j_s} \prod_{s \notin v_2(G)} \varepsilon_{j'_s} \int f(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}, y) f(\xi_{j'_1,1}^{(\bar{\delta}_1)}, \dots, \xi_{j'_k,k}^{(\bar{\delta}_k)}, y) \rho(dy), \quad f \in \mathcal{F}, \quad (5.7)$$

where $\delta_s = 1$ if $s \in V_1$, $\delta_s = -1$ if $s \notin V_1$, and $\bar{\delta}_s = 1$ if $s \in V_2$, $\bar{\delta}_s = -1$ if $s \notin V_2$.

With the help of these random variables we can write that

$$\bar{W}(f) = \sum_{G \in \mathcal{G}, V_1, V_2 \subset \{1, \dots, k\}} (-1)^{|V_1|+|V_2|} H_{n,k}(f|G, V_1, V_2) \quad \text{for all } f \in \mathcal{F},$$

because

$$\int \bar{I}_{n,k}^{(V_1, \varepsilon)}(f, y) \bar{I}_{n,k}^{(V_2, \varepsilon)}(f, y) \rho(dy) = \sum_{G \in \mathcal{G}} H_{n,k}(f|G, V_1, V_2), \quad \text{for all } V_1, V_2 \subset \{1, \dots, k\}.$$

Since the number of terms in this sum is less than $2^{4k} k!$, it implies that Lemma 4B has the following corollary:

Corollary of Lemma 4B. *Let a set of functions \mathcal{F} satisfy the conditions of Proposition 5. Then there exists some $A_0 = A_0(k)$ such that if the integrals $H_{n,k}(f)$, $f \in \mathcal{F}$, determined by this class of functions \mathcal{F} have a good tail behaviour at level $T^{(2k+1)/2k}$ for some $T \geq A_0$, then the inequality*

$$P \left(\sup_{f \in \mathcal{F}} H_{n,k}(f) > A^2 n^{2k} \sigma^{2(k+1)} \right) \leq 2 \sum_{G \in \mathcal{G}, V_1, V_2 \subset \{1, \dots, k\}} P \left(\sup_{f \in \mathcal{F}} |H_{n,k}(f|G, V_1, V_2)| > \frac{A^2}{2^{4k+1} k!} n^{2k} \sigma^{2(k+1)} \right) + 2^{2k+1} n^{k-1} e^{-A^{1/2k} n \sigma^2 / k} \quad (5.8)$$

holds with the random variables $H_{n,k}(f)$ and $H_{n,k}(f|G, V_1, V_2)$ defined in formulas (5.3'') and (5.7) for all $A \geq T$.

The proof of Lemmas 4A and 4B uses the result of the following Lemma 5 which states that certain random vectors have the same distribution.

Lemma 5. *Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ be a sequence of independent random variables, $P(\varepsilon_s = 1) = P(\varepsilon_s = -1) = \frac{1}{2}$, $1 \leq s \leq n$, which is independent also of $2k$ fixed independent copies $\xi_{1,s}^{(1)}, \dots, \xi_{n,s}^{(1)}$ and $\xi_{1,s}^{(-1)}, \dots, \xi_{n,s}^{(-1)}$, $1 \leq s \leq k$, of a sequence ξ_1, \dots, ξ_n of independent μ distributed random variables.*

a) *Let \mathcal{F} be a class of functions which satisfies the conditions of Proposition 4. With the help of the above random variables introduce the independent U -statistic*

$$\bar{I}_{n,k}^V(f) = \frac{1}{k!} \sum_{\substack{1 \leq j_s \leq n, s=1, \dots, k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} f\left(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}\right), \quad f \in \mathcal{F}, \quad (5.9)$$

for all sets $V \subset \{1, \dots, k\}$ and functions $f \in \mathcal{F}$ together with its ‘randomized version’

$$\bar{I}_{n,k}^{(V,\varepsilon)}(f) = \frac{1}{k!} \sum_{\substack{1 \leq j_s \leq n, s=1, \dots, k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} \varepsilon_{j_1} \cdots \varepsilon_{j_k} f\left(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}\right), \quad f \in \mathcal{F}, \quad (5.9')$$

where $\delta_s = \pm 1$, $1 \leq s \leq k$, $\delta_s = 1$ if $s \in V$, and $\delta_s = -1$ if $s \notin V$.

Then the sets of random variables

$$S(f) = \sum_{V \subset \{1, \dots, k\}} (-1)^{|V|} \bar{I}_{n,k}^V(f), \quad f \in \mathcal{F}, \quad (5.10)$$

and sets of random variables

$$\bar{S}(f) = \sum_{V \subset \{1, \dots, k\}} (-1)^{|V|} \bar{I}_{n,k}^{(V,\varepsilon)}(f), \quad f \in \mathcal{F}, \quad (5.10')$$

have the same joint distribution.

b) *Let \mathcal{F} be the class of functions satisfying Proposition 5. For all functions $f \in \mathcal{F}$ and $V \subset \{1, \dots, k\}$ consider the independent U -statistics determined by the random variables $\xi_{1,s}^{(1)}, \dots, \xi_{n,s}^{(1)}$ and $\xi_{1,s}^{(-1)}, \dots, \xi_{n,s}^{(-1)}$, $1 \leq s \leq k$ by formula (5.3), and define with their help the random variables*

$$W(f) = \int \left[\sum_{V \subset \{1, \dots, k\}} (-1)^{|V|} \bar{I}_{n,k}^V(f, y) \right]^2 \rho(dy), \quad f \in \mathcal{F}. \quad (5.11)$$

Then the random vectors $\{W(f): f \in \mathcal{F}\}$ defined in (5.11) and $\{\bar{W}(f): f \in \mathcal{F}\}$ defined in (5.5) have the same distribution.

Proof of Lemma 5. Let us consider Part a) of Lemma 5. I claim that for all $M \in \{1, \dots, n\}$ the conditional distribution of the random vector in (5.10') under the condition that $\varepsilon_j = 1$ if $j \in M$ and $\varepsilon_j = -1$ if $\varepsilon_j \in \{1, \dots, n\} \setminus M$ agrees with the distribution of the vector in (5.10). Since the distribution of the vector in (5.10) does not change if we exchange the random variables $\xi_{j,s}^{(1)}$ and $\xi_{j,s}^{(-1)}$ in it if $j \notin M$, $1 \leq s \leq k$, and do not exchange them otherwise, it is enough to understand that the random vector we get from the vector in (5.10) after this transformation agrees with the random vector in (5.10') if we write $\varepsilon_j = 1$ for $j \in M$ and $\varepsilon_j = -1$ for $j \notin M$ in it. These random vectors really agree (not only in distribution) since for all functions $f \in \mathcal{F}$ both vectors have a component which is the sum of terms of the form $f(\xi_{j_1,1}^{(\delta_{j_1})}, \dots, \xi_{j_k,k}^{(\delta_{j_k})})$, $\delta_{j_s} = \pm 1$, $1 \leq s \leq k$, multiplied with an appropriate power of (-1) , and this power equals the number of -1 components in the sequence $\delta_{j_1}, \dots, \delta_{j_k}$ plus the cardinality of the set $\{j_1, \dots, j_k\} \cap M$. Part b) of the lemma can be proved in the same way, hence it is omitted.

Lemma 4A will be proved with the help of part a) of Lemma 5 and the following Lemma 6A.

Lemma 6A. *Let us consider a class of functions \mathcal{F} satisfying the conditions of Proposition 4, and the random variables $\bar{I}_{n,k}^V(f)$, $f \in \mathcal{F}$, $V \subset \{1, \dots, k\}$, defined in formula (5.1). Let $\mathcal{B} = \mathcal{B}(\xi_{1,s}^{(1)}, \dots, \xi_{n,s}^{(1)}; 1 \leq s \leq k)$ denote the σ -algebra generated by the random variables $\xi_{1,s}^{(1)}, \dots, \xi_{n,s}^{(1)}$, $1 \leq s \leq k$, taking part in the definition of the random variables $\bar{I}_{n,k}^V(f)$. For all $V \subset \{1, \dots, k\}$, $V \neq \{1, \dots, k\}$, there exists a number $A_0 = A_0(k) > 0$ such that the inequality*

$$P \left(\sup_{f \in \mathcal{F}} E \left(\bar{I}_{n,k}^V(f)^2 \mid \mathcal{B} \right) > 2^{-(3k+3)} A^2 n^{2k} \sigma^{2k+2} \right) < n^{k-1} e^{-A^{1/(2k-1)} n \sigma^2 / k} \quad (5.12)$$

holds for all $A \geq A_0$.

Proof of Lemma 6A. Let us first consider the case $V = \emptyset$. Then $E \left(\bar{I}_{n,k}^\emptyset(f)^2 \mid \mathcal{B} \right) = E \left(\bar{I}_{n,k}^\emptyset(f)^2 \right) \leq \frac{n!}{k!} \sigma^2 \leq n^{2k} \sigma^{2k+2}$ for all $f \in \mathcal{F}$. In the above calculation we exploited that the functions $f \in \mathcal{F}$ are canonical, and this implies certain orthogonalities, and beside this the inequality $n\sigma^2 \geq 1$ holds. The above relation implies inequality (5.12) for $V = \emptyset$ for all $\omega \in \Omega$ if the number A_0 is chosen sufficiently large.

To avoid some complications in the notation let us restrict our attention to the sets $V = \{1, \dots, u\}$, $1 \leq u < k$, and prove relation (5.12) for such sets. For this goal let us introduce the random variables

$$\bar{I}_{n,k}^V(f, j_{u+1}, \dots, j_k) = \frac{1}{k!} \sum_{\substack{1 \leq j_s \leq n, s=1, \dots, u \\ j_s \neq j_{s'} \text{ if } s \neq s', 1 \leq s, s' \leq k}} f \left(\xi_{j_1,1}^{(1)}, \dots, \xi_{j_u,u}^{(1)}, \xi_{j_{u+1},u+1}^{(-1)}, \dots, \xi_{j_k,k}^{(-1)} \right),$$

for all $f \in \mathcal{F}$, i.e. we fix some indices j_{u+1}, \dots, j_k , $1 \leq j_s \leq n$, $u+1 \leq s \leq k$, $j_s \neq j_{s'}$ if $s \neq s'$, and sum up only those terms in the sum defining $\bar{I}_{n,k}^V(f)$ which contain

$\xi_{j_{u+1}, u+1}^{(-1)}, \dots, \xi_{j_k, k}^{(-1)}$ in their last $k - u$ coordinates. Then we can write

$$\begin{aligned} E(\bar{I}_{n,k}^V(f)^2 | \mathcal{B}) &= E \left(\left(\sum_{\substack{1 \leq j_s \leq n \\ j_s \neq j_{s'} \text{ if } s \neq s'}} E(\bar{I}_{n,k}^V(f, j_{u+1}, \dots, j_{u_k})^2 | \mathcal{B}) \right)^2 \right) \\ &= \sum_{\substack{1 \leq j_s \leq n \\ j_s \neq j_{s'} \text{ if } s \neq s'}} E(\bar{I}_{n,k}^V(f, j_{u+1}, \dots, j_{u_k})^2 | \mathcal{B}). \end{aligned} \quad (5.13)$$

The last relation follows from the identity

$$E(\bar{I}_{n,k}^V(f, j_{u+1}, \dots, j_{u_k}) \bar{I}_{n,k}^V(f, j'_{u+1}, \dots, j'_{u_k}) | \mathcal{B}) = 0$$

if $(j_{u+1}, \dots, j_k) \neq (j'_{u+1}, \dots, j'_k)$, which relation holds, since f is a canonical function.

It follows from relation (5.13) that

$$\begin{aligned} &\left\{ \omega : \sup_{f \in \mathcal{F}} E(\bar{I}_{n,k}^V(f)^2 | \mathcal{B})(\omega) > 2^{-(3k+3)} A^2 n^{2k} \sigma^{2k+2} \right\} \\ &\subset \bigcup_{\substack{1 \leq j_s \leq n \\ j_s \neq j_{s'} \text{ if } s \neq s'}} \left\{ \omega : \sup_{f \in \mathcal{F}} E(\bar{I}_{n,k}^V(f, j_{u+1}, \dots, j_{u_k})^2 | \mathcal{B})(\omega) > \frac{A^2 n^{2k} \sigma^{2k+2}}{2^{(3k+3)} n^{k-u}} \right\}. \end{aligned} \quad (5.14)$$

The probability of the events in the union at the right-hand side of (5.14) can be estimated with the help of the corollary of Proposition 5 with parameter $u < k$ instead of k . (We may assume that Proposition 5 holds for $u < k$.) This corollary yields that

$$P \left(\sup_{f \in \mathcal{F}} E(\bar{I}_{n,k}^V(f, j_{u+1}, \dots, j_{u_k})^2 | \mathcal{B}) > \frac{A^2 \sigma^{2k+2} n^{k+u}}{2^{(2k+3)}} \right) \leq e^{-A^{-1/(2u+1)}(n-u)\sigma^2}. \quad (5.15)$$

Indeed, the expression $E(\bar{I}_{n,k}^V(f, j_{u+1}, \dots, j_{u_k})^2 | \mathcal{B})$ can be calculated in the following way: Take the independent U -statistic

$$\bar{I}_{n,k}^V(f, x_{u+1}, \dots, x_k) = \frac{1}{k!} \sum_{\substack{j_s \in \{1, \dots, n\} \setminus \{j_{u+1}, \dots, j_k\}, \\ s=1, \dots, u, j_s \neq j_{s'} \text{ if } s \neq s'}} f(\xi_{j_1, 1}^{(1)}, \dots, \xi_{j_u, u}^{(1)}, x_{u+1}, \dots, x_k), \quad (5.16)$$

of order u with sample size $n - k + u$, and integrate the square of this function with respect to the variables x_{u+1}, \dots, x_k by the measure μ^{k-u} . Hence the expression at the left-hand side of (5.15) can be bounded by means of Proposition 5 if we apply it for our class of functions \mathcal{F} considering them as functions on $(X^u \times Y, \mathcal{X}^u \times \mathcal{Y}, \mu^u \times \rho)$ with $(Y, \mathcal{Y}, \rho) = (X^{k-u}, \mathcal{X}^{k-u}, \mu^{k-u})$. (A small inaccuracy was committed in the above

statement because to define the expression in (5.16) as a U -statistic we should have divided by $u!$ instead of $k!$. But this causes no real problem.)

We get inequality (5.15) from Proposition 5 by replacing the level $\frac{A^2 \sigma^{2k+2} n^{k+u}}{2^{(3k+3)}}$ in the probability at the left-hand side by $A^2(n-u)^{2u} \sigma^{2u+2} < \frac{A^2 \sigma^{2k+2} n^{k+u}}{2^{(2k+2)}}$. The last inequality really holds if the constant K is chosen sufficiently large in the condition $n\sigma^2 > K \log n$ of Proposition 4.

Relations (5.14) and (5.15) imply that

$$P \left(\sup_{f \in \mathcal{F}} E(\bar{I}_{n,k}^V(f)^2 | \mathcal{B})(\omega) > 2^{-(3k+3)} A^2 n^{2k} \sigma^{2k+2} \right) \leq n^{k-u} e^{-A^{-1/(2u+1)}(n-u)\sigma^2},$$

and $u \leq k-1$. Hence also inequality (5.12) holds.

Now we prove Lemma 4A.

Proof of Lemma 4A. We show with the help of Lemmas 2 and Lemma 6A that

$$P \left(\sup_{f \in \mathcal{F}} n^{k/2} |\bar{I}_{n,k}(f)| > A n^{k/2} \sigma^{k+1} \right) < 2P \left(\sup_{f \in \mathcal{F}} |S(f)| > \frac{A}{2} n^k \sigma^{k+1} \right) + 2^k n^{k-1} e^{-A^{1/(2k-1)} n \sigma^2 / k} \quad (5.17)$$

with the function $S(f)$ defined in (5.10). To prove relation (5.17) introduce the random variables $Z(f) = (-1)^k \bar{I}_{n,k}^{\{1, \dots, k\}}(f)$ and $\bar{Z}(f) = \sum_{V \subset \{1, \dots, k\}, V \neq \{1, \dots, k\}} (-1)^{|V|+1} \bar{I}_{n,k}^V(f)$ for all $f \in \mathcal{F}$, the σ -algebra \mathcal{B} considered in Lemma 6A and the set

$$B = \bigcap_{\substack{V \subset \{1, \dots, k\} \\ V \neq \{1, \dots, k\}}} \left\{ \omega : \sup_{f \in \mathcal{F}} E(\bar{I}_{n,k}^V(f)^2 | \mathcal{B})(\omega) \leq 2^{-(3k+3)} A^2 n^{2k} \sigma^{2k+2} \right\}.$$

Observe that $S(f) = Z(f) - \bar{Z}(f)$, $f \in \mathcal{F}$, $B \in \mathcal{B}$, and by Lemma 6A the inequality $1 - P(B) \leq 2^k n^{k-1} e^{-A^{1/(2k-1)} n \sigma^2 / k}$ holds. Hence to prove relation (5.17) as a consequence of Lemma 2 it is enough to show that

$$P \left(|\bar{Z}(f)| > \frac{A}{2} n^k \sigma^{k+1} \middle| \mathcal{B} \right) (\omega) \leq \frac{1}{2} \quad \text{for all } f \in \mathcal{F} \quad \text{if } \omega \in B. \quad (5.18)$$

But $P(|\bar{I}_{n,k}(f)| > 2^{-(k+1)} A n^k \sigma^{k+1} | \mathcal{F})(\omega) \leq 2^{-(k+1)}$ for all $f \in \mathcal{F}$ if $\omega \in B$ by the ‘conditional Chebishev inequality’, hence relation (5.18) holds.

Lemma 4A follows from relation (5.17), part a of Lemma 5 and the observation that the random vectors $\{\bar{I}_{n,k}^{(V, \varepsilon)}(f)\}$, $f \in \mathcal{F}$, defined in (5.9’) have the same distribution for

all $V \subset \{1, \dots, k\}$ as the random vector $\bar{I}_{n,k}^\varepsilon(f)$, $f \in \mathcal{F}$, considered in the formulation of Lemma 4A. Hence

$$P \left(\sup_{f \in \mathcal{F}} |S(f)| > \frac{A}{2} n^k \sigma^{k+1} \right) \leq 2^k P \left(\sup_{f \in \mathcal{F}} |\bar{I}_{n,k}^\varepsilon(f)| > 2^{-(k+1)} A n^k \sigma^{k+1} \right).$$

In the proof of Lemma 4B we apply following Lemma 6B which is a version of Lemma 6A.

Lemma 6B. *Let us consider a class of functions \mathcal{F} satisfying the conditions of Proposition 5 and the random variables $\bar{I}_{n,k}^V(f, y)$, $f \in \mathcal{F}$, $V \subset \{1, \dots, k\}$, defined in formula (5.3). Let $\mathcal{B} = \mathcal{B}(\xi_{1,s}^{(1)}, \dots, \xi_{n,s}^{(1)}; 1 \leq s \leq k)$ denote the σ -algebra generated by the random variables $\xi_{1,s}^{(1)}, \dots, \xi_{n,s}^{(1)}$, $1 \leq s \leq k$, taking part in the definition of the random variables $\bar{I}_{n,k}^V(f, y)$ and $H_{n,k}^V(f)$.*

- a) *For all $V \subset \{1, \dots, k\}$, $V \neq \{1, \dots, k\}$, there exists a number $A_0 = A_0(k) > 0$ such that the inequality*

$$P \left(\sup_{f \in \mathcal{F}} E(H_{n,k}^V(f) | \mathcal{B}) > 2^{-(4k+4)} A^{(2k-1)/k} n^{2k} \sigma^{2k+2} \right) < n^{k-1} e^{-A^{1/2k} n \sigma^2 / k}. \quad (5.19)$$

holds for all $A \geq A_0$.

- b) *Given two subsets $V_1, V_2 \subset \{1, \dots, k\}$ of the set $\{1, \dots, k\}$ define the random integrals*

$$H_{n,k}^{(V_1, V_2)}(f) = \int |\bar{I}_{n,k}^{V_1}(f, y) \bar{I}_{n,k}^{V_2}(f, y)| \rho(dy), \quad f \in \mathcal{F},$$

with the help of the functions $\bar{I}_{n,k}^V(f, y)$ defined in (5.3). If at least one of the sets V_1 and V_2 is not the set $\{1, \dots, k\}$, then there exists some number $A_0 = A_0(k) > 0$ such that if the integrals $H_{n,k}(f)$, $f \in \mathcal{F}$, determined by this class of functions \mathcal{F} have a good tail behaviour at level $T^{(2k+1)/2k}$ for some $T \geq A_0$, then the inequality

$$P \left(\sup_{f \in \mathcal{F}} E(H_{n,k}^{(V_1, V_2)}(f) | \mathcal{B}) > 2^{-(2k+2)} A^2 n^{2k} \sigma^{2k+2} \right) < 2n^{k-1} e^{-A^{1/2k} n \sigma^2 / k}. \quad (5.20)$$

holds for all $A \geq T$.

Proof of Lemma 6B. Part a) of Lemma 6B can be proved in the same way as Lemma 6A, only the formulas applied in the proof become a little bit more complicated. Hence I omit the proof. (The difference between the power of the parameter A at the right-hand side of formulas (5.19) and (5.12) appear, since the left-hand side of (5.19) contains the term $A^{(2k-1)/2k}$ and not A^2 .) Part b) will be proved with the help of Part a) and the inequality

$$\sup_{f \in \mathcal{F}} E(H_{n,k}^{(V_1, V_2)}(f) | \mathcal{B}) \leq \left(\sup_{f \in \mathcal{F}} E(H_{n,k}^{V_1}(f) | \mathcal{B}) \right)^{1/2} \left(\sup_{f \in \mathcal{F}} E(H_{n,k}^{V_2}(f) | \mathcal{B}) \right)^{1/2}$$

which follows from the Schwarz inequality applied for integrals with respect to conditional distributions. Let us assume that $V_1 \neq \{1, \dots, k\}$. The last inequality implies that

$$\begin{aligned} & P \left(\sup_{f \in \mathcal{F}} E(H_{n,k}^{(V_1, V_2)}(f) | \mathcal{B}) > 2^{-(2k+2)} A^2 n^{2k} \sigma^{2k+2} \right) \\ & \leq P \left(\sup_{f \in \mathcal{F}} E(H_{n,k}^{V_1}(f) | \mathcal{B}) > 2^{-(4k+4)} A^{(2k-1)/k} n^{2k} \sigma^{2k+2} \right) \\ & \quad + P \left(\sup_{f \in \mathcal{F}} E(H_{n,k}^{V_2}(f) | \mathcal{B}) > A^{(2k+1)/k} n^{2k} \sigma^{2k+2} \right) \end{aligned}$$

Hence the estimate (5.19) for $V = V_1$ together with the inequality

$$P \left(\sup_{f \in \mathcal{F}} E(H_{n,k}^{V_2}(f) | \mathcal{B}) > A^{(2k+1)/k} n^{2k} \sigma^{2k+2} \right) \leq n^{k-1} e^{-A^{1/2k} n \sigma^2 / k}$$

which follows from Part a) if $V_2 \neq \{1, \dots, n\}$ (in this case the level $A^{(2k+1)/k} n^{2k} \sigma^{2k+2}$ can be replaced by $2^{-(4k+4)} A^{(2k-1)/k} n^{2k} \sigma^{2k+2}$ in the probability we consider) and from the conditions of Part b) if $V_2 = \{1, \dots, k\}$ imply relation (5.20).

Now I prove Lemma 4B.

Proof of Lemma 4B. By Part b) of Lemma 5 it is enough to prove that relation (5.6) holds if the random variables $\bar{W}(f)$ are replaced in it by the random variables $W(f)$ defined in formula (5.11). We shall prove this by applying Lemma 2 with the choice of $Z(f) = H_{n,k}^{(\bar{V}, \bar{V})}(f)$, $\bar{V} = \{1, \dots, k\}$, $\bar{Z}(f) = W(f) - Z(f)$, $f \in \mathcal{F}$, $\mathcal{B} = \mathcal{B}(\xi_{1,s}^{(1)}, \dots, \xi_{n,s}^{(1)}; 1 \leq s \leq k)$, and the set

$$B = \bigcap_{\substack{(V_1, V_2): V_j \subset \{1, \dots, k\}, j=1,2 \\ V_1 \neq \{1, \dots, k\} \text{ or } V_2 \neq \{1, \dots, k\}}} \left\{ \omega : \sup_{f \in \mathcal{F}} E(H_{n,k}^{(V_1, V_2)}(f) | \mathcal{B})(\omega) \leq 2^{-(2k+2)} A^2 n^{2k} \sigma^{2k+2} \right\}.$$

By Lemma 6B $1 - P(B) \leq 2^{2k+1} n^{k-1} e^{-A^{1/2k} n \sigma^2 / k}$, and to prove Lemma 4B with the help of Lemma 2 it is enough to show that

$$P \left(|\bar{Z}(f)| > \frac{A^2}{2} n^{2k} \sigma^{2(k+1)} \middle| \mathcal{B} \right) (\omega) \leq \frac{1}{2} \quad \text{for all } f \in \mathcal{F} \text{ if } \omega \in B.$$

To prove this relation observe that

$$E(|\bar{Z}(f)| | \mathcal{B}) \leq \sum_{\substack{(V_1, V_2): V_j \subset \{1, \dots, k\}, j=1,2 \\ V_1 \neq \{1, \dots, k\} \text{ or } V_2 \neq \{1, \dots, k\}}} E(H_{n,k}^{(V_1, V_2)}(f) | \mathcal{B}), \leq \frac{A^2}{4} n^{2k} \sigma^{2k+2} \quad \text{if } \omega \in B$$

for all $f \in \mathcal{F}$. Hence the ‘conditional Markov inequality’ implies that

$$P \left(|\bar{Z}(f)| > \frac{A^2}{2} n^{2k} \sigma^{2k+2} \middle| \mathcal{B} \right) \leq \frac{1}{2} \quad \text{if } \omega \in B \quad \text{and } f \in \mathcal{F}.$$

Lemma 4B is proved.

The proof of Propositions 4 and 5

The proof of Propositions 4 and 5 for general $k \geq 1$ with the help of the symmetrization lemmas 4A and 4B is similar to the proof of Proposition 4 in the case $k = 1$ presented in Section 4. The proof applies an induction procedure with respect to the parameter k . In the proof of Proposition 4 for parameter k we may assume that Propositions 4 and 5 hold for $k' < k$. In the proof of Proposition 5 we may also assume that Proposition 4 holds for the parameter k .

In the proof of Proposition 4 let us introduce (with the notation of this proposition) the functions

$$S_{n,k}^2(f)(x_{j,s}, 1 \leq j \leq n, 1 \leq s \leq k) = \frac{1}{k!} \sum_{\substack{1 \leq j_s \leq n, s=1, \dots, k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} f^2(x_{j_1,1}, \dots, x_{j_k,k}), \quad f \in \mathcal{F}, \quad (6.1)$$

where $x_{j,s} \in X$, $1 \leq j \leq n$, $1 \leq s \leq k$. Fix some number $A > T$ and define the set H

$$H = H(A) = \left\{ (x_{j,s}, 1 \leq j \leq n, 1 \leq s \leq k), \right. \\ \left. \sup_{f \in \mathcal{F}} S_{n,k}^2(f)(x_{j,s}, 1 \leq j \leq n, 1 \leq s \leq k) > 2^k A^{4/3} n^k \sigma^2 \right\}. \quad (6.2)$$

We want to show that

$$P(\{\omega : (\xi_{j,s}(\omega), 1 \leq j \leq n, 1 \leq s \leq k) \in H\}) \leq 2^k e^{-A^{2/3k} n \sigma^2} \quad \text{if } A \geq T. \quad (6.3)$$

Relation (6.3) will be proved by means of the Hoeffding decomposition of the U -statistics with kernel functions $f^2(x_1, \dots, x_k)$, $f \in \mathcal{F}$, and by the estimation of the sum this decomposition yields. More explicitly, write

$$f^2(x_1, \dots, x_k) = \sum_{V \subset \{1, \dots, k\}} f_V(x_j, j \in V) \quad (6.4)$$

with $f_V(x_j, j \in V) = \prod_{j \notin V} P_j \prod_{j \in V} Q_j f^2(x_1, \dots, x_k)$, where P_j and Q_j are the operators

P_μ and Q_μ defined in formulas (2.6) and (2.6'') if $(Y_1 \times Z \times Y_2, \mathcal{Y}_1 \times \mathcal{Z} \times \mathcal{Y}_2)$ is the k -fold product (X^k, \mathcal{X}^k) of the measure space (X, \mathcal{X}) in these definitions, Z is the j -th component in these products, and Y_1 is the product of the components before and Y_2 is the product of the components after this component. (Relation (6.4) follows from the

identity $f^2 = \prod_{j=1}^k (P_j + Q_j) f^2$ if the multiplications are carried out in this formula. In

the calculation we exploit that the operators P_j and $P_{j'}$ are commutative if $j \neq j'$, and the same relation holds for the pairs P_j and $Q_{j'}$ or Q_j and $P_{j'}$ or Q_j and $Q_{j'}$.)

The identity $S_{n,k}^2(f)(\xi_{j,r}, 1 \leq j \leq n, 1 \leq r \leq k) = k! I_{n,k}(f^2)$ holds for all $f \in \mathcal{F}$, and by writing the Hoeffding decomposition (6.4) for each term $f^2(\xi_{j_1,1}, \dots, \xi_{j_k,k})$ in the

expression $I_{n,k}(f^2)$ we get that

$$\begin{aligned} & P \left(\sup_{f \in \mathcal{F}} S_{n,k}^2(f)(\xi_{j,s}, 1 \leq j \leq n, 1 \leq s \leq k) > 2^k A^{4/3} n^k \sigma^2 \right) \\ & \leq \sum_{V \subset \{1, \dots, k\}} P \left(\sup_{f \in \mathcal{F}} n^{k-|V|} |\bar{I}_{n,|V|}(f_V)| > A^{4/3} n^k \sigma^2 \right) \end{aligned} \quad (6.5)$$

with the functions f_V in (6.4). We want to give a good estimate for all terms in the sum at the right-hand side in (6.5). For this goal we show that the classes of functions $\{f_V : f \in \mathcal{F}\}$ satisfy the conditions of Proposition 4 for all $V \subset \{1, \dots, k\}$.

The functions f_V are canonical for all $V \subset \{1, \dots, k\}$. (This follows from the commutativity relations between the operators P_j and Q_j mentioned before, the identity $P_j Q_j = 0$ and the fact that the canonical property of the function can be expressed in the form $P_j f_V = 0$ for all $j \in V$.) We have $|f^2(x_1, \dots, x_k)| \leq 2^{-2(k+1)}$. The norm of Q_j as a map from the L_∞ space to L_∞ space is less than 2, the norm of P_j is less than

1, hence $\left| \sup_{x_j \in X, j \in V} f_V(x_j, j \in V) \right| \leq 2^{-(k+2)} \leq 2^{-(k+1)}$ for all $V \subset \{1, \dots, k\}$. We have

$$\int f^4(x_1, \dots, x_k) \mu(dx_1) \dots \mu(dx_k) \leq 2^{-(k+1)} \sigma^2, \text{ hence } \int f_V^2(x_j, j \in V) \prod_{j \in V} \mu(dx_j) \leq$$

$2^{-(k+1)} \sigma^2 \leq \sigma^2$ for all $V \subset \{1, \dots, k\}$ by Lemma 1. Finally, to check that the class of functions $\mathcal{F}_V = \{f_V : f \in \mathcal{F}\}$ is L_2 -dense with exponent L and parameter D observe that for all probability measures ρ on (X^k, \mathcal{X}^k) and pairs of functions $f, g \in \mathcal{F}$ $\int (f^2 - g^2)^2 d\rho \leq 2^{-2k} \int (f - g)^2 d\rho$. This implies that if $\{f_1, \dots, f_m\}$, $m \leq D\varepsilon^{-L}$, is an ε -dense subset of \mathcal{F} in the space $L_2(X^k, \mathcal{X}^k, \rho)$, then the set of functions $\{2^k f_1^2, \dots, 2^k f_m^2\}$ is an ε -dense subset of the class of functions $\mathcal{F}' = \{2^k f^2 : f \in \mathcal{F}\}$. Then by Lemma 1 for all $V \subset \{1, \dots, k\}$ the set of functions $\{(f_1)_V, \dots, (f_m)_V\}$ is an ε -dense subset of the class of functions \mathcal{F}_V in the space $L_2(X^k, \mathcal{X}^k, \rho)$. This means that \mathcal{F}_V is also L_2 -dense with exponent L and parameter D .

For $V = \emptyset$ the relation $f_V = \int f^2(x_1, \dots, x_k) \mu(dx_1) \dots \mu(dx_k) \leq \sigma^2$ holds, and $I_{|V|}(f_{|V|}) = f_V \leq \sigma^2$. Therefore the term corresponding to $V = \emptyset$ in the sum at the right-hand side of (6.5) equals zero if $A_0 \geq 1$ in the conditions of Proposition 4. The terms corresponding to sets V , $1 \leq |V| \leq k$ in these sums satisfy the inequality

$$\begin{aligned} & P \left(\sup_{f \in \mathcal{F}} |\bar{I}_{n,|V|}(f_V)| > A^{4/3} n^{|V|} \sigma^2 \right) \\ & \leq P \left(\sup_{f \in \mathcal{F}} |\bar{I}_{n,|V|}(f_V)| > A^{4/3} n^{|V|} \sigma^{|V|+1} \right) \leq e^{-A^{2/3k} n \sigma^2} \quad \text{if } 1 \leq |V| \leq k. \end{aligned}$$

This inequality follows from the inductive hypothesis if $|V| < k$, and in the case $V = \{1, \dots, k\}$ from the inequality $A \geq T$ and the assumption that U -statistics determined by a class of functions satisfying the conditions of Proposition 4 have a good

tail behaviour at level $T^{4/3}$. The last relation together with formula (6.5) imply relation (6.3).

By conditioning the probability $P\left(\left|\bar{I}_{n,k}^\varepsilon(f)\right| > 2^{-(k+2)}An^{k/2}\sigma^{k+1}\right)$ with respect to the random variables $\xi_{j,s}$, $1 \leq j \leq n$, $1 \leq s \leq k$ we get with the help of Proposition A that

$$\begin{aligned} & P\left(\left|\bar{I}_{n,k}^\varepsilon(f)\right| > 2^{-(k+2)}An^k\sigma^{k+1} \mid \xi_{j,s}(\omega) = x_{j,s}, 1 \leq j \leq n, 1 \leq s \leq k\right) \\ & \leq C \exp\left\{-B\left(\frac{A^2n^{2k}\sigma^{2(k+1)}}{2^{2k+4}S_{n,k}^2(x_{j,s}, 1 \leq j \leq n, 1 \leq s \leq k)}\right)^{1/k}\right\} \\ & \leq Ce^{-2^{-3-4/k}BA^{2/3k}n\sigma^2} \quad \text{for all } f \in \mathcal{F} \quad \text{if } \{x_{j,s}, 1 \leq j \leq n, 1 \leq s \leq k\} \notin H. \end{aligned} \quad (6.6)$$

Given some points $x_{j,s}$, $1 \leq j \leq n$, $1 \leq s \leq k$, define the probability measures ρ_s , $1 \leq s \leq k$, uniformly distributed on the set $x_{j,s}$, $1 \leq j \leq s$, i.e. $\rho_s(x_{j,s}) = \frac{1}{n}$, $1 \leq j \leq n$, and their product $\rho = \rho_1 \times \dots \times \rho_k$. If f is a function on (X^k, \mathcal{X}^k) such that $\int f^2 d\rho \leq \delta^2$ with some $\delta > 0$, then $|f(x_{j,s})| \leq \delta n^{k/2}$ for all $1 \leq s \leq k$, $1 \leq j \leq n$, and $P\left(\left|\bar{I}_{n,k}^\varepsilon(f)\right| > \delta n^{3k/2} \mid \xi_{j,s} = x_{j,s}, 1 \leq j \leq n, 1 \leq s \leq k\right) = 0$. Choose the numbers $\bar{\delta} = An^{-k/2}2^{-(k+2)}\sigma^{k+1}$ and $\delta = 2^{-(k+2)}n^{-k-1/2} \leq \bar{\delta}$. (The inequality $\delta \leq \bar{\delta}$ holds, since $A \geq A_0 \geq 1$, and $\sigma \geq n^{-1/2}$.) Choose a δ -dense set $\{f_1, \dots, f_m\}$ in the $L_2(X^k, \mathcal{X}^k, \rho)$ space with $m \leq D\delta^{-L} \leq 2^{(k+2)L}n^{\beta+(k+1/2)L}$ elements. Then formula (6.6) implies that

$$\begin{aligned} & P\left(\sup_{f \in \mathcal{F}} \left|\bar{I}_{n,k}^\varepsilon(f)\right| > 2^{-(k+1)}An^k\sigma^{k+1} \mid \xi_{j,s}(\omega) = x_{j,s}, 1 \leq j \leq n, 1 \leq s \leq k\right) \\ & \leq \sum_{j=1}^m P\left(\left|\bar{I}_{n,k}^\varepsilon(f_j)\right| > 2^{-(k+2)}An^k\sigma^{k+1} \mid \xi_{j,s}(\omega) = x_{j,s}, 1 \leq j \leq n, 1 \leq s \leq k\right) \\ & \leq C2^{(k+2)L}n^{\beta+(k+1/2)L}e^{-2^{-3-4/k}BA^{2/3k}n\sigma^2} \quad \text{if } \{x_{j,s}, 1 \leq j \leq n, 1 \leq s \leq k\} \notin H. \end{aligned} \quad (6.7)$$

Relations (6.3) and (6.7) imply that

$$\begin{aligned} & P\left(\sup_{f \in \mathcal{F}} \left|\bar{I}_{n,k}^\varepsilon(f)\right| > 2^{-(k+1)}An^k\sigma^{k+1}\right) \\ & \leq C2^{(k+2)L}n^{\beta+(k+1/2)L}e^{-2^{-3-4/k}BA^{2/3k}n\sigma^2} + 2^k e^{-A^{2/3k}n\sigma^2} \quad \text{if } A \geq T. \end{aligned} \quad (6.8)$$

Proposition 4 follows from the estimates (5.2) and (6.8) if the constants A_0 and K in the condition $n\sigma^2 \geq K((L + \beta) \log n + 1)$ are chosen sufficiently large. In this case the upper bound these estimate yields for the probability at the left-hand side of (3.8) is smaller than $e^{-A^{2/3k}n\sigma^2}$.

Let us turn to the proof of Proposition 5. By formula (5.8) it is enough to show that

$$P \left(\sup_{f \in \mathcal{F}} |H_{n,k}(f|G, V_1, V_2)| > \frac{A^2}{2^{4k+1}k!} n^{2k} \sigma^{2(k+1)} \right) \leq e^{-A^{1/2k} n \sigma^2} \quad (6.9)$$

for all $G \in \mathcal{G}$ and $V_1, V_2 \in \{1, \dots, k\}$ if $A \geq A_0$.

with the random variables $H_{n,k}(f|G, V_1, V_2)$ defined in formula (5.7). Let us first prove (6.9) in the case when $|e(G)| = k$, i.e. all vertices of the diagram G are an end-point of some edge, and the expression $H_{n,k}(f|G, V_1, V_2)$ contains no ‘symmetrizing term’ ε_j . By the Schwarz inequality

$$|H_{n,k}(f|G, V_1, V_2)| \leq \left(\sum_{\substack{j_1, \dots, j_k, 1 \leq j_s \leq n, \\ j_s \neq j_{s'} \text{ if } s \neq s'}} \int f^2(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}, y) \rho(dy) \right)^{1/2} \left(\sum_{\substack{j_1, \dots, j_k, 1 \leq j_s \leq n, \\ j_s \neq j_{s'} \text{ if } s \neq s'}} \int f^2(\xi_{j_1,1}^{(\bar{\delta}_1)}, \dots, \xi_{j_k,k}^{(\bar{\delta}_k)}, y) \rho(dy) \right)^{1/2}, \quad (6.10)$$

for such diagrams G , where $\delta_s = 1$ if $s \in V_1$, $\delta_s = -1$ if $s \notin V_1$, and $\bar{\delta}_s = 1$ if $s \in V_2$, $\bar{\delta}_s = -1$ if $s \notin V_2$. Hence

$$\begin{aligned} & \left\{ \omega : \sup_{f \in \mathcal{F}} |H_{n,k}(f|G, V_1, V_2)(\omega)| > \frac{A^2}{2^{4k+1}k!} n^{2k} \sigma^{2(k+1)} \right\} \\ & \subset \left\{ \omega : \sup_{f \in \mathcal{F}} \sum_{\substack{j_1, \dots, j_k, 1 \leq j_s \leq n, \\ j_s \neq j_{s'} \text{ if } s \neq s'}} \int f^2(\xi_{j_1,1}^{(\delta_1)}(\omega), \dots, \xi_{j_k,k}^{(\delta_k)}(\omega), y) \rho(dy) > \frac{A^2 n^{2k} \sigma^{2(k+1)}}{2^{4k+1}k!} \right\} \\ & \cup \left\{ \omega : \sup_{f \in \mathcal{F}} \sum_{\substack{j_1, \dots, j_k, 1 \leq j_s \leq n, \\ j_s \neq j_{s'} \text{ if } s \neq s'}} \int f^2(\xi_{j_1,1}^{(\bar{\delta}_1)}(\omega), \dots, \xi_{j_k,k}^{(\bar{\delta}_k)}(\omega), y) \rho(dy) > \frac{A^2 n^{2k} \sigma^{2(k+1)}}{2^{4k+1}k!} \right\}. \end{aligned}$$

The last relation implies that

$$\begin{aligned} & P \left(\sup_{f \in \mathcal{F}} |H_{n,k}(f|G, V_1, V_2)| > \frac{A^2}{2^{4k+1}k!} n^{2k} \sigma^{2(k+1)} \right) \\ & \leq 2P \left(\sup_{f \in \mathcal{F}} \sum_{\substack{j_1, \dots, j_k, 1 \leq j_s \leq n, \\ j_s \neq j_{s'} \text{ if } s \neq s'}} h_f(\xi_{j_1,1}, \dots, \xi_{j_k,k}) > \frac{A^2 n^{2k} \sigma^{2(k+1)}}{2^{4k+1}k!} \right) \quad (6.11) \end{aligned}$$

with $h_f(x_1, \dots, x_k) = \int f^2(x_1, \dots, x_k, y) \rho(dy)$, $f \in \mathcal{F}$. (In this upper bound we could get rid of the terms δ_j and $\bar{\delta}_j$, i.e. on the dependence of the expression $H_{n,k}(f|G, V_1, V_2)$ on the sets V_1 and V_2 , since the probability of the events in the previous formula do not depend on these terms.)

I claim that

$$P \left(\sup_{f \in \mathcal{F}} |\bar{I}_{n,k}(h_f)| \geq An^k \sigma^2 \right) \leq 2^k e^{-A^{1/2k} n \sigma^2} \quad \text{for } A \geq A_0 \quad (6.12)$$

if the constant A_0 and K are chosen sufficiently large in Proposition 5. Relation (6.12) together with the relation $\frac{n^{2k} \sigma^{2(k+1)}}{2^{4k+1} k!} \geq n^k \sigma^2$ imply that the probability at the right-hand side of (6.11) can be bounded by $2^{k+1} e^{-A^{1/2k} n \sigma^2}$, and the estimate (6.9) holds in the case $|e(G)| = k$. Relation (6.12) can be proved similarly to formula (6.3) in the proof of Proposition 4. It is not difficult to check that $0 \leq \int h_f(x_1, \dots, x_k) \mu(dx_1) \dots \mu(dx_k) \leq \sigma^2$, $\sup |h_f(x_1, \dots, x_k)| \leq 2^{-2(k+1)}$, and the class of functions $\mathcal{H} = \{2^k h_f, f \in \mathcal{F}\}$ is an L_2 -dense class with exponent L and parameter D . This means that by applying the Hoeffding decomposition of the functions h_f , $f \in \mathcal{F}$, similarly to formula (6.4) we get such sets of functions $(h_f)_V$, $f \in \mathcal{F}$ for all $V \subset \{1, \dots, k\}$ which satisfy the conditions of Proposition 4. Hence a natural adaptation of the estimate given for the expression at the right-hand side of (6.5) yields the proof of formula (6.12). Let us observe that by our inductive hypothesis the result of Proposition 4 holds also for k , and this allows us to carry out the estimates we need also for the class of functions $(h_f)_V$, $f \in \mathcal{F}$, with $V = \{1, \dots, k\}$ if $A \geq A_0$.

In the case $e(G) < k$ formula (6.9) will be proved with the help of Proposition A. To carry out this proof first an appropriate expression will be introduced and bounded for all sets $V_1, V_2 \subset \{1, \dots, k\}$ and diagrams $G \in \mathcal{G}$ such that $|e(G)| < k$. To define the expression we shall bound first some notations will be introduced.

Let us consider the set $J_0(G) = J_0(G, k, n)$,

$$J_0(G) = \{(j_1, \dots, j_k, j'_1, \dots, j'_k) : 1 \leq j_s, j'_s \leq n, 1 \leq s, s' \leq k, j_s \neq j_{s'} \text{ if } s \neq s', \\ j'_s \neq j'_{s'} \text{ if } s \neq s', j_s = j'_{s'} \text{ if } (s, s') \in G, j_s \neq j'_{s'} \text{ if } (s, s') \notin G\},$$

the set of those sequences $(j_1, \dots, j_k, j'_1, \dots, j'_k)$ which appear as indices in the summation in formula (5.7). I give a partition of $J_0(G)$ appropriate for our purposes.

For this aim let us first define the sets $M_1 = M_1(G) = \{s(1), \dots, s(k - |e(G)|)\} = \{1, \dots, k\} \setminus v_1(G)$, $s(1) < \dots < s(k - |e(G)|)$, and $M_2 = M_2(G) = \{\bar{s}(1), \dots, \bar{s}(k - |e(G)|)\} = \{1, \dots, k\} \setminus v_2(G)$, $\bar{s}(1) < \dots < \bar{s}(k - |e(G)|)$, the sets of those vertices of the first and second row of the diagram G in increasing order from which no edges start. Let us also introduce the set $V(G) = V(G, n, k)$,

$$V(G) = \{(j_{s(1)}, \dots, j_{s(k-|e(G)|)}, j'_{\bar{s}(1)}, \dots, j'_{\bar{s}(k-|e(G)|)}) : 1 \leq j_{s(p)}, j'_{\bar{s}(p)} \leq n, \\ 1 \leq p \leq k - |e(G)|, j_{s(p)} \neq j_{s(p')}, j'_{\bar{s}(p)} \neq j'_{\bar{s}(p')} \text{ if } p \neq p', 1 \leq p, p' \leq k - |e(G)|\},$$

which is the set consisting from the restriction of the coordinates of the vectors

$$(j_1, \dots, j_k, j'_1, \dots, j'_k) \in J_0(G)$$

to $M_1 \cup M_2$. Given a vector $v \in V(G)$ let $v(r)$, $1 \leq r \leq k - |e(G)|$, and $\bar{v}(r)$, $1 \leq r \leq k - |e(G)|$, denote its coordinates corresponding to the set M_1 and M_2 respectively. Put

$$\begin{aligned} E_G(v) = \{ & (j_1, \dots, j_k, j'_1, \dots, j'_k) : 1 \leq j_s \leq n, \text{ if } s \in v_1(G), 1 \leq j'_s \leq n \text{ if } s \in v_2(G), \\ & j_s \neq j_{s'}, j'_s \neq j'_{s'} \text{ if } s \neq s', j_s = j_{s'} \text{ if } (s, s') \in G \text{ and } j_s \neq j'_{s'} \text{ if } (s, s') \notin G \\ & j_{s(r)} = v(r), j'_{\bar{s}(r)} = \bar{v}(r), 1 \leq r \leq k - |e(G)|\}, \quad v \in V(G), \end{aligned}$$

where $\{s(1), \dots, s(k - |e(G)|)\} = M_1$, $\{\bar{s}(1), \dots, \bar{s}(k - |e(G)|)\} = M_2$ in the last line of this definition. The set $E_G(v)$ contains those vectors in $J_0(G)$ whose coordinates in $M_1 \cup M_2$ are prescribed by the vector $v \in V(G)$ and the remaining coordinates are chosen freely.

Now we define the partition

$$J_0(G) = \bigcup_{v \in V(G)} E_G(v).$$

of the set $J_0(G)$.

The inequality

$$P\left(S(\mathcal{F}|G, V_1, V_2) > A^{8/3} n^{2k} \sigma^4\right) \leq 2^{k+1} e^{-A^{2/3k} n \sigma^2} \quad \text{if } A \geq A_0 \text{ and } e(G) < k \quad (6.13)$$

will be proved for the random variable

$$\begin{aligned} S(\mathcal{F}|G, V_1, V_2) = \sup_{f \in \mathcal{F}} \sum_{v \in V(G)} \left(\sum_{(j_1, \dots, j_k, j'_1, \dots, j'_k) \in E_G(v)} \int f(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}, y) \right. \\ \left. f(\xi_{j'_1,1}^{(\bar{\delta}_1)}, \dots, \xi_{j'_k,k}^{(\bar{\delta}_k)}, y) \rho(dy) \right)^2, \end{aligned} \quad (6.13')$$

where $\delta_s = 1$ if $s \in V_1$, $\delta_s = -1$ if $s \notin V_1$, and $\bar{\delta}_s = 1$ if $s \in V_2$, $\bar{\delta}_s = -1$ if $s \notin V_2$.

To prove formula (6.13) let us first fix some $v \in V(G)$ and apply the Schwarz inequality. It yields that

$$\begin{aligned} & \left(\sum_{(j_1, \dots, j_k, j'_1, \dots, j'_k) \in E_G(v)} \int f(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}, y) f(\xi_{j'_1,1}^{(\bar{\delta}_1)}, \dots, \xi_{j'_k,k}^{(\bar{\delta}_k)}, y) \rho(dy) \right)^2 \\ & \leq \left(\sum_{(j_1, \dots, j_k, j'_1, \dots, j'_k) \in E_G(v)} \int f^2(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}, y) \rho(dy) \right) \\ & \quad \left(\sum_{(j_1, \dots, j_k, j'_1, \dots, j'_k) \in E_G(v)} \int f^2(\xi_{j'_1,1}^{(\bar{\delta}_1)}, \dots, \xi_{j'_k,k}^{(\bar{\delta}_k)}, y) \rho(dy) \right). \end{aligned}$$

for all $v \in V(G)$. Summing up these inequalities for all $v \in V(G)$ we get that

$$\begin{aligned}
S(\mathcal{F}|G, V_1, V_2) &\leq \sup_{f \in \mathcal{F}} \sum_{v \in V(G)} \left(\sum_{(j_1, \dots, j_k, j'_1, \dots, j'_k) \in E_G(v)} \int f^2(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}, y) \rho(dy) \right) \\
&\quad \left(\sum_{(j_1, \dots, j_k, j'_1, \dots, j'_k) \in E_G(v)} f^2(\xi_{j'_1,1}^{(\bar{\delta}_1)}, \dots, \xi_{j'_k,k}^{(\bar{\delta}_k)}, y) \rho(dy) \right) \quad (6.14) \\
&\leq \sup_{f \in \mathcal{F}} \left(\sum_{(j_1, \dots, j_k, j'_1, \dots, j'_k) \in J_0(G)} \int f^2(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}, y) \rho(dy) \right) \\
&\quad \sup_{f \in \mathcal{F}} \left(\sum_{(j_1, \dots, j_k, j'_1, \dots, j'_k) \in J_0(G)} f^2(\xi_{j'_1,1}^{(\bar{\delta}_1)}, \dots, \xi_{j'_k,k}^{(\bar{\delta}_k)}, y) \rho(dy) \right)
\end{aligned}$$

To check formula (6.14) we have to observe that by multiplying the inner sum at the left-hand side of this inequality each term $f^2(\xi_{j_1,1}^{(\delta_1)}, \dots, \xi_{j_k,k}^{(\delta_k)}, y) f^2(\xi_{j'_1,1}^{(\bar{\delta}_1)}, \dots, \xi_{j'_k,k}^{(\bar{\delta}_k)}, y)$ appears only once. (In particular, it is determined which index $v \in V(G)$ has to be taken in the outer sum to get this term. The coordinates of this vector v agree with the coordinates of the vector $j = (j_1, \dots, j_k, j'_1, \dots, j'_k)$ in $M_1 \cup M_2$, with the coordinates of the vector j which correspond to those vertices from which no edges of the diagram G start.) Beside this, all these products appear if the multiplications at the right-hand expression are carried out.

Relation (6.14) implies that

$$P(S(\mathcal{F}|G, V_1, V_2)) > A^{8/3} n^{2k} \sigma^4 \leq 2P \left(\sup_{f \in \mathcal{F}} \bar{I}_{n,k}(h_f) > A^{4/3} n^k \sigma^2 \right)$$

with $h_f(x_1, \dots, x_k) = \int f^2(x_1, \dots, x_k, y) \rho(dy)$. (Here we exploited that in the last formula $S(\mathcal{F}|G, V_1, V_2)$ is bounded by the product of two random variables whose distributions do not depend on the sets V_1 and V_2 .) Thus to prove inequality (6.13) it is enough to show that

$$2P \left(\sup_{f \in \mathcal{F}} \bar{I}_{n,k}(h_f) > A^{4/3} n^k \sigma^2 \right) \leq 2^{k+1} e^{-A^{2/3k}} \quad \text{if } A \geq A_0. \quad (6.15)$$

Actually formula (6.15) has been already proved, only formula (6.12) has to be applied, and the parameter A has to be replaced by $A^{4/3}$ in it.

The proof of Proposition 5 can be completed similarly to Proposition 4. It follows from Proposition A that

$$\begin{aligned}
P \left(|H_{n,k}(f|G, V_1, V_2)| > \frac{A^2}{2^{4k+2} k!} n^{2k} \sigma^{2(k+1)} \left| \xi_{j,s}^{\pm 1}, 1 \leq j \leq n, 1 \leq s \leq k \right. \right) (\omega) \\
\leq C e^{-B2^{-(4+2/k)} (k!)^{-1/k} A^{2/3k} n \sigma^2} \quad \text{if } S(\mathcal{F}|G, V_1, V_2)(\omega) \leq A^{8/3} n^{2k} \sigma^4 \\
\text{for all } f \in \mathcal{F}, G \in \mathcal{G}, |e(G)| < k, \quad \text{and } V_1, V_2 \in \{1, \dots, k\} \quad \text{if } A \geq A_0. \quad (6.16)
\end{aligned}$$

Indeed, in this case the conditional probability considered in (6.16) can be bounded by $C \exp \left\{ -B \left(\frac{A^4 n^{4k} \sigma^{4(k+1)}}{2^{8k+4} (k!)^2 S(\mathcal{F}|G, V_1, V_2)} \right)^{1/2j} \right\} \leq C \exp \left\{ -B \left(\frac{A^{4/3} n^{2k} \sigma^{4k}}{2^{8k+4} (k!)^2} \right)^{1/2j} \right\}$, where $2j = 2k - 2|e(G)|$, the number of vertices of the diagram G from which no edges start. Since $j \leq k$, $n\sigma^2 \geq 1$, and also $\frac{A^{4/3}}{2^{8k+4} (k!)^2} \geq 1$ if A_0 is chosen sufficiently large the above calculation implies relation (6.16).

Let us show that also the inequality

$$P \left(\sup_{f \in \mathcal{F}} |H_{n,k}(f|G, V_1, V_2)| > \frac{A^2}{2^{4k+1} k!} n^{2k} \sigma^{2(k+1)} \left| \xi_{j,s}^{\pm 1}, 1 \leq j \leq n, 1 \leq s \leq k \right. \right) (\omega) \\ \leq C n^{(3k+1)L/2+\beta} e^{-BA^{2/3k} n \sigma^2 / 2^{(4+2/k)} (k!)^{1/k}} \text{ if } S(\mathcal{F}|G, V_1, V_2)(\omega) \leq A^{8/3} n^{2k} \sigma^4 \\ \text{for all } G \in \mathcal{G}, |e(G)| < 1, \text{ and } V_1, V_2 \in \{1, \dots, k\} \text{ if } A \geq A_0 \quad (6.17)$$

holds.

To deduce formula (6.17) let us fix an elementary event $\omega \in \Omega$ which satisfies the relation $S(\mathcal{F}|G, V_1, V_2)(\omega) \leq A^{8/3} n^{2k} \sigma^4$, two sets $V_1, V_2 \subset \{1, \dots, k\}$, a diagram G , consider the points $x_{j,s}^{(\pm 1)} = x_{j,s}^{(\pm 1)}(\omega) = \xi_{j,s}^{(\pm 1)}(\omega)$, $1 \leq j \leq n$, $1 \leq s \leq k$, and introduce with their help the following probability measures: For all $1 \leq s \leq k$ define the probability measures $\nu_s^{(1)}$ which are uniformly distributed on the points $x_{j,s}^{(\delta_s)}$, $1 \leq j \leq n$, and $\nu_s^{(2)}$ which are uniformly distributed on the points $x_{j,s}^{(\bar{\delta}_s)}$, $1 \leq j \leq n$, where $\delta_s = 1$ if $s \in V_1$, $\delta_s = -1$ if $s \notin V_1$, and similarly $\bar{\delta}_s = 1$ if $s \in V_2$ and $\bar{\delta}_s = -1$ if $s \notin V_2$. Let us consider the product measures $\alpha_1 = \nu_1^{(1)} \times \dots \times \nu_k^{(1)} \times \rho$, $\alpha_2 = \nu_1^{(2)} \times \dots \times \nu_k^{(2)} \times \rho$ on the product space $(X^k \times Y, \mathcal{X}^k \times \mathcal{Y})$, where ρ is that probability measure on (Y, \mathcal{Y}) which appears in Proposition 5, together with the measure $\alpha = \frac{\alpha_1 + \alpha_2}{2}$. Given two functions $f \in \mathcal{F}$ and $g \in \mathcal{F}$ we give an upper bound for $|H_{n,k}(f|G, V_1, V_2)(\omega) - H_{n,k}(g|G, V_1, V_2)(\omega)|$ if $\int (f - g)^2 d\alpha \leq \delta$ with some $\delta > 0$. (This bound does not depend on the ‘randomizing terms’ $\varepsilon_j(\omega)$ in the definition of the random variable $H_{n,k}(\cdot|G, V_1, V_2)$.)

In this case $\int (f - g)^2 d\alpha_j \leq 2\delta^2$, and

$$\int |f(x_{1,j_1}^{(\delta_1)}, \dots, x_{k,j_k}^{(\delta_k)}, y) - g(x_{1,j_1}^{(\delta_1)}, \dots, x_{k,j_k}^{(\delta_k)}, y)|^2 \rho(dy) \leq 2\delta^2 n^k, \\ \int |f(x_{1,j_1}^{(\delta_1)}, \dots, x_{k,j_k}^{(\delta_k)}, y) - g(x_{1,j_1}^{(\delta_1)}, \dots, x_{k,j_k}^{(\delta_k)}, y)| \rho(dy) \leq \sqrt{2}\delta n^{k/2}$$

for all $1 \leq s \leq k$, and $1 \leq j_s \leq n$, and the same result holds if all δ_s is replaced by $\bar{\delta}_s$, $1 \leq s \leq k$. Since $|f| \leq 1$ for $f \in \mathcal{F}$, the condition $\int (f - g)^2 d\alpha \leq \delta^2$ implies that

$$\int |f(\xi_{j_1,1}^{(\delta_1)}(\omega), \dots, \xi_{j_k,k}^{(\delta_k)}(\omega), y) f(\xi_{j'_1,1}^{(\bar{\delta}_1)}(\omega), \dots, \xi_{j'_k,k}^{(\bar{\delta}_k)}(\omega), y) \rho(dy) \\ - g(\xi_{j_1,1}^{(\delta_1)}(\omega), \dots, \xi_{j_k,k}^{(\delta_k)}(\omega), y) g(\xi_{j'_1,1}^{(\bar{\delta}_1)}(\omega), \dots, \xi_{j'_k,k}^{(\bar{\delta}_k)}(\omega), y) \rho(dy)| \leq 2\sqrt{2}\delta n^{k/2}$$

for all vectors $(j_1, \dots, j_k, j'_1, \dots, j'_k)$ which appear as an index in the summation in (5.7), and

$$|H_{n,k}(f|G, V_1, V_2)(\omega) - H_{n,k}(g|G, V_1, V_2)(\omega)| \leq 2\sqrt{2}\delta n^{5k/2}$$

if the originally fixed $\omega \in \Omega$ is considered.

Put $\bar{\delta} = \frac{A^2 n^{-k/2} \sigma^{2(k+1)}}{2^{(4k+7/2)} k!}$, and $\delta = n^{-(3k+1)/2} \leq \bar{\delta}$ (since $\sigma \geq \frac{1}{\sqrt{n}}$ and we may assume that $A \geq A_0$ is sufficiently large), choose a δ -dense subset $\{f_1, \dots, f_m\}$ in the $L_2(X^k \times Y, \mathcal{X}^k \times Y, \alpha)$ space with $m \leq D\delta^{-L} \leq n^{(3k+1)L/2+\beta}$ elements. Relation (6.16) for these functions together with the above estimates yield formula (6.17).

It follows from relations (6.13) and (6.17) that

$$P\left(\sup_{f \in \mathcal{F}} |H_{n,k}(f|G, V_1, V_2)| > \frac{A^2}{2^{4k+1} k!} n^{2k} \sigma^{2(k+1)}\right) \leq 2^{k+1} e^{-A^2/3k} n \sigma^2 \\ + C n^{(3k+1)L/2+\beta} e^{-BA^2/3k} n \sigma^2 / 2^{(4+2/k)} (k!)^{1/k} \quad \text{if } A \geq A_0$$

for all $V_1, V_2 \subset \{1, \dots, k\}$ also in the case $|e(G)| \leq k-1$. This means that relation (6.9) holds also in this case if the constants A_0 and K are chosen sufficiently large in Proposition 5. Proposition 5 is proved.

Appendix. The proof of Proposition A

The proof will be based on the hypercontractive inequality for Rademacher functions. Let me first recall this result.

The hypercontractive inequality for Rademacher functions. *Let us consider the measure spaces (X, \mathcal{X}, μ) and $(Y, \mathcal{Y}, \nu) = (X, \mathcal{X}, \mu)$ defined as $X = \{-1, 1\}$, \mathcal{X} contains all subsets of X , and $\mu(\{1\}) = \mu(\{-1\}) = \frac{1}{2}$. Given a real number $\gamma > 0$ let us define the linear operator \mathbf{T}_γ which maps the real (or complex) valued functions on the space X to the real (or complex) valued functions on the space Y , and satisfies the relations $\mathbf{T}_\gamma r_0 = r_0$, and $\mathbf{T}_\gamma r_1 = \gamma r_1$, where $r_0(1) = r_0(-1) = 1$, and $r_1(1) = 1, r_1(-1) = -1$. For all $n = 1, 2, \dots$ let us consider the n -fold product $(X^n, \mathcal{X}^n, \mu^n)$ and $(Y^n, \mathcal{Y}^n, \nu^n)$ together with the n -fold product of the operator \mathbf{T}_γ^n (i.e. \mathbf{T}_γ^n is the linear transformation for which $\mathbf{T}_\gamma^n(f_1(x_1) \cdots f_n(x_n)) = (\mathbf{T}_\gamma f_1(x_1) \cdots \mathbf{T}_\gamma f_n(x_n))$ for all functions $f_s, 1 \leq s \leq n$, on the space (X, \mathcal{X}, μ)). For all $n = 1, 2, \dots$ the transformation \mathbf{T}_γ^n from the space $L_p(X^n, \mathcal{X}^n, \mu^n)$ to the space $L_q(Y^n, \mathcal{Y}^n, \nu^n)$ has the norm 1 if $1 < p \leq q < \infty$, and $\gamma \leq \sqrt{\frac{p-1}{q-1}}$.*

The following corollary of the hypercontractive inequality is useful for us.

Corollary of the hypercontractive inequality. *Let $\varepsilon_1, \dots, \varepsilon_n$ be independent identically distributed random variables $P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = \frac{1}{2}$, $1 \leq j \leq n$, fix some real numbers $a(j_1, \dots, j_k)$ for all indices (j_1, \dots, j_k) such that $1 \leq j_s \leq n$, $1 \leq s \leq k$, and $j_s \neq j_{s'}$ if $s \neq s'$, and define the random variable*

$$Z = \sum_{\substack{1 \leq j_s \leq n, 1 \leq s \leq k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} a(j_1, \dots, j_k) \varepsilon_{j_1} \cdots \varepsilon_{j_k}.$$

The inequality

$$E|Z|^q \leq \left(\frac{q-1}{p-1}\right)^{kq/2} (E|Z|^p)^{q/p} \quad \text{if } 1 < p \leq q < \infty$$

holds.

Proof of the Corollary. Let us define the function

$$f(x_1, \dots, x_n) = \sum_{\substack{1 \leq j_s \leq n, 1 \leq s \leq k \\ j_s \neq j_{s'} \text{ if } s \neq s'}} a(j_1, \dots, j_k) r_1(x_{j_1}) \cdots r_1(x_{j_k})$$

on the space $(X^n, \mathcal{X}^n, \mu^n)$. Observe that $\mathbf{T}_\gamma^n f = \gamma^k f$ for this function f and all $\gamma > 0$, and $E|Z|^p = \|f\|_p^p$, $E|Z|^q = \|f\|_q^q$. Fix some numbers $1 < p \leq q \leq \infty$, and put $\gamma = \sqrt{\frac{p-1}{q-1}}$. Since the norm of \mathbf{T}_γ^n as a transformation from the space $L_p(X^n, \mathcal{X}^n, \mu^n)$ to the space $L_q(Y^n, \mathcal{Y}^n, \nu^n)$ is 1, $\|\mathbf{T}_\gamma^n f\|_q = \gamma^k \|f\|_q \leq \|f\|_p$. The above relations imply that $(E|Z|^q)^{1/q} \leq \left(\frac{q-1}{p-1}\right)^{k/2} (E|Z|^p)^{1/p}$ in this case, and this is what we had to show.

Applying the corollary with $p = 2$ and some $q > 2$ we get that

$$E|Z|^q \leq (q-1)^{kq/2} (EZ^2)^{q/2} \leq q^{kq/2} (EZ^2)^{q/2} = q^{kq/2} \bar{S}^q$$

with

$$\bar{S}^2 = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} \left(\sum_{\pi \in \Pi_k} a((j_{\pi(1)}, \dots, j_{\pi(k)})) \right)^2,$$

where Π_k denotes the set of all permutations of the set $\{1, \dots, k\}$. Observe that

$$\left(\sum_{\pi \in \Pi_k} a((j_{\pi(1)}, \dots, j_{\pi(k)})) \right)^2 \leq k! \sum_{\pi \in \Pi_k} a^2((j_{\pi(1)}, \dots, j_{\pi(k)})) \quad \text{for all } 1 \leq j_1 < \dots < j_k \leq n,$$

hence $\bar{S}^2 \leq k! S^2$, and $E|Z|^q \leq q^{kq/2} (k!)^{q/2} S^q$ with the number S^2 defined in (3.4). Thus the Markov inequality implies that

$$P(|Z| > x) \leq \left(q^{k/2} \frac{\sqrt{k!} S}{x} \right)^q \quad \text{for all } x > 0 \quad \text{and } q \geq 2.$$

Choose the number q as the solution of the equation $q \left(\frac{\sqrt{k!} S}{x} \right)^{2/k} = \frac{1}{e}$. Then we get that $P(|Z| > x) \leq \exp \left\{ -B \left(\frac{x}{S} \right)^{2/k} \right\}$ with $B = \frac{k}{2e(k!)^{1/k}}$, provided that $q = \frac{1}{e k!^{1/k}} \left(\frac{x}{S} \right)^{2/k} \geq 2$, i.e. $B \left(\frac{x}{S} \right)^{2/k} \geq k$. By multiplying the above upper bound with $C = e^k$ we get such an estimate for $P(|Z| > x)$ which holds for all $x > 0$.

References:

- 1.) Alexander, K. (1987) The central limit theorem for empirical processes over Vapnik–Červonenkis classes. *Ann. Probab.* **15**, 178–203
- 2.) Arcones, M. A. and Giné, E. (1994) U -processes indexed by Vapnik–Červonenkis classes of functions with application to asymptotics and bootstrap of U -statistics with estimated parameters. *Stoch. Proc. Appl.* **52**, 17–38
- 3.) Beckner, W. (1975) Inequalities in Fourier Analysis. *Ann. Math.* **102**, 159–182
- 4.) de la Peña, V. H. and Giné, E. (1999) Decoupling From dependence to independence. Springer series in statistics. Probability and its application. Springer Verlag, New York, Berlin, Heidelberg
- 5.) de la Peña, V. H. and Montgomery–Smith, S. (1995) Decoupling inequalities for the tail-probabilities of multivariate U -statistics. *Ann. Probab.*, **25**, 806–816
- 6.) Gross, L. (1975) Logarithmic Sobolev inequalities. *Amer. J. Math.* **97**, 1061–1083
- 7.) Major, P. (1988) On the tail behaviour of the distribution function of multiple stochastic integrals. *Probability Theory and Related Fields*, **78**, 419–435
- 8.) Major, P. (2003) An estimate about multiple stochastic integrals with respect to a normalized empirical measure. submitted to *Probability Theory and Related Fields*
- 9.) Pollard, D. (1984) Convergence of Stochastic Processes. Springer–Verlag, New York

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