THE LARGE-SCALE LIMIT OF DYSON'S HIERARCHICAL VECTOR-VALUED MODEL AT LOW TEMPERATURES. THE NON-GAUSSIAN CASE.

PART II. DESCRIPTION OF THE LARGE-SCALE LIMIT.

[1]Bleher, P. M. and [2]Major, P.

[1]Keldysh Institute of Applied Mathematics,
Moscow A–47 Miusskaya Square 4, U.S.S.R.
[2]Mathematical Institute of the Hungarian Academy
of Sciences, Budapest H–1364 P.O.B. 127, Hungary

1. Introduction. In this paper we investigate the large-scale limit of the equilibrium state of Dyson's hierarchical vector valued p dimensional, $p\ge 2$, model with parameter c, $1<c<\sqrt{2}$, at low temperatures. More precisely, in Theorem 1 we construct a probability measure $\bar\mu=\bar\mu(T)$ on $(R^p)^{\mathbf Z}$ with $\mathbf Z=\{1,2,\dots\}$ which is an equilibrium state of the model. In Theorem 2 we determine the large-scale limit of a $\bar\mu$ distributed random field together with the right scaling, i.e. we prove that if

$$\sigma=\left\{\sigma(j)=\left(\sigma^{(1)}(j),\dots,\sigma^{(p)}(j)\right)\in R^p,\, j\in\mathbf Z\right\}$$

is a $\bar\mu$ distributed random field then the finite dimensional distributions of the random fields

$$\mathcal R_n\sigma=\left\{\left(\mathcal R_n\sigma^{(1)}(j),\dots,\mathcal R_n\sigma^{(p)}(j)\right)\in R^p,\, j\in\mathbf Z\right\}, \tag{1.1}$$

$$\mathcal R_n\sigma^{(1)}(j)=c^n\,2^{-n}\sum_{k=(j-1)2^n+1}^{j2^n}\left[\sigma^{(1)}(k)-E\sigma^{(1)}(k)\right],\quad j\in\mathbf Z, \tag{1.2}$$

$$\mathcal R_n\sigma^{(s)}(j)=c^{n/2}2^{-n}\sum_{k=(j-1)2^n+1}^{j2^n}\sigma^{(s)}(k),\quad j\in\mathbf Z,\quad s=2,\dots,p \tag{1.3}$$

tend to those of a limit random field, and describe the finite dimensional distributions of this limit field.

The distributions of the fields $\mathcal R_n\sigma$ defined in (1.1)–(1.3) are called the renormalizations of the distribution $\bar\mu$ of the underlying field σ. More precisely, they are its renormalization with parameters $\alpha=1-\frac{\log c}{\log 2}$ in the first coordinate and $\bar\alpha=1-\frac12\frac{\log c}{\log 2}$ in the coordinates $s=2,\dots,p$, because we multiplied by $2^{-n\alpha}$ in

(1.2), by $2^{-n\bar\alpha}$ in (1.3), and the number of summands is 2^n in these formulas. If the finite dimensional distributions of the fields $\mathcal{R}_n\sigma$ converge to those of a limit field then this limit field, more precisely its distribution, is called the large-scale limit of the measure $\bar\mu$.

Given some $h\in R^1$, $h\ge 0$, and positive integer N let the Gibbs measure $\mu_N^h=\mu_N^h(T,t)$ be defined on $(R^p)^{2^N}$ with density function

$$p_N^h(x_1,\dots,x_{2^N})=p_N^h(x_1,\dots,x_{2^N},t,T)\,,\qquad x_j=(x_j^{(1)},\dots,x_j^{(p)})\in R^p\,,\quad j=1,\dots,2^N\,,$$

(1.4)
$$p_N^h(x_1,\dots,x_{2^N})=Z_N^{-1}(T,t,h)\exp\left\{-\frac{1}{T}\left(-\sum_{i=1}^{2^N-1}\sum_{j=i+1}^{2^N}U(i,j)x_ix_j-h\sum_{j=1}^{2^N}x_j^{(1)}\right)\right\}\prod_{j=1}^{2^N}p(x_j,t)\,,$$

where

$$Z_N=\int\exp\left\{-\frac{1}{T}\left(-\sum_{i=1}^{2^N-1}\sum_{j=i+1}^{2^N}U(i,j)x_ix_j-h\sum_{j=1}^{2^N}x_j^{(1)}\right)\right\}\prod_{j=1}^{2^N}p(x_j,t)\,dx_j$$

is the grand partition function, and $p(x,t)$ is defined in (1.3) of Part I. Let $p_N^h(x)$ denote the density function of the average $2^{-N}\sum_{j=1}^{2^N}\sigma(j)$ of the μ_N^h distributed random vector $(\sigma(1),\dots,\sigma(2^n))$. Put $\mu_N=\mu_N^h$, $p(x_1,\dots,x_{2^N})=p_N^h(x_1,\dots,x_{2^N})$ and $p_N(x)=p_N^h(x)$ in the case $h=0$.

In Part I we have described the asymptotic behaviour of the above defined density function $p_N(x)$. The result of Theorem A formulated below are contained in Theorems 1, 2 and Lemma 13 of Part I. Let us consider the integral equation

(1.5)
$$g(x)=\left(\frac{2}{c\sqrt{\pi}}\right)^{p-1}\int_{R^p}\exp(-v^2)g\left(\frac{x}{c}+u+\frac{v^2}{2}\right)g\left(\frac{x}{c}-u+\frac{v^2}{2}\right)dudv\,,\qquad x,\ u\in R^1\,,\quad v\in R^{p-1}\,,$$

where v^2 denotes scalar product. In Part I we have proved that equation (1.5) has a unique non-trivial (i.e. not identically zero) solution in the class of functions $\mathcal{A}=\left\{g,\int e^{tx}|g(x)|dx<\infty\ \text{if}\ t<t_0(g),\ t_0(g)>0\right\}$. In this work we consider this function as the solution of equation (1.5). It is a density function which is positive for all x. Since the function $p_n(x,t)$ depends on x only through $|x|$ we can define a function $\bar p_n(z)=\bar p_n(z,t,T)$, $z\in R^1$, such that $p_n(x)=\bar p_n(|x|)$ for all $x\in R^p$. Now we formulate the following

Theorem A. *If $1<c<\sqrt2$ then there exist some thresholds $T_0=T_0(c)>0$ such that for all $0<T<T_0$, $0<t<t_0$, $t_0=t_0(c)$, (t is the parameter of $p(x,t)$ in formula (1.3) of Part I) the following relations hold:*

There are some $M = M(c,T,t) > 0$ *and* $n_0 = n_0(c,T,t) > 0$ *such that for* $n > n_0$

(1.6)
$$\begin{aligned} c^{-n}p_n(x,T) &= c^{-n}\bar{p}_n(|x|,T) \\ &= B\exp\left\{-\frac{a_0c^n}{T}M(|x|-M)\right\}g\left(\frac{a_1c^n}{T}M(|x|-M)\right)(1+r_n(x)) \end{aligned}$$

for $-\eta nc^{-n} < |x|-M < \eta n^{1/\alpha}c^{-n}$ *with some* $B = B(c,T,t) > 0,\ \eta = \eta(c,T,t) > 0$, *and the error term* $r_n(x)$ *satisfies the inequality*

$$|r_n(x)| \le Kq^n \tag{1.7}$$

with some $K > 0$ *and* $0 < q < 1$ *depending on* $c,\ T$ *and* t. *In formula (1.6)* $g(x)$ *is the solution of the equation (1.5),* $a_0 = \frac{2}{2-c},\ \ a_1 = a_0 + 1$ *and* $\alpha = \frac{\log 2}{\log c}$.

$$c^{-n}\bar{p}_n(x,T) \le Kq^n\exp\left\{-L(c^n|x-M|)^{2+\delta}\right\} \quad \text{if } x > M + \eta n^{1/\alpha}c^{-n} \tag{1.8}$$

with some $\delta > 0$, $K > 0$ *and* $L > 0$ *which depend on* c, T *and* t. *The solution of the equation (1.6) satisfies the inequality*

$$0 < g(x) < C\exp(-Ax^\alpha) \quad \text{for } x > 0 \tag{1.9}$$

with some $C > 0,\ \ A > 0$. *We have*

$$c^{-n}\bar{p}_n(x,T) \le C_1\exp\{-C_2c^n|x-M|\} \quad \text{for} \quad 0 < x < M \tag{1.10}$$

with some $C_1 > 0,\ \ C_2 > 0$ *depending on* $c,\ T$ *and* t. *We also have*

$$M^2 = \frac{a_0 - T}{tT} + R(t,T)$$

with some $|R(t,T)| \le$ *const., and such that* $R(t,T) \to 0$ *and* $T \to 0$.

Given some integers $N \ge k \ge 0$ we define the probability measure $\mu^h_{k,N}$ as the projection of the measure μ^h_N to the first 2^k coordinates, i.e. $\mu^h_{k,N}$ is a probability measure on $(R^p)^{2^k}$, and for all measurable $A \subset (R^p)^{2^k}$ $\mu^h_{k,N}(A) = \mu^h_N(A) = \mu^h_N\left(A \times (R^p)^{2^N-2^k}\right)$. Our first result is the following

Theorem 1. *Let the conditions of Theorem A be satisfied. Consider an arbitrary sequence of real numbers* h_N, $N = 0,1,2,\dots$ *such that*

$$\frac{2}{2-c}\frac{M}{T}\left(\frac{c}{2}\right)^N \le \frac{h_N}{T} \le D\left(\frac{c}{2}\right)^N \tag{1.11}$$

with some $\infty > D > \frac{2}{2-c}\frac{M}{T}$, *where* M *and* T *are the same as in Theorem A. Then the measures* $\mu^{h_N}_N$ *tend to a probability measure* $\bar{\mu} = \bar{\mu}(t,T,c)$ *on* $(R^p)^{\mathbf{Z}}$. *More precisely, for all* $k \ge 0$ *the measures* $\mu^{h_N}_{k,N}$ *converge to the projection of* $\bar{\mu}$ *to the first* 2^k *coordinates in variational metric as* $N \to \infty$. *The measure* μ *does not depend on the choice of sequences* h_N.

Then we prove the following

Theorem 2. *Let* $\sigma = \left\{\sigma(n) = \left(\sigma^{(1)}(n), \dots, \sigma^{(p)}(n)\right) \in R^p,\ n \in \mathbf{Z}\right\}$ *be a* $\bar\mu$ *distributed random field with the distribution* $\bar\mu$ *defined in Theorem 1. Then the finite dimensional distributions of the random fields* $R_n\sigma$ *defined in (1.1), (1.2), (1.3) tend to those of a random field* $Y = \left(Y(n) = \left(Y^{(1)}(n), \dots, Y^{(p)}(n)\right) \in R^p,\ n \in \mathbf{Z}\right)$. *For all* $k \ge 0$ *the density function* $h_k(x_1, \dots, x_{2^k})$, $x_j = \left(x_j^{(1)}, \dots, x_j^{(p)}\right) \in R^p$, *of the random vector* $(Y(1), \dots, Y(2^k))$ *is given by the formula*

(1.12)

$$h_k(x_1, \dots, x_{2^k}) = C(k) \exp\left\{-\frac{1}{T}\sum_{s=2}^{p}\left(\frac{1}{2-c}\sum_{j=1}^{2^k} x_j^{(s)2} - \frac{2-c}{c}\left(\frac{c}{4}\right)^k \left(\sum_{j=1}^{2^k} x_j^{(s)}\right)^2 + \sum_{i=1}^{2^k-1}\sum_{j=i+1}^{2^k} U(i,j) x_i^{(s)} x_j^{(s)}\right)\right\} \prod_{j=1}^{2^k} g\left(\frac{4-c}{(2-c)T}\left(M x_j^{(1)} + \frac{1}{2}\sum_{s=2}^{p} x_j^{(s)2}\right)\right),$$

where the function g *is defined in (1.5), the constant* M *is the same as in Theorem A, and* $C(k)$ *is an appropriate norming constant.*

In Appendix E we prove the following

Theorem B. *The measure* $\bar\mu = \bar\mu(T, t, c)$ *constructed in Theorem 1 is a Gibbs state with Hamiltonian* $\mathcal{H}$ *and free measure* ν *defined in formulas (1.1)–(1.3) of Part I at temperature* T.

Theorem B is very plausible. Its proof depends on a rather standard limiting procedure in statistical physics literature. Nevertheless, we have found no result which could have been directly applied in our case. We present the proof of Theorem B in Appendix E.

Let us discuss the role of condition (1.11) in Theorem 1. The lower bound

$$h_N > \frac{2}{2-c} M \left(\frac{2}{c}\right)^N \tag{1.11′}$$

is essential in Theorem 1, it is needed to get a pure state with magnetization in the direction $e_1 = (1, 0, \dots, 0)$ for the limit measure $\bar\mu$. If it were violated we would get a Gibbs state with Hamiltonian $\mathcal{H}$ again for the limit, but this Gibbs state would be a mixture of Gibbs states with different directions of magnetization, and it is not natural to renormalize such a mixture. On the other hand the upper bound for h_N in (1.11) seems not to be essential. We believe that the same limit measure $\bar\mu$ would be obtained for any sequence h_N, $h_N > 0$ satisfying (1.11′) or with the help of the double limiting procedure $\mu^h = \lim_{N\to\infty} \mu_N^h$, $h > 0$, $\bar\mu = \lim_{h\to 0} \mu^h$. This second way was chosen to construct the equilibrium state in the case $\sqrt{2} < c < 2$ in paper [5]. However, to prove these statements we would need a large deviation result on the behaviour of $p_n(x)$ which is stronger than Theorem A. Since we are not able to prove such a result we have proved Theorem 1 under the condition (1.11), but we think that this is not an essential restriction.

In formula (1.12) we have a quadratic form inside the exponent. This means that the random variables $Y^{(\ell)}(j)$, $\ell = 2, \dots, p$ appearing in Theorem 2 are jointly Gaussian. We describe the structure of this limit field in more detail. The random fields $\{\mathcal{R}_n\sigma^{(s)}(j),\ j\in\mathbf{Z}\}$, $s=2,\dots,p$, and $\{M\mathcal{R}_n\sigma^{(1)}(j)+\frac12\sum\limits_{s=2}^{p}\mathcal{R}_n\sigma^{(s)}(j)^2,\ j\in\mathbf{Z}\}$ tend to independent random fields as $n\to\infty$. The limit of the random fields $\{\mathcal{R}_n(\sigma^{(s)}(j)),\ j\in\mathbf{Z}\}$ is the (disregarding a multiplying factor) unique Gaussian self-similar field with self-similarity parameter $1-\frac12\frac{\log c}{\log 2}$, whose distribution is invariant under all permutations of the index set $\mathbf{Z}$ which preserves the hierarchical distance $d(i,j)$. The random fields $\{M\mathcal{R}_n\sigma^{(1)}(j)+\frac12\sum_{s=2}^{p}\mathcal{R}_n\sigma^{(s)}(j)\,,\ j\in\mathbf{Z}\}$ tend to a random field consisting of independent random variables with the density function $\frac{2-c}{(4-c)T}g(\frac{4-c}{(2-c)T}x)$. This is a quadratic functional of a Gaussian field (see Lemma 12 in Part I).

The above result can also be interpreted in the following way: Given a $\bar\mu$ distributed random field $\sigma(n)$, $n\in\mathbf{Z}$, define the absolute value of the appropriately normalized partial sums $|\mathcal{R}_n|\sigma(j)=c^n2^{-n}\left(|\mathcal{R}_n\sum_{k=(j-1)2^n+1}^{j2^n}\sigma(k)|-M\right)$, $j\in\mathbf{Z}$. Then the random fields $\mathcal{R}_n\sigma^{(s)}(j)$, $j=2,\dots,p$, and the random fields $|\mathcal{R}_n|\sigma(j)$ tend in distribution to independent random fields. The limit of $\mathcal{R}_n\sigma^{(s)}(j)$ $j=2,\dots,p$ is Gaussian, and the limit of $|\mathcal{R}_n|\sigma(j)$ consists of independent random variables. This follows immediately from the above description of the limit behaviour of the fields $\mathcal{R}_n\sigma$ together with the observation that $|\mathcal{R}_n|\sigma(j)-(\mathcal{R}_n\sigma^{(1)}(j)+\frac{1}{2M}\sum_{s=2}^{p}\mathcal{R}_n\sigma^{(s)}(j)^2)\Rightarrow 0$ stochastically as $n\to\infty$.

We believe that the above property is a special case of a more general law. Let us remark that an analogous statement also holds in the case $\sqrt2<c<2$, but this is a degenerate case. It follows from the results of [5] that if $\{\sigma(j),\,j\in\mathbf{Z}\}$ is a $\bar\mu=\bar\mu(c)$, $\sqrt2<c<2$ distributed random field with the equilibrium state $\bar\mu$ constructed in [5] then the random fields

$$|\mathcal{R}_n|(\sigma)(j)=2^{-n/2}\left(\left|\sum_{k=(j-1)2^n+1}^{j2^n}\sigma(k)\right|-M\right),\quad j\in\mathbf{Z},$$

have the same limit as the random fields

$$\mathcal{R}_n\sigma^{(1)}(j)=2^{-n/2}\sum_{k=(j-1)2^n+1}^{j2^n}\left(\sigma^{(1)}(k)-M\right)$$

as $n\to\infty$, since in this case $|\mathcal{R}_n|\sigma(j)-\mathcal{R}_n\sigma^{(1)}(j)\Rightarrow 0$. This limit consists of independent (Gaussian) random variables which is also independent of the limit of the random fields $\mathcal{R}_n\sigma^{(j)}$, $s=2,\dots,p$.

The method of this paper is very similar to that of [5]. The two main steps in the proofs consist of the description of the limit behaviour of the function $p_n(x)$ done in Part I, and a good asymptotic formula for the Radon–Nikodym derivatives $\frac{d\mu_{n,N}^{h_N}}{d\mu_n}$. Then an appropriate limiting procedure supplies the proof of Theorems 1 and 2. The investigation of the Radon–Nikodym derivatives can be considered as an adaptation of the method of [5] to the present case. The main difference between the two cases is that now $p_n(x)$ is not asymptotically Gaussian. But although the Radon–Nikodym derivative $\frac{d\mu_{n,N}^{h_N}}{d\mu_n}$ depends on $p_n(x)$, its asymptotic behaviour does not. As we shall see, in the investigation of the asymptotic behaviour of the above Radon–Nikodym derivative we only need some estimates on the tail behaviour of $p_n(x)$, but not its explicit form. This is the reason why we can adapt the method of [5].

2. On the basic estimates needed in the proof. Reduction to integral equations. We need a good asymptotic formula for the Radon–Nikodym derivative $\frac{d\mu^{h_N}_{n,N}}{d\mu_n}$. It can be expressed exactly with the help of the following formulas:

$$\frac{d\mu^{h_N}_{n,N}}{d\mu_n}(x_1,\dots,x_{2^n})=f^{h_N}_{n,N}\left(2^{-n}\sum_{j=1}^{2^n}x_j\right),\quad n\le N \tag{2.1}$$

$$f^{h_N}_{N,N}(x)=K(N,h_N)\exp\left(\frac{2^N h_N x^{(1)}}{T}\right) \tag{2.2}$$

$$f^{h_N}_{n,N}(x)=K(n,N,h_N)S_n f^{h_N}_{n+1,N}(x) \tag{2.3}$$

with

$$S_n f(x)=\int_{R^p}\exp\left(\frac{c^n}{T}xy\right)f\left(\frac{x+y}{2}\right)p_n(y)\,dy \tag{2.3'}$$

where $K(n,N,h_N)$ are appropriate norming factors, xy denotes scalar product, and p_n is the density function appearing in Theorem A. For scalar valued models formulas (2.1)–(2.3′) are proved in the main formula in [4]. The proof for the vector valued case is the same, but since the proof in [4] is a bit complicated we present it in Appendix C.

Let us define the sequences $g_n=g_n(N,h_N)$ and $A_n=A_n(N,h_N)$ by the recursive relations

$$g_N=g_N(N,h_N)=\frac{2^N h_N}{T} \tag{2.4}$$

(2.4′)

$$g_n=g_n(N,h_N)=\frac{g_{n+1}}{2}+\frac{c^n}{T}M\qquad\text{for } n<N$$

(2.5)

$$A_N=A_N(N,h_N)=0$$

(2.5′)

$$A_n=A_n(N,h_N)=\frac{A_{n+1}}{4}+\frac{(\frac{c^n}{T}+\frac{A_{n+1}}{2})^2}{\frac{2c^n}{T}+\frac{g_{n+1}}{M}-A_{n+1}}\qquad\text{for } n<N\,,$$

where M and T are the same as in Theorem A. In Section 7 of [6] we have claimed that

$$f_n(x)=f^h_{n,N}(x)\sim K_n\exp\left\{g_n(x^{(1)}-M)+A_n\sum_{s=2}^{p}x^{(s)2}\right\},$$

and have given a heuristic explanation. In the following Proposition 1 we formulate this result in a more precise form. For the sake of simpler notation we assume that $R^p = R^2$. From now on $C,\ C_1,\ K,\ L$ etc. denote appropriate constants. The same letter may denote different constants in different formulas. Let us define the domains

(2.6) $$\Omega_n^1 = \{x \in R^2,\ \big||x|-M\big| < c^{-0.4n}\ ,\ |x^{(2)}| < c^{-0.4n}\ ,\ x^{(1)} > 0\}$$

(2.6′) $$\Omega_n^2 = \left\{x \in R^2,\ \big||x|-M\big| < c^{-0.4n}\right\} - \Omega_n^1$$

(2.6″) $$\Omega_n^3 = \left\{x \in R^2,\ \big||x|-M\big| \ge c^{-0.4n}\right\}\ .$$

Clearly $\Omega_n^1 \cup \Omega_n^2 \cup \Omega_n^3 = R^2$. Now we formulate the following

Proposition 1. *For all q, $c^{-0,2} < q < 1$, there is some $n_0 = n_0(T,M,c,D,q)$ such that if (1.11) holds and $N \ge n \ge n_0$ then the Radon–Nikodym derivative $f_n(x) = f_{n,N}^{h_N}(x)$ appearing in (2.1) satisfies the following relations:*
a) In the domain Ω_n^1

(2.7) $$f_n(x) = L_n \exp\left\{g_n\left(x^{(1)}-M\right) + A_n x^{(2)2} + \varepsilon_n(x)\right\}$$

with

$$\sup_{x\in\Omega_n^1} |\varepsilon_n(x)| \le q^n\ .$$

b) In the domain Ω_n^2

(2.8) $$0 \le f_n(x) \le L_n \exp\left\{g_n(|x|-M) - \left(\frac{g_n}{2M} - A_n\right)c^{-0,8n} + q^n\right\}\ .$$

c) In the domain Ω_n^3

(2.9) $$0 \le f_n(x) \le L_n \exp\left\{\frac{g_n}{2M}(|x|^2 - M^2)\right\}\ ,$$

where the numbers A_n and g_n are defined in (2.4)–(2.5′), and $L_n = L_n(N, h_N)$ is an appropriate norming constant.

We also prove the following result which is a slight modification of Lemma 1 in [5].

Lemma 1. *Let us choose some integer N and $h_N > 0$. Define the sequences g_n and A_n, $0 \le n \le N$, by formulas (2.4)–(2.5′) and put $\bar g_n = c^{-n} g_n$, $\bar A_n = c^{-n} A_n$. If h_N satisfies relation (1.11) then $\bar g_N \ge \bar g_{N-1} \ge \cdots \ge \bar g_0 \ge \bar g$ and $0 = \bar A_N \le \bar A_{N-1} \le \cdots \le \bar A_0 \le \bar A$ with $\bar g = \frac{2}{2-c}\frac{M}{T}$, and $\bar A = \frac{2-c}{cT}$. If the relations $N > N_0$ and $N > n^B$ also hold with some appropriate $N_0 = N_0(c,M,T,D)$ and $B = B(c,M,T,D)$ then $|\bar g_n - \bar g| < 4^{-n}$, $|\bar A_n - \bar A| < 4^{-n}$.*

Proposition 1 together with the characterization of the asymptotic behaviour of the sequences g_n and A_n made in Lemma 1 supplies a good asymptotic formula for the Radon–Nikodym derivative f_n. Here Ω_n^1 is the typical region, where we have a good asymptotic formula, in Ω_n^2 and Ω_n^3 we have only given an upper bound. Actually we are interested in the density function

$$p_n\left(2^{-n}\sum_{j=1}^{2^n} x_j\right) f_{n,N}^{h_N}\left(2^{-n}\sum_{j=1}^{2^n} x_j\right)$$

of the measure $\mu_{n,N}^{h_N}$. The tail behaviour of the functions $p_n(x)$ and $f_n(x)$ together show that $2^{-n}\sum_{j=1}^{2^n} x_j$ is contained in Ω_n^3 with a negligible small $\mu_{n,N}^{h_N}$ probability. It is contained in Ω_n^2 also with a small probability, since in this domain $f_n(x)$ is small. To see it, let us observe that by Lemma 1

$$\frac{g_n}{2M} - A_n \ge c^n \left(\frac{\bar g}{2M} - \bar A\right) = \frac{c^n}{T}\left(\frac{1}{2-c} - \frac{2-c}{c}\right) = \frac{c^n}{T}\frac{(4-c)(c-1)}{2-c} > 0\,,$$

hence the term $-\left(\frac{g_n}{2M} - A_n\right) c^{-0.8n}$ in the exponent of (2.8), makes this upper bound (2.8) sufficiently small for our purposes.

In Section 7 of [6] we have given a heuristic argument for formula (2.7). The following remark explains the content of the estimate (2.8).

Remark. If $x \in \Omega_n^1$ then

$$\begin{aligned} |x| &= \left(x^{(1)2} + x^{(2)2}\right)^{1/2} = x^{(1)}\left(1 + \frac{x^{(2)2}}{x^{(1)2}}\right)^{1/2} = x^{(1)} + \frac{x^{(2)2}}{2x^{(1)}} + O(x^{(2)4}) \\ &= x^{(1)} + \frac{x^{(2)2}}{2M} + O\left(x^{(2)4} + x^{(2)2}|x^{(1)} - M|\right) = x^{(1)} + \frac{x^{(2)2}}{2M} + O(c^{-1.2n})\,, \end{aligned}$$

hence

$$\begin{aligned} &\exp\left\{g_n(|x| - M) - \left(\frac{g_n}{2M} - A_n\right) x^{(2)2}\right\} \\ &\qquad = \exp\left\{g_n(x^{(1)} - M) + A_n x^{(2)2} + O(c^{-0.2n})\right\}\,. \end{aligned}$$

The above calculation shows that on the boundary of the domains Ω_n^1 and Ω_n^2 the right-hand side of formulas (2.7) and (2.8) have the same magnitude. The estimate (2.8) expresses the fact that this is the worst region, where the weakest upper bound can be given for $f_n(x)$ in Ω_n^2.

With the help of Proposition 1, Lemma 1 and Theorem A we are able to carry out a limiting procedure which supplies Theorem 1. Moreover, it yields the following Proposition 2. Let $\bar\mu_n$ denote the projection of the measure $\bar\mu$ constructed in Theorem 1 to the first 2^n coordinates, i.e. let $\bar\mu_n$ be the measure on $(R^2)^{2^n}$ defined by the relation $\bar\mu_n(A) = \bar\mu\left(A \times (R^2)^\infty\right)$ for all measurable sets $A \subset (R^2)^{2^n}$. The following result holds true:

Proposition 2. *For all q, $c^{-0.2} < q < 1$, there is some $n_0 = n_0(c,T,M,q)$ such that for all $n \ge n_0$ the measure $\bar\mu_n$ is absolute continuous with respect to the measure μ_n, and its Radon–Nikodym derivative satisfies the relations:*

$$\frac{d\bar\mu_n}{d\mu_n}(x_1,\dots,x_{2^n}) = \bar f_n\left(2^{-n}\sum_{i=1}^{2^n} x_i\right) \tag{2.10}$$

and
a) For $x \in \Omega_n^1$

$$\bar f_n(x) = L_n \exp\left\{c^n \bar g(x^{(1)} - M) + c^n \bar A x^{(2)2} + \varepsilon_n(x)\right\} \tag{2.11}$$

with

$$\sup_{x\in\Omega_n^1} |\varepsilon_n(x)| \le q^n . \tag{2.11$'$}$$

b) For $x \in \Omega_n^2$

$$0 \le \bar f_n(x) \le L_n \exp\left\{c^n \bar g(|x| - M) - c^{0.2n}\left(\frac{\bar g}{2M} - \bar A\right) + q^n\right\} . \tag{2.12}$$

c) For $x \in \Omega_n^3$

(2.13)

$$0 \le \bar f_n(x) \le L_n \exp\left\{\frac{\bar g c^n}{4M}(x^2 - M^2)\right\} \qquad \text{if } 0 < x < M - c^{-0.4n} ,$$

(2.13′)

$$0 \le \bar f_n(x) \le L_n \exp\left\{\frac{\bar g c^n}{M}(x^2 - M^2)\right\} \qquad \text{if } x > M + c^{-0.4n} ,$$

with $\bar g = \frac{2}{2-c}\frac{M}{T}$, $\bar A = \frac{2-c}{cT}$ and an appropriate norming constant L_n. This norming constant satisfies the relation

$$C_1 < c^{-n/2} L_n < C_2 \qquad \text{with some } 0 < C_1 < C_2 < \infty .$$

Theorem 2 can be deduced from Proposition 2 and Theorem A.

Let us finally remark that the function $f_n(x) = f_{n,N}^{h_N}(x)$ clearly satisfies Proposition 1 for $n = N$, since in this case $f_N(x) = L_N \exp\{g_N(x^{(1)} - M)\}$. Hence Proposition 1 follows from Lemma 1 and the following

Proposition 1′. *For all q, $c^{-0.2} < q < 1$, there exists some $n_0 = n_0(T,M,c,D,q)$ such that if for $n \ge n_0$ the function $f(x)$ satisfies the following relations with some $\bar g c^{n+1} < g_{n+1} \le D c^{n+1}$, $0 \le A_{n+1} \le \bar A c^{n+1}$, $\bar g = \frac{2}{2-c}\frac{M}{T}$, $\bar A = \frac{2-c}{cT}$, $D > \bar g$:*
a) For $x \in \Omega_{n+1}^1$

$$f(x) = \exp\left\{g_{n+1}(x^{(1)} - M) + A_{n+1} x^{(2)2} + \varepsilon_{n+1}(x)\right\} \tag{2.15}$$

$$\sup_{x\in\Omega_{n+1}^1} |\varepsilon_{n+1}(x)| \le q^{n+1} . \tag{2.15$'$}$$

b) For $x \in \Omega^2_{n+1}$

$$0 \le f(x) \le \exp\left\{g_{n+1}(|x|-M) + \left(\frac{g_{n+1}}{2M} - A_{n+1}\right) c^{-0.8(n+1)} + q^{n+1}\right\}\,; \tag{2.16}$$

c) For $x \in \Omega^3_{n+1}$

$$0 \le f(x) \le \exp\left\{\frac{g_{n+1}}{2M}(|x|^2 - M^2)\right\} \tag{2.17}$$

then the function $S_n f(x)$ *defined by (2.3′) satisfies, with the constants* g_n *and* A_n *defined by (2.4′) and (2.5′) with the above* g_{n+1} *and* A_{n+1}*, the following relations with some appropriate norming constant* L_n*:*
a) In the domain Ω^1_n

$$S_n f(x) = L_n \exp\left\{g_n(x^{(1)} - M) + A_n x^{(2)2} + \varepsilon_n(x)\right\} \tag{2.18}$$

with

$$\sup_{x\in\Omega^1_n} |\varepsilon_n(x)| \le q^n\,. \tag{2.18′}$$

b) In the domain Ω^2_n

$$0 \le S_n f(x) \le L_n \exp\left\{g_n(|x|-M) - \left(\frac{g_n}{2M} - A_n\right) c^{-0.8n} + q^n\right\}\,. \tag{2.19}$$

c) In the domain Ω^3_n

$$0 \le S_n f(x) \le L_n \exp\left\{\frac{g_n}{2M}(|x|^2 - M^2)\right\}\,. \tag{2.20}$$

3. The proof of Lemma 1. The proof is a modification of that given for Lemma 1 in [5]. Simple calculation shows that $\bar g_n - \bar g = (\frac{c}{2})^{N-n}(\bar g_N - \bar g)$. The statements of Lemma 1 about the sequence g_n follow from this identity. To investigate $\bar A_n$ let us introduce the function

$$T(x,g) = \frac{c}{4}x + \frac{\left(\frac{1}{T} + \frac{c}{2}x\right)^2}{\frac{2}{T} + \frac{cg}{M} - cx}\,, \quad x \in R^1\,, \quad g \in R^1\,.$$

Clearly, $\bar A_n = T(\bar A_{n+1}, \bar g_{n+1})$. On the other hand $T(\bar A, \bar g) = \bar A$, and some calculation shows that T has the following monotonicity properties: $T(x,g) < T(x,g')$ if $0 < x < \bar A$ and $g > g' > \bar g$; and $T(x',g) > T(x,g)$ if $0 < x < x' < A$ and $g > \bar g$. (These properties follow e.g. from the relations

$$\frac{\partial}{\partial g} T(x,g) = -\frac{c}{M}\frac{\left(\frac{1}{T} + \frac{c}{2}x\right)^2}{\left(\frac{2}{T} + \frac{c}{M}g - cx\right)^2} < 0\,,$$

and

$$\frac{\partial}{\partial x}T(x,g)=\frac{c\left(\frac{2}{T}+\frac{cg}{2M}\right)^2}{\left(\frac{2}{T}+\frac{c}{M}g-cx\right)^2}>0\,,$$

and the fact that $T(x,g)$ has no singularity in the domain $\{(x,g),\, 0<x<\bar A,\, g>\bar g\}$.) We have $0<\bar A_{N-1}<\bar A$, since $\bar A_{N-1}=T(0,\bar g_N)>0$ and $\bar A_{N-1}=T(0,\bar g_N)<T(\bar A,\bar g)=\bar A$. Then we get by induction that $0\le \bar A_{n+1}<\bar A_n<\bar A$ implies that $0<\bar A_n<\bar A_{n-1}<\bar A$. Indeed, $\bar A_{n-1}=T(\bar A_n,\bar g_n)<T(\bar A,\bar g)=\bar A$, and $\bar A_{n-1}=T(\bar A_n,\bar g_n)>T(\bar A_{n+1},\bar g_{n+1})=\bar A_n$, as we have claimed.

The conditions $N>N_0$ and $N>n^B$ with sufficiently large N_0 and B imply that $|\bar g_\ell-\bar g|<10^{-n}$ for all $0<\ell\le\sqrt{N}$, and $\bar A-5^{-n}<A^*<\bar A$, where A^* is the smaller solution of the equation $T(x,g^*)=x$ with $g^*=\bar g_{\sqrt{N}}$. Indeed, the last equation is a small perturbation of the equation $T(x,\bar g)=x$, which has two solutions $A_1=\bar A$ and $A_2=\frac{1}{(2-c)T}>\bar A$. Hence the solutions of the equation $T(x,g^*)=x$ are very close to A_1 and A_2. We claim that the monotonicity properties of the sequences $\bar g_n$ and $\bar A_n$ and the function $T(x,g)$ imply that $\bar A>\bar A_n\ge T_{g^*}^{\sqrt{N}-n}(0)$, where $T_{g^*}^k$ denotes the k-th iteration of the function $T(x,g^*)$ with fixed g^* in the variable x. Indeed, $\bar A>\bar A_n$, $0<\bar A_{\sqrt{N}}<\bar A$, and we get by induction that for all $\ell\ge 0$ $\bar A_{\sqrt{N}-\ell}\ge T_{g^*}^{\ell}(0)$, which implies the required statement with $\ell=\sqrt{N}-n$.

To complete the proof of Lemma 1 it is enough to show that $T_{g^*}^n(0)$ tends exponentially fast in n to the smaller solution A^* of the equation $T_{g^*}(x)=x$. Since $T(x,g^*)$ is a convex increasing function (in the variable x) it is enough to show that $\frac{\partial T(x,g^*)}{\partial x}\le\alpha<1$ for $x=A^*$ if (A^*,g^*) is in a small neighbourhood of the point $(\bar A,\bar g)$. But this follows from the continuity of the function $\frac{\partial T(x,g)}{\partial x}$, and the fact that its value in the point $(\bar A,\bar g)$ equals $c^{-1}<1$. Lemma 1 is proved.

4. Some preparatory remarks to the proof of Proposition 1′. We shall prove the following estimates under the conditions of Proposition 1′.

Put

$$S_n^i f(x)=\int_{\{y,\,\frac{x+y}{2}\in\Omega_{n+1}^i\}}\exp\left(\frac{c^n}{T}xy\right)f\left(\frac{x+y}{2}\right)p_n(y)\,dy\,,\quad i=1,2,3\,.$$

Then we have with some appropriate $\varepsilon'=\varepsilon'(c)$, $\varepsilon'>0$, and the same q as in Proposition 1′:

In the domain $x\in\Omega_n^1$

$$S_n^1 f(x)=L_n\exp\left\{g_n(x^{(1)}-M)+A_n x^{(2)2}+\bar\varepsilon_n(x)+\hat\varepsilon_n(x)\right\}\tag{4.1}$$

with

$$\sup_{x\in\Omega_n^1}|\bar\varepsilon_n(x)|\le q^{n+1}\,,\qquad \sup_{x\in\Omega_n^1}|\hat\varepsilon_n(x)|\le Kc^{-0.2n}\,,\tag{4.1′}$$

where K does not depend on n,

$$L_n \geq c^{-n} \exp\left(\frac{c^n}{T} M^2\right), \tag{4.1''}$$

$$S_n^2 f(x) \leq L_n \exp\left\{g_n(x^{(1)} - M) + A_n x^{(2)2} - \varepsilon' c^{0.2n}\right\}, \tag{4.2}$$

$$S_n^3 f(x) \leq L_n \exp\left\{g_n(x^{(1)} - M) + A_n x^{(2)2} - \frac{1}{6} c^{n/2}\right\}. \tag{4.3}$$

In the domain $x \in \Omega_n^2$

(4.4)
$$S_n^1 f(x) \leq L_n \exp\left\{g_n(|x| - M) - \left(\frac{g_n}{2M} - A_n\right) c^{-0.8n} + q^{n+1} + K c^{-0.2n}\right\},$$

(4.5)
$$S_n^2 f(x) \leq L_n \exp\left\{g_n(|x| - M) - \left(\frac{g_n}{2M} - A_n\right) c^{-0.8n} - \varepsilon' c^{0.2n}\right\},$$

(4.6)
$$S_n^3 f(x) \leq L_n \exp\left\{g_n(|x| - M) - \left(\frac{g_n}{2M} - A_n\right) c^{-0.8n} - \frac{1}{6} c^{n/2}\right\}.$$

In the domain $x \in \Omega_n^3$

$$S_n f(x) \leq L_n \exp\left\{\frac{g_n}{2M}(|x|^2 - M^2)\right\}. \tag{4.7}$$

We show that these estimates imply Proposition $1'$. Indeed, for $x \in \Omega_n^1$

$$S_n f(x) = S_n^{(1)} f(x) + S_n^{(2)} f(x) + S_n^{(3)} f(x) = L_n \exp\{g_n(x^{(1)} - M) + A_n x^{(2)2} + \varepsilon_n(x)\}$$

with

$$\sup_{x \in \Omega_n^1} |\varepsilon_n(x)| \leq \sup_{x \in \Omega_n^1} |\bar{\varepsilon}_n(x)| + \sup_{x \in \Omega_n^1} |\hat{\varepsilon}_n(x)| + 2\exp(-\varepsilon' c^{0.2n}) + 2\exp\left(-\frac{1}{6} c^{n/2}\right).$$

(Here we have exploited that $e^t < 1 + 2|t|$ for small t.) Hence

$$\sup_{x \in \Omega_n^1} |\varepsilon_n(x)| \leq q^{n+1} + K c^{-0.2n} + 2\exp(-\varepsilon' c^{0.2n}) + 2\exp\left(-\frac{1}{6} c^{n/2}\right) \leq q^n$$

if $c^{-0.2} < q < 1$, and $n \geq n_0(q, D, \varepsilon')$.

For $x \in \Omega_n^2$ we have analogously

$$\begin{aligned} S_n f(x) &= S_n^1 f(x) + S_n^2 f(x) + S_n^3 f(x) \\ &\leq L_n \exp\Bigg\{ g_n(|x| - M) - \left(\frac{g_n}{2M} - A_n\right) c^{-0.8n} + q^{n+1} + K c^{-0.2n} \\ &\qquad + 2\exp(-\varepsilon' c^{0.2n}) + 2\exp\left(-\frac{1}{6} c^{n/2}\right)\Bigg\} \\ &\leq L_n \exp\left\{g_n(|x| - M) - \left(\frac{g_n}{2M} - A_n\right) c^{-0.8n} + q^n\right\}, \end{aligned}$$

as we have claimed in Proposition $1'$. For $x \in \Omega_n^3$ (4.7) contains the needed estimate.

The above estimates will be proved in the next Section. In this Section we prove two lemmas which we need during the proof. Put

$$S_n^\varepsilon f(x) = \int_{R^2 - V_n^\varepsilon(x)} \exp\left(\frac{c^n}{T} xy\right) f_n\left(\frac{x+y}{2}\right) p_n(y)\, dy \tag{4.8}$$

with

$$V_n^\varepsilon(x) = \left\{ y \in R^2\,, \quad \big||y| - M\big| \le \varepsilon c^{-0.4n}\,, \quad xy \ge |x||y| - \varepsilon c^{-0.4n} \right\}\,, \tag{4.8$'$}$$

where xy denotes scalar product.

Lemma 2. *There is some $\varepsilon_0 = \varepsilon_0(c)$ and $n_0 = n_0(T, M, D, c, \varepsilon)$ such that if $n \ge n_0$, $0 < \varepsilon < \varepsilon_0$, $0 \le g_{n+1} < Dc^{n+1}$ and*

$$0 \le f(x) \le \exp\left\{ \frac{g_{n+1}}{2M} (|x|^2 - M^2) \right\} \qquad \text{for all } x \in R^2 \tag{4.9}$$

then

a) $$0 \le S_n f(x) \le c^n \exp\left\{ \frac{c^n}{T} M^2 + \frac{g_n}{2M}(|x|^2 - M^2) - \frac{c^n}{3T}(|x| - M)^2 \right\}\,, \quad x \in R^2$$

and

b) $$0 \le S_n^\varepsilon f(x) \le \exp\left\{ \frac{c^n}{T} M^2 + g_n(|x| - M) - c^{n/2} \right\} \qquad \text{if } \big||x| - M\big| < c^{-0.4n}$$

with $g_n = \frac{g_{n+1}}{2} + \frac{c^n}{T} M$.

Lemma 3. *There is some $n_0 = n_0(T, M, D, c)$ such that if for $n \ge n_0$*

$$\begin{aligned} 0 \le f(x) &\le \exp\{g_{n+1}(|x| - M)\} && \text{for } \big||x| - M\big| < c^{-0.4(n+1)} \\ f(x) &= 0 && \text{for } \big||x| - M\big| \ge c^{-0.4(n+1)} \end{aligned}$$

then

$$0 \le S_n f(x) \le K \exp\left\{ \frac{c^n}{T} M^2 + g_n(|x| - M) \right\} \qquad \text{for } \ ||x| - M| \le c^{-0.4n}$$

with

$$g_n = \frac{g_{n+1}}{2} + \frac{c^n}{T} M$$

and some $K = K(T, M, D, c) > 0$.

Proof of Lemma 2.
Part a). We have

$$0 \le S_n f(x) \le \int \exp\left\{ \frac{c^n}{T} xy + \frac{g_{n+1}}{2M} \left(\left| \frac{x+y}{2} \right|^2 - M^2 \right) \right\} p_n(y)\, dy\,.$$

Clearly, $\max_{|y|=r} xy$ is taken in the point $y=\frac{x}{|x|}r$, and it equals $|x|r$. Similarly $\max_{|y|=r}(x+y)^2=(|x|+r)^2$. Hence

(4.10)
$$0\le S_nf(x)\le 2\pi\int_0^\infty \exp\left\{\frac{c^n}{T}|x|r+\frac{g_{n+1}}{2M}\left[\left(\frac{|x|+r}{2}\right)^2-M^2\right]\right\} r\bar p_n(r)\,dr\,.$$

Let us split up the integral into two parts, $\int_0^M$ and $\int_M^\infty$. Put

$$f_n(t)=c^{-n}\bar p_n(M+c^{-n}t)\,. \tag{4.11}$$

It follows from (1.10) that

$$f_n(t)\le C_1\exp(-C_2|t|)\qquad \text{for } -c^nM<t<0 \tag{4.11$'$}$$

and from (1.8), (1.7) and (1.9) that

$$f_n(t)\le C_3\exp(-t^2)\qquad \text{for } t>0 \tag{4.11$''$}$$

with some appropriate $C_1>0$, $C_2>0$ and $C_3>0$. (Relations (1.7) and (1.9) are needed in the domain $0<|x|<\eta n^{1/\alpha}c^{-n}$.)

First we estimate the integral $\int_0^M$. For $0\le r\le M$ we have

$$\frac{c^n}{T}|x|r+\frac{g_{n+1}}{2M}\left(\left(\frac{|x|+r}{2}\right)^2-M^2\right)\le \frac{c^n}{T}|x|M+\frac{g_{n+1}}{2M}\left(\left(\frac{|x|+M}{2}\right)^2-M^2\right)\,.$$

Hence (4.11) and (4.11$'$) imply that

$$\begin{aligned}\int_0^M \dots\,dr &\le C_1\exp\left\{\frac{c^n}{T}|x|M+\frac{g_{n+1}}{2}\left[\left(\frac{|x|+M}{2}\right)^2-M^2\right]\right\}\\ &\qquad\int_{-c^nM}^0 (M+c^{-n}t)e^{-C_2|t|}\,dt\\ &\le C_4M\exp\left\{\frac{c^n}{T}|x|M+\frac{g_{n+1}}{2M}\left[\left(\frac{|x|+M}{2}\right)^2-M^2\right]\right\}\,.\end{aligned}$$

Simple calculation yields the identity

$$\begin{aligned}&\frac{c^n}{T}|x|M+\frac{g_{n+1}}{2M}\left[\left(\frac{|x|+M}{2}\right)^2-M^2\right]\\ &\qquad=\frac{c^n}{T}M^2+\frac{g_n}{2M}(|x|^2-M^2)-\left(\frac{c^n}{2T}+\frac{g_{n+1}}{8M}\right)(|x|-M)^2\,.\end{aligned}$$

Then, since $C_4M<c^{n/2}$ we get that

(4.12)
$$\int_0^M \dots\,dr\le c^{n/2}\exp\left\{\frac{c^n}{T}M^2+\frac{g_n}{2M}(|x|^2-M^2)+\left(\frac{c^n}{2T}+\frac{g_{n+1}}{8M}\right)(|x|-M)^2\right\}\,.$$

Let us estimate $\int_M^\infty \dots\, dr$. We make the change of variables $r = M + c^{-n}t$, introduce $f_n(t) = c^{-n}\bar p_n(M + c^{-n}t)$, $\bar g_n = c^{-n} g_n$ and $\bar g_{n+1} = c^{-(n+1)} g_{n+1}$. Since

(4.13)
$$\begin{aligned}
&\frac{c^n}{T}|x|r + \frac{g_{n+1}}{2M}\left[\left(\frac{|x|+r}{2}\right)^2 - M^2\right] \\
&= \frac{c^n}{T}|x|M + \frac{g_{n+1}}{2M}\left[\left(\frac{|x|+M}{2}\right)^2 - M^2\right] \\
&\quad \frac{c^n}{T}|x|(r-M) + \frac{g_{n+1}}{8M}(r-M)(2|x|+r+M) \\
&= \frac{c^n}{T}M^2 + \frac{g_n}{2M}(|x|^2 - M^2) - \left(\frac{c^n}{2T} + \frac{g_{n+1}}{8M}\right)(|x|-M)^2 \\
&\quad + \left(\frac{c^n}{T} + \frac{g_{n+1}}{4M}\right)|x|(r-M) + \frac{g_{n+1}}{4}(r-M) + \frac{g_{n+1}}{8M}(r-M)^2
\end{aligned}$$

hence
(4.14)
$$\int_M^\infty \dots\, dr = \exp\left\{\frac{c^n}{T}M^2 + \frac{g_n}{2M}(|x|^2 - M^2) - \left(\frac{c^n}{2T} + \frac{g_{n+1}}{8M}\right)(|x|-M)^2\right\} J_n(|x|)$$

with

(4.14′)
$$\begin{aligned}
J_n(|x|) &= \int_0^\infty \exp\Big\{\left(\frac{1}{T} + \frac{c}{4M}\bar g_{n+1}\right)|x|t \\
&\qquad + \frac{c}{4}\bar g_{n+1}t + c\frac{c^{-n}}{8M}\bar g_{n+1}t^2\Big\}(M + c^{-n}t) f_n(t)\, dt \\
&= \int_0^\infty \exp\Big\{\left(\frac{1}{T} + \frac{c}{4M}\bar g_{n+1}\right)(|x|-M)t \\
&\qquad + \left(\frac{M}{T} + \frac{c}{2}\bar g_{n+1}\right)t + \frac{c^{-n+1}}{8M}\bar g_{n+1}t^2\Big\}(M + c^{-n}t) f_n(t)\, dt\,.
\end{aligned}$$

Relation (4.11″) implies that

$$\begin{aligned}
J_n(|x|) &\le C_5 \int_0^\infty \exp\left\{\left(\frac{1}{T} + \frac{c}{4M}\bar g_{n+1}\right)(|x|-M)t - \frac{t^2}{2}\right\} dt \\
&\le C_6\{\exp C_7(|x|-M)^2\}\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_M^\infty \dots\, dr \le C_6 \exp\Big\{&\frac{c^n}{T}M^2 + \frac{g_n}{2M}(|x|^2 - M^2) \\
&- \left(\frac{c^n}{2T} + \frac{g_{n+1}}{8M}\right)(|x|-M)^2 + C_7(|x|-M)^2\Big\}\,.
\end{aligned}$$

Since $\frac{c^n}{2T}+\frac{g_{n+1}}{8M}-C_7\ge\frac{c^n}{3T}$ and $C_6\le c^{n/2}$ for large n, hence

$$\int_M^\infty\dots dr\le c^{n/2}\exp\left\{\frac{c^n}{T}M^2+\frac{g_n}{2M}(|x|^2-M^2)-\frac{c^n}{3T}(|x|-M)^2\right\}.$$

This inequality together with (4.12) imply that

(4.15)

$$S_nf(x)\le 4\pi c^{n/2}\exp\left\{\frac{c^n}{T}M^2+\frac{g_n}{2M}(|x|^2-M^2)-\frac{c^n}{3T}(|x|-M)^2\right\}\le$$

$$\le c^n\exp\left\{\frac{c^n}{T}M^2+\frac{g_n}{2M}(|x|^2-M^2)-\frac{c^n}{3T}(|x|-M)^2\right\}$$

as we have claimed.

Part b). Let us introduce

$$R_n^\varepsilon f(x)=\int_{\{y,\,||y|-M|\ge\varepsilon c^{-0.4n}\}}\exp\left(\frac{c^n}{T}xy\right)f\left(\frac{x+y}{2}\right)p_n(y)\,dy$$

and

$$Q_n^\varepsilon f(x)=\int_{\{y,\,||y|-M|<\varepsilon c^{-0.4n},\,xy\le|x||y|-\varepsilon c^{-0.4n}\}}\dots\,dy\,.$$

Clearly, $S_n^\varepsilon f(x)=R_n^\varepsilon f(x)+Q_n^\varepsilon f(x)$, and by (4.10)

$$0\le R_n^\varepsilon f(x)\le 2\pi\left[\int_0^{M-\varepsilon c^{-0.4n}}+\int_{M+\varepsilon c^{-0.4n}}^\infty\right]$$
$$\exp\left\{\frac{c^n}{T}|x|r+\frac{g_{n+1}}{2M}\left(\left(\frac{|x|+r}{2}\right)^2-M^2\right)\right\}r\bar p_n(r)\,dr\,.$$

Moreover, similarly to part a), we get by using (4.13) and the observation

$$\left(\frac{c^n}{T}+\frac{g_{n+1}}{4M}\right)|x|(r-M)+\frac{g_{n+1}}{4}(r-M)+\frac{g_{n+1}}{8M}(r-M)^2$$
$$<\frac{g_{n+1}}{4}(r-M)\big(1+\frac{r-M}{2M}\big)<-2c^{n/2}\quad\text{if }-M<r-M<-c^{-0.4n}$$

that

(4.16)

$$\int_0^{M-\varepsilon c^{-0.4n}}\dots\,dr$$
$$\le C_1\exp\left\{-2c^{n/2}+\frac{c^n}{T}M^2+\frac{g_n}{2M}(|x|^2-M^2)-\left(\frac{c^n}{2T}+\frac{g_{n+1}}{8M}\right)(|x|-M)^2\right\}$$
$$\int_0^{M-\varepsilon c^{-0.4n}}r\bar p_n(r)\,dr$$
$$\le c^{-n/2}\exp\left\{-2c^{n/2}+\frac{c^n}{T}M^2+\frac{g_n}{2M}(|x|^2-M^2)\right.$$
$$\left.-\left(\frac{c^n}{2T}+\frac{g_{n+1}}{8M}\right)(|x|-M)^2\right\}$$
$$\le c^{-n/2}\exp\left\{-2c^{n/2}+\frac{c^n}{T}M^2+\frac{g_n}{2M}(|x|^2-M^2)\right\},$$

and by (4.13) and (4.11″), similarly to (4.14), (4.14′)

$$\int_{M+\varepsilon c^{-0.4n}}^{\infty} \dots\, dr \le \exp\Big\{\frac{c^n}{T}M^2+\frac{g_n}{2M}(|x|^2-M^2)-\Big(\frac{c^n}{2T}+\frac{g_{n+1}}{8M}\Big)(|x|-M)^2\Big\}J_n^{\varepsilon}(|x|)$$

with

$$J_n^{\varepsilon}(|x|)=C_5\int_{\varepsilon c^{0.6n}}^{\infty} c^n \exp\Big\{\Big(\frac{1}{T}+\frac{c}{4M}\bar g_{n+1}\Big)(|x|-M)t+\Big(\frac{M}{T}+\frac{c}{2}\bar g_{n+1}\Big)t-\frac{t^2}{2}\Big\}\,dt\,.$$

Since $||x|-M|\le c^{-0.4n}$, and $\bar g_{n+1}<D$

$$\sup_{t\ge 0}\Big\{\Big(\frac{1}{T}+\frac{c}{4M}\bar g_{n+1}\Big)(|x|-M)t+\Big(\frac{M}{T}+\frac{c}{2}\bar g_{n+1}\Big)t-\frac{t^2}{6}\Big\}\le C'\,,$$

and therefore

$$J_n^{\varepsilon}(|x|)\le C_6\int_{\varepsilon c^{0.6n}}^{\infty} c^n\exp\Big(-\frac{t^2}{3}\Big)\,dt\le \exp(-\varepsilon' c^{1.2n})\le c^{-n}\exp(-2c^{n/2})$$

with some $\varepsilon'=\varepsilon'(\varepsilon)>0$. Hence

$$\int_{M+\varepsilon c^{-0.4n}}^{\infty}\dots\, dr\le c^{-n}\exp\Big\{\frac{c^n}{T}M^2+\frac{g_n}{2M}(|x|^2-M^2)-2c^{n/2}\Big\}.$$

The last inequality together with (4.16) imply that

$$R_n^{\varepsilon}f(x)\le 4\pi c^{-n/2}\exp\Big\{-2c^{n/2}+\frac{c^n}{T}M^2+\frac{g_n}{2M}(|x|^2-M^2)\Big\}. \tag{4.17}$$

Now we estimate $Q_n^{\varepsilon}f(x)$. We have

$$0\le Q_n^{\varepsilon}f(x)\le\int_{\substack{\{y,||y|-M|\le c^{-0.4n},\\ xy\le|x||y|-\varepsilon c^{-0.4n}\}}}\exp\Big\{\frac{c^n}{T}xy+\frac{g_{n+1}}{2M}\Big[\Big(\frac{x+y}{2}\Big)^2-M^2\Big]\Big\}p_n(y)\,dy\,.$$

Since for $|y|=r$ $\;xy\le r|x|-\varepsilon c^{-0.4n}$ and $\big(\frac{x+y}{2}\big)^2\le\big(\frac{|x|+r}{2}\big)^2$ in the last integral, hence

$$\begin{aligned}Q_n^{\varepsilon}f(x)&\le 2\pi\int_0^{\infty}\exp\Big\{\frac{c^n}{T}\left(|x|r-\varepsilon c^{-0.4n}\right)+\frac{g_{n+1}}{2M}\Big[\Big(\frac{|x|+r}{2}\Big)^2-M^2\Big]\Big\}r\bar p_n(r)\,dr\\ &\le 2\pi\exp\Big(-\frac{\varepsilon c^{0.6n}}{T}\Big)\int_0^{\infty}\exp\Big\{\frac{c^n}{T}|x|r+\frac{g_{n+1}}{2M}\Big[\Big(\frac{|x|+r}{2}\Big)^2-M^2\Big]\Big\}\\ &\qquad r\bar p_n(r)\,dr\,.\end{aligned}$$

The last integral has already appeared in (4.10). We have estimated it in Part a), and bounded it by the right hand side of (4.15). Hence

$$\begin{aligned}Q_n^{\varepsilon}f(x)&\le 2\pi c^n\exp\Big\{-\frac{\varepsilon}{T}c^{0.6n}+\frac{c^n}{T}M^2+\frac{g_n}{2M}\left(|x|^2-M^2\right)\Big\}\\ &\le\frac12\exp\Big\{-2c^{n/2}+\frac{c^n}{T}M^2+\frac{g_n}{2M}\left(|x|^2-M^2\right)\Big\}.\end{aligned}$$

This inequality together with (4.17) imply that

$$S_n^\varepsilon f(x) \le \exp\left\{-2c^{n/2} + \frac{c^n}{T} M^2 + \frac{g_n}{2M}\left(|x|^2 - M^2\right)\right\}.$$

Since $\frac{1}{2M}(|x|^2 - M^2) = (|x| - M) + \frac{1}{2M}(|x| - M)^2$ and $\frac{g_n}{2M}(|x| - M)^2 \le Dc^{0.2n}$ if $||x| - M| \le c^{-0.4n}$, hence the last inequality implies that under the conditions of Part b)

$$\begin{aligned} S_n^\varepsilon f(x) &\le \exp\left\{-2c^{n/2} + \frac{c^n}{T} M^2 + \frac{g_n}{2M}(|x| - M) + Kc^{0.2n}\right\} \\ &\le \exp\left\{\frac{c^n}{T} M^2 + g_n(|x| - M) - c^{n/2}\right\}, \end{aligned}$$

as we have claimed.

Proof of Lemma 3.

$$0 \le S_n f(x) \le \int_{\{y,\, ||\frac{x+y}{2}| - M| \le c^{-0.4(n+1)}\}} \exp\left\{\frac{c^n}{T} xy + g_{n+1}\left(\left|\frac{x+y}{2}\right| - M\right)\right\} p_n(y)\, dy\,.$$

Since

$$\max_{|y|=r}\left\{\frac{c^n}{T} xy + g_{n+1}\left(\left|\frac{x+y}{2}\right| - M\right)\right\} \le \frac{c^n}{T}|x|r + g_{n+1}\left[\left(\frac{|x|+r}{2}\right) - M\right],$$

hence

$$S_n f(x) \le 2\pi \int_0^\infty \exp\left\{\frac{c^n}{T}|x|r + g_{n+1}\left(\frac{|x|+r}{2} - M\right)\right\} r\bar p_n(r)\, dr\,.$$

Writing $\frac{c^n}{T}|x|r = \frac{c^n}{T}M^2 + \frac{c^n}{T}M(|x| - M + r - M) + \frac{c^n}{T}(|x| - M)(r - M)$ we get that

$$\begin{aligned} S_n f(x) \le 2\pi \exp&\left\{\frac{c^n}{T} M^2 + g_n(|x| - M)\right\} \\ &\int_0^\infty \exp\left\{g_n(r - M) + \frac{c^n}{T}(|x| - M)(r - M)\right\} r\bar p_n(r)\, dr\,. \end{aligned}$$

The change of variables $r = M + c^{-n}t$ and the introduction of $\bar g_n = c^{-n} g_n$ yields that

$$\begin{aligned} S_n f(x) \le 2\pi \exp&\left\{\frac{c^n}{T} M^2 + g_n(|x| - M)\right\} \\ &\int_{-c^n M}^\infty \exp\left\{\bar g_n t + \frac{(|x| - M)}{T} t\right\} (M + c^{-n}t) f_n(t)\, dt\,. \end{aligned}$$

Since $0 < \bar{g}_n < D$, and $||x| - M| < c^{-0.4n}$ relations (4.11′) and (4.11″) imply that for large n

$$\int_{-c^n M}^{\infty} \exp\left\{\bar{g}_n t + \frac{1}{T}(|x| - M)t\right\}(M + c^{-n}t)f_n(t)\,dt < \bar{K}$$

with some $\bar{K} > 0$ independent of n. Hence

$$\begin{aligned} S_n f(x) &\le 2\pi\bar{K}\exp\left\{\frac{c^n}{T}M^2 + g_n(|x| - M)\right\} \\ &\le K\exp\left\{\frac{c^n}{T}M^2 + g_n(|x| - M)\right\}, \end{aligned}$$

as we have claimed.

5. The proof of Proposition 1′. In this Section we prove the estimates (4.1)–(4.7) which imply Proposition 1′.

a) The estimation of $S_n^1 f(x)$ for $x \in \Omega_n^1$.
It follows from (2.15) and (2.15′) that

$$S_n^1 f(x) = \int_{\{y,\ \frac{x+y}{2} \in \Omega_{n+1}^1\}} \exp\left\{\frac{c^n}{T}xy + g_{n+1}\left(\frac{x^{(1)} + y^{(1)}}{2} - M\right) + A_{n+1}\left(\frac{x^{(2)} + y^{(2)}}{2}\right)^2 + \varepsilon_{n+1}\left(\frac{x+y}{2}\right)\right\}p_n(y)\,dy\,.$$

Hence

$$S_n^1 f(x) = \exp(\bar{\varepsilon}_n(x)) \int_{\{y,\ \frac{x+y}{2} \in \Omega_{n+1}^1\}} \exp\left\{\frac{c^n}{T}xy + g_{n+1}\left(\frac{x^{(1)} + y^{(1)}}{2} - M\right) + A_{n+1}\left(\frac{x^{(2)} + y^{(2)}}{2}\right)^2\right\}p_n(y)\,dy\,.$$

with some

(5.1) $$\sup_{x \in \Omega_n^1} |\bar{\varepsilon}_n(x)| \le q^{n+1}\,.$$

Let us rewrite the last expression in polar coordinate system. We get that

(5.2) $$S_n^1 f(x) = \exp(\bar{\varepsilon}_n(x)) \int_0^{\infty} I_n(r)\bar{p}_n(r)\,dr$$

with

$$I_n(r) = \int_{\{\varphi \in \Gamma_n(r,x)\}} r\exp\left\{\frac{c^n}{T}\left(x^{(1)}y^{(1)} + x^{(2)}y^{(2)}\right) + g_{n+1}\left(\frac{x^{(1)} + y^{(1)}}{2} - M\right) + A_{n+1}\left(\frac{x^{(2)} + y^{(2)}}{2}\right)^2\right\}d\varphi$$

where $y^{(1)} = r\cos\varphi$, $y^{(2)} = r\sin\varphi$, $-\pi < \varphi < \pi$, $y = (y^{(1)}, y^{(2)})$ and $\Gamma_n(r,x) = \{\varphi,\ \frac{x+y}{2} \in \Omega^1_{n+1}\}$. We shall express $I_n(r)$ as an asymptotically Gaussian integral with respect to φ. For this aim we give some bounds on $x^{(1)}$, $x^{(2)}$, $y^{(1)}$ and $y^{(2)}$ if $x \in \Omega^1_n$ and $\frac{x+y}{2} \in \Omega^1_{n+1}$. We have

$$\begin{aligned} &|x^{(2)}| < c^{-0.4n} \\ &|x^{(1)} - M| < \frac{3}{2} c^{-0.4n} \end{aligned} \qquad \text{if} \quad x \in \Omega^1_n\,. \tag{5.3}$$

The second relation in (5.3) holds, since

$$\begin{aligned} |x^{(1)} - M| &= \left|(|x|^2 - x^{(2)2})^{1/2} - M\right| = \left| |x| - \frac{x^{(2)2}}{2|x|} + O(x^{(2)4}) - M \right| \\ &\le \left||x| - M\right| + \left|\frac{x^{(2)2}}{2|x|} + O(x^{(2)4})\right| \le c^{-0.4n} + Kc^{-0.8n} \le \frac{3}{2}c^{-0.4n}\,. \end{aligned}$$

Similarly, if $\frac{x+y}{2} \in \Omega^1_{n+1}$ then $|\frac{x^{(2)}+y^{(2)}}{2}| < c^{-0.4n}$, and $|\frac{x^{(1)}+y^{(1)}}{2} - M| < 2c^{-0.4n}$. Hence

$$|y^{(2)}| \le |-x^{(2)}| + |x^{(2)} + y^{(2)}| \le 3c^{-0.4n}, \tag{5.4}$$

$$|y^{(1)} - M| = |M - x^{(1)}| + |x^{(1)} + y^{(1)} - 2M| \le 6c^{-0.4n} \tag{5.4$'$}$$

and

(5.4$''$)

$$\begin{aligned} |r - M| = ||y| - M| &= \left|(y^{(1)2} + y^{(2)2})^{1/2} - M\right| \\ &= \left|y^{(1)} + \frac{y^{(2)2}}{2y^{(1)}} + O(y^{(2)4}) - M\right| \le 10c^{-0.4n} \end{aligned}$$

if $x \in \Omega^1_n$ and $\frac{x+y}{2} \in \Omega^1_{n+1}$. In particular, (5.4$''$) implies that

$$I_n(r) = 0 \qquad \text{if } |r - M| > 10c^{-0.4n}, \text{ and } x \in \Omega^1_n\,. \tag{5.5}$$

Furthermore, $|\varphi| \le 2|\sin\varphi| = \frac{2}{r}|y^{(2)}| = O(c^{-0.4n})$ and $y^{(1)} = r(1-\cos\varphi)$ $= r(1-\varphi^2/2) + O(c^{-1.6n})$, $y^{(2)} = r\sin\varphi = r\varphi + O(c^{-1.2n})$ and $r = M + O(c^{-0.4n})$ if $\varphi \in \Gamma_n(r,x)$. Hence

$$\begin{aligned} I_n(r) &= \left(1 + O(c^{-0.2n})\right) \int_{\{\varphi \in \Gamma_n(r,x)\}} M \exp\left\{ \frac{c^n}{T}\left(x^{(1)} r\left(1 - \frac{\varphi^2}{2}\right) + x^{(2)} r\varphi\right) \right. \\ &\qquad \left. + g_{n+1}\left(\frac{x^{(1)} + r(1 - \frac{\varphi^2}{2})}{2} - M\right) + A_{n+1}\left(\frac{x^{(2)} + r\varphi}{2}\right)^2 \right\} d\varphi \\ &= \left(1 + O(c^{-0.2n})\right) M \exp\left\{ \frac{c^n}{T} r x^{(1)} + \frac{g_{n+1}}{2}\left(\frac{x^{(1)} + r}{2} - M\right) + \frac{A_{n+1}}{4} x^{(2)2} \right\} \\ &\qquad \int_{\{\varphi \in \Gamma_n(r,x)\}} \exp\left\{ -\left(\frac{c^n}{2T} x^{(1)} + \frac{g_{n+1}}{4} - r\frac{A_{n+1}}{4}\right) r\varphi^2 \right. \\ &\qquad\qquad \left. + \left(\frac{c^n}{T} + \frac{A_{n+1}}{2}\right) x^{(2)} r\varphi \right\} d\varphi\,. \end{aligned}$$

Moreover, since $c^n x^{(1)} r\varphi^2 = c^n M^2\varphi^2 + O(c^{-0.2n})$, $g_{n+1} r\varphi^2 = g_{n+1} M\varphi^2 + O(c^{-0.2n})$, $A_{n+1} r^2\varphi^2 = A_{n+1} M^2\varphi^2 + O(c^{-0.2n})$ and $(\frac{c^n}{T} + \frac{A_{n+1}}{2})x^{(2)} r\varphi = (\frac{c^n}{T} + \frac{A_{n+1}}{2})x^{(2)} M\varphi + O(c^{-0.2n})$ in our case (observe that $g_{n+1} \le Dc^{n+1}$ and $A_{n+1} \le \bar{A}c^{n+1}$), hence we make an error of order $O(c^{-0.2n})$ by substituting $x^{(1)}$ and r by M in the last integral, i.e.

$$I_n(r) = \left(1 + O(c^{-0.2n})\right) M \exp\left\{\frac{c^n}{T} r x^{(1)} + \frac{g_{n+1}}{2}\left(\frac{x^{(1)} + r}{2} - M\right) + \frac{A_{n+1}}{4} x^{(2)2}\right\}$$
$$\int_{\{\varphi\in\Gamma_n(r,x)\}} \exp\left\{(\frac{c^n}{2T} + \frac{g_{n+1}}{4M} - \frac{A_{n+1}}{4}) M^2\varphi^2 + (\frac{c^n}{T} + \frac{A_{n+1}}{2}) M x^{(2)}\varphi\right\} d\varphi\,,$$

or equivalently

$$I_n(r) = \left(1 + O(c^{-0.2n})\right) \exp\left\{\frac{c^n}{T} r x^{(1)} + \frac{g_{n+1}}{2}\left(\frac{x^{(1)} + r}{2} - M\right)\right.$$
$$\left. + \frac{A_{n+1}}{4} x^{(2)2} + \frac{(\frac{c^n}{T} + \frac{A_{n+1}}{2})^2 x^{(2)2}}{4(\frac{c^n}{2T} + \frac{g_{n+1}}{4M} - \frac{A_{n+1}}{4})}\right\}$$
$$\int_{\{\varphi\in\Gamma_n(r,x)\}} M \exp\left\{-M^2(\frac{c^n}{2T} + \frac{g_{n+1}}{4M} - \frac{A_{n+1}}{4})(\varphi - \gamma_n x^{(2)})^2\right\} d\varphi\,,$$

with

$$\gamma_n = \frac{(\frac{c^n}{T} + \frac{A_{n+1}}{2})}{(\frac{c^n}{2T} + \frac{g_{n+1}}{4M} - \frac{A_{n+1}}{4})}\,.$$

By $(2.5')$ we get from this relation that

(5.6)

$$I_n(r) = \left(1 + O(c^{-0.2n})\right) \exp\left\{\frac{c^n}{T} r x^{(1)} + \frac{g_{n+1}}{2}\left(\frac{x^{(1)} + r}{2} - M\right) + A_n x^{(2)2}\right\}$$
$$\int_{\{\varphi\in\Gamma_n(r,x)\}} M \exp\left\{-M^2(\frac{c^n}{2T} + \frac{g_{n+1}}{4M} - \frac{A_{n+1}}{4})\left(\varphi - \gamma_n x^{(2)}\right)^2\right\} d\varphi\,.$$

Since

$$(5.7)\qquad (\frac{c^n}{2T} + \frac{g_{n+1}}{4M} - \frac{A_{n+1}}{4}) \ge c^n\left(\frac{1}{2T} + c\frac{\bar{g}}{4M} - c\frac{\bar{A}}{4}\right) = Kc^n$$

with $K = \frac{1}{2T} + c\frac{\bar{g}}{4M} - c\frac{\bar{A}}{4} = \frac{c(4-c)}{4(2-c)T} > 0$ relation (5.6) implies that

$$(5.8)\qquad I_n(r) \le \sqrt{\pi}\left(1 + O(c^{-0.2n})\right)(\frac{c^n}{2T} + \frac{g_{n+1}}{4M} - \frac{A_{n+1}}{4})^{-1/2}$$
$$\exp\left\{\frac{c^n}{T} r x^{(1)} + \frac{g_{n+1}}{2}\left(\frac{x^{(1)} + r}{2} - M\right) + A_n x^{(2)2}\right\}$$

Moreover, we claim that

(5.9)

$$I_n(r) = \left(1+O(c^{-0.2n})\right)\sqrt{\pi}\,\left(\frac{c^n}{2T}+\frac{g_{n+1}}{4M}-\frac{A_{n+1}}{4}\right)^{-1/2} \exp\left\{\frac{c^n}{T}rx^{(1)}+\frac{g_{n+1}}{2}\left(\frac{x^{(1)}+r}{2}-M\right)+A_nx^{(2)2}\right\} \quad \text{if } |r-M|<\bar\varepsilon c^{-0.4n}$$

with some approppriate $\bar\varepsilon>0$. Because of (5.6) and (5.7) to prove (5.9) it is enough to show that there is some $\varepsilon=\varepsilon_0(\bar\varepsilon)>0$ such that

$$\{\varphi\colon |\varphi-\gamma_nx^{(2)}|<\varepsilon_0c^{-0.4n}\}\subset\Gamma_n(x,r) \qquad \text{if } |r-M|<\bar\varepsilon c^{-0.4n}\,. \tag{5.10}$$

Since $A_{n+1}\le\bar Ac^{n+1}$ and $g_{n+1}\ge\bar gc^{n+1}$

$$\gamma_n=\frac{(\frac{c^n}{T}+\frac{A_{n+1}}{2})}{2(\frac{c^n}{2T}+\frac{g_{n+1}}{4M}-\frac{A_{n+1}}{4})}\le\frac{1+\frac{2-c}{2}}{(1+\frac{c}{2-c}-\frac{2-c}{2})M}=\frac{2-c}{cM}\,,$$

and we prove (5.10) by showing that

$$\left\{\frac{x^{(1)}+r\cos\varphi}{2},\ \frac{x^{(2)}+r\sin\varphi}{2}\right\}\in\Omega^1_{n+1} \tag{5.10$'$}$$

if $|\varphi-\gamma_nx^{(2)}|<\varepsilon_0c^{-0.4n}$, $|r-M|\le\bar\varepsilon c^{-0.4n}$ and $x\in\Omega^1_n$. But in this case

$$\begin{aligned}\left|\frac{x^{(2)}+r\sin\varphi}{2}\right|&\le\frac12\left(|x^{(2)}|+r|\varphi|\right)\le\frac12\left(|x^{(2)}|+r(\gamma_n|x^{(2)}|+|\varphi-\gamma_nx^{(2)}|)\right)\\&\le\frac12\left[c^{-0.4n}+(M+\bar\varepsilon c^{-0.4n})\left(\frac{2-c}{cM}c^{-0.4n}+\varepsilon_0c^{-0.4n}\right)\right]\\&\le\frac{1}{\sqrt c}c^{-0.4n}\le c^{-0.4(n+1)}\end{aligned}$$

if $\varepsilon_0>0$ and $\bar\varepsilon>0$ are sufficiently small. We also get with the help of (5.3′) that

$$\begin{aligned}\left|\frac{x^{(1)}+r\cos\varphi}{2}-M\right|&\le\left|\frac{x^{(1)}-M}{2}\right|+\frac r2|\cos\varphi-1|+\left|\frac{r-M}{2}\right|\\&\le\left|\frac{x^{(1)}-M}{2}\right|+\frac{r\varphi^2}{2}+\left|\frac{r-M}{2}\right|\\&\le\left|\frac{x^{(1)}-M}{2}\right|+\frac r2\left(\gamma_n|x^{(2)}|+\varepsilon_0c^{-0.4n}\right)^2+\left|\frac{r-M}{2}\right|\\&\le\left(\frac34+\bar\varepsilon\right)c^{-0.4n}+Kc^{-0.8n}\le c^{-0.1}c^{-0.4(n+1)}\end{aligned}$$

if $\varepsilon_0>0$ and $\bar\varepsilon>0$ are sufficiently small. The above estimates imply (5.10′) hence also (5.10). Now we can estimate the term $\int_0^\infty I_n(r)\bar p_n(r)\,dr$. Relation (5.9) yields that

$$\int_{|r-M|<\bar\varepsilon c^{-0.4n}}I_n(r)\bar p_n(r)\,dr=\left(1+O(c^{-0.2n})\right)\sqrt{\pi}\left(\frac{c^n}{2T}+\frac{g_{n+1}}{4M}-\frac{A_{n+1}}{4}\right)^{-1/2}J_{n,\bar\varepsilon} \exp\left\{\frac{c^n}{T}x^{(1)}M+\frac{g_{n+1}}{4}\left(x^{(1)}-M\right)+A_nx^{(2)2}\right\}$$

with

$$J_{n,\bar\varepsilon} = J_{n,\bar\varepsilon}(x^{(1)}) = \int_{|r-M|<\bar\varepsilon c^{-0.4n}} \exp\left\{\frac{c^n}{T}x^{(1)}(r-M) + \frac{g_{n+1}}{4}(r-M)\right\}\bar p_n(r)\,dr$$
$$= \int_{|t|<\bar\varepsilon c^{0.6n}} \exp\left\{\frac{tx^{(1)}}{T} + c\frac{\bar g_{n+1}}{4}t\right\} f_n(t)\,dt$$

with the function f defined in (4.11). On the other hand, by (5.8) and (5.5)

$$\int_{|r-M|>\bar\varepsilon c^{-0.4n}} I_n(r)\bar p_n(r)\,dr$$
$$\le \left(1+O(c^{-0.2n})\right)\sqrt{\pi}\left(\frac{c^n}{2T} + \frac{g_{n+1}}{4M} - \frac{A_{n+1}}{4}\right)^{-1/2}$$
$$\exp\left\{\frac{c^n}{T}x^{(1)}M + \frac{g_{n+1}}{4}(x^{(1)}-M) + A_n x^{(2)2}\right\}\bar J_{n,\bar\varepsilon}$$

with

$$\bar J_{n,\bar\varepsilon} = \int_{10c^{0.6n}>|t|>\bar\varepsilon c^{0.6n}} \exp\left\{\frac{tx^{(1)}}{T} + c\frac{\bar g_{n+1}}{4}t\right\} f_n(t)\,dt\,.$$

Let us remark that $J_{n,\bar\varepsilon} = J_{n,\bar\varepsilon}(x^{(1)})$ depends on $x^{(1)}$. We show that this dependence is very weak. Namely, since

$$\left|\frac{d}{dx^{(1)}}J_{n,\bar\varepsilon}(x^{(1)})\right| = \int_{|t|<\bar\varepsilon c^{0.6n}} \exp\left\{\frac{tx^{(1)}}{T} + c\frac{\bar g_{n+1}}{4}t\right\}\frac{|t|}{T}f_n(t)\,dt \le C < \infty$$

by (4.11′) and (4.11″), and for $x^{(1)} = M$ the expression

$$J_{n,\bar\varepsilon}(M) = \int_{|t|<\bar\varepsilon c^{0.6n}} \exp\left\{\frac{tM}{T} + c\frac{\bar g_{n+1}}{4}t\right\} f_n(t)\,dt$$

satisfies the relation

$$0 < K_1 < J_{n,\bar\varepsilon}(M) < K_2 < \infty \tag{5.11}$$

because of (4.11′), (4.11″), the inequality $0 \le \bar g_{n+1} \le D$ and the relation $f_n(t) \ge$ *const.* > 0 for $|t| < 1$ that follows from Theorem A. Hence

$$J_{n,\bar\varepsilon}(x^{(1)}) = \left(1+O(c^{-0.4n})\right)J_{n,\bar\varepsilon}(M) \qquad \text{if } x \in \Omega_n^1\,.$$

Similarly,

$$0 \le \bar J_{n,\bar\varepsilon} = O\left(\exp(-Kc^{0.6n})\right)\,.$$

The above relations imply that

$$\int I_n(r)\bar p_n(r)\,dr = \left(1+O(c^{-0.2n})\right)\sqrt{\pi}\left(\frac{c^n}{2T} + \frac{g_{n+1}}{4M} - \frac{A_{n+1}}{4}\right)^{-1/2}J_{n,\bar\varepsilon}(M)$$
$$\exp\left\{\frac{c^n}{T}M^2 + g_n(x^{(1)}-M) + A_n x^{(2)2}\right\}\,.$$

The last formula together with (5.1) and (5.2) imply (4.1) with

$$L_n = \sqrt{\pi}\left(\frac{c^n}{2T} + \frac{g_{n+1}}{4M} - \frac{A_{n+1}}{4}\right)^{-1/2}J_{n,\bar\varepsilon}(M)\exp\left(\frac{c^n}{T}M^2\right)\,.$$

Since $\left(\frac{c^n}{2T} + \frac{g_{n+1}}{4M} - \frac{A_{n+1}}{4}\right) <$ *const.*c^n relation (5.11) and the last formula imply (4.1″).

b) The estimation of $S_n^1 f(x)$ for $x \in \Omega_n^2$.

We divide Ω_n^2 to two subsets $\bar\Omega_n^2$ and $\hat\Omega_n^2$, where we apply different arguments. Put

$$\Omega_{n,\varepsilon}^1 = \left\{x, \big||x|-M\big| \le c^{-0.4n}, |x^{(2)}| \le (1+\varepsilon)c^{-0.4n},\, x^{(1)}>0\right\},\ \varepsilon>0,$$
$$\bar\Omega_n^2 = \Omega_{n,\varepsilon}^1 - \Omega_n^1\,,$$

and

$$\hat\Omega_n^2 = \left\{x, \big||x|-M\big| \le c^{-0.4n}\right\} - \Omega_{n,\varepsilon}^1\,.$$

Clearly, $\Omega_n^2 = \bar\Omega_n^2 \cup \hat\Omega_n^2$. The domain $\Omega_{n,\varepsilon}^1$ is a slight enlargement of Ω_n^1. It is not difficult to see by analizing the proof of relation (4.1) that for sufficiently small $\varepsilon>0$

$$S_n^1 f(x) = L_n \exp\left\{g_n(x^{(1)}-M) + A_n x^{(2)2} + \bar\varepsilon(x) + \hat\varepsilon(x)\right\}$$

if $x \in \Omega_{n,\varepsilon}^1$ with some $|\bar\varepsilon_n(x)| < q^{n+1}$ and $\hat\varepsilon_n(x) < Kc^{-0.2n}$. Since $x^{(1)}-M = |x|-M-\frac{x^{(2)2}}{2M}+O(c^{-1.2n})$ and $|x^{(2)}| \ge c^{-0.4n}$ for $x \in \bar\Omega_n^2$, the above relation implies that

$$\begin{aligned} S_n^1 f(x) = L_n \exp\bigg\{ & g_n(|x|-M) - \left(\frac{g_n}{2M}-A_n\right)x^{(2)2} \\ & + \bar\varepsilon_n(x) + \hat\varepsilon_n(x) + O\left(c^{-0.2n}\right) biggr\} \\ \le L_n \exp & \left\{g_n(|x|-M) - \left(\frac{g_n}{2M}-A_n\right)c^{-0.8n} + q^{n+1} + Kc^{-0.2n}\right\} \end{aligned}$$

in this case, what we had to show. For $x \in \hat\Omega_n^2$ we define

$$S_n^\varepsilon f(x) = \int_{R^2 - V_n^\varepsilon(x)} \exp\left(\frac{c^n}{T}xy\right) f\left(\frac{x+y}{2}\right) p_n(y)\,dy$$

and

$$T_n^\varepsilon f(x) = \int_{\{y,\ \frac{x+y}{2}\in\Omega_{n+1}^1,\ y\in V_n^\varepsilon(x)\}} \exp\left(\frac{c^n}{T}xy\right) f\left(\frac{x+y}{2}\right) p_n(y)\,dy\,,$$

where $V_n^\varepsilon(x)$ is defined in (4.8′), and $\varepsilon>0$ is appropriately chosen. The function $S_n^\varepsilon f(x)$ will be bounded with the help of Part b) of Lemma 2, and $T_n^\varepsilon f(x)$ similarly to $S_n^1 f(x)$ in the case $x \in \Omega_n^1$. To apply Lemma 2 first we show that under the conditions of Proposition 1′

$$(5.12) \qquad 0 \le f(x) \le 2\exp\left\{\frac{g_{n+1}}{2M}\left(|x|^2-M^2\right)\right\} \qquad \text{for all } x \in R^2\,.$$

For $x \in \Omega_{n+1}^1$

$$x^{(1)}-M = |x|-M-\frac{x^{(2)2}}{2|x|}+O\left(x^{(2)4}\right) = |x| - \frac{x^{(2)2}}{2M} - M + O\left(c^{-1.2n}\right)\,,$$

hence

$$0 \le f(x) \le \exp\left\{ g_{n+1}(|x|-M) - \left(\frac{g_{n+1}}{2M} - A_{n+1}\right) x^{(2)2} + \varepsilon_{n+1}(x) + O(c^{-0.2n}) \right\} ,$$

and since $\frac{g_{n+1}}{2M} - A_{n+1} > 0$ hence

$$0 \le f(x) \le \frac{3}{2} \exp\{g_{n+1}(|x|-M)\} .$$

This inequality together with the relation

$$|x| - M = \frac{1}{2M}\left[(|x|^2 - M^2) - (|x|-M)^2\right] \le \frac{1}{2M}\left(|x|^2 - M^2\right)$$

imply (5.12) for $x \in \Omega^1_{n+1}$. Similarly, for $x \in \Omega^2_{n+1}$ the relation

$$\begin{aligned} 0 \le f(x) &\le \exp\left\{ g_{n+1}(|x|-M) - \left(\frac{g_{n+1}}{2M} - A_{n+1}\right) c^{-0.8(n+1)} + q^{n+1} \right\} \\ &\le \frac{3}{2} \exp\{g_n(|x|-M)\} \end{aligned}$$

implies (5.12), and this relation also holds for $x \in \Omega^3_{n+1}$ by relation (2.17).

By (5.12) part b) of Lemma 2 can be applied for $\frac{1}{2}f(x)$, and it yields that

$$S^\varepsilon_n f(x) \le 2\exp\left\{ \frac{c^n}{T} M^2 + g_n(|x|-M) - c^{n/2} \right\} \qquad \text{if } x \in \Omega^2_n \ \ (\text{or } x \in \Omega^1_n) .$$

Since $L_n \ge c^{-n}\exp\{\frac{c^n}{T}M^2\}$, and $(\frac{g_n}{2M} - A_n)c^{-0.8n} = O(c^{0.2n}) \le \frac{1}{3}c^{n/2}$, the last inequality implies that

(5.13)
$$S^\varepsilon_n f(x) \le L_n \exp\left\{ g_n(|x|-M) - \left(\frac{g_n}{2M} - A_n\right) c^{-0.8n} - \frac{1}{2}c^{n/2} \right\} \qquad \text{if } x \in \Omega^2_n \ \ (\text{or } x \in \Omega^1_n) .$$

To estimate $T^\varepsilon_n f(x)$ first we show that if $x \in \Omega^2_n$, $y \in V^\varepsilon_n(x)$ and $\frac{x+y}{2} \in \Omega^1_{n+1}$ then

(5.14)
$$\begin{aligned} &|x^{(2)}| < 2\sqrt{\varepsilon}c^{-0.2n}, \ \ |y^{(2)}| < 2\sqrt{\varepsilon}c^{-0.2n}, \ \ |x^{(1)} - M| \le 2\sqrt{\varepsilon}c^{-0.2n}, \\ &|y^{(1)} - M| \le 2\sqrt{\varepsilon}c^{-0.2n}, \quad \big||y| - M\big| \le \varepsilon c^{-0.4n} . \end{aligned}$$

Indeed, in this case $(x,y) \ge |x||y| - \varepsilon c^{-0.4n}$, and

$$\begin{aligned} 0 \le |x-y|^2 &= |x|^2 + |y|^2 - 2(x,y) \le |x|^2 + |y|^2 - 2|x||y| + \varepsilon c^{-0.4n} \\ &= (|x| - |y|)^2 + \varepsilon c^{-0.4n} \le 2(|x|-M)^2 + 2(|y|-M)^2 + \varepsilon c^{-0.4n} \\ &\le 2(1+\varepsilon)c^{-0.8n} + \varepsilon c^{-0.4n} \le 2\varepsilon c^{-0.4n} . \end{aligned}$$

Hence $|\frac{x^{(2)}-y^{(2)}}{2}| \le \sqrt{\varepsilon}c^{-0.2n}$ and $\frac{1}{2}|(x^{(1)}-M)-(y^{(1)}-M)| \le \sqrt{\varepsilon}c^{-0.2n}$. Since $\frac{x+y}{2}\in\Omega^1_{n+1}$, $|\frac{x^{(2)}+y^{(2)}}{2}| \le c^{-0.4(n+1)} \le \sqrt{\varepsilon}c^{-0.2n}$ and $\frac{1}{2}|(x^{(1)}-M)+(y^{(1)}-M)| \le \frac{3}{2}c^{-0.4(n+1)} \le \sqrt{\varepsilon}c^{-0.2n}$ by (5.3). These relations together with the definition of $V^\varepsilon_n(x)$ imply (5.14) that enables us to estimate $T^\varepsilon_n f(x)$ similarly to $S^1_n f(x)$ for $x\in\Omega^1_n$.

We get that

$$T^\varepsilon_n f(x) \le (1+q^{n+1})\int_0^\infty I_n(r)\bar{p}_n(r)\,dr \tag{5.15}$$

with

$$\begin{aligned} I_n(r) = \int_{\{\varphi\in\Gamma_n(r,x)\}} r\exp\biggl\{\frac{c^n}{T}\left(x^{(1)}y^{(1)}+x^{(2)}y^{(2)}\right) \\ + g_{n+1}\left(\frac{x^{(1)}+y^{(1)}}{2}-M\right)A_{n+1}\left(\frac{x^{(2)}+y^{(2)}}{2}\right)^2\biggr\}\,d\varphi\,, \end{aligned} \tag{5.16}$$

where $y^{(1)}=r\cos\varphi$, $y^{(2)}=r\sin\varphi$, $y=(y^{(1)},y^{(2)})$ and

$$\Gamma_n(r,x)=\{\varphi\colon \frac{x+y}{2}\in\Omega^1_{n+1}, y\in V^\varepsilon_n(x)\}.$$

Observe that by (5.14) $|r-M|<\varepsilon c^{-0.4n}$, $|\varphi|\le 2|\sin\varphi| = \frac{2}{r}|y^{(2)}| \le \frac{2}{r}\sqrt{\varepsilon}c^{-0.2n}$. Let us make the change of variables $z=\sin\varphi$ in the integral $I_n(r)$. We have $y^{(2)}=rz$, $y^{(1)}=r(1-\frac{y^{(2)2}}{r^2})^{1/2} = r(1-\frac{y^{(2)2}}{2r^2})+O(y^{(2)4})$, $z\le 2\frac{\sqrt{\varepsilon}}{r}c^{-0.2n}$, hence $|y^{(1)}-r(1-\frac{z^2}{2})|\le K\varepsilon^2c^{-0.8n}$ with some $K>0$ independent of ε. These relations imply that if $\varphi\in\Gamma_n(r,x)$ then

$$\begin{aligned} &\frac{c^n}{T}\left(x^{(1)}y^{(1)}+x^{(2)}y^{(2)}\right)+g_{n+1}\left(\frac{x^{(1)}+y^{(1)}}{2}-M\right)+A_{n+1}\left(\frac{x^{(2)}+y^{(2)}}{2}\right)^2 \\ &\quad\le \frac{c^n}{T}r\left(x^{(1)}\left(1-\frac{z^2}{2}\right)+x^{(2)}z\right)+g_{n+1}\left(\frac{x^{(1)}+r(1-\frac{z^2}{2})}{2}-M\right) \\ &\qquad + A_{n+1}\left(\frac{x^{(2)}+rz}{2}\right)^2 + K\varepsilon^2c^{0.2n} \end{aligned}$$

with some $K>0$. Since $\frac{dz}{d\varphi}>\frac{1}{2}$ and $|r-M|<\varepsilon c^{-0.4n}$ if $\Gamma_n(r,x)$ is not empty, the above inequality together with (5.16) imply that

$$I_n(r)=0 \qquad \text{if } |r-M|\ge\varepsilon c^{-0.4n} \tag{5.17}$$

and

$$I_n(r)\le 3\exp(K\varepsilon^2c^{0.2n})\hat{I}_n(r) \qquad \text{for } |r-M|<\varepsilon c^{-0.4n} \tag{5.17$'$}$$

with

$$\hat{I}_n(r) = \int_{-\infty}^{\infty} M \exp\left\{ \frac{c^n}{T} r \left(x^{(1)} \left(1 - \frac{z^2}{2} \right) + x^{(2)} z \right) \right.$$
$$\left. + g_{n+1} \left(\frac{x^{(1)} + r(1 - \frac{z^2}{2})}{2} - M \right) + A_{n+1} \left(\frac{x^{(1)} + rz}{2} \right)^2 \right\} dz\,.$$

The expression $\hat{I}_n(r)$ can be calculated explicitly, and we get that for $|r - M| < \varepsilon c^{-0.4n}$

$$\hat{I}_n = \sqrt{\pi} M \left(\frac{c^n}{2T} r x^{(1)} + \frac{g_{n+1}}{4} r - \frac{A_{n+1}}{4} r^2 \right)^{-1/2}$$
$$\exp\left\{ \frac{c^n x^{(1)} r}{T} + g_{n+1} \left(\frac{x^{(1)} + r}{2} - M \right) + A_n(x^{(1)}, r) x^{(2)2} \right\}$$

with

$$A_n(x^{(1)}, r) = \frac{A_{n+1}}{4} + \frac{(\frac{c^n}{T} + \frac{A_{n+1}}{2})^2}{2(\frac{c^n x^{(1)}}{Tr} + \frac{g_{n+1}}{2r} - \frac{A_{n+1}}{2})}\,.$$

Hence

$$\hat{I}_n(r) \le K c^{-n/2} \exp\left\{ \frac{c^n x^{(1)} r}{T} + g_{n+1} \left(\frac{x^{(1)} + r}{2} - M \right) + A_n(x^{(1)}, r) x^{(2)2} \right\}$$
$$= K c^{-n/2} \exp\left\{ \frac{c^n}{T} M^2 + g_n(x^{(1)} - M) + A_n(x^{(1)}, r) x^{(2)2} \right.$$
$$\left. + \left(\frac{c^n x^{(1)}}{T} + \frac{g_{n+1}}{2} \right)(r - M) \right\}.$$

Observe that $A_n = A_n(M, M)$, and if $x \in \hat{\Omega}_n^2$, $\; y \in V_n^\varepsilon(x)$ and $\frac{x+y}{2} \in \Omega_n^1$ then

$$|A_n(x^{(1)}, r) - A_n| \le K_0 c^n (|x^{(1)} - M| + |r - M|)$$
$$\left| x^{(1)} - M - \left(|x| - M - \frac{x^{(2)2}}{2M} \right) \right| \le K_1 x^{(2)2} (x^{(2)2} + |x^{(1)} - M|)\,,$$

hence (5.17′) implies that
(5.18)

$$I_n(r) \le \bar{K} c^{-n/2} \exp\left\{ K \varepsilon^2 c^{0.2n} + \frac{c^n}{T} M^2 + g_n(x^{(1)} - M) + A_n x^{(2)2} \right.$$
$$\left. + K_0 c^n (|x^{(1)} - M| + |r - M|) x^{(2)2} + \left(\frac{c^n x^{(1)}}{T} + \frac{g_{n+1}}{2} \right)(r - M) \right\}$$
$$\le \bar{K} c^{-n/2} \exp\left\{ K \varepsilon^2 c^{0.2n} + \frac{c^n}{T} M^2 + g_n(|x| - M) - \left(\frac{g_n}{2M} - A_n \right) x^{(2)2} \right.$$
$$+ \left[K_0 c^n (|x^{(1)} - M| + |r - M|) + K_1 g_n (x^{(2)2} + |x^{(1)} - M|) \right] x^{(2)2}$$
$$\left. + \left(\frac{c^n x^{(1)}}{T} + \frac{g_{n+1}}{2} \right)(r - M) \right\}$$

if $|r-M|\le \varepsilon c^{-0.4n}$. Since $\frac{g_n}{2M}-A_n>\alpha c^n$ with some $\alpha>0$ and $x^{(2)2}>(1+\varepsilon)c^{-0.4n}$ if $x\in\hat\Omega_n^2$

$$\begin{aligned}K\varepsilon^2c^{0.2n}&+\left[K_0c^n(|x^{(1)}-M|+|r-M|)+K_1g_n(x^{(2)2}+|x^{(1)}-M|)\right]x^{(2)2}\\&-\left(\frac{g_n}{2M}-A_n\right)x^{(2)2}\\&=K\varepsilon^2c^{0.2n}-\left(\frac{g_n}{2M}-A_n+O\left(c^{0.8n}\right)\right)x^{(2)2}\\&\le K\varepsilon^2c^{0.2n}-\left(\frac{g_n}{2M}-A_n+O\left(c^{0.8n}\right)\right)(1+\varepsilon)^2c^{-0.8n}\\&\le-\left(\frac{g_n}{2M}-A_n\right)(1+\varepsilon)c^{-0.8n}\end{aligned}$$

if $\varepsilon>0$ is chosen sufficiently small. This relation together with (5.18) imply that

$$\begin{aligned}I_n(r)\le Kc^{-n/2}\exp\biggl\{&\frac{c^n}{T}M^2+g_n(|x|-M)-(1+\varepsilon)\left(\frac{g_n}{2M}-A_n\right)c^{-0.8n}\\&+\left(\frac{c^nx^{(1)}}{T}+\frac{g_{n+1}}{2}\right)(r-M)\biggr\}\quad\text{if } |r-M|<\varepsilon c^{-0.4n}\,.\end{aligned}$$

With the help of this inequality, the estimate we have on the function $\bar p_n(r)$ and (5.17) we can bound the integral in (5.15) from above. We get that

$$T_n^\varepsilon f(x)\le L_n\exp\left\{g_n(|x|-M)-\left(1+\frac{\varepsilon}{2}\right)\left(\frac{g_n}{2M}-A_n\right)c^{-0.8n}\right\}.$$

Since $S_n^1f(x)\le S_n^\varepsilon f(x)+T_n^\varepsilon f(x)$ the last inequality together with (5.13) imply (4.4) for $x\in\hat\Omega_n^2$.

c) The estimation of $S_n^2f(x)$ for $x\in\Omega_n^1$ and $x\in\Omega_n^2$.
We have

$$\begin{aligned}0\le S_n^2f(x)&=\int_{\{y,\,\frac{x+y}{2}\in\Omega_{n+1}^2\}}\exp\left(\frac{c^n}{T}xy\right)f\left(\frac{x+y}{2}\right)p_n(y)\,dy\\&\le\exp\left\{-\left(\frac{g_{n+1}}{2M}-A_{n+1}\right)c^{-0.8(n+1)}+q^{n+1}\right\}\\&\qquad\int_{\{y,\,\frac{x+y}{2}\in\Omega_{n+1}^2\}}\exp\left\{\frac{c^n}{T}xy+g_{n+1}\left(\frac{|x+y|}{2}-M\right)\right\}p_n(y)\,dy\,.\end{aligned}$$

It follows from Lemma 3 that for $x\in\Omega_n^1\cup\Omega_n^2$

$$\begin{aligned}0\le S_n^2f(x)\le K\exp\biggl\{&-\left(\frac{g_{n+1}}{2M}-A_{n+1}\right)c^{-0.8(n+1)}\\&+q^{n+1}+\frac{c^n}{T}M^2+g_n(|x|-M)\biggr\}.\end{aligned}$$

By Lemma 1 $g_{n+1}c^{-(n+1)} \geq g_n c^n$, $A_n c^{-n} \geq A_{n+1}c^{-(n+1)}$ i.e. $g_{n+1} \geq c g_n$ and $cA_n \geq A_{n+1}$. Hence

$$\left(\frac{g_{n+1}}{2M} - A_{n+1}\right)c^{-0.8(n+1)} \geq \left(\frac{g_n}{2M} - A_n\right)c^{-0.8n}c^{0.2} \geq (1+\varepsilon)\left(\frac{g_n}{2M} - A_n\right)c^{-0.8n}$$

if $0 < \varepsilon < c^{0.2} - 1$. Since $\varepsilon(\frac{g_n}{2M} - A_n)c^{-0.8n} > \varepsilon' c^{0.2n}$ with some appropriate $\varepsilon'(\varepsilon) > 0$ we get that

$$\left(\frac{g_{n+1}}{2M} - A_{n+1}\right)c^{-0.8(n+1)} \geq \left(\frac{g_n}{2M} - A_n\right)c^{-0.8n} + \varepsilon' c^{0.2n}$$

Therefore

(5.19)

$$\begin{aligned} S_n^2 f(x) &\leq K \exp\left\{\frac{c^n}{T}M^2 + g_n(|x| - M) - \left(\frac{g_n}{2M} - A_n\right)c^{-0.8n} + q^{n+1} - \varepsilon' c^{0.2n}\right\} \\ &\leq L_n \exp\left\{g_n(|x| - M) - \left(\frac{g_n}{2M} - A_n\right)c^{-0.8n} - \frac{\varepsilon'}{2}c^{0.2n}\right\}. \end{aligned}$$

This is estimate (4.5) (with $\frac{\varepsilon'}{2}$ instead of ε'). For $x \in \Omega_n^1$

$$g_n(x^{(1)} - M) + A_n x^{(2)2} = g_n(|x| - M) - \left(\frac{g_n}{2M} - A_n\right)x^{(2)2} + O(c^{-0.2n}),$$

hence (5.19) implies that for $x \in \Omega_n^1$

$$S_n^2 f(x) \leq L_n \exp\left\{g_n(x^{(1)} - M) + A_n x^{(2)2} - \frac{\varepsilon'}{4}c^{0.2n}\right\},$$

and this is relation (4.2) (with $\frac{\varepsilon'}{4}$).

d) The estimation of $S_n^3 f(x)$ for $x \in \Omega_n^1$ and $x \in \Omega_n^2$.
Clearly

$$S_n^3 f(x) = S_n^\varepsilon f(x) + \int_{\{y,\ \frac{x+y}{2} \in \Omega_{n+1}^3\} \cap V_n^\varepsilon(x)} \exp\left(\frac{c^n}{T}xy\right) f\left(\frac{x+y}{2}\right) p_n(y)\, dy,$$

where $S_n^\varepsilon f(x)$ and V_n^ε are defined in (4.8) and (4.8′). The term $S_n^\varepsilon f(x)$ is bounded in (5.13). On the other hand, we claim that there is some $\varepsilon_0 = \varepsilon_0(c)$ such that if $0 < \varepsilon < \varepsilon_0$ and $x \in \Omega_n^1 \cup \Omega_n^2$ then the set $\{y, \frac{x+y}{2} \in \Omega_{n+1}^3\} \cap V_n^\varepsilon(x)$ is empty, hence the last integral is zero. We have to show that if $\big||x| - M\big| < c^{-0.4n}$, $y \in V_n^\varepsilon$, i.e. $\big||y| - M\big| < \varepsilon c^{-0.4n}$ and $xy > |x||y| - \varepsilon c^{-0.4n}$ then $\frac{x+y}{2} \notin \Omega_{n+1}^3$, i.e. $-c^{-0.4(n+1)} \leq |\frac{x+y}{2}| - M \leq c^{-0.4(n+1)}$.

Estimation from above:

$$\left|\frac{x+y}{2}\right| - M \leq \frac{|x| - M}{2} + \frac{|y| - M}{2} \leq \frac{1+\varepsilon}{2}c^{-0.4n} \leq c^{-0.4(n+1)}$$

if $\frac{1+\varepsilon}{2} \le c^{-0.4}$, what holds for sufficiently small ε.

Estimation from below:

$$\begin{aligned}\left|\frac{x+y}{2}\right|^2 - M^2 &= \frac{|x|^2+|y|^2+2xy}{4} - M^2 \\ &\ge \frac{|x|^2+|y|^2+2|x||y|-2\varepsilon c^{-0.4n}}{4} - M^2 = \left(\frac{|x|+|y|}{2}\right)^2 - M^2 - \frac{\varepsilon}{2}c^{-0.4n} \\ &\ge \left(\frac{M-c^{-0.4n}+M-\varepsilon c^{-0.4n}}{2}\right)^2 - M^2 - \frac{\varepsilon}{2}c^{-0.4n} \\ &\ge -(1+\varepsilon)Mc^{-0.4n} - \frac{\varepsilon}{2}c^{-0.4n} = -M\left(1+\varepsilon+\frac{\varepsilon}{2M}\right)c^{-0.4n}\,.\end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{x+y}{2}\right| &\ge \left(M^2 - M\left(1+\varepsilon+\frac{\varepsilon}{2M}\right)c^{-0.4n}\right)^{1/2} \\ &\ge M - \frac{(1+\varepsilon+\frac{\varepsilon}{2M})}{2}c^{-0.4n} - Kc^{-0.8n} \\ &= M - \left[\left(\frac{1}{2}+\frac{\varepsilon}{2}\left(1+\frac{1}{2M}\right)\right) + Kc^{-0.4n}\right]c^{-0.4n} \ge M - c^{-0.4(n+1)}\end{aligned}$$

if $\frac{1}{2}+\frac{\varepsilon}{2}(1+\frac{1}{2M}) + Kc^{-0.4n} \le c^{-0.4}$, what we had to show.

Hence $S_n^3 f(x) = S_n^\varepsilon f(x)$, and (5.13) implies (4.6). To prove (4.3) we still have to remark that for $x \in \Omega_n^1$

$$\begin{aligned}&g_n(|x|-M) - \left(\frac{g_n}{2M} - A_n\right)c^{-0.8n} - \frac{1}{2}c^{n/2} \\ &\le g_n(|x|-M) - \left(\frac{g_n}{2M} - A_n\right)x^{(2)2} - \frac{1}{2}c^{n/2} \le g_n(x^{(1)}-M) + A_n x^{(2)2} - \frac{1}{6}c^{n/2}\,.\end{aligned}$$

e) The estimation of $S_n f(x)$ for $x \in \Omega_n^3$.

We get from (5.12) and Part a) of Lemma 2 that

$$0 \le S_n f(x) \le 2c^n \exp\left\{\frac{c^n}{T}M^2 + \frac{g_n}{2M}(|x|^2-M^2) - \frac{c^n}{3T}(|x|^2-M^2)\right\}\,.$$

For $x \in \Omega_n^3$ $\frac{c^n}{3T}(|x|^2-M^2) \ge \frac{c^{0.2n}}{3T}$, hence (4.1″) implies that

$$\begin{aligned}0 \le S_n f(x) &\le 2c^n \exp\left\{\frac{c^n}{T}M^2 + \frac{g_n}{2M}(|x|^2-M^2) - \frac{c^{0.2n}}{3T}\right\} \\ &\le L_n \exp\left\{\frac{g_n}{2M}(|x|^2-M^2)\right\}\,,\end{aligned}$$

i.e. relation (4.7) holds, as we have claimed. Proposition 1′ is proved.

6. The proof of Theorem 1 and Proposition 2. Existence of the thermodinamical limit. First we show with the help of Proposition 1, Lemma 1 and Theorem A that for all q, $c^{-0.2} < q < 1$, there exist some thresholds n_0 and $N_0(n,q)$ for $n \ge n_0$ such that if $n \ge n_0$ and $N \ge N_0(n,q)$ then

$$\frac{d\mu_{n,N}^{h_N}}{d\mu_n}(x_1,\dots,x_{2^n}) = f_{n,N}^{h_N}\left(2^{-n}\sum_{j=1}^{2^n} x_j\right) \tag{6.1}$$

with

$$f_{n,N}^{h_N}(x) = L_n \exp\left\{\bar g c^n(x^{(1)}-M) + \bar A c^n x^{(2)2} + \varepsilon_n(x)\right\} \qquad \text{for } x\in\Omega_n^1\,, \tag{6.2}$$

where

$$\sup_{x\in\Omega_n^1} |\varepsilon_n(x)| \le q^n\,, \tag{6.2'}$$

$$f_{n,N}^{h_N}(x) \le L_n \exp\left\{\bar g c^n(|x|-M) - \left(\frac{\bar g}{2M}-\bar A\right)c^{0.2n} + q^n\right\} \qquad \text{for } x\in\Omega_n^2 \tag{6.3}$$

$$f_{n,N}^{h_N}(x) \le L_n \exp\left\{\frac{\bar g c^n}{M}(|x|^2-M^2)\right\} \qquad \text{if } \ x > M + c^{-0.4n} \tag{6.4}$$

$$f_{n,N}^{h_N}(x) \le L_n \exp\left\{\frac{\bar g c^n}{2M}(|x|^2-M^2)\right\} \qquad \text{if } \ 0 < x < M - c^{-0.4n} \tag{6.4'}$$

with some appropriate norming constant L_n which satisfies the relation

$$C_1 < c^{-n/2} L_n < C_2 \qquad \text{with some } \ 0 < C_1 < C_2 < \infty\,. \tag{6.5}$$

Indeed, Proposition 1 and Lemma 1 imply (6.1)–(6.4′) with some norming constant $L_n = L_n(N,h_N)$. (In the domain Ω_n^3 we have divided the cases $|x|^2 - M^2 > 0$ and $|x|^2 - M^2 < 0$, since here we apply that $\bar g c^n < g_n < 2\bar g c^n$.) It remains to prove (6.5) and to show that L_n can be chosen independently of N and h_N. For this aim we observe that

$$1 = \mu_{n,N}^{h_N}((R^2)^{2^n}) = \int f_{n,N}^{h_N}(x)p_n(x)\,dx = \int_{\Omega_n^1} + \int_{\Omega_n^2} + \int_{\Omega_n^3}\,. \tag{6.6}$$

By applying the change of variables $r = M + c^{-n}t$ and by using the function $f_n(t)$ defined in (4.11) we get that

$$\begin{aligned}
\int_{\Omega_n^3} f_{n,N}^{h_N}(x)p_n(x)\,dx \le& \int_0^{M-c^{-0.4n}} L_n(N,h_N)\exp\left\{\frac{\bar g c^n}{2M}\left(r^2-M^2\right)\right\} r\bar p_n(r)\,dr \\
&+ \int_{M+c^{-0.4n}}^{\infty} L_n(N,h_N)\exp\left\{\frac{\bar g c^n}{M}\left(r^2-M^2\right)\right\} r\bar p_n(r)\,dr \\
=& L_n(N,h_N)\left[\int_{-c^nM}^{-c^{0.6n}} \exp\left\{\frac{\bar g}{2M}\left(2Mt+\frac{t^2}{c^n}\right)\right\}\left(M+c^{-n}t\right) f_n(t)\,dt \right.\\
&\left. + \int_{c^{0.6n}}^{\infty} \exp\left\{\frac{\bar g}{M}\left(2Mt+\frac{t^2}{c^n}\right)\right\}\left(M+c^{-n}t\right) f_n(t)\,dt\right].
\end{aligned}$$

Relations (4.11′) and (4.11″) imply that

$$\int_{\Omega_n^3} = L_n(N,h_N) O\left(\exp(-Kc^{0.6n})\right) \qquad \text{with some } K>0\,. \tag{6.7}$$

Similarly,

$$\int_{\Omega_n^2} \le L_n(N,h_N)\exp\left\{-\frac{1}{2}\left(\frac{\bar g}{2M}-\bar A\right)c^{0.2n}\right\}. \tag{6.7′}$$

(Observe that $\frac{\bar g}{2M}-\bar A>0$.)

Define the number T_n,

$$T_n=\int_{\Omega_n^1}\exp\left\{\bar g c^n(x^{(1)}-M)+\bar A c^n x^{(2)2}\right\}p_n(x)\,dx\,. \tag{6.8}$$

It follows from Theorem A that

$$C_1c^{-n/2}<T_n<C_2c^{-n/2} \qquad \text{with some } 0<C_1<C_2<\infty\,. \tag{6.9}$$

Indeed, since the expression in the exponent of (6.8) can be written in the form

$$\bar g c^n(x^{(1)}-M)+\bar A c^n x^{(2)2}=\bar g c^n(|x|-M)-\left(\frac{\bar g}{2M}-\bar A\right)c^n x^{(2)2}+O\left(c^{-0.2n}\right),$$

we get (6.9) by integrating (6.8) first by the variable $x^{(2)}$. Some calculation with the help of (6.2) and (6.2′) shows that

$$\left|\int_{\Omega_n^1} f_{n,N}^{h_N}(x)p_n(x)\,dx-L_n(N,h_N)T_n\right|\le L_n(N,h_N)T_nq^n\,. \tag{6.10}$$

Relations (6.6), (6.7), (6.7′) and (6.10) imply that

$$1=L_n(N,h_N)T_n\left(1+\varepsilon_n+O(\exp(-c^{0.1n}))\right), \qquad \text{and } \varepsilon_n\le q^n\,.$$

The last relation implies that relations (6.1)–(6.4′) remain valid if we choose $L_n=T_n^{-1}$, and this L_n satisfies (6.5) by (6.9).

We prove Theorem 1 with the help of (6.1)–(6.5). Fix some integer $k\ge 0$, and define for all $n\ge k$ and measurable sets $A\subset(R^2)^{2^k}$ the cylindrical set $A(n)=A\times(R^2)^{2^n-2^k}\subset(R^2)^{2^n}$. Put

$$\tilde\mu_n(A)=L_n\int_{\tilde A(n)}\exp\left\{\bar g c^n 2^{-n}\sum_{j=1}^{2^n}(x_j^{(1)}-M)+\bar A c^n 4^{-n}\left(\sum_{j=1}^{2^n}x_j^{(2)}\right)^2\right\} \mu_n(\,dx_1\ldots,dx_{2^n})$$

with $\tilde{A}(n) = A(n) \cap \{(x_1, \dots, x_{2^n}),\ 2^{-n}\sum_{j=1}^{2^n} x_j \in \Omega_n^1\}$. We claim that if $n > n_0$ and $N > N_0(n,q)$ then

$$\left|\tilde{\mu}_n(A(n)) - \mu_{n,N}^{h_N}(A(n))\right| \le Kq^n \tag{6.11}$$

with some $K > 0$ independent of the set A. Indeed,

$$\begin{aligned}\left|\tilde{\mu}_n(A(n)) - \mu_{n,N}^{h_N}(A(n))\right| &\le \int_{\Omega_n^2 \cup \Omega_n^3} f_{n,N}^{h_N}(x) p_n(x)\, dx \\ &+ \int_{\tilde{A}(n)} \mu_n(\,dx_1 \dots, dx_{2^n}) \Bigg| f_{n,N}^{h_N}\left(2^{-n}\sum_{j=1}^{2^n} x_j\right) \\ &- L_n \exp\left\{\bar{g}c^n 2^{-n}\sum_{j=1}^{2^n}(x_j^{(1)} - M) + \bar{A}c^n 4^{-n}\left(\sum_{j=1}^{2^n} x_j^{(2)}\right)^2\right\}\Bigg| = I_1 + I_2\,.\end{aligned}$$

It follows from (6.5), (6.7) and (6.7′) that

$$I_1 \le \exp\left(-c^{0.1n}\right)\,. \tag{6.12}$$

On the other hand we increase the term I_2 by enlarging the domain of integration to the set $\{(x_1, \dots, x_{2^n}),\ 2^{-n}\sum_{j=1}^{2^n} x_j \in \Omega_n^1\}$. Hence

$$I_2 \le \int_{\Omega_n^1} \left| f_{n,N}^{h_N}(x) - L_n \exp\left\{\bar{g}c^n(x^{(1)} - M) + \bar{A}c^n x^{(2)2}\right\}\right| p_n(x)\, dx\,.$$

The last inequality together with (6.2) and (6.2′) imply that

$$I_2 \le 2q^n \mu_{n,N}^{h_N}\left((R^2)^{2^n}\right) = 2q^n\,.$$

The last inequality together with (6.12) imply (6.11). Since for all $k \ge 0$ and measurable sets $A \in (R^2)^{2^k}$, $k \le n \le N$ we have $\mu_{n,N}^{h_N}(A(n)) = \mu_{k,N}^{h_N}(A)$, relation (6.11) implies that for all $\varepsilon > 0$ there is some $N_0(\varepsilon)$ such that for $N > N_0(\varepsilon)$ and $N' > N_0(\varepsilon)$ the relation $|\mu_{n,N}^{h_N}(A) - \mu_{k,N'}^{h_{N'}}(A)| < \varepsilon$ holds true. Let us emphasize that the threshold $N(\varepsilon)$ does not depend on the set A. Hence the last relation means that the limit $\bar{\mu}_k(A) = \lim_{N\to\infty} \mu_{k,N}^{h_N}(A)$ exists, and the convergence is uniform in A. This implies that $\mu_{k,N}^{h_N} \to \bar{\mu}_k$ in variational metric. To complete the proof of Theorem 1 we have to show that the measure $\bar{\mu}_k$ does not depend on the sequence h_N. But it is not difficult to see with the help of (6.11) that this statement holds, since $\bar{\mu}_k(A) = \lim_{n\to\infty} \tilde{\mu}(A(n))$, and the right hand side of the last expression does not depend on h_N.

Proof of Proposition 2. Let $n > n_0$ and $N > N_0(n,q)$. Relations (6.1)–(6.5) hold for such pairs n and N. By Theorem 1 the measures $\mu_{n,N}^{h_N}$ converge in variational metric to the projection $\bar{\mu}_n$ of the measure $\bar{\mu}$ to $(R^2)^{2^n}$ as $N \to \infty$. Since all measures are absolute continuous with respect to the measure μ_n the above convergence is equivalent to the convergence of the Radon–Nikodym derivatives $\frac{d\mu_{n,N}^{h_N}}{d\mu_n} = f_{n,N}^{h_N}$ to $\frac{d\bar{\mu}_n}{d\mu_n} = \bar{f}$ in L_1 norm in the space $((R^2)^{2^n}, \mu_n)$ as $N \to \infty$. Since all functions $f_{n,N}^{h_N}(x)$ satisfy (6.1)–(6.5) for $N > N_0(n,q)$, their limit $\bar{f}$ also has this property. Hence Proposition 2 holds true.

7. The proof of Theorem 2. Existence of the large-scale limit. First we need some results about the transformation $Q_n = Q_n(k)$ of probability measures on $(R^2)^{2^{n+k}}$ to probability measures on $(R^2)^{2^k}$ to be defined below. First we define a transformation $Q_n = Q_n(k)$, $Q_n\colon (R^2)^{2^{n+k}} \to (R^2)^{2^k}$ in the following way: For all $(x_1, \dots, x_{2^{n+k}})$, $x_j \in R^2$, $j = 1, \dots, 2^{n+k}$

$$Q_n(x_1, \dots, x_{2^{n+k}}) = (y_1, \dots, y_{2^k}), \qquad y_j = 2^{-n} \sum_{l=(j-1)2^n+1}^{j2^n} x_l, \qquad j = 1, \dots, 2^k.$$

This transformation induces the transformation Q_n of probability measures on $(R^2)^{2^{n+k}}$ to probability measures on $(R^2)^{2^k}$ in a natural way. Namely, if ν is a probability measure on $(R^2)^{2^{n+k}}$ and $(\eta(1), \dots, \eta(2^{n+k}))$ is a ν distributed vector then $Q_n\nu$ is the distribution of the random vector $Q_n(\eta(1), \dots, \eta(2^{n+k}))$. In Theorem 2 we have to study an appropriately rescaled version of the measure $Q_n\bar\mu_{n+k}$. It is not difficult to see that relation (2.10) implies that

$$\frac{dQ_n\bar\mu_{n+k}}{dQ_n\mu_{n+k}}(x_1, \dots, x_{2^k}) = \bar f_{n+k}\left(2^{-k} \sum_{j=1}^{2^k} x_j\right). \tag{7.1}$$

We formulate below Theorem C which follows from the relatively simple Theorem 1 in [4]. For the sake of completeness we present its proof in Appendix B.

Theorem C. *The above defined measure $Q_n\mu_{n+k} = Q_n\mu_{n+k}(T, t)$ has a density function $h_k(x_1, \dots, x_{2^k})$ of the form*

$$h_k(x_1, \dots, x_{2^k}) = L(T, t, n, k) \exp\left\{-\frac{1}{T}\mathcal{H}_k\left(c^{n/2}x_1, \dots, c^{n/2}x_{2^k}\right)\right\} \prod_{j=1}^{2^k} p_n(x_j)$$
$$x_j \in R^2, \quad j = 1, \dots, 2^k.$$

Here $\mathcal{H}_k$ is the Hamiltonian function defined in (1.2′) of Part I, $p_n(x)$ is the function appearing in Theorem A, and L is an appropriate norming constant.

Formula (7.1) and Theorem C enable us to express the density function of the random vector $\{\mathcal{R}_n\sigma^{(1)}(j), \mathcal{R}_n\sigma^{(2)}(j), 1 \le j \le 2^k\}$ with the help of the functions $p_n(x)$ and $f_n(x)$, where the sequence $\{\sigma(j), j \in \mathbf{Z}\}$ is $\bar\mu$ distributed, and $\mathcal{R}_n\sigma^{(1)}$, $\mathcal{R}_n\sigma^{(2)}$ are defined in (1.2) and (1.3) of Part II. This density equals to

(7.2)

$$h_{n,k}(x_1, \dots, x_{2^k})$$
$$= L_{n,k}\bar f_{n+k}\left(2^{-k} \sum_{j=1}^{2^k} \tilde x_j\right) \exp\left\{-\frac{1}{T}\mathcal{H}_k\left(c^{n/2}\tilde x_1, \dots, c^{n/2}\tilde x_{2^k}\right)\right\} \prod_{j=1}^{2^k} p_n(\tilde x_j)$$

with

$$\tilde x = \tilde x(x) = \left(M + c^{-n}x^{(1)}, c^{-n/2}x^{(2)}\right) \qquad \text{for } x = \left(x^{(1)}, x^{(2)}\right). \tag{7.2′}$$

Let us define the sets $W_n\subset R^2$ and $\bar W_n\subset R^2$ by the formulas

$$\begin{aligned}\bar W_n&=\left\{(x^{(1)},x^{(2)}),\ M-\frac{n\eta}{c^n}<|x|<M+\frac{\eta n^{1/\alpha}}{c^n},\ |x^{(2)}|<c^{-0.45n},\ x^{(1)}>0\right\}\\ W_n&=\left\{(x^{(1)},x^{(2)}),\ \tilde x(x)\in\bar W\right\},\end{aligned}$$

where η and α are the same constants as in Theorem A, and $\tilde x(x)$ is defined in (7.2′). We shall show that there is some $n_0>0$ and $0<q<1$ such that

$$P\Big(\big(\mathcal R_n\sigma^{(1)}(j),\ \mathcal R_n\sigma^{(2)}(j)\big)\notin W_n\Big)\le q^n\qquad \text{if } n\ge n_0 \tag{7.3}$$

for a $\bar\mu$ distributed random field $\sigma(j)$, $j\in\mathbf Z$, and give a good asymptotic formula for the expression $h_{n,k}(x_1,\dots,x_{2^k})$ defined in (7.2) if $x_j\in W_n$ for all $1\le j\le 2^k$. First we prove (7.3).

$$\begin{aligned}&P\Big(\big(\mathcal R_n\sigma^{(1)}(j),\ \mathcal R_n\sigma^{(2)}(j)\big)\notin W_n\Big)=P\bigg(2^{-n}\sum_{l=1}^{2^k}\sigma(k)\notin\bar W_n\bigg)\\ &\qquad=\int_{R^2-\bar W_n}\bar f_n(x)p_n(x)\,dx=\int_{\Omega_n^1-\bar W_n}+\int_{\Omega_n^2}+\int_{\Omega_n^3}=I_1+I_2+I_3\,.\end{aligned}$$

We get similarly to the estimates (6.7) and (6.7′) that

$$\begin{aligned}&I_3\le\exp\left(-Cc^{0.6n}\right),\\ &I_2\le const.\exp\left\{-\left(\frac{\bar g}{2M}-\bar A\right)c^{0.2n}\right\}\le\exp\left(-Cc^{0.2n}\right)\end{aligned}$$

The term I_1 has to be estimated a little more carefully.

Define

$$\tilde\Omega_n^1=\left\{(x^{(1)},x^{(2)}),\ \big||x|-M\big|<c^{-0.4n},\ |x^{(2)}|<c^{-0.45n},\ x^{(1)}>0\right\},$$

and write

$$I_1=I_{1,1}+I_{1,2}=\int_{\Omega_n^1-\tilde\Omega_n^1}+\int_{\tilde\Omega_n^1-\bar W_n}.$$

We get, similarly to the estimation of I_3 and I_2 that $I_{1,1}\le\exp(-Kc^{0.1n})$, and we can write by (2.11) and (2.11′) that

$$\begin{aligned}I_{1,2}&\le 2L_n\int_{\tilde\Omega_n^1-\bar W_n}\exp\left\{\bar gc^n(x^{(1)}-M)+\bar Ac^nx^{(2)2}\right\}p_n(x)\,dx\\ &\le 3L_n\int_{\tilde\Omega_n^1-\bar W_n}\exp\left\{\bar gc^n(|x|-M)-\left(\frac{\bar g_n}{2M}-A\right)c^nx^{(2)2}\right\}p_n(x)\,dx\,.\end{aligned}$$

Then integrating first by $x^{(2)}$ we get that

$$I_{1,2}\le KL_nc^{-n/2}\left[\int_{M-c^{-0.4n}}^{M-\eta nc^{-n}}+\int_{M+\eta n^{1/\alpha}c^{-n}}^{M+c^{-0.4n}}\right]\exp\left(\bar gc^n(r-M)\right)\bar p_n(r)\,dr\le q^n$$

with the help of relations (1.8) and (1.10) in Theorem A. Let us emphasize that it was the multiplying term q^n in (1.8) that enabled us to give an exponentially small bound for the second term in the last integral. The above estimates imply (7.3).

To estimate the expression (7.2) in the case $x_j \in W_n,\ j=1,2,\dots,2^k$, we make some preparatory remarks. Put

$$\ell_n(x) = c^n(|\tilde x(x)| - M) = c^n\left\{\left[(M+c^{-n}x^{(1)})^2 + c^{-n}x^{(2)2}\right]^{1/2} - M\right\}$$

We have

(7.4)
$$\ell_n(x) = x^{(1)} + \frac{x^{(2)2}}{2M} + O\left(c^{-n}\left(x^{(2)4} + |x^{(1)}|x^{(2)2}\right)\right)$$
$$= x^{(1)} + \frac{x^{(2)2}}{2M} + O\big(c^{-0.8n}\big) \qquad \text{if} \quad x \in W_n\,,$$

because, as it is not difficult to see, $c^{-n/2}|x^{(2)}| < c^{-0.45n}$, and $M - 2c^{-0.9n} < M + c^{-n}x^{(1)} < M + 2c^{-0.9n}$ if $x \in W_n$. We show with the help of Theorem A and (7.4) that

(7.5)
$$p_n(\tilde x) = \exp\left\{-\frac{a_0}{T}\left(Mx^{(1)} + \frac{x^{(2)2}}{2}\right)\right\} g\left(\frac{a_1}{T}\left(Mx^{(1)} + \frac{x^{(2)2}}{2}\right)\right)(1 + O(q^n))$$

with some $0<q<1$ for $x \in W_n$ if $\eta > 0$ is chosen sufficiently small in Theorem A. Indeed, by Theorem A

$$p_n(\tilde x) = \exp\left\{-\frac{a_0 M}{T}\ell_n(x)\right\} g\left(\frac{a_1 M}{T}\ell_n(x)\right)(1+O(q^n))\,,$$

$$\exp\left\{-\frac{a_0 M}{T}\ell_n(x)\right\} = \exp\left\{-\frac{a_0}{T}\left(Mx^{(1)} + \frac{x^{(2)2}}{2}\right) + O\left(c^{-0.8n}\right)\right\}\,,$$

and

(7.6)
$$\left|g\left(\frac{a_1}{T}M\ell_n(x)\right) - g\left(\frac{a_1}{T}(Mx^{(1)} + \frac{x^{(2)2}}{2})\right)\right| = O(c^{-0.8n})$$

by (7.4) and the boundedness of the function $\frac{d}{dx}g(x)$. (See Lemma 13 in Part I.) On the other hand, by Lemma 17 of Part I the relation $-2\eta n < x^{(1)} + \frac{x^{(2)2}}{2M} < 2\eta n^{1/\alpha}$ if $x \in W_n$, which holds because of the definition of W_n, and the inequality

$$\left|c^n\big(|\tilde x(x)| - M\big) - \left(x^{(1)} + \frac{x^{(2)2}}{2M}\right)\right| \le Kc^{-0.8n}$$

we have

$$g\left(\frac{a_1}{T}\left(Mx^{(1)} + \frac{x^{(2)2}}{2}\right)\right) > c^{-0.3n}$$

if $x \in W_n$, and η is chosen in Theorem A sufficiently small. Hence (7.6) can be rewritten as

$$g\left(\frac{a_1}{T}M\ell_n(x)\right) = g\left(\frac{a_1}{T}\left(Mx^{(1)} + \frac{x^{(2)2}}{2})\right)\left(1 + O(c^{-n/2})\right)\right) .$$

The above relations imply (7.5).

We also claim that

(7.7)

$$\mathcal{H}_k\left(c^{n/2}\tilde{x}_1, \dots, c^{n/2}\tilde{x}_{2^k}\right) = -\sum_{i=1}^{2^k-1}\sum_{j=i+1}^{2^k} U(i,j)\left[M(x_i^{(1)} + x_j^{(1)}) + x_i^{(2)}x_j^{(2)} + c^n M^2\right] + O(q^n)$$

and

$$\text{(7.8)}\quad \bar{f}_{n+k}\left(2^{-k}\sum_{j=1}^{2^k}\tilde{x}_j\right) = L_{n+k}\exp\left\{\frac{c^k}{2^k}\bar{g}\sum_{j=1}^{2^k}x_j^{(1)} + \frac{c^k}{4^k}\bar{A}\left(\sum_{j=1}^{2^k}x_j^{(2)}\right)^2 + O(q^n)\right\}$$

if $x_j \in W_n$, $j = 1, \dots, 2^k$.

Indeed,

$$\mathcal{H}_k\left(c^{n/2}\tilde{x}_1, \dots, c^{n/2}\tilde{x}_{2^k}\right) = -\sum_{i=1}^{2^k-1}\sum_{j=i+1}^{2^k} U(i,j)\left[c^n(M + c^{-n}x_i^{(1)})(M + c^{-n}x_j^{(1)}) + x_i^{(2)}x_j^{(2)}\right] ,$$

hence to prove (7.7) it is enough to remark that in the last expression the terms $c^{-n}x_i^{(1)}x_j^{(1)}$ are negligibly small, since $c^{-n}x_i^{(1)}x_j^{(1)} = O(c^{-0.8n})$ if $x_i \in W_n$ and $x_j \in W_n$.

To prove (7.8) we have to show that $2^{-k}\sum_{j=1}^{2^k}\tilde{x}_j \in \Omega^1_{n+k}$ if $\tilde{x}_j \in \bar{W}_n$ for all $j = 1, \dots, 2^k$ and then apply Proposition 2. We can write with the notation $\tilde{x}_j = (\tilde{x}_j^{(1)}, \tilde{x}_j^{(2)})$ that $|2^{-k}\sum_{j=1}^{2^k}\tilde{x}_j^{(2)}| < c^{-0.45n} \le c^{-0.4n}4^{-k}$, and $|2^{-k}\sum_{j=1}^{2^k}\tilde{x}_j^{(1)} - M| \le 2c^{-0.45n} \le 4^{-k}c^{-0.4n}$ if n is sufficiently large (k is fixed, $n \to \infty$) and $\tilde{x}_j \in W_n$ for $j = 1, \dots, 2^k$. These relations imply that $2^{-k}\sum_{j=1}^{2^k}\tilde{x}_j \in \Omega^1_{n+k}$. We get, by putting (7.5), (7.7) and (7.8) into (7.2) that

$$\begin{aligned} h_{n,k}(x_1, \dots, x_{2^k}) = \bar{L}_{n,k}\exp\Bigg\{&\frac{c^k}{2^k}\bar{g}\sum_{j=1}^{2^k}x_1^{(j)} + \frac{c^k}{4^k}\bar{A}\left(\sum_{j=1}^{2^k}x_j^{(2)}\right)^2 \\ &- \frac{1}{T}\sum_{i=1}^{2^k-1}\sum_{j=i+1}^{2^k}U(i,j)\left(Mx_i^{(1)} + Mx_j^{(1)} + x_i^{(2)}x_j^{(2)}\right) - \sum_{j=1}^{2^k}\frac{a_0}{T}\left(Mx_j^{(1)} + \frac{x_j^{(2)2}}{2}\right)\Bigg\} \\ &\prod_{j=1}^{2^k} g\left(\frac{a_1}{T}\left(Mx_j^{(1)} + \frac{x_j^{(2)2}}{2}\right)\right)(1 + O(q^n)) \end{aligned}$$

with some $0 < q < 1$ if $x_j \in W_n$, $j = 1, 2, \ldots, 2^k$.

Simple calculation shows that $-\sum_{i=1}^{2^k} U(i,j) = \frac{1-(c/2)^k}{1-c/2}$, hence the coefficient of $x_j^{(1)}$, $(c/2)^k \bar{g} + \frac{M}{T} \sum_{j=1}^{2^k} U(i,j) - \frac{a_0 M}{T}$ equals zero, and

$$h_{n,k}(x_1, \ldots, x_{2^k}) = h_k(x_1, \ldots, x_{2^k})\big(1 + O(q^n)\big) \tag{7.9}$$

if $x_j \in W_n$, $j = 1, 2, \ldots, 2^k$, where h_k is defined in (1.12) (with $p = 2$). It is not difficult to see that (7.3) also holds with a random vector with density function (1.12). Hence (7.3) and (7.9) imply Theorem 2.

8. Some open problems and conjectures. Dyson [12] has defined a more general class of models than that considered in this work. He defined, with the help of a real function $\varphi\colon \mathbf{Z} \to R^1$, models with the Hamiltonian function

$$\mathcal{H}(\sigma) = -\sum_{i \in \mathbf{Z}} \sum_{\substack{j \in \mathbf{Z} \\ j > i}} \varphi(d(i,j)) S_n^(f(x) i) \sigma(j), \qquad \sigma = \{\sigma(i),\ i \in \mathbf{Z}\}\,, \tag{8.1}$$

where $d(\cdot,\cdot)$ denotes the hierarchical distance on $\mathbf{Z}$ given in formula (1.1) of Part I. In this work we have considered models in the special case $\varphi(x) = |x|^{-a}$ with $a = 2 - \frac{\log c}{\log 2}$. One question we are going to discuss here is that which are the functions φ for which Dyson's model with the Hamiltonian (8.1) has a phase transition at low temperatures. In the boundary case some more delicate phenomena appear which we also want to discuss. The behaviour of vector and scalar-valued models is different. First we discuss the vector-valued case.

The quantities $M_n = M_n(T)$ considered in Part I can be defined in a natural way in the general case. The arguments of Part I suggest that the relation

$$M_{n+1} = M_n - \frac{1}{4^n M_n \varphi(2^n)} \tag{8.2}$$

holds true. The existence or non-existence of phase transition depends on whether $M = \lim_{n\to\infty} M_n$ equals to zero or not if T is small, i.e. if M_0 is large. Hence formula (8.2) suggests that a phase transition at low temperatures occurs if and only if $\sum \frac{1}{4^n \varphi(2^n)}$ is convergent. Dyson has formulated the same conjecture in [13] and proved its convergent part in the special case when $\sigma(i) \in R^3$. He has also solved the problem for scalar-valued models. He proved that there is a phase transition at low temperatures if $\varphi(n) \ge C \frac{\log\log n}{n^2}$ with some $C > 0$, and there is none if $\varphi(n) \frac{\log\log n}{n^2} \to 0$. Moreover, in the boundary case $\varphi(n) = C \frac{\log\log n}{n^2}$ the following Thouless effect occurs: There is some critical parameter $T_{cr.}$ such that $M(T) = \lim_{n\to\infty} M_n(T) > 0$ for $T \le T_{cr.}$ and $M(T) = 0$ for $T > T_{cr.}$. The quantity $M(T)$ has a physical content, it is called the spontaneous magnetization. The interesting feature of the above result is that it states that the function $M(T)$ has a discontinuity at $T = T_{cr.}$. This particular behaviour of the spontaneous magnetization appears only in the boundary case $\varphi(n) = C \frac{\log\log n}{n^2}$. On the other hand, the Thouless effect occurs in some other models too, like in the one-dimensional Ising model with $\frac{1}{|x-y|^2}$ interaction, in one-dimensional percolation models if the probability of the event that the points i and j are connected has the order $C(T)|i-j|^{-2}$, e.t.c.. In recent time several interesting papers appeared on this subject, (see e.g. [1], [2], [17]). On the other hand, there are some other interesting phenomena connected with the Thouless effect, like the irregular behaviour of the correlation function, whose investigation requires essentially new ideas.

The appearance of phase transitions and the Thouless effect in scalar-valued models are connected with the behaviour of the sequence M_n. The quantity M_{n+1} can be expressed asymptotically in a simple way with the help of M_n and the function φ in scalar-valued models too. But this formula is essentially different from his vector-valued counterpart, namely

$$M_{n+1} \sim M_n \left[1 - \exp\left\{ -\frac{1}{T} M_n^2 4^n \varphi(2^n) \right\} \right] . \tag{8.3}$$

In the particular case $\varphi(n) = \frac{\log \log n}{n^2}$ we have

$$_{n+1} \sim M_n \left[1 - \exp\left\{ -\frac{1}{T} M_n^2 \log n \right\} \right] . \tag{8.3$'$}$$

Formula (8.3′) may help us to understand the cause of the Thouless effect, at least at a heuristic level. If $M_n(T) < \sqrt{T}$ for some n then relation (8.3) implies that $M(T) = \lim_{n\to\infty} M_n(T) = 0$, hence either $M(T) \geq \sqrt{T}$ or $M(T) = 0$. Since $M(T) \neq 0$ for small T, this relation implies the discontinuity of the function $M(T)$. In vector-valued models relation (8.2) does not suggest such a behaviour. We expect however that some delicate effects appear in this case too, and we are going to study them in the future.

Let us remark that the study of existence or non-existence of phase transitions at low temperatures seems to be an essentially simpler problem than the study of the Thouless effect and related questions. In the first problem it is enough to consider sufficiently low temperatures, and in the case of vector-valued models with Hamiltonian function of the form (8.1) for instance the method of the present paper works without any essential changes. In the second problem however, one has to study the behaviour of the model near the critical temperature, and this requires more work and new ideas.

Another problem we are going to discuss here is the description of the large-scale limit of vector-valued equilibrium states with translation invariant Hamiltonian function. We have discussed its scalar-valued counterpart in Section 8 of our paper [6], and formulated our conjectures about it.

Let us consider vector-valued models on the d-dimensional integer lattice with Hamiltonian function

$$\mathcal{H}(\sigma) = -\sum_{\substack{|i-j|=1 \\ i,j\in\mathbf{Z}^d}} \sigma(i)\sigma(j) - \sum_{i\in\mathbf{Z}^d} p(\sigma(i))$$

with

$$p(x) = -\frac{t}{4}|x|^4 - \frac{|x|^2}{2}, \qquad t > 0 ,$$

and the Lebesgue measure on R^p with some $p \geq 2$ as the free measure of the model. The expression $\sigma(i)\sigma(j)$ in the above formulas denotes scalar product.

Let $\{X(k) = (X_k^{(1)}, \dots, X_k^{(p)}), k \in \mathbf{Z}^d\}$ be a random field with the distribution of a (pure) equlibrium state with the above Hamiltonian function at a certain temperature T. If $d \geq 3$ then there exists a spontaneous magnetization at sufficiently low temperatures. This is proved with the help of the infrared bounds (see e.g. [14]). In case of phase transition we consider that pure state for which the direction of the spontaneous magnetization is $e_1 = (1, 0, \dots, 0)$, i.e. $E\,X_k^{(1)} = M > 0$, and $E\,X_k^{(s)} = 0$ for $s = 2, \dots, p$.

Define the "renormalized" random fields $\{Y_k(N) = (Y_k(N)^1, \dots, Y_k(N)^p)\}$, $N = 1, 2, \dots$ by the formulas

$$Y_k(N)^1 = A(N)^{-1} \sum_{j \in D_k(N)} (X_j^{(1)} - EX_j^{(1)}) \tag{8.4}$$

$$Y_k(N)^s = B(N)^{-1} \sum_{j \in D_k(N)} X_j^{(s)}, \qquad s = 2, \dots, p, \tag{8.4'}$$

with

$$D_k(N) = \Big\{ j = (j^{(1)}, \dots, j^{(d)}) \in \mathbf{Z}^d, \quad k^{(r)}N + 1 \leq j^{(r)} \leq (k^{(r)} + 1)N, \\ r = 1 \dots, d \Big\}.$$

We are interested in the question that for which choice of $A(N)$ and $B(N)$ the fields $Y_k(N)$ have a non-trivial limit, i.e. a limit which is not concentrated on a single configuration. We also want to describe the distribution of the limit field.

Dyson's hierarchical model with parameter c, $1 < c < 2$ can be considered as an approximation of translation invariant models with nearest neighbour interaction on the d-dimensional lattice $\mathbf{Z}^d$ with $d = \frac{2}{1 - \log_2 c}$. (See paper [21] for a discussion of this approximation.) It must be admitted that the above approximation is made only at a heuristic level, but it helps us to get a better understanding about the behaviour of the large-scale limit. On the basis of the present work and [5] we can formulate the following conjectures about the large-scale limit of translation invariant models at low temperatures.

The behaviour of the large-scale limit is different in the cases $d > 4$, $d = 4$ and $d < 4$, and they correspond to the cases $c > \sqrt{2}$, $c = \sqrt{2}$ and $c < \sqrt{2}$ in Dyson's hierarchical model. In accordance with [5] we expect that for $d > 4$ the large-scale limit exists at low temperatures with $A(N) = N^{d/2}$ and $B(N) = N^{(d+2)/2}$. The limit field $\{Y_k = (Y_k^{(1)}, \dots, Y_k^{(p)}),\ k \in \mathbf{Z}^d\}$ is such that the fields $\{Y_k^{(j)},\ k \in \mathbf{Z}^d\}$, $j = 1, 2, \dots, p$ are independent, $\{Y_k^{(1)},\ k \in \mathbf{Z}^d\}$ consists of independent identically distributed Gaussian random variables with zero mean, $\{Y_k^{(s)},\ k \in \mathbf{Z}^d\}$, $s = 2, \dots, p$, are massless free Gaussian fields, i.e. they have the same distribution as the field

$$Y_k = C \int \frac{\exp(ikx)}{|x|} \prod_{j=1}^{d} \left\{ \frac{\exp(ix^{(j)}) - 1}{ix^{(j)}} \right\} W(\,dx), \qquad k \in \mathbf{Z}^d,$$

with some $C>0$, where $W(\,dx)$ is a complex-valued white noise field on R^d with the conjugation property $W(x)=\overline{W(-x)}$. (For the definition of $W(x)$ see e.g. [16].) The situation is similar in the case $d=4$, i.e. the investigation of Dyson's model suggests a similar behaviour with the only difference that a logarithmic factor appears in the normalizing term $A(n)$. More precisely, for $d=4$ the fields $Y_k^{(N)}$ defined in (8.4) and (8.4$'$) with $A(N)=N^2\sqrt{\log N}$ and $B(N)=N^{(d+2)/2}=N^3$ have a Gaussian limit as $N\to\infty$ which consists of independent components $s=1,\dots,p$, similarly to the case $d>4$. The limit of the fields $Y_k^{(s)}$, $s=2,\dots,p$, is a massless free field.

The result of the present work motivates the following conjecture for $d=3$.

Conjecture. *For $d=3$ the large-scale limit exists at low temperatures with the normalizations $A(N)=N^2$ and $B(N)=N^{5/2}$. The large-scale limit has the same distribution as the random field $\{Y_k=(Y_k^{(1)},\dots,Y_k^{(p)}),\ k\in\mathbf{Z}^3\}$ defined by the formulas*

$$(8.5)\qquad Y_k^{(s)}=C\int\frac{\exp(ikx)}{|x|}\prod_{j=1}^{3}\frac{\exp(ix^{(j)})-1}{ix^{(j)}}W_s(\,dx),\quad k\in\mathbf{Z}^3\,,\ s=2,\dots,p$$

and

(8.6)

$$Y_1^{(s)}=-\frac{C^2}{2M}\sum_{s=2}^{p}\iint\frac{\exp(ik(x+y))}{|x||y|}\prod_{j=1}^{3}\left\{\frac{\exp[i(x^{(j)}+y^{(j)})]-1}{i(x^{(j)}+y^{(j)})}\right\}W_s(\,dx)W_s(\,dy),\qquad k\in\mathbf{Z}^3\,,$$

where $C>0$ is an appropriate positive constant, and $W_s(\,dx)$, $s=1,\dots,p$ are independent complex valued white noise fields on R^3 with the conjugation property $W(x)=\overline{W(-x)}$. (For the definition of two-fold stochastic integrals with respect to a Gaussian field see e.g. [16]. Such an integral appears in the definition of $Y_1^{(s)}$.)

The fields $Y_s^{(k)}$, $s=2,\dots,p$ defined in (8.5) are massless free fields, the field defined in (8.6) belongs to the class of self-similar fields constructed in Dobrushin's paper [11]. It is a quadratic functional of a Gaussian field, just as the corresponding field in Dyson's model for $1<c<\sqrt{2}$.

The large-scale limit of the equilibrium state in Dyson's model described in Theorem 2 of Part II has the following independence property: The random variables $Y_k^{(1)}+\frac{1}{2M}\sum_{s=2}^{p}Y_k^{(s)2}$ are independent for different k. This independence property does not hold for their translation invariant counterpart defined in (8.5) and (8.6). It cannot be preserved, because translation invariant models have less symmetry. Nevertheless, the following non-rigorous argument shows some analogy between the behaviour of the fields defined in (8.5) and (8.6) and the above mentioned independence property. In this non-rigorous argument we consider the limit field appearing in the Conjecture as the discretization of a generalized field.

Let $\delta(t)$ denote the Dirac-delta function in the point t, and consider the generalized field which at $\delta(t)$ takes the value $Y(\delta(t)) = \big(Y^{(1)}(\delta(t)), \dots, Y^{(s)}(\delta(t))\big)$,

$$Y^{(s)}(\delta(t)) = C \int \frac{\exp(itx)}{|x|} W^{(s)}(\,dx)\,, \qquad s = 2, \dots, p\,,$$

$$Y^{(1)}(\delta(t)) = -\frac{C^2}{2M} \sum_{s=2}^{p} \iint \frac{\exp(it(x+y))}{|x||y|} W^{(s)}(\,dx)\, W^{(s)}(\,dy)\,.$$

Actually this definition is not correct, since the above stochactic integrals are meaningless because of the divergence of the integrals $\int_{R^3} \frac{dx}{|x|^2}$ and $\int_{R^3}\int_{R^3} \frac{dx\,dy}{|x|^2|y|^2}$. But the integral

$$Y^{(1)}(\varphi) = \int Y^{(1)}(\delta(t))\varphi(t)\,dt = -\frac{C^2}{2M} \sum_{s=2}^{p} \iint \frac{\tilde\varphi(x+y)}{|x||y|} W^{(s)}(\,dx)\, W^{(s)}(\,dy)\,.$$

and

$$Y^{(s)}(\varphi) = \int Y^{(s)}(\delta(t))\varphi(t)\,dt = C \int \frac{\tilde\varphi(x)}{|x|} W^{(s)}(\,dx), \quad 2 \le s \le p\,,$$

are meaningful for nice funtions φ. In particular, they are meaningful for the indicator functions of the unit cubes $\prod_{i=1}^{3} [k_i, k_i + 1)$ which we denote by φ_k if $k = (k_1, k_2, k_3)$. The random field appearing in the Conjecture can be considered as the discretization of the above defined generalized field if we identify Y_k with $Y(\varphi(k))$.

A formal application of Itô's formula (see e.g. [16]) would supply the relation $Y^{(1)}(\delta(t)) - \frac{1}{2M}\sum_{s=2}^{p} Y^{(s)}(\delta(t))^2 = const.$, and this can be considered as the analogue of the independence property of the large-scale limit of Dyson's model for the above defined generalized field $Y(\delta(\cdot))$. On the other hand by our Conjecture the discretization of this generalized field is the large-scale limit of the three-dimensional translation invariant vector-valued model at low temperatures.

Let us finally discuss the cases $d = 1$ and $d = 2$. The case $d = 1$ is rather simple. In this case there is no phase transition, and if $\{X_k,\ k \in \mathbf{Z}^1\}$ is a random field with the distribution function of an equilibrium state at any temperature then it satisfies the central limit theorem with the usual normalization. The case $d = 2$ is more delicate. In this case the dimension p, $(\sigma(i) \in R^p)$, also plays an important role. In this case there is no symmetry breaking, but for $d = 2$, $p = 2$ a more delicate phenomenon, the so-called Kosterlitz–Thouless effect occurs. (See [15]). This means that at low temperatures the correlation function decreases rather slowly, only power-like. Hence a non-trivial large-scale limit should appear in this case. For $d = 2$, $p \ge 3$ it is expected that the Kosterlitz–Thouless effect does not occur, but for the time being it is proved only at a physical level (see[18]). Hence, it is expected that for $d = 2$, $p > 2$ the large-scale limit has the same (trivial) behaviour as for $d = 1$.

When typing the final version of this work the authors learned about some recent results about the Kosterlitz–Thouless effect (see [22], [23]). The arguments of these works, also supported by computer simulation, suggest that the situation in two-dimensional translation invariant models is essentially different from what we had expected. In particular, the difference between the cases $p = 2$ and $p > 2$ in the models we have discussed at the end of this Section does nevertheless not occur.

Appendix

Appendix A. The proof of the basic recursive relations (2.1) and (2.1′) in Part I. Formula (2.1′) immediately follows from (1.4) in Part I with $n=0$. To prove (2.1) let us first observe that the recursive relation

(A1)
$$\mathcal{H}_{n+1}\left(x_1,\dots,x_{2^{n+1}}\right)=\mathcal{H}_n\left(x_1,\dots,x_{2^n}\right)+\mathcal{H}_n\left(x_{2^n+1},\dots,x_{2^{n+1}}\right) - c^n\left(2^{-n}\sum_{i=1}^{2^n}x_i\right)\left(2^{-n}\sum_{j=2^n+1}^{2^{n+1}}x_j\right)$$

holds for $n\ge 0$, where

$$\mathcal{H}_n(x_1,\dots,x_{2^n})=-\sum_{i=1}^{2^n}\sum_{j=i+1}^{2^n}U(i,j)x_ix_j\ ,$$

and $U(i,j)$ is defined by (1.1) and (1.2′) in Part I. By relation (1.4) in Part I

(A2)
$$p_{n+1}(x,T)=\frac{1}{Z_{n+1}(T,t)}\int\exp\biggl\{-\frac{1}{T}\mathcal{H}_n(x_1,\dots,x_{2^{n+1}})\biggr\} \delta\biggl(2^{-(n+1)}\sum_{i=1}^{2^{n+1}}x_i-x\biggr)\prod_{i=1}^{2^{n+1}}p(x_i)\,dx_i\ ,$$

where $Z_{n+1}(T,t)$ is an appropriate norming constant, and $\delta\bigl(2^{-(n+1)}\sum_{i=1}^{2^{n+1}}x_i-x\bigr)$ means that integration in (A2) is taken on the hyperplane $2^{-(n+1)}\sum_{i=1}^{2^{n+1}}x_i=x$ with respect to the Lebesgue measure. Let us fix some number u, and calculate the integral on the right-hand side of (A2) by integrating first on the hyperplane defined by the relations $2^{-n}\sum_{i=1}^{2^n}x_i=x+u$ and $2^{-n}\sum_{i=2^n+1}^{2^{n+1}}x_i=x-u$ and then by integrating by u. We get with the help of relations (A1) and (A2) that

$$\begin{aligned}p_n(x,T)&=\frac{1}{Z_{n+1}(T,t)}\int\exp\left\{\frac{c^n}{T}(x+u)(x-u)\right\}\\&\quad\left[\int\exp\left\{-\frac{1}{T}\mathcal{H}_n(x_1,\dots,x_{2^n})\right\}\delta\left(2^{-n}\sum_{i=1}^{2^n}x_i-(x+u)\right)\prod_{i=1}^{2^n}p(x_i)\,dx_i\right]\\&\quad\left[\int\exp\left\{-\frac{1}{T}\mathcal{H}_n(x_{2^n+1},\dots,x_{2^{n+1}})\right\}\delta\biggl(2^{-n}\sum_{i=2^n+1}^{2^{n+1}}x_i-(x-u)\biggr)\right.\\&\qquad\left.\prod_{i=2^n+1}^{2^{n+1}}p(x_i)\,dx_i\right]du\\&=C_n\int\exp\left\{\frac{c^n}{T}(x^2-u^2)\right\}p_n(x+u)p_n(x-u)\,du\ ,\end{aligned}$$

as we have claimed.

Appendix B. The proof of Theorem C. Since the measure μ_{n+k} has the density function

$$\frac{1}{Z_{n+k}}\exp\left\{-\frac{1}{T}\mathcal{H}_{n+k}(z_1,\dots,z_{2^{n+k}})\right\}\prod_{i=1}^{2^{n+k}}p(z_i)\,,$$

the density function of the measure $Q_n\mu_{n+k}$, the function $h_k(x_1,\dots,x_{2^k})$ equals to

(B1)

$$h_k(x_1,\dots,x_{2^k})=\frac{1}{Z_{n+k}}\int\exp\left\{-\frac{1}{T}\mathcal{H}_{n+k}(z_1,\dots,z_{2^{n+k}})\right\}\prod_{l=1}^{2^k}\delta\left(2^{-n}\sum_{j=(l-1)2^n+1}^{l2^n}z_j-x_l\right)\prod_{i=1}^{2^{n+k}}p(z_i)\,dz_i\,,$$

where $\prod_{l=1}^{2^k}\delta(2^{-n}\sum_{j=(l-1)2^n+1}^{l2^n}z_j-x_l)$ in the integral (B1) means that integration is taken on the hyperplane defined by the relations $2^{-n}\sum_{j=(l-1)2^n+1}^{l2^n}z_j=x_l$, $l=1,\dots,2^k$, with respect to the Lebesgue measure. The special structure of the hierarchical distance implies that

$$\mathcal{H}_{n+k}(z_1,\dots,z_{2^{n+k}})=\sum_{l=1}^{2^k}\mathcal{H}_n(z_{(l-1)2^n+1},\dots,z_{l2^n})-\sum_{i=1}^{2^k}\sum_{j=i+1}^{2^k}c^nU(i,j)\bar z_i\bar z_j\,,$$

with $\bar z_i=2^{-n}\sum_{p=(i-1)2^n+1}^{i2^n}z_p,\;\; i=1\dots,2^k$.

Hence relation (B1) can be rewritten as

$$\begin{aligned}h_k(x_1,\dots,x_{2^k})&=\frac{1}{Z_{n+k}}\exp\left\{\frac{1}{T}\sum_{i=1}^{2^k}\sum_{j=i+1}^{2^k}c^nU(i,j)x_ix_j\right\}\\&\quad\cdot\prod_{l=1}^{2^k}\int\exp\left\{-\frac{1}{T}\mathcal{H}_n(z_{(l-1)2^n+1},\dots,z_{l2^n})\right\}\\&\qquad\delta\left(2^{-n}\sum_{j=(l-1)2^n+1}^{l2^n}z_j-x_l\right)\prod_{j=(l-1)2^n+1}^{l2^n}p(z_j)\,dz_j\\&=C_{k,n}\exp\left\{-\frac{1}{T}\mathcal{H}_k(c^{n/2}x_1,\dots,c^{n/2}x_{2^k})\right\}\prod_{l=1}^{2^k}p_n(x_l,T)\,,\end{aligned}$$

as we have claimed.

Appendix C. The calculation of the Radon–Nikodym derivatives. The proof of formulas (2.1)—(2.3′) in Part II. For $n=N$ relations (2.1) and (2.2) of Part II immediately follow from formula (1.4) in Part II. Hence it is enough to prove our relations by induction from $n+1$ to n. Clearly,

$$P_{n+1}(x_1,\dots,x_{2^{n+1}}) = C_n P_n(x_1,\dots,x_{2^n}) P_n(x_{2^n+1},\dots,x_{2^{n+1}}) \exp\left\{\frac{c^n}{T}\left(2^{-n}\sum_{j=1}^{2^n} x_j\right)\left(2^{-n}\sum_{j=2^n+1}^{2^{n+1}} x_j\right)\right\}$$

with some norming constant C_n. Given some measurable set $A\subset (R^p)^{2^n}$ define the cylindrical set $\tilde A\subset (R^p)^{2^{n+1}}$ as $\tilde A = A\times (R^p)^{2^n}$. By our inductive hypothesis for $n+1$

$$\begin{aligned}\mu_{n,N}^{h_N}(A) &= \int_{\tilde A} f_{n+1,N}^{h_N}\left(2^{-(n+1)}\sum_{j=1}^{2^{n+1}} x_j\right) P_{n+1}(x_1,\dots,x_{2^{n+1}})\,dx_1\dots dx_{2^{n+1}} \\ &= C_n\int_{\tilde A} f_{n+1,N}^{h_N}\left(2^{-n}\left(\sum_{j=1}^{2^n}\frac{x_j}{2}+\sum_{j=2^n+1}^{2^{n+1}}\frac{x_j}{2}\right)\right) \\ &\qquad P_n(x_1,\dots,x_{2^n})P_n(x_{2^n+1},\dots,x_{2^{n+1}}) \\ &\qquad \exp\left\{\frac{c^n}{T}\left(2^{-n}\sum_{j=1}^{2^n} x_j\right)\left(2^{-n}\sum_{j=2^n+1}^{2^{n+1}} x_j\right)\right\} dx_1\dots dx_{2^{n+1}}\ .\end{aligned}$$

Let us calculate the last integral by first integrating on the hyperplanes where $x_1,\dots,x_{2^n}$ and $y=\frac{1}{2^n}\sum_{j=2^n+1}^{2^{n+1}} x_j$ are fixed. Since $P_n(x_{2^n+1},\dots,x_{2^{n+1}})$ is the only term in the integrand which is not constant on such a hyperplane, and its integral equals $p_n(y)$ on it, we get that

$$\begin{aligned}\mu_{n,N}^{h_N}(A) = C_n\int_{A\times R^p} & f_{n+1,N}^{h_N}\left(2^{-n}\Big(\sum_{j=1}^{2^n}\frac{x_j}{2}+\frac{y}{2}\Big)P_n(x_1,\dots,x_{2^n})\right)p_n(y) \\ & \exp\Big\{\frac{c^n}{T}2^{-n}(\sum_{j=1}^{2^n}x_j)y\Big\}\,dx_1\dots\,dx_{2^n}dy\ .\end{aligned}$$

Hence we get, by integrating first by the variable y that

$$\mu_{n,N}^{h_N}(A) = C_n'\int_A S_n f_{n+1,N}^{h_N}\left(2^{-n}\sum_{j=1}^{2^n}x_j\right)P_n(x_1,\dots,x_{2^n})\,dx_1\dots\,dx_{2^n}\ .$$

Since this relation holds for all measurable sets $A\subset (R^p)^{2^n}$, it implies our inductive hypothesis for n.

Appendix D. On limit Gibbs states. Here we briefly describe the definition of limit Gibbs states (also called equilibrium states in the literature,) and discuss some important questions related to this definition. Limit Gibbs states are defined with the help of a Hamiltonian (often called energy) function, a free measure and a physical parameter, the temperature T. The Hamiltonian function is a formal series. Let us have a subset $\mathbf{Z}\subset\mathbf{Z}^d$ of the d-dimensional integer lattice and a closed set $K\subset R^p$ in the p-dimensional Euclidedan space. We consider a Hamiltonian function $\mathcal{H}(\sigma)$ of the form

$$\mathcal{H}(\sigma)=-\sum_{i,j\in\mathbf{Z}}U(i,j)\sigma(i)\sigma(j),\quad \sigma=\{\sigma(j),\ \sigma(j)\in K,\ j\in\mathbf{Z}\},$$

where $U(\cdot,\cdot)$ is a given function, $U:\ \mathbf{Z}\times\mathbf{Z}\to R^1$, and $\sigma(i)\sigma(j)$ denotes scalar product. (There is a more general definition of Hamiltonian functions, but this special class is sufficiently large for our purposes.) Given some finite set $V\subset\mathbf{Z}$, we define the energy function $\mathcal{H}_V(\sigma)$ as

$$\text{(D1)}\qquad \mathcal{H}_V(\sigma)=-\sum_{i,j\in V}U(i,j)\sigma(i)\sigma(j)\,,\qquad \sigma=\{\sigma(i),\ i\in V\},$$

and the conditional energy in V with respect to a configuration $\bar\sigma$ in $\mathbf{Z}-V$ as

$$\text{(D2)}\qquad \mathcal{H}_V(\sigma|\bar\sigma)=\mathcal{H}_V(\sigma)-\sum_{i\in V}\sum_{j\in\mathbf{Z}-V}U(i,j)\sigma(i)\bar\sigma(j),$$
$$\sigma=\{\sigma(i),\ i\in V\},\ \bar\sigma=\{\bar\sigma(i),\ i\in\mathbf{Z}-V\},$$

provided that the last sum is convergent. Given some $h\in R^p$, we also introduce the energy of a configuration σ in the volume V with respect to the external field h as

$$\text{(D3)}\qquad \mathcal{H}_V^h(\sigma)=\mathcal{H}_V(\sigma)-h\sum_{i\in V}\sigma(i).$$

Given some finite set $V\subset\mathbf{Z}$, a configuration $\bar\sigma=\{\bar\sigma(j),j\in\mathbf{Z}-V\}$ outside V, a Hamiltonian function $\mathcal{H}(\sigma)$) and a free measure $P(\,dx)$ on K, we define the Gibbs measure in volume V with respect to the external field $\bar\sigma$ at temperature T as the probability measure $\mu_{V,T}(\,\cdot\,|\bar\sigma)$ on K^V given by the formula

$$\text{(D4)}\qquad \mu_{V,T}(\sigma\in A|\bar\sigma)=\frac{1}{Z(V,T,\bar\sigma)}\int_A\exp\left\{-\frac{1}{T}\mathcal{H}_V(\sigma|\bar\sigma)\right\}\prod_{j\in V}P(\,d\sigma(j)),$$

where $A\in K^V$ is an arbitrary measurable set, $\sigma=\{\sigma(j),j\in V\}$ and $Z(V,T,\bar\sigma)$ is an appropriate norming constant, provided that the above expression is meaningful. Now we formulate the following

Definition of Gibbs states. *A probability measure $\bar\mu$ is a Gibbs state with Hamiltonian function $\mathcal{H}$ and free measure P at temperature T if a $\bar\mu$ distributed random field $\sigma(j)$, $j\in\mathbf{Z}$ satisfies the following relation: For any finite set V and measurable set $A\subset K^V$ the conditional probability of the event $\sigma\in A$, $\sigma=\{\sigma(j),j\in V\}$ with* *respect to the condition* $\{\sigma(j)=\bar\sigma,\ j\in\mathbf{Z}-V\}$ *with a configuration* $\bar\sigma=\{\bar\sigma(j),\ j\in\mathbf{Z}-V\}$ *equals to*

$$\bar\mu(\sigma\in A\,|\,\{\sigma(j),\ j\in\mathbf{Z}-V\}=\bar\sigma)=\mu_{V,T}(A\,|\,\bar\sigma)$$

with $\bar\mu$ *probability one, where* $\mu_{V,T}$ *is defined in (D4).*

The question arises whether Gibbs states on $K^{\mathbf{Z}}$ exist, and whether they are unique. A natural way to construct Gibbs states is to carry out the following procedure. Choose an increasing family of sets $V_n\subset\mathbf{Z}$, $\cup V_n=\mathbf{Z}$, fix a configuration $\bar\sigma=\bar\sigma^{(n)}=\{\bar\sigma(j),\ j\in\mathbf{Z}-V_n\}$ for each V_n, and consider the measures $\mu_{V_n,T}(\,\cdot\,|\bar\sigma)$ defined in (D4). Prove that under some mild restrictions there is a convergent subsequence of this sequence, and the limit of this subsequence is a Gibbs state. The problem about the uniqueness of Gibbs states is closely related to the question whether, in dependence of the choice of the external configuration $\bar\sigma^{(n)}$, different limits can appear in the above construction. A slightly different, and often useful approach is to choose a sequence $h_n\in R^n$, $h_n\to 0$, and try to construct Gibbs states as the limit of a sequence of measures of the form $\mu^{h_n}_{V_n,T}$, where we define the probability measure $\mu^h_{V,T}$ as

(D5)

$$\mu^h_{V,T}(A)=\frac{1}{Z(V,T,h)}\int_A \exp\left\{-\frac{1}{T}\left(\sum_{i\in V}\sum_{j\in V}U(i,j)x_ix_j-h\sum_{i\in V}x_i\right)\right\}\prod_{i\in V}P(dx_i)\,.$$

If K is a compact subset of R^p then standard results in probability theory imply the compactness of the measures $\mu_{V_n,T}(\,\cdot\,|\bar\sigma)$ or of $\mu^{h_n}_{V_n,T}$ in weak topology, i.e. the existence of a convergent subsequence in this topology. (See e.g. [3].) Nevertheless, there are many interesting models, where the set K is non-compact (e.g. $K=R^p$,) and in such cases a hard analysis is needed to prove the existence of such a convergent subsequence. (See e.g. [10] or [20] as an example.) In order to prove that the limit of the sequence of measures $\mu_{V_n,T}(\,\cdot\,|\bar\sigma)$ (or $\mu^{h_n}_{V_n,T}$) is really a Gibbs state it is worth while to rewrite the definition of Gibbs states in an equivalent integral form. Let $f=f(x_{j_1},\dots,x_{j_k})$ and $g=g(x_{l_1},\dots,x_{l_k})$, $x_{j_i}\in R^p$, $x_{l_i}\in R^p$, be two bounded and continuous functions with finitely many arguments, $V=\{j_1,\dots,j_k\}$, $W=\{l_1,\dots,l_k\}$, $V\subset\mathbf{Z}$, $W\subset\mathbf{Z}$ such that $V\cap W=\emptyset$.

The measure $\bar\mu$ on $K^{\mathbf{Z}}$ is a Gibbs state if and only if

$$\text{(D6)}\qquad \int_{K^{\mathbf{Z}}} fg\,d\bar\mu=\int_{K^{\mathbf{Z}-V}}\mu(f)g\,d\bar\mu$$

for all functions f and g with the above properties, where

(D7)

$$\mu(f)=\mu(f)(\bar\sigma)=\int_{K^V} f(\sigma(j_1),\dots,\sigma(j_k))\ \mu_{V,T}(\,d\sigma|\bar\sigma)$$
$$\sigma=\{\sigma(j),j\in V\},\quad \bar\sigma=\{\sigma(j),j\in\mathbf{Z}-V\},$$

and $\mu_{V,T}$ is defined in (D4).

Let us consider an arbitrary sequence of sets $V_n \subset \mathbf{Z}$, $\cup V_n = \mathbf{Z}$, and numbers $h_n \in R^p$, $h_n \to 0$. Some calculation shows that for sufficiently large n (if $V \subset V_n$, $W \subset V_n$)

(D8) $$\int_{K^{V_n}} fg\, d\mu^{h_n}_{V_n,T} = \int_{K^{V_n - V}} \mu^{h_n}(f) g\, d\mu^{h_n}_{V_n,T}$$

with

$$\mu^{h_n}(f) = \mu^{h_n}_{V,V_n,T}(f; \sigma(j), j \in V_n - V)$$
$$= \frac{\int f(\sigma(j_1), \dots, \sigma(j_k)) \exp\{-\frac{1}{T}(\mathcal{H}_{V,V_n}(\sigma) - h_n \sum\limits_{i\in V} \sigma(i))\} \prod\limits_{i\in V} P(\,d\sigma(i))}{\int \exp\{-\frac{1}{T}(\mathcal{H}_{V,V_n}(\sigma) - h_n \sum\limits_{i\in V} \sigma(i))\} \prod\limits_{i\in V} P(\,d\sigma(i))},$$

where

$$\mathcal{H}_{V,V_n}(\sigma) = \sum_{i\in V} \sum_{j\in V_n} \sigma(i)\sigma(j)\ .$$

If the sequence $\mu^{h_n}_{V_n,T}$ tends weakly to the measure $\bar{\mu}$ then the left-hand side of (D8) converges to that of (D6). Hence to prove that the limit measure $\bar{\mu}$ is a Gibbs state it suffices to establish the convergence of the right-hand side of (D8) to that of (D6). If the Hamiltonian function has a finite range interaction, i.e. there is some number $r > 0$ such that $U(i,j) = 0$ if $|i-j| \ge r$ then it is not difficult to see that $\mu^{h_n}(f)(\bar{\sigma}) \to \mu(f)(\bar{\sigma})$, and the required convergence can be proved with the help of this relation. In case of infinite range interaction one must be more careful, especially if the state space K is non-compact. Dyson's model which we are investigating is such a model. In Theorem 1 of Part II we have proved the weak convergence of the measures $\mu^{h_n}_N$ to $\bar{\mu}$. (Actually, we have proved a stronger form of convergence.) In Appendix E we prove Theorem B, i.e. we show that the limit measure is a Gibbs state. In the proof we approximate Dyson's model with a model with finite range interaction, and this enables us to carry out the required limiting procedure. In Appendix E we restrict ourselves to Dyson's model, although the argument also works in more general cases.

Appendix E. The proof of Theorem B. We apply the argument of Appendix D. The proof of Theorem B can be completed by showing that also in the case of Dyson's model the right-hand side of (D8) tends to that of (D6). We formulate this statement in more detail.

It suffices to consider the case when $V = \{1, 2, \dots, 2^k\}$, $W = \{2^k + 1, 2, \dots, 2^m\}$ with some $0 \le k < m$, i.e. $f = f(x_1, \dots, x_{2^k})$, $g = g(x_{2^k+1}, \dots, x_{2^m})$ and $V_N = \{1, 2, \dots, 2^N\}$. (We apply the notation of Appendix D.) Introduce the functions

(E1)
$$p^h_{k,N}(x_1, \dots, x_{2^k} | x_{2^k+1}, \dots, x_{2^N})$$
$$= \frac{\exp\{-\frac{1}{T}(\sum_{i=1}^{2^k} \sum_{j=i+1}^{2^N} U(i,j) x_i x_j - h \sum_{i=1}^{2^k} x_i)\}}{\int \exp\{-\frac{1}{T}(\sum_{i=1}^{2^k} \sum_{j=i+1}^{2^N} U(i,j) x_i x_j - h \sum_{i=1}^{2^k} x_i)\} \prod_{i=1}^{2^k} p(x_i)\, dx_i},$$
$$x_j \in R^p,\ j = 1, \dots, 2^N$$

and

$$\text{(E1}'\text{)}\qquad p_k(x_1,\dots,x_{2^k}|x_{2^k+1},\dots) = \frac{\exp\bigl\{-\frac{1}{T}\sum_{i=1}^{2^k}\sum_{j=i+1}^{\infty}U(i,j)x_ix_j\bigr\}}{\int\exp\bigl\{-\frac{1}{T}\sum_{i=1}^{2^k}\sum_{j=i+1}^{\infty}U(i,j)x_ix_j\bigr\}\prod_{i=1}^{2^k}p(x_i)\,dx_i},\qquad x_j\in R^p,\ j=1,\dots,$$

where the function $U(\cdot,\cdot)$ is defined in (1.2) and $p(x)$ in (1.3) of Part I. Put

(E2)

$$\mu^h_{k,N}(f)(x_{2^k+1},\dots,x_{2^N}) = \int f(x_1,\dots,x_{2^k})\, p^h_{k,N}(x_1,\dots,x_{2^k}|x_{2^k+1},\dots,x_{2^N})\prod_{i=1}^{2^k}p(x_i)\,dx_i$$

and

$$\text{(E2}'\text{)}\quad \mu_k(f)(x_{2^k+1},\dots) = \int f(x_1,\dots,x_{2^k})p_k(x_1,\dots,x_{2^k}|x_{2^k+1},\dots)\prod_{i=1}^{2^k}p(x_i)\,dx_i.$$

The convergence of the right hand side of (D8) to that of (D6) is equivalent to the relation

$$\text{(E3)}\qquad \lim_{N\to\infty}\int g\mu^{h_N}_{k,n}(f)\,d\mu^{h_N}_N = \int g\mu_k(f)\,d\bar\mu$$

in our case, where $\mu^{h_N}_N$ and $\bar\mu$ are the same probability measures on $(R^p)^{2^N}$ and $(R^p)^{\mathbf{Z}}$ as in Theorem 1 of Part I.

To prove this relation let us introduce the sets $A(K,k,n,N)$ and $A(K,k,n)$,where $K\in R^1,\ \ K>0,\ \ k,n,N\in\mathbf{Z}$ and $k<n<N$, defined by the formulas

$$A(K,k,n,N) = \bigl\{(x_{2^k+1},\dots,x_{2^N}),\ x_j\in R^p,\ j=2^k+1,\dots 2^N,\ |x_j|<K \text{ if } 2^k<j\le 2^n, \text{ and } |x_j|<2^{l\alpha} \text{ if } 2^l<j\le 2^{l+1},\ l=n,\dots,N-1\bigr\}$$

and

$$A(K,k,n) = \bigl\{(x_{2^k+1},\dots),\ x_j\in R^p,\ j=2^k+1,\dots,\ |x_j|<K \text{ if } 2^k<j\le 2^n, \text{ and } |x_j|<2^{l\alpha} \text{ if } 2^l<j\le 2^{l+1},\ l=n,n+1,\dots\bigr\},$$

where $\alpha=\frac34-\frac12\frac{\log c}{\log 2}$.

We claim that for all $\varepsilon>0$ some $n=n(\varepsilon)$ and $K=K(\varepsilon,n)$ can be chosen in such a way that

$$\text{(E4)}\qquad \mu^{h_N}_N\bigl((x_{2^k+1},\dots,x_{2^N})\notin A(K,k,n,N)\bigr)<\varepsilon$$

and

$$\text{(E4}'\text{)} \qquad \bar\mu\big((x_{2^k+1},\dots,)\notin A(K,k,n,)\big)<\varepsilon\,.$$

To prove relations (E4) and (E4′) let us first observe that there is a universal constant C, $C>0$, such that if $S_n^(f(x)j)$ is the j-th coordinate of a $\mu_N^{h_N}$ or $\bar\mu$ distributed random variable then the inequality $E\sigma^2(j)<C$ holds for all $j\in\mathbf{Z}$ and measures $\mu_N^{h_N}$ and $\bar\mu$. This can be seen by observing that the argument of Section 6 in Part II actually implies that all moments of the j-th coordinate $\sigma(j)$ of a $\mu_N^{h_N}$ distributed random vector σ converge to the corresponding moment of the j-th coordinate of a $\bar\mu$ distributed random vector as $N\to\infty$. Then we get,by exploiting that $1-2\alpha>0$ that

$$\mu_N^{h_N}\left\{(x_{2^k+1},\dots,x_{2^N})\notin A(K,k,n,N)\right\}\le C\left(\frac{2^n-2^k}{K^2}+\sum_{j=n}^{N}2^{j(1-2\alpha)}\right)<\varepsilon$$

if first n and then K is chosen sufficiently large. The proof of (E4′) is the same. (We remark that relations (E4) and (E4′) hold with arbitrary $\alpha>0$ in the definition of the sets $A(\cdot,\cdot,\cdot)$. To prove it we have to apply the stronger statement $E\sigma^{2k}<C_k$ for all $k\ge1$. This observation is needed if we want to prove Theorem B in the case $\sqrt{2}<c<2$ too.)

We claim that for all $\varepsilon>0$ there is some $N_0=N_0(\varepsilon,K,n)$ such that for $N>N_0$

(E5)

$$\left|p_{k,N}^{h_n}(x_1,\dots,x_{2^k}|x_{2^k+1}\dots,x_{2^N})-p_{k,N_0}^{h_n}(x_1,\dots,x_{2^k}|x_{2^k+1}\dots,x_{2^{N_0}})\right|$$
$$<\varepsilon\left(\sum_{i=1}^{2^k}|x_i|\right)\exp\left\{(2Kn+1)\sum_{i=1}^{2^k}|x_i|\right\}$$

and

(E5′)

$$\left|p_k(x_1,\dots,x_{2^k}|x_{2^k+1}\dots)-p_{k,N_0}^{0}(x_1,\dots,x_{2^k}|x_{2^k+1}\dots,x_{2^{N_0}})\right|$$
$$<\varepsilon\left(\sum_{i=1}^{2^k}|x_i|\right)\exp\left\{(2Kn+1)\sum_{i=1}^{2^k}|x_i|\right\}$$

if $(x_{2^k+1},\dots,x_{2^N})\in A(K,k,n,N)$ and $(x_{2^k+1},\dots)\in A(K,k,n)$. (The constants K and n in formulas (E5) and (E5′) are the same as in the definition of the sets $A(K,k,n,N)$.) First we show that (E5) and (E5′) together with (E4) and (E4′) imply (E3), hence also Theorem B. Indeed, since $p_n(x)$ decreases at infinity faster than $\exp(-x^2/2)$ hence (E5) and (E5′) imply that

$$\left|\mu_{k,N}^{h_N}(f)(x_{2^k+1},\dots,x_{2^N})-\mu_{k,N_0}^{0}(f)(x_{2^k+1},\dots,x_{2^{N_0}})\right|<const.\varepsilon$$

and

$$\left|\mu_k(f)(x_{2^k+1},\dots) - \mu^0_{k,N_0}(f)(x_{2^k+1},\dots,x_{2^{N_0}})\right| < \text{const.}\varepsilon\ .$$

if $(x_{2^k+1},\dots) \in A(K,k,n)$. This implies that $(x_{2^k+1},\dots,x_{2^N}) \in A(K,k,n,N)$. Since this relation holds on a set of $1-\varepsilon$ $\mu_N^{h_N}$ resp. $\bar\mu$ probability by (E4) and (E4′), the functions f, g, $\mu^{h_N}_{k,N}(f)$ and $\mu(f)$ are bounded, hence an error less than *const.*ε is committed if $\mu^{h_N}_{k,N}$ and $\mu_k(f)$ is replaced by μ^0_{k,N_0} in formula (E3). After this replacement relation (E3) holds, because the projections of the measures $\mu^{h_N}_N$ to $(R^p)^{2^N}$ converge to the projection of $\bar\mu$ to the same subspace. Since $\varepsilon>0$ can be chosen arbitrary small, relation (E3) holds in its original form.

We prove only (E5) the proof of (E5′) being the same. Let us first observe that for any $\eta>0$ there is some $N_0 = N_0(K,n,\eta)$ such that for $N \ge N_0$ and $(x_{2^k+1},\dots,x_{2^N}) \in A(K,k,n,N)$

(E6)

$$\begin{aligned}
&\left|\exp\left\{-\frac{1}{T}\Big(\sum_{i=1}^{2^k}\sum_{j=i+1}^{2^N} U(i,j)x_ix_j - h_N\sum_{i=1}^{2^k} x_i\Big)\right\} - \exp\left\{-\frac{1}{T}\sum_{i=1}^{2^k}\sum_{j=i+1}^{2^{N_0}} U(i,j)x_ix_j\right\}\right| \\
&= \exp\left\{-\frac{1}{T}\sum_{i=1}^{2^k}\sum_{j=i+1}^{2^N} U(i,j)x_ix_j\right\} \\
&\qquad \left|\exp\left\{-\frac{1}{T}\left(\sum_{i=1}^{2^k}\sum_{j=2^{N_0}+1}^{2^N} U(i,j)x_ix_j - h_N\sum_{i=1}^{2^k} x_i\right)\right\} - 1\right| \\
&\le \eta\frac{\sum_{i=1}^{2^k}|x_i|}{T}\exp\left\{\frac{2Kn+\eta}{T}\sum_{i=1}^{2^k}|x_i|\right\}.
\end{aligned}$$

In the last relation we have applied the inequality $|e^x-1| \le |x|e^{|x|}$ together with the relations $h_N < \eta/2$, $|\sum_{j=2^{N_0}+1}^{2^N} U(i,j)x_j| \le |\sum_{j=N_0}^{\infty} 2^{j(\alpha-1)}\left(\frac{c}{4}\right)^j| < \eta/2$ and $|\sum_{j=i+1}^{2^N} U(i,j)x_j| \le 2Kn$ if $N > N_0$, $j > 2^k$, $(x_{2^k+1},\dots,x_{2^N}) \in A(K,k,n,N)$ and N_0 is sufficently large. Integrating inequality (E6) with respect to the measure $\prod_{i=1}^{2^k} p(x_i)\,dx_i$ we get that

(E7)

$$\begin{aligned}
&\left|\int \exp\left\{-\frac{1}{T}\left(\sum_{i=1}^{2^k}\sum_{j=i+1}^{2^N} U(i,j)x_ix_j - h_N\sum_{i=1}^{2^k} x_i\right)\right\}\prod_{i=1}^{2^k} p(x_i)\,dx_i\right. \\
&\qquad \left. - \int \exp\left\{-\frac{1}{T}\sum_{i=1}^{2^k}\sum_{j=i+1}^{2^{N_0}} U(i,j)x_ix_j\right\}\prod_{i=1}^{2^k} p(x_i)\,dx_i\right| < const.\eta\ ,
\end{aligned}$$

where the *const.* may depend on K and n. In formulas (E6) and (E7) we have shown that both the numerators and the denominators of the functions $p^{h_N}_{k,N}$ and

p^0_{k,N_0} defined in (E1) are close to each other. The number η can be chosen arbitrary small in these estimates by fixing first n then $K=K(n)$ and finally $N_0=N_0(K,n)$ in an appropriate way. Moreover, given some appropriately chosen n and K the number $\eta>0$ can be taken arbitrary small if $N_0=N_0(K,n,\eta)$ is sufficiently large. Hence we prove (E5) by showing that

$$\int \exp\{-\frac{1}{T}\sum_{i=1}^{2^k}\sum_{j=i+1}^{2^{N_0}} U(i,j)x_ix_j\}\prod_{i=1}^{2^k} p(x_i)\,dx_i > D \tag{E8}$$

with some $D>0$ on the set $A(K,k,n,N_0)$, i.e. the integral in (E8) is separated from zero. Here the constant D may depend on K and n but not on N_0. Relation (E8) holds, since if $|x_j|<1$, $j=1,2,\dots,2^k$ and $(x_{2^k+1},\dots,x_{2^N})\in A(K,k,n,N_0)$ then the integrand in (E8) is separated from zero.

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